Covariant quantization of general relativity in four dimensional space time

http://www.donquijote.co.uk/blog/don-quixote-de-la-mancha

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14/Dec./2017 @Nikhef

YK, Journal of Mathematical Physics 58 (9) (2017) 092502
YK, arXiv:1703.05574

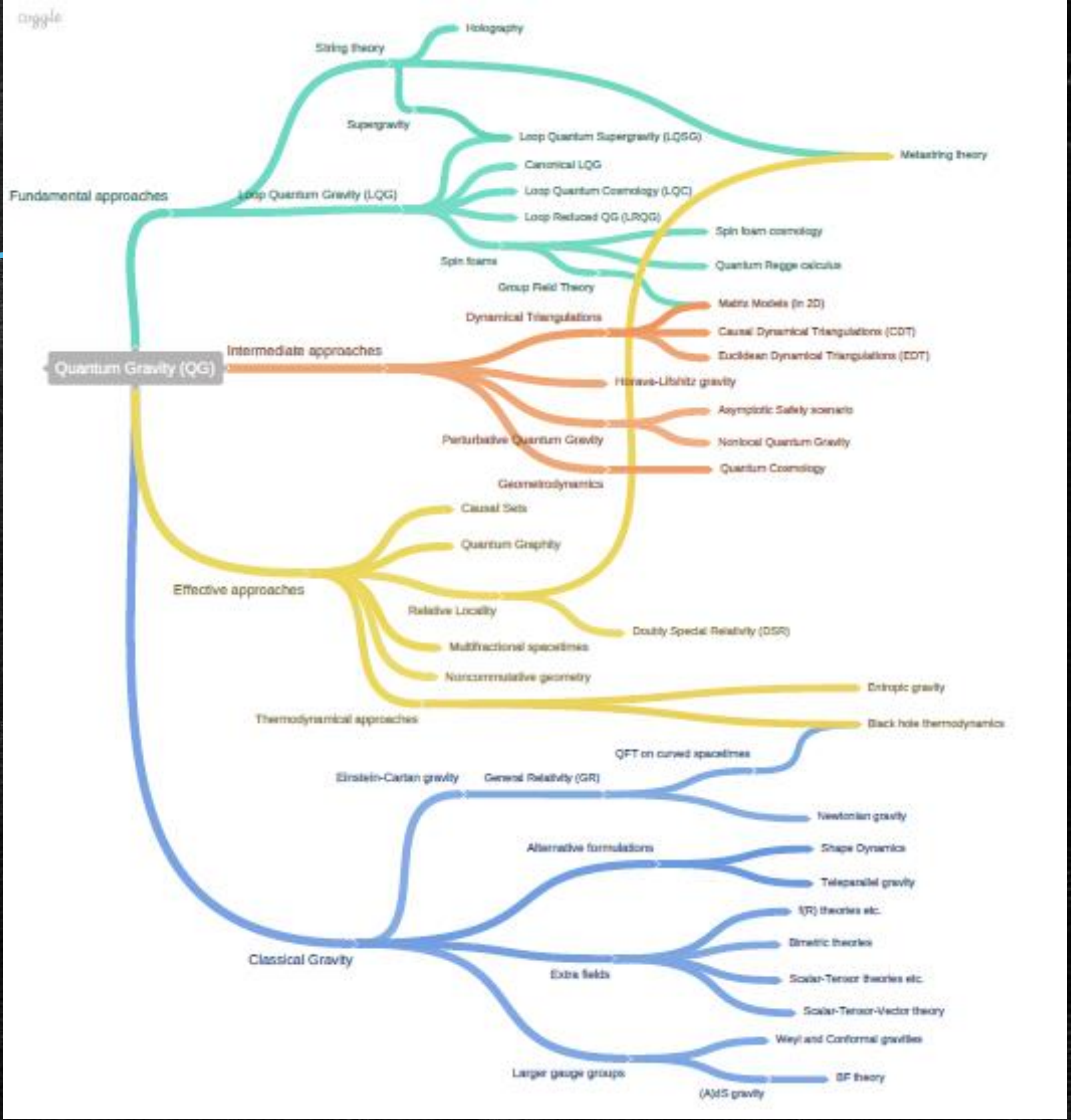
Outline

- Introduction
  - List of quantum GR
  - Our strategy
- Mathematical set-up
- Translation symmetry in GR
  - What can we learn from a three-dimensional case?
- Canonical quantization
  - Nakanishi-Kugo-Ojima formalism
  - Physical vacuum, S-matrix, Unitarity
- Summary

Introduction

Introduction

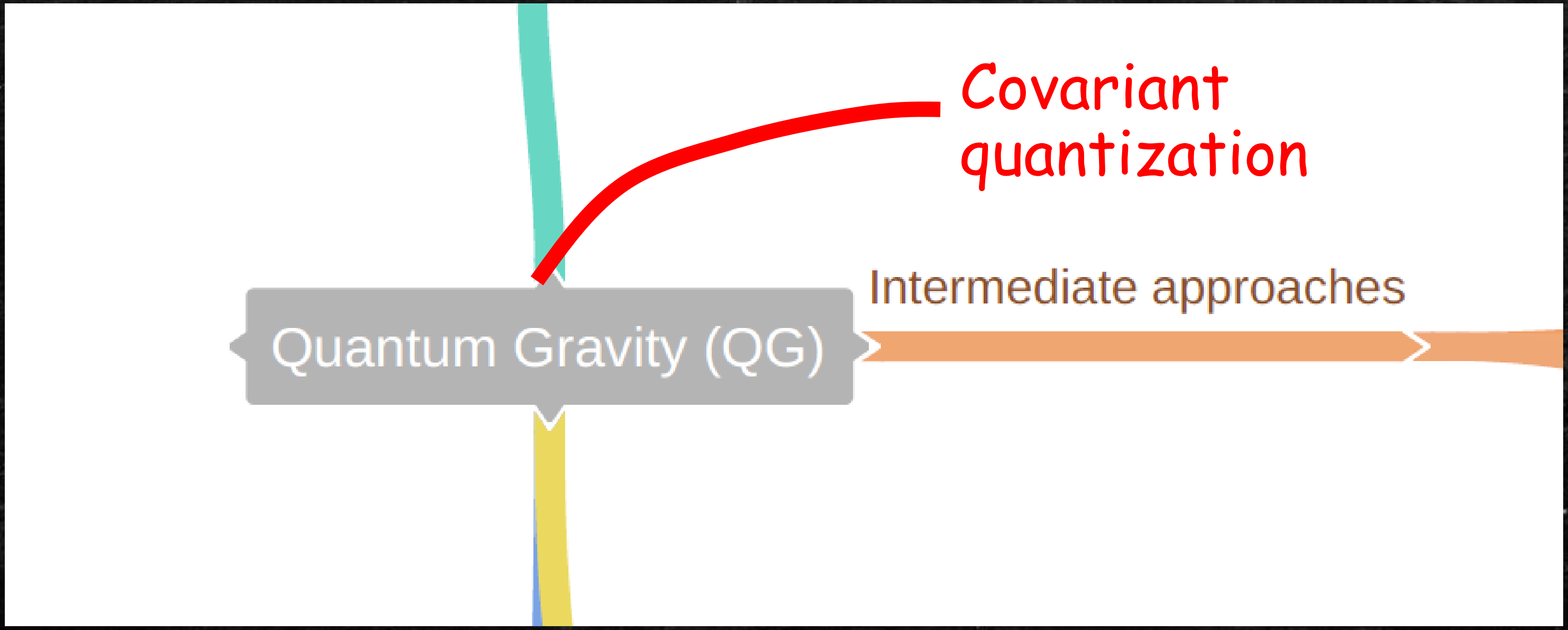
Outline of the taxonomy of approaches to quantum gravity



J. Mielczarek, T. Trzesniewski
arXive:1708.07445

Introduction

Conservative as mush as possible
Walk the high road



Introduction

- Our strategy
  - Rely only on experimentally established theories:
    - Quantum field theory & General relativity
  - Do not assume any experimentally unconfirmed things:
    - Supersymmetry → $GL(1,3)/SO(1,3) + (SU_c(3)\otimes SU_w(2)\otimes U_y(1))$
    - Extra dimension → space time dimension = 4
    - String / D-brane / AdS/CFT / Multiverse…

Introduction

- Our strategy
  - Nothing other than QGR → No ToE
  - Experimentally testable
    - The experiment has already been done.

Introduction

- Our strategy
  - Particle physicist's view

Riemannian Metric tensor: $g_{\mu\nu}^{(R)}$

Solution of Einstein eq.: $g_{\mu\nu}^{(E)}$

→ General Relativity: $g_{\mu\nu}^{(R)} \equiv g_{\mu\nu}^{(E)}$

Introduction

- Our strategy
  - Particle physicist's view

$$\left[\begin{array}{ll} \text{Riemannian Metric tensor:} & g_{\mu\nu}^{(R)} \\ \text{Solution of Einstein eq.:} & g_{\mu\nu}^{(E)} \end{array}\right.$$

→ General Relativity: $g_{\mu\nu}^{(R)} \equiv g_{\mu\nu}^{(E)}$

Quantization ↓

$$g_{\mu\nu}^{(R)} =< g_{\mu\nu}^{(Q)} >$$

Mathematical set-up

Mathematical set-up

$$\begin{cases} \text{Global Manifold} & \mathscr{M} & GL(1,3) \\ \text{Local Lorentz Manifold} & \mathcal{M} & SO(1,3) \end{cases}$$

Mathematical set-up

Global Manifold $\mathscr{M}$ $GL(1,3)$

Local Lorentz Manifold $\mathcal{M}$ $SO(1,3)$

Model of the universe

↑

general principle
of
relativity

Mathematical set-up

Global Manifold $\mathscr{M}$ $GL(1,3)$

Local Lorentz Manifold $\mathcal{M}$ $SO(1,3)$

Einstein's Equivalent principle!

Tangent (flat) space

Mathematical set-up

Global Manifold $\mathscr{M}$ $GL(1,3)$

Local Lorentz Manifold $\mathcal{M}$ $SO(1,3)$

Einstein's Equivalent principle!

Line element :

$$ds^2 = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu$$

vierbein :

$$\mathcal{E} : V \to \tilde{V} : V^\mu \mapsto \tilde{V}^a(x) = \mathcal{E}^a_\mu(x)V^\mu$$

$$\mathscr{M} \longleftrightarrow \mathcal{M}$$

Metric tensor:

$$g_{\mu_1\mu_2}(x)\ \mathcal{E}^{\mu_1}_a(x)\ \mathcal{E}^{\mu_2}_b(x) = \eta_{ab}$$

$$\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$\mathscr{M}$ $\mu, \nu, \rho, \cdots$

$\mathcal{M}$ $a, b, c, \cdots$

Mathematical set-up

Vierbein formalism

$GL(1,3)$ Covariant derivative : What is parallel ?

$$\nabla_\mu T^a{}_b = \partial_\mu T^a{}_b + \omega_\mu{}^a{}_c T^c{}_b - \omega_\mu{}^c{}_b T^a{}_c$$

Spin connection :

$$\omega_\mu{}^a{}_b = \mathcal{E}^a_\nu \Gamma^\nu{}_{\mu\rho}\, \mathcal{E}^\rho_b - \left(\partial_\mu \mathcal{E}^a_\rho\right) \mathcal{E}^\rho_b$$

Levi-Civita connection

Mathematical set-up

Vierbein formalism

Differential forms (Fraktur letters)

Vierbein form : $\mathfrak{e}^a \;=\; \mathcal{E}^a_\mu dx^\mu$

Spin form : $\mathfrak{w}^{ab} = dx^\mu \omega_\mu{}^a{}_c \, \eta^{cb}$

Torsion form : $\mathfrak{T}^a = d\mathfrak{e}^a + \mathfrak{w}^a{}_b \wedge \mathfrak{e}^b$

Curvature form : $\mathfrak{R}^{ab} = d\mathfrak{w}^{ab} + \mathfrak{w}^a{}_c \wedge \mathfrak{w}^{cb}$

$$\mathfrak{a} \wedge \mathfrak{b} = (-1)^{p+q}\, \mathfrak{b} \wedge \mathfrak{a}$$

$$\mathfrak{a} \in \bigwedge^p, \;\; \mathfrak{b} \in \bigwedge^q$$

Mathematical set-up

Vierbein formalism

Differential forms (Fraktur letters)

volume form :

$$\mathfrak{v} = \frac{1}{4!}\epsilon_{a_1a_2a_3a_4}\mathfrak{e}^{a_1} \wedge \mathfrak{e}^{a_2} \wedge \mathfrak{e}^{a_3} \wedge \mathfrak{e}^{a_4}$$

surface form :

$$\mathfrak{S}_{ab} = \frac{1}{2}\epsilon_{abc_1c_2}\mathfrak{e}^{c_1} \wedge \mathfrak{e}^{c_2}$$

GL(1,4) invariant

$$\int \mathfrak{v}$$

Mathematical set-up

Vierbein formalism

SO(1,3) covariant total derivative :

$$d_{\mathfrak{w}}\mathfrak{F}^{abc\cdots} = d\mathfrak{F}^{abc\cdots} + \mathfrak{w}^{a}{}_{z} \wedge \mathfrak{F}^{zbc\cdots} + \mathfrak{w}^{b}{}_{z} \wedge \mathfrak{F}^{azc\cdots} + \cdots$$

Torsion form:

$$d_{\mathfrak{w}}\mathfrak{e}^{a} = \mathfrak{T}^{a}$$

$$d_{\mathfrak{w}}\mathfrak{T}^{a} = \mathfrak{R}^{ab_1} \wedge \mathfrak{e}^{b_2}\ \eta_{b_1 b_2}$$

Bianchi identity:

$$d_{\mathfrak{w}}\mathfrak{R}^{ab} = 0$$

Mathematical set-up

Vierbein formalism

$SO(1,3)$ constant tensors :

(Flat) Metric tensor : $\eta_{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

Levi-Civita tensor : ϵ_{abcd} $(\epsilon_{0123} = +1)$

Completely anti-symmetric tensor

Mathematical set-up

Vierbein formalism

Lagrangian

$$\mathfrak{L}_G = \frac{1}{4}\epsilon_{....}\mathfrak{R}^{\cdot\cdot}(\mathfrak{w}) \wedge \mathfrak{e}^{\cdot} \wedge \mathfrak{e}^{\cdot} - \Lambda\ \mathfrak{v}$$

Action integral

$$\mathscr{I}_G = \frac{1}{k_{\mathrm{E}}}\int_\Sigma \mathfrak{L}_G, \qquad \boxed{\kappa_E = 4\pi G}$$

$$= \frac{1}{2k_{\mathrm{E}}}\int_\Sigma \left(\mathfrak{R}^{\cdot\cdot} - \frac{\Lambda}{3!}\mathfrak{e}^{\cdot} \wedge \mathfrak{e}^{\cdot}\right) \wedge \mathfrak{S}_{..}$$

Euler-Lagrange eq. of motion

$$\mathfrak{T}^a = d\mathfrak{e}^a + \mathfrak{w}^a{}_{.} \wedge \mathfrak{e}^{\cdot} = 0$$

Torsion-less equation

$$\epsilon_{a\cdots}\left(\frac{1}{2}\mathfrak{R}^{\cdot\cdot} \wedge \mathfrak{e}^{\cdot} - \frac{\Lambda}{3!}\mathfrak{e}^{\cdot} \wedge \mathfrak{e}^{\cdot} \wedge \mathfrak{e}^{\cdot}\right) = 0$$

Einstein equation

Mathematical set-up

Vierbein formalism

Differential forms (Fraktur letters)

Connection form : $\mathfrak{A}$

Curvature form : $\mathfrak{F} = d\mathfrak{A} + \mathfrak{A} \wedge \mathfrak{A}$

Chern classes: $$\sum_{j=0} c_j(\mathfrak{F})t^j = det\left(I + i\frac{\mathfrak{F}}{2\pi}t\right)$$

Mathematical set-up

Vierbein formalism

Differential forms (Fraktur letters)

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Chern classes:

$$\sum_{j=0} c_j(\mathfrak{F}) t^j = det\left(I + i\frac{\mathfrak{F}}{2\pi}t\right)$$

Topological invariant

↓

Chern-Wiel theory

Translation Symmetry

YK, Journal of Mathematical Physics 58 (9) (2017) 092502

Translation Symmetry

Rely only on experimentally established theories:
QFT& GR

We need something new!

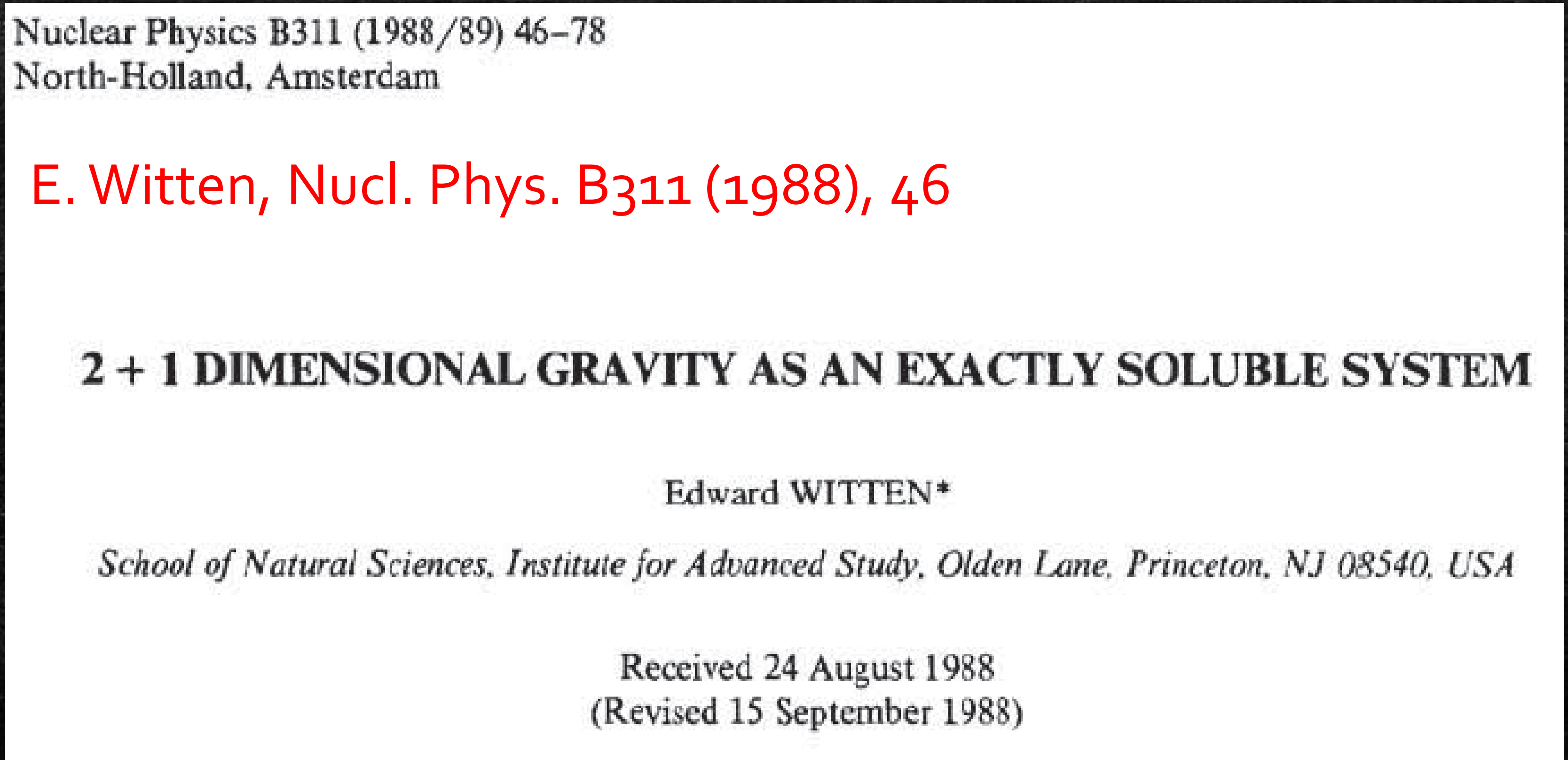
- Hint:

In (1+2) dimension, quantum general relativity exists.

Translation Symmetry

- Hint:

In (1+2) dimension, quantum general relativity exists.



Nuclear Physics B311 (1988/89) 46–78
North-Holland, Amsterdam

E. Witten, Nucl. Phys. B311 (1988), 46

2 + 1 DIMENSIONAL GRAVITY AS AN EXACTLY SOLUBLE SYSTEM

Edward WITTEN*

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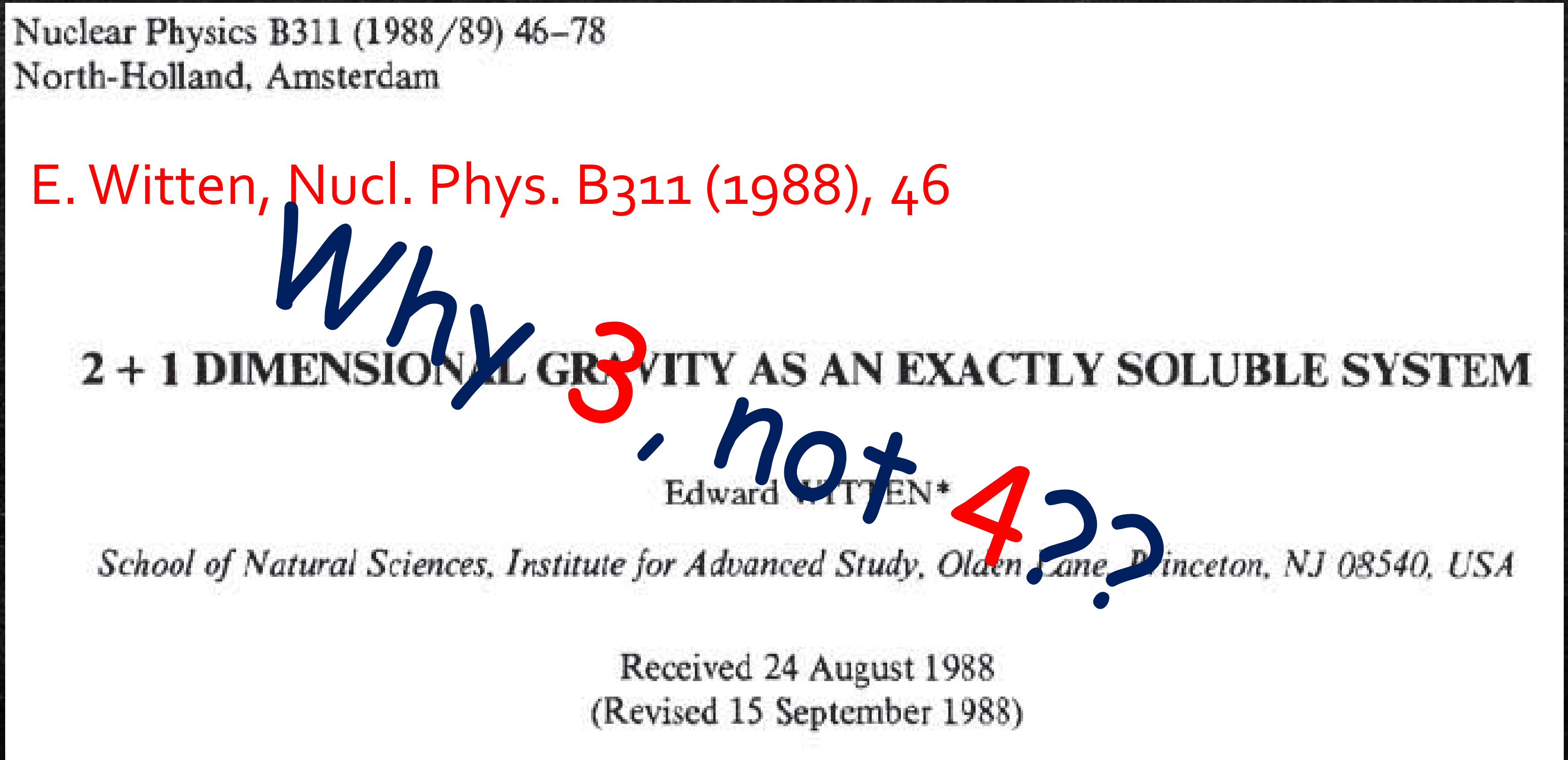
Received 24 August 1988
(Revised 15 September 1988)

Translation Symmetry

- Hint:

In (1+2) dimension, quantum general relativity exists.



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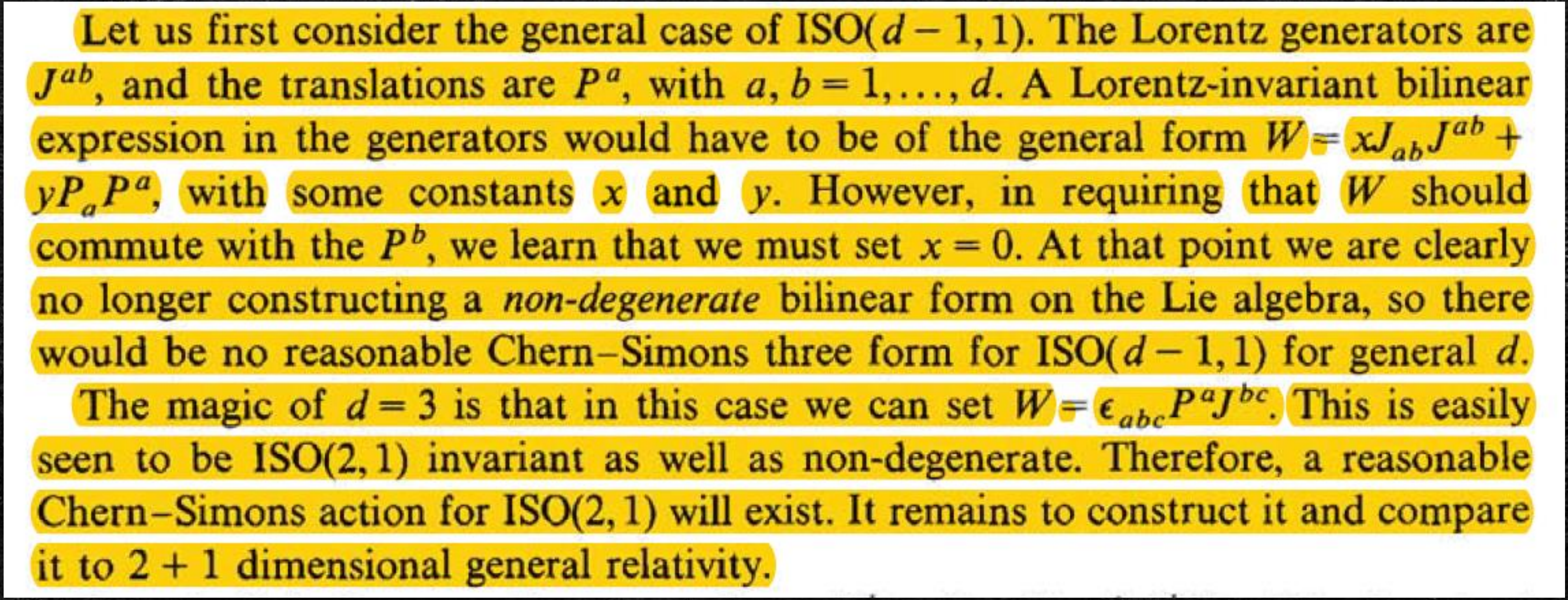
Why 3, not 4??

Translation Symmetry

- Hint:

E. Witten, Nucl. Phys. B311 (1988), 46



Let us first consider the general case of ISO$(d-1,1)$. The Lorentz generators are J^{ab}, and the translations are P^a, with $a, b = 1, \ldots, d$. A Lorentz-invariant bilinear expression in the generators would have to be of the general form $W = xJ_{ab}J^{ab} + yP_aP^a$, with some constants x and y. However, in requiring that W should commute with the P^b, we learn that we must set $x = 0$. At that point we are clearly no longer constructing a *non-degenerate* bilinear form on the Lie algebra, so there would be no reasonable Chern–Simons three form for ISO$(d-1,1)$ for general d.

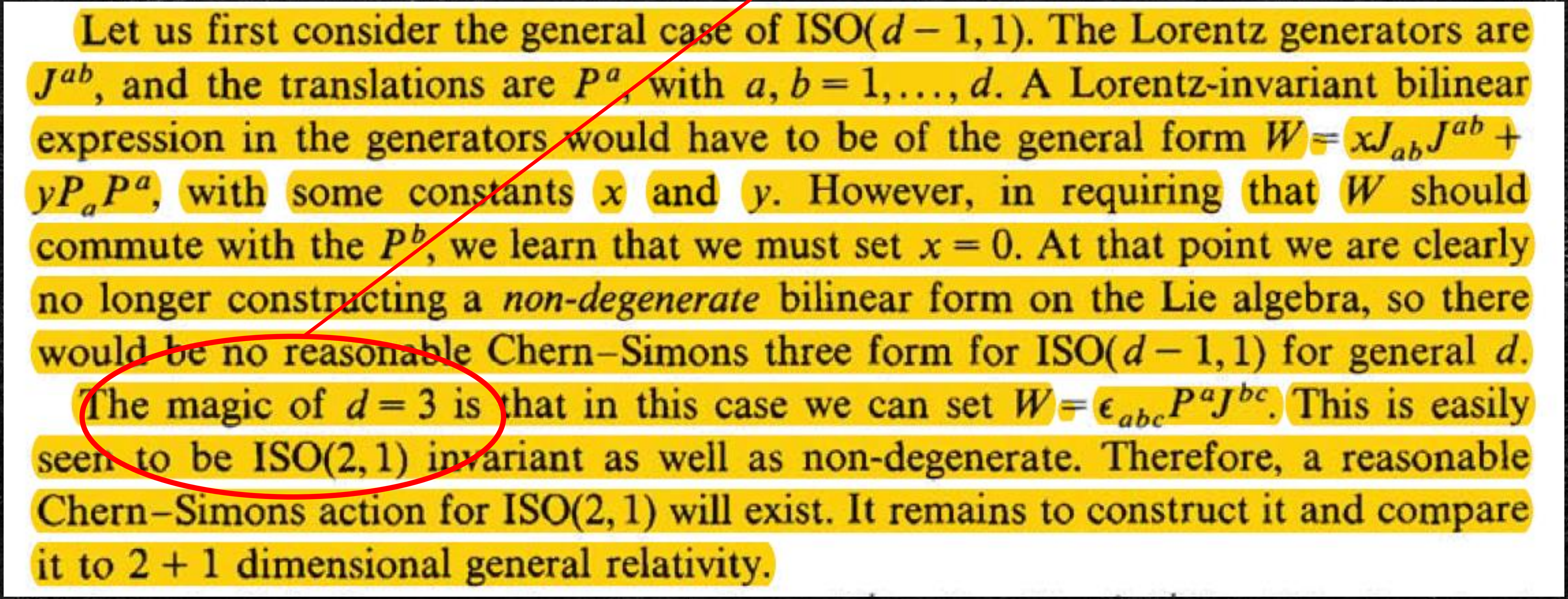
The magic of $d = 3$ is that in this case we can set $W = \epsilon_{abc}P^aJ^{bc}$. This is easily seen to be ISO(2, 1) invariant as well as non-degenerate. Therefore, a reasonable Chern–Simons action for ISO(2, 1) will exist. It remains to construct it and compare it to 2 + 1 dimensional general relativity.

Translation Symmetry

- Hint:

Poincare Symmetry

E. Witten, Nucl. Phys. B311 (1988), 46



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Translation Symmetry

- Geometrical symmetry in local QFT
  - Poincaré symmetry = Lorentz transformation + Translation

Translation Symmetry

- Geometrical symmetry in local QFT
  - Poincaré symmetry = Lorentz transformation + Translation

$$V'^{\mu} = \Lambda^{\mu}_{\nu} V^{\nu} + a^{\mu}$$

$$I_{gauge} = -\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}dx^4$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$$

Physics

A_μ: Potnsial

$F_{\mu\nu}$: Field Strength

Translation Symmetry

Connection form : $\mathfrak{A}$

Curvature form : $\mathfrak{F} = d\mathfrak{A} + \mathfrak{A} \wedge \mathfrak{A}$

$$I_{gauge} = -\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}dx^4$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$$

Geometry

A_μ: Connection

$F_{\mu\nu}$: Curvature

Translation Symmetry

- Geometrical symmetry in local QFT
  - Poincaré symmetry = Lorentz transformation + Translation

$$ISO(1,3) = SO(1,3) \ltimes T^4$$

Translation Symmetry

- Geometrical symmetry in local QFT
  - Poincaré symmetry = Lorentz transformation + Translation

$$ISO(1,3) = SO(1,3) \ltimes T^4$$

- Geometrical symmetry in general relativity
  - General Linear Symmetry:

$$GL(1,3) \supset SO(1,3)$$

Einstein's Equivalent principle!

Translation Symmetry

- Geometrical symmetry in general relativity
  - General coordinate transformation:

$$V'^{\mu} = \frac{dx'^{\mu}}{dx^{\nu}}(x)\,V^{\nu}$$

$$\boxed{I_{gauge} = -\frac{1}{4}\int \sqrt{-g}\,g_{\mu_1\mu_2}(x)\,g_{\nu_1\nu_2}(x)F^{\mu_1\nu_1}F^{\mu_2\nu_2}dx^4}$$

No translation symmetry!

Translation Symmetry

- We need something new!
  - Hint: E. Witten, Nucl. Phys. B311 (1988), 46

$SO(1,d\text{-}1)$

Generator : $J_{ab}\,,P_a$

rotation translation

Lie algebra

$$[P_a, P_b] = 0,$$

$$[J_{ab}, P_c] = -\eta_{ac}P_b + \eta_{bc}P_a,$$

$$[J_{ab}, J_{cd}] = -\eta_{ac}J_{bd} + \eta_{bc}J_{ad} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}$$

Translation Symmetry

- We need something new!
  - Hint: E. Witten, Nucl. Phys. B311 (1988), 46

$SO(1,d\text{-}1)$

Generator : $J_{ab}\, , P_a$

rotation translation

Bi-liner Invariants : $W = \alpha\, J_{ab}J^{ab} + \beta P_a P^a$

Translation Symmetry

- We need something new!
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SO(1,d-1)

Generator : $J_{ab}\,,P_a$

rotation translation

$\beta\,\eta_{ab}P^aP^b$

Bi-liner Invariants : $W=\alpha\,J_{ab}J^{ab}+\beta P_aP^a$

$\epsilon_{abc\cdots}$?

Translation Symmetry

- We need something new!
  - Hint: E. Witten, Nucl. Phys. B311 (1988), 46

    SO(1,2)

    Generator : J_{ab}, P_a

Bi-liner Invariants : $W = \alpha J_{ab} J^{ab} + \beta P_a P^a + \gamma \epsilon_{abc} J^{ab} P^c$

Magic for three dimension

Translation Symmetry

- We need something new!
  - **Hint:** E. Witten, Nucl. Phys. B311 (1988), 46

    $SO(1,2)$

Generator : J_{ab}, P_a

Bi-liner Invariants : $W = \alpha\, J_{ab} J^{ab} + \beta P_a P^a \;\; + \gamma\, \epsilon_{abc}\, J^{ab} P^c$

Four dimension **Three dimension**

$$\mathfrak{L}_G = \frac{1}{4}\epsilon_{....}\mathfrak{R}^{\cdot\cdot}(\mathfrak{w}) \wedge \mathfrak{e}^{\cdot} \wedge \mathfrak{e}^{\cdot} \longrightarrow \mathfrak{L}_G = \frac{1}{4}\epsilon_{...}\mathfrak{R}^{\cdot\cdot}(\mathfrak{w}) \wedge \mathfrak{e}^{\cdot}$$

Translation Symmetry

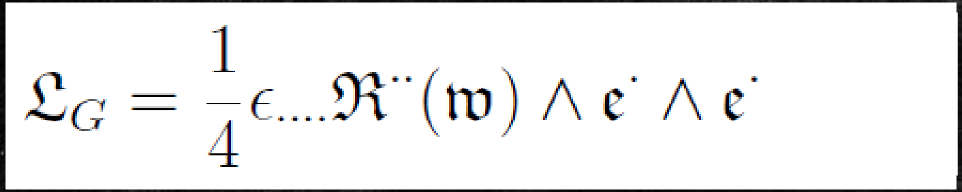
- We need something new!
  - Hint: E. Witten, Nucl. Phys. B311 (1988), 46

$SO(1,3)$

Generator : J_{ab}, P_{ab}

Bi-liner Invariants : $W = \alpha J_{ab} J^{ab} + \beta P_a P^a + \gamma \epsilon_{abcd} J^{ab} P^{cd}$

Four dimension



$$\mathfrak{L}_G = \frac{1}{4}\epsilon_{....}\mathfrak{R}^{..}(\mathfrak{w}) \wedge \mathfrak{e}^{.} \wedge \mathfrak{e}^{.}$$

$$\boxed{P_{ab}\ ?}$$

Translation Symmetry

Co-translation:

Translation

Generator : $P_{ab} \quad = \quad P_a \iota_b$ Contraction

Contraction : $\iota_a \quad = \quad \iota_{\xi^a}, \qquad \xi^a \quad = \quad \eta^{ab}\mathcal{E}_b^{\mu}\partial_{\mu},$

$$\iota_a e^b = \varepsilon_a^{\mu}\varepsilon_{\nu}^{b}\delta_{\mu}^{\nu} = \delta_a^b$$

Operator : $\delta_{CT} = \xi^{\cdot}\delta_T \iota.$

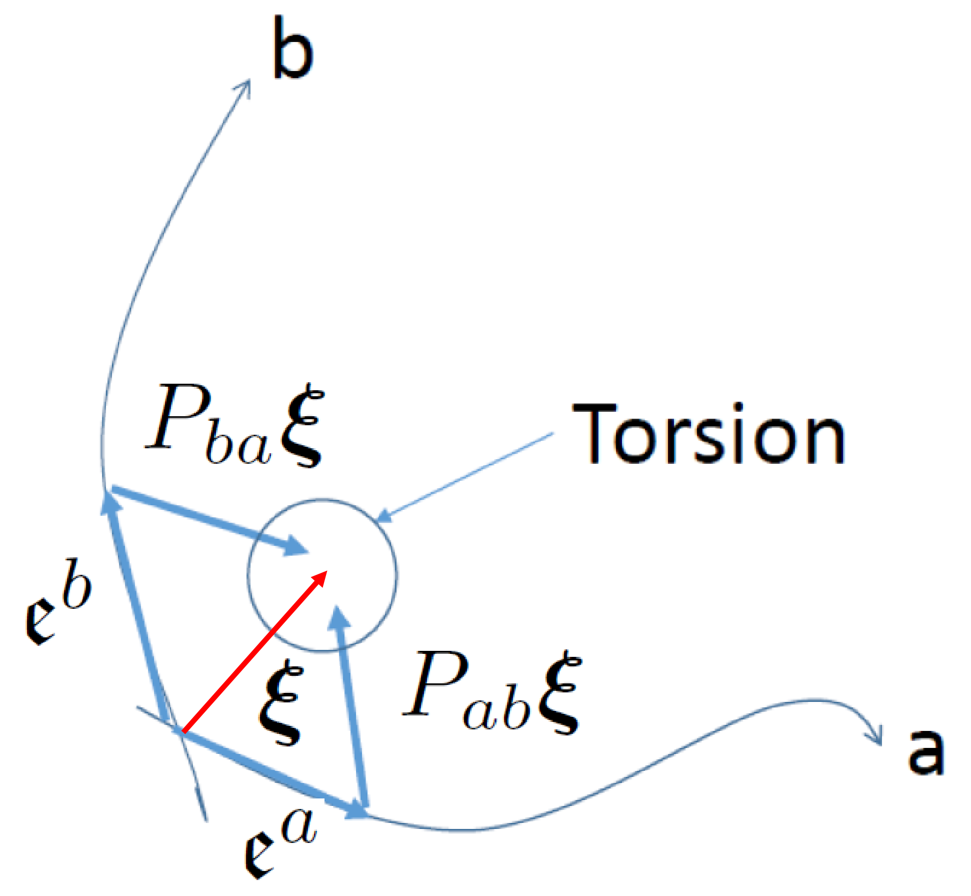
Translation Symmetry

Co-translation:

$$\begin{aligned} \boldsymbol{\xi} &= \xi_\cdot \mathfrak{e}^\cdot \\ \iota_a \boldsymbol{\xi} &= \xi_a \end{aligned}$$

$$\delta_{CT} \mathfrak{L}_4 = 0$$

Intuitive image:



Translation Symmetry

Three dimension

Principal connection: $\mathfrak{A}_0 = \{J_{ab}, P_c\} \times \{\mathfrak{w}^{ab}, \mathfrak{e}^c\} = J_{..}\mathfrak{w}^{..} + P_{.}\mathfrak{e}^{.}$

Principal curvature : $\mathfrak{F}_0 = d\mathfrak{A}_0 + \mathfrak{A}_0 \wedge \mathfrak{A}_0 = J_{..}\mathfrak{R}^{..} + P_{.}\mathfrak{T}^{.}$

$$\longrightarrow \quad \delta_T \mathfrak{L}_3 \quad = \quad 0$$

Four dimension

Principal connection: $\mathfrak{A} = \{J_{ab}, P^{cd}\} \times \{\mathfrak{w}^{ab}, \mathfrak{S}_{cd}\} = J_{..}\mathfrak{w}^{..} + P^{..}\mathfrak{S}_{..},$

Principal curvature : $\mathfrak{F} = d\mathfrak{A} + \mathfrak{A} \wedge \mathfrak{A} = J_{..}\mathfrak{R}^{..} + P^{..}d_{\mathfrak{w}}\mathfrak{S}_{..},$

$$\longrightarrow \quad \delta_{CT} \mathfrak{L}_4 \quad = \quad 0$$

Translation Symmetry

What we can learn from this result:

Phase space

Fundamental variables : $\{\mathfrak{w}^{ab}, \mathfrak{S}_{cd}\} \longrightarrow \{\mathfrak{w}, \mathfrak{M} = \mathfrak{S}\}$

Lagrangian : $\mathfrak{L}_G(\mathfrak{w}, \mathfrak{e}) \quad = \quad \frac{1}{2}\mathfrak{S}_{..} \wedge \left(\mathfrak{R}^{..} - \cancel{\frac{\Lambda}{3!}\mathfrak{S}^{..}}\right)$

Generalized momentum : $\mathfrak{M}_{ab} = \frac{\delta \mathfrak{L}_G}{\delta\left(d\mathfrak{w}^{ab}\right)} = \mathfrak{S}_{ab}$

$\delta_{CT}\mathfrak{v}_4 \quad \neq \quad 0$

Hamiltonian:

$$\begin{aligned}\mathfrak{H}_G \quad &= \quad \frac{1}{2}\mathfrak{M}_{..} \wedge d\mathfrak{w}^{..} - \mathfrak{L}_G, \\ &= \quad -\frac{1}{2}\mathfrak{S}_{..} \wedge \mathfrak{w}^{\cdot}{}_{*} \wedge \mathfrak{w}^{*\cdot}\end{aligned}$$

Translation Symmetry

What we can learn from this result:

Phase space :

$$\{\mathfrak{w}, \mathfrak{M} = \mathfrak{S}\}$$

Hamiltonian:

$$\mathfrak{H}_G = -\frac{1}{2}\mathfrak{S}_{\cdot\cdot} \wedge \mathfrak{w}^{\cdot}{}_{*} \wedge \mathfrak{w}^{\cdot *}$$

Energy-momentum tensor (three form)

$$^{*}t_d = \frac{1}{\kappa}(\omega^{ab} \wedge \omega^{cf} \wedge E_f - \omega^{af} \wedge \omega^{fb} \wedge E^c)\varepsilon_{abcd}$$

Frè, P.: Gravity, a Geometrical Course: Volume 1: Development of the Theory and Basic Physical Applications. Gravity, a Geometrical Course. Springer Netherlands (2012)

Translation Symmetry

What we can learn from this result:

Phase space : $\{\mathfrak{w}, \mathfrak{M} = \mathfrak{S}\}$

Hamiltonian: $\mathfrak{H}_G = -\frac{1}{2}\mathfrak{S}_{..} \wedge \mathfrak{w}^{\cdot}{}_{*} \wedge \mathfrak{w}^{\cdot *}$

We found correct canonical variables!

Translation Symmetry

What we can learn from this result:

Cosmological constant: $\Lambda \int_{\mathcal{M}_4} \mathfrak{v} = \pi^2 \int_{\tilde{\mathcal{M}}_5} c_2 \quad \in \mathbb{R}/\mathbb{Z}$

Canonical Quantization

YK, arXiv:1703.05574

Canonical Quantization

Poisson brackets:

$$\{\mathfrak{a}, \mathfrak{b}\}_{\mathrm{PB}} = \frac{\delta \mathfrak{a}}{\delta \mathfrak{w}^{\cdot\cdot}} \wedge \frac{\delta \mathfrak{b}}{\delta \mathfrak{S}_{..}} - \frac{\delta \mathfrak{b}}{\delta \mathfrak{w}^{\cdot\cdot}} \wedge \frac{\delta \mathfrak{a}}{\delta \mathfrak{S}_{..}}$$

Canonical equation of motion:

$$\begin{aligned} \{\mathfrak{w}^{\bullet\bullet}, \mathfrak{w}^{\bullet\bullet}\}_{\mathrm{PB}} &= \{\mathfrak{S}_{\bullet\bullet}, \mathfrak{S}_{\bullet\bullet}\}_{\mathrm{PB}} = 0, \\ \{\mathfrak{w}^{a_1 a_2}, \mathfrak{S}_{b_1 b_2}\}_{\mathrm{PB}} &= \delta^{[a_1}_{b_1}\delta^{a_2]}_{b_2}, \end{aligned}$$

Poisson bracket ⟶ Commutation relation

Canonical variable ⟶ Operator

Canonical Quantization

Classical equations of motion

Canonical equation of motion:

$$\left\{\begin{aligned} \frac{\delta \mathfrak{H}_G}{\delta \mathfrak{w}^{\bullet\bullet}} &= -d\mathfrak{M}_{\bullet\bullet}, \\ \frac{\delta \mathfrak{H}_G}{\delta \mathfrak{M}_{ab}} &= -\epsilon_{\cdot\cdot\cdot\cdot}\left(\epsilon_{ab\cdot *}\mathfrak{e}^{*}\right)^{-1} \wedge \mathfrak{e}^{\cdot} \wedge \mathfrak{w}^{\cdot}{}_{*} \wedge \mathfrak{w}^{*\cdot} = d\mathfrak{w}^{ab}, \end{aligned}\right.$$

PB representation of EoM:

$$\left\{\begin{aligned} \epsilon_{a\cdots}\left\{\mathfrak{w}^{\cdot\cdot}, \mathfrak{H}_G\right\}_{\mathrm{PB}} \wedge \mathfrak{e}^{\cdot} &= -\epsilon_{a\cdots}\mathfrak{w}^{\cdot}{}_{*} \wedge \mathfrak{w}^{*\cdot} \wedge \mathfrak{e}^{\cdot} = \epsilon_{a\cdots} d\mathfrak{w}^{\cdot\cdot} \wedge \mathfrak{e}^{\cdot}, \\ \left\{\mathfrak{S}_{a\cdot}, \mathfrak{H}_G\right\}_{\mathrm{PB}} &= -\left(-\mathfrak{w}^{b\cdot} \wedge \mathfrak{S}_{\cdot a}\right) = d\mathfrak{S}_{a\cdot}, \end{aligned}\right.$$

Canonical Quantization

Commutation relation:

$$\begin{cases} \left[\widehat{\mathfrak{w}}^{\bullet\bullet}, \widehat{\mathfrak{w}}^{\bullet\bullet}\right] = \left[\widehat{\mathfrak{S}}_{\bullet\bullet}, \widehat{\mathfrak{S}}_{\bullet\bullet}\right] = 0 \\ \left[\widehat{\mathfrak{w}}^{a_1 a_2}(x), \widehat{\mathfrak{S}}_{b_1 b_2}(y)\right] = -i\delta^{(4)}(x-y)\delta^{[a_1}_{b_1}\delta^{a_2]}_{b_2} \end{cases}$$

Operator:

$$\begin{cases} \widehat{\mathfrak{w}}^{\bullet\bullet} = \mathfrak{w}^{\bullet\bullet}, \\ \widehat{\mathfrak{S}}_{\bullet\bullet} = i\dfrac{\delta}{\delta\mathfrak{w}^{\bullet\bullet}}, \end{cases}$$

Schrödinger eq.:

$$\widehat{\mathfrak{H}}_G|\Psi(\omega)\rangle = E_G|\Psi(\omega)\rangle$$

Canonical Quantization

Commutation relation:

$$\left[\widehat{\mathfrak{w}}^{\bullet\bullet},\widehat{\mathfrak{w}}^{\bullet\bullet}\right] = \left[\widehat{\mathfrak{S}}_{\bullet\bullet},\widehat{\mathfrak{S}}_{\bullet\bullet}\right] = 0$$

$$\left[\widehat{\mathfrak{w}}^{a_1a_2}(x),\widehat{\mathfrak{S}}_{b_1b_2}(y)\right] = -i\delta^{(4)}(x-y)\delta^{[a_1}_{b_1}\delta^{a_2]}_{b_2}$$

Operator:

$$\begin{aligned}\widehat{\mathfrak{w}}^{\bullet\bullet} &= \mathfrak{w}^{\bullet\bullet},\\ \widehat{\mathfrak{S}}_{\bullet\bullet} &= i\frac{\delta}{\delta\mathfrak{w}^{\bullet\bullet}},\end{aligned}$$

Schrödinger eq.:

$$\widehat{\mathfrak{H}}_G|\Psi(\omega)\rangle = E_G|\Psi(\omega)\rangle$$

Whole story is not so simple.

Gauge fixing!!

Canonical Quantization

Symmetry (gauge) fixing

e.g. $\text{Path Integral} \propto \sum_{\mathfrak{w}\in\Omega} \exp\left(i\int \mathfrak{L}_G\right)$

Ω : set of all possible spin forms

$\Omega_{gauge} \subset \Omega$: gauge transformed subset

$$\sum_{\mathfrak{w}\in\Omega_{gauge}} \exp\left(i\int \mathfrak{L}_G\right) = \sum_{\mathfrak{w}\in\Omega_{gauge}} (const.) = (Vol.) \times (const.) \to \infty$$

Canonical Quantization

Symmetry (gauge) fixing

Quantum Lagrangian: $$\mathfrak{L}_{QG} \quad = \quad \mathfrak{L}_{G} + \mathfrak{L}_{GF} + \mathfrak{L}_{FP}$$

Lagrange multiplier method

- Classical Gravity
- Gauge fixing
- Faddeev-Popov ghost

Naknishi-Kugo-Ojima covariant Quantization

Canonical Quantization

(Naknishi-Kugo-Ojima Quantization)

Lagrangian:

$$\mathfrak{L}_{QG} = \mathfrak{L}_{G} + \mathfrak{L}_{GF} + \mathfrak{L}_{FP},$$

$$\begin{cases} \mathfrak{L}_{G} & = & \frac{1}{2}\left(\mathfrak{R}^{\cdot\cdot} \qquad\qquad\qquad \right) \wedge \mathfrak{S}_{\cdot\cdot}, \\ \mathfrak{L}_{GF} & = & -\frac{1}{2}\left(d\mathfrak{b}^{\cdot\cdot} + \alpha \mathfrak{b}^{\cdot}{}_{*} \wedge \mathfrak{b}^{*\cdot}\right) \wedge \mathfrak{S}_{\cdot\cdot}, \\ \mathfrak{L}_{FP} & = & -\frac{i}{2}\left(d\tilde{\mathfrak{c}}^{\cdot\cdot} + \alpha \tilde{\mathfrak{c}}^{\cdot}{}_{*} \wedge \mathfrak{b}^{*\cdot}\right) \wedge \mathfrak{c}^{\star} \wedge \overline{\mathfrak{e}}_{\star\cdot\cdot}, \end{cases}$$

Canonical Quantization
(Naknishi-Kugo-Ojima Quantization)

- Fields: $\{\mathfrak{w}^{ab}, \mathfrak{S}_{ab}, \mathfrak{b}^{ab}, \mathfrak{c}^{a}, \tilde{\mathfrak{c}}^{a}\}$
- Classical Gravity: $\mathfrak{L}_G(\mathfrak{w}, \mathfrak{S})$
- Gauge fixing: $\mathfrak{L}_{GF}(\mathfrak{b})$
- Faddeev-Popov ghost: $\mathfrak{L}_{FP}(\mathfrak{b}, \mathfrak{c}, \tilde{\mathfrak{c}})$

anti-ghost field

ghost field

auxiliary field

$$\begin{aligned}
\left[\widehat{\mathfrak{w}}^{a_1a_2}(x), \widehat{\mathfrak{S}}_{b_1b_2}(y)\right] &= -i\delta^{(4)}(x-y)\delta^{[a_1}_{b_1}\delta^{a_2]}_{b_2},\\
\left[\widehat{\mathfrak{b}}^{a_1a_2}(x), \widehat{\mathfrak{S}}_{b_1b_2}(y)\right] &= -i\delta^{(4)}(x-y)\delta^{[a_1}_{b_1}\delta^{a_2]}_{b_2},\\
\left\{\widehat{\tilde{\mathfrak{c}}}^{a_1a_2}(x), \widehat{\mathfrak{c}^{\cdot} \wedge \bar{\mathfrak{e}}_{\cdot b_1b_2}}(y)\right\} &= -i\delta^{(4)}(x-y)\delta^{[a_1}_{b_1}\delta^{a_2]}_{b_2}.
\end{aligned}$$

Others are zero

Canonical Quantization

(Naknishi-Kugo-Ojima Quantization)

BRS symmetry

Fields: $\left\{\mathfrak{w}^{ab}, \mathfrak{S}_{ab}, \mathfrak{b}^{ab}, \mathfrak{c}^{a}, \tilde{\mathfrak{c}}^{a}\right\}$

BRS symmetry: $\delta_{BRS}\left[\mathfrak{L}_{QG}\right] \quad = \quad 0$

Nilpotency: $\delta_{BRS}\left[\delta_{BRS}\left[\left\{\mathfrak{w}^{ab}, \mathfrak{S}_{ab}, \mathfrak{b}^{ab}, \mathfrak{c}^{a}, \tilde{\mathfrak{c}}^{a}\right\}\right]\right] \quad = \quad \{0, 0, 0, 0, 0\}$

Canonical Quantization

(Naknishi-Kugo-Ojima Quantization)

BRS transformation:

$$\begin{cases} \delta_{\mathrm{BRS}}\left[\mathfrak{e}^{a}\right] = \mathfrak{c}^{a} \\ \delta_{\mathrm{BRS}}\left[\mathfrak{w}^{ab}\right] = \mathfrak{w}^{a\cdot}\,\chi^{b}{}_{\cdot} + \mathfrak{w}^{\cdot b}\,\chi^{a}{}_{\cdot} - d\chi^{a}{}_{\cdot}\,\eta^{b\cdot} \\ \delta_{\mathrm{BRS}}\left[\mathfrak{b}^{ab}\right] = \delta_{\mathrm{BRS}}\left[\mathfrak{c}^{a}\right] = 0, \\ \delta_{\mathrm{BRS}}\left[\tilde{\mathfrak{c}}^{ab}\right] = i\mathfrak{b}^{ab}. \end{cases}$$

$$\Rightarrow \begin{cases} \delta_{BRS}\left[\mathfrak{L}_{QG}\right] = 0 \\ \delta_{BRS}\left[\delta_{BRS}\left[\bullet\right]\right] = 0 \end{cases}$$

Canonical Quantization
(Naknishi-Kugo-Ojima Quantization)

Noether current/charge: $\delta_{BRS}\left[\mathfrak{L}_{QG}\right] = 0$

$$\left[\ \widehat{\mathfrak{Q}}_{\mathfrak{b}} \quad \widehat{\mathfrak{Q}}_{\tilde{\mathfrak{c}}}\ \right]$$

BRS-charge

$$\left\{\begin{aligned} d\widehat{\mathfrak{Q}}_{\mathfrak{b}} &= d\widehat{\mathfrak{Q}}_{\tilde{\mathfrak{c}}} = 0, \\ \delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}_{\tilde{\mathfrak{c}}}\right] &= -\widehat{\mathfrak{Q}}_{\mathfrak{b}}, \\ \delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}_{\mathfrak{b}}\right] &= 0. \\ \widehat{\mathfrak{Q}}_{\mathfrak{b}}^{2} &= 0, \\ \left[\widehat{\mathfrak{Q}}_{\tilde{\mathfrak{c}}}, \widehat{\mathfrak{Q}}_{\mathfrak{b}}\right] &= -i\widehat{\mathfrak{Q}}_{\mathfrak{b}} \end{aligned}\right.$$

Canonical Quantization

(Naknishi-Kugo-Ojima Quantization)

Hilbert Space

Hilbert Space: $$\mathcal{H} = \left\{ |\Psi\rangle \,\middle|\, \int |\Psi(\bullet)|^2 < \infty,\ \bullet \in \left\{\mathfrak{w}^{ab}, \mathfrak{b}^{ab}, \mathfrak{c}^{a}, \tilde{\mathfrak{c}}^{a}\right\} \right\}$$

Norm: $$\langle \mathfrak{w} | \mathfrak{w} \rangle = \frac{1}{2} \int_{\Sigma} \mathfrak{w}^{\cdot}{}_{*} \wedge \mathfrak{w}^{*\cdot} \wedge \mathfrak{S}_{..}$$

Integration measure

Canonical Quantization
(Naknishi-Kugo-Ojima Quantization)

Hilbert Space

Hilbert Space:

$$\mathcal{H} = \left\{ |\Psi\rangle \,\middle|\, \int |\Psi(\bullet)|^2 < \infty,\ \bullet \in \{\mathfrak{w}^{ab}, \mathfrak{b}^{ab}, \mathfrak{c}^a, \tilde{\mathfrak{c}}^a\} \right\}$$

Norm:

$$\langle \mathfrak{w}|\mathfrak{w}\rangle = \frac{1}{2}\int_{\Sigma} \mathfrak{w}^{\cdot}{}_{*} \wedge \mathfrak{w}^{*\cdot} \wedge \mathfrak{S}_{..}$$

Integration measure

$$\begin{matrix} > \\ \langle \blacksquare|\blacksquare\rangle = 0 \\ < \end{matrix}$$

Canonical Quantization

(Naknishi-Kugo-Ojima Quantization)

Hilbert Space: $$\mathcal{H} = \left\{ |\Psi\rangle \;\middle|\; \int |\Psi(\bullet)|^2 < \infty, \; \bullet \in \left\{ \mathfrak{w}^{ab}, \mathfrak{b}^{ab}, \mathfrak{c}^{a}, \tilde{\mathfrak{c}}^{a} \right\} \right\}$$

Norm: $$\langle \mathfrak{w} | \mathfrak{w} \rangle = \frac{1}{2} \int_{\Sigma} \mathfrak{w}^{\cdot}{}_{*} \wedge \mathfrak{w}^{*\cdot} \wedge \mathfrak{S}_{..} \longrightarrow \langle \blacksquare | \blacksquare \rangle \overset{>}{\underset{<}{=}} 0$$

$$\mathcal{H}_{phys} = \left\{ |\Psi_{phys}\rangle \;\middle|\; \langle \Psi_{phys} | \Psi_{phys} \rangle \geq 0 \right\}$$

Non degenerate

$$\langle \Psi_{phys} | \Psi_{phys} \rangle = 0$$

$$\Updownarrow$$

$$|\Psi_{phys}\rangle = 0$$

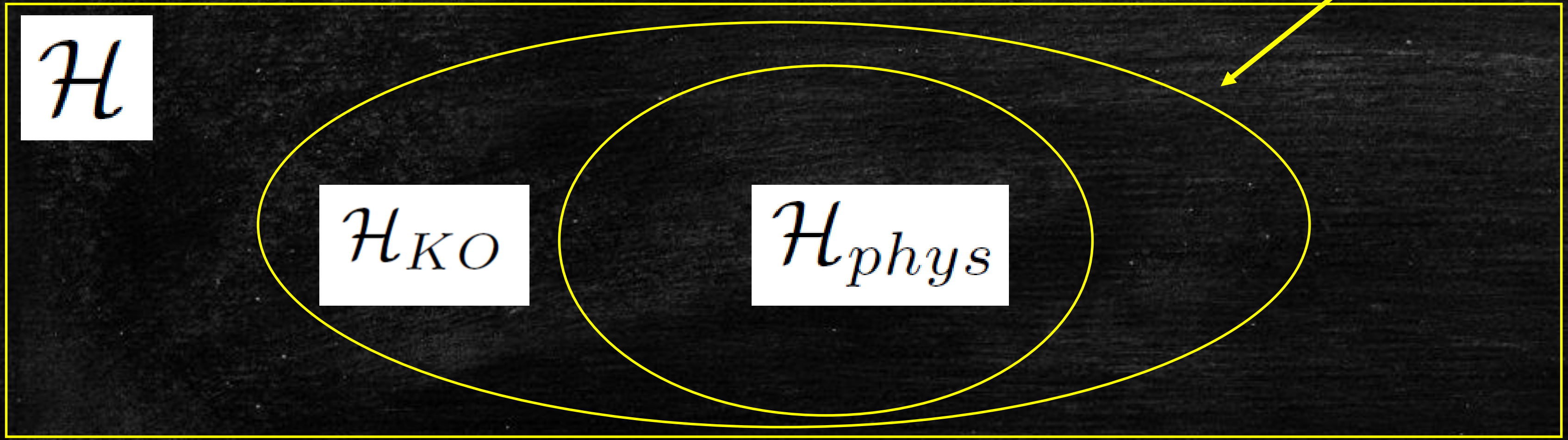
Canonical Quantization

(Naknishi-Kugo-Ojima Quantization)

Physical states: $\mathcal{H}_{phys} \quad = \quad \left\{ |\Psi_{phys}\rangle \;\middle|\; \langle\Psi_{phys}|\Psi_{phys}\rangle \geq 0 \right\}$

K-O states: $\mathcal{H}_{KO} \quad = \quad \left\{ |\Psi_{KO}\rangle \;\middle|\; \widehat{\mathfrak{Q}}_{\mathfrak{b}}|\Psi_{KO}\rangle = \widehat{\mathfrak{Q}}_{\tilde{\mathfrak{c}}}|\Psi_{KO}\rangle = 0 \right\}$

cf. $\widehat{\mathfrak{Q}}_{\mathfrak{b}}|\Psi_{phys}\rangle = \widehat{\mathfrak{Q}}_{\tilde{\mathfrak{c}}}|\Psi_{phys}\rangle = 0$



Canonical Quantization

(Naknishi-Kugo-Ojima Quantization)

$$|\Psi_{KO}\rangle \quad = \quad |\Psi_{phys}\rangle + \widehat{\mathfrak{Q}}_{\mathfrak{b}}|\Phi\rangle \longleftarrow \widehat{\mathfrak{Q}}_{\mathfrak{b}}|\Phi\rangle \neq 0$$

$$\begin{aligned} \langle\Psi_{KO}\big|\Psi_{KO}\rangle \quad &= \quad \langle\Psi_{phys}\big|\Psi_{phys}\rangle \\ d|\Psi_{KO}\rangle \quad &= \quad d|\Psi_{phys}\rangle + d\widehat{\mathfrak{Q}}_{\mathfrak{b}}|\Phi\rangle \end{aligned}$$

Canonical Quantization

(Naknishi-Kugo-Ojima Quantization)

$$|\Psi_{KO}\rangle \quad = \quad |\Psi_{phys}\rangle + \widehat{\mathfrak{Q}}_{\mathfrak{b}}|\Phi\rangle \longleftarrow \widehat{\mathfrak{Q}}_{\mathfrak{b}}|\Phi\rangle \neq 0$$

$$\begin{aligned} \big\langle\Psi_{KO}\big|\Psi_{KO}\big\rangle \quad &= \quad \big\langle\Psi_{phys}\big|\Psi_{phys}\big\rangle \\ d|\Psi_{KO}\rangle \quad &= \quad d|\Psi_{phys}\rangle + d\cancel{\widehat{\mathfrak{Q}}_{\mathfrak{b}}}|\Phi\rangle = d|\Psi_{phys}\rangle \end{aligned}$$

Canonical Quantization
(Naknishi-Kugo-Ojima Quantization)

$$|\Psi_{KO}\rangle \quad = \quad |\Psi_{phys}\rangle + \widehat{\mathfrak{Q}}_{\mathfrak{b}}|\Phi\rangle \quad \longleftarrow \quad \widehat{\mathfrak{Q}}_{\mathfrak{b}}|\Phi\rangle \neq 0$$

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$\Rightarrow$ Unitarity of S-matrix (Kugo-Ojima Theorem)

Canonical Quantization
(Naknishi-Kugo-Ojima Quantization)

- ○ Quantum Lagrangian
- ○ Commutation relation
  - ✓ Operator
- ○ Schrödinger equation
- ○ Physical Hilbert space
  - ✓ BRS-transformation
  - ✓ BRS-charge

$$\mathfrak{L}_{QG} \quad = \quad \mathfrak{L}_G + \mathfrak{L}_{GF} + \mathfrak{L}_{FP}$$

Canonical Quantization

(Naknishi-Kugo-Ojima Quantization)

- Quantum Lagrangian
- Commutation relation
  - ✓ Operator
- Schrödinger equation
- Physical Hilbert space
  - ✓ BRS-transformation
  - ✓ BRS-charge

$$\begin{cases} \left[\widehat{\mathfrak{w}}^{\bullet\bullet}, \widehat{\mathfrak{w}}^{\bullet\bullet}\right] = \left[\widehat{\mathfrak{S}}_{\bullet\bullet}, \widehat{\mathfrak{S}}_{\bullet\bullet}\right] = 0 \\ \left[\widehat{\mathfrak{w}}^{a_1a_2}(x), \widehat{\mathfrak{S}}_{b_1b_2}(y)\right] = -i\delta^{(4)}(x-y)\delta^{[a_1}_{b_1}\delta^{a_2]}_{b_2} \end{cases}$$

Canonical Quantization

(Naknishi-Kugo-Ojima Quantization)

- Quantum Lagrangian
- Commutation relation
    - ✓ Operator
- Schrödinger equation
- Physical Hilbert space
    - ✓ BRS-transformation
    - ✓ BRS-charge

$$\begin{aligned} \widehat{\mathfrak{w}}^{\bullet\bullet} &= \mathfrak{w}^{\bullet\bullet}, \\ \widehat{\mathfrak{S}}_{\bullet\bullet} &= i\frac{\delta}{\delta\mathfrak{w}^{\bullet\bullet}}, \end{aligned}$$

Canonical Quantization
(Naknishi-Kugo-Ojima Quantization)

- ○ Quantum Lagrangian
- ○ Commutation relation
  - ✓ Operator
- ○ Schrödinger equation
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  - ✓ BRS-transformation
  - ✓ BRS-charge

$$\widehat{\mathfrak{H}}_G|\Psi(\omega)\rangle \quad = \quad E_G|\Psi(\omega)\rangle$$

Canonical Quantization

(Naknishi-Kugo-Ojima Quantization)

- ○ Quantum Lagrangian
- ○ Commutation relation
  - ✓ Operator
- ○ Schrödinger equation
- ○ Physical Hilbert space
  - ✓ BRS-transformation
  - ✓ BRS-charge

$$\delta_{BRS}\left[\mathfrak{L}_{QG}\right] = 0$$

$$\left[\ \widehat{\mathfrak{Q}}_{\mathfrak{b}} \quad \widehat{\mathfrak{Q}}_{\tilde{\mathfrak{c}}}\ \right]$$

Canonical Quantization
(Naknishi-Kugo-Ojima Quantization)

- Quantum Lagrangian
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Canonical Quantization
(Naknishi-Kugo-Ojima Quantization)

- Quantum Lagrangian
- Commutation relation
  - ✓ Operator
- Schrödinger equation
- Physical Hilbert space
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Formal construction

↓

Concrete calculations
Applications

Canonical Quantization
(Naknishi-Kugo-Ojima Quantization)

- ○ Quantum Lagrangian
- ○ Commutation relation
  - ✓ Operator
- ○ Schrödinger equation
- ○ Physical Hilbert space
  - ✓ BRS-transformation
  - ✓ BRS-charge

Formal construction

Concrete calculations
Applications

Next step !!

Canonical Quantization

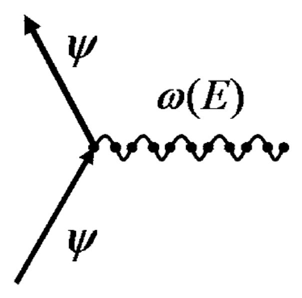
Where is divergence (non-renormalizability)?

We don't use perturbation:

$$g_{\mu\nu} \quad = \quad \eta_{\mu\nu} + \sqrt{\kappa_E} h_{\mu\nu}$$

✗

We don't treat graviton:

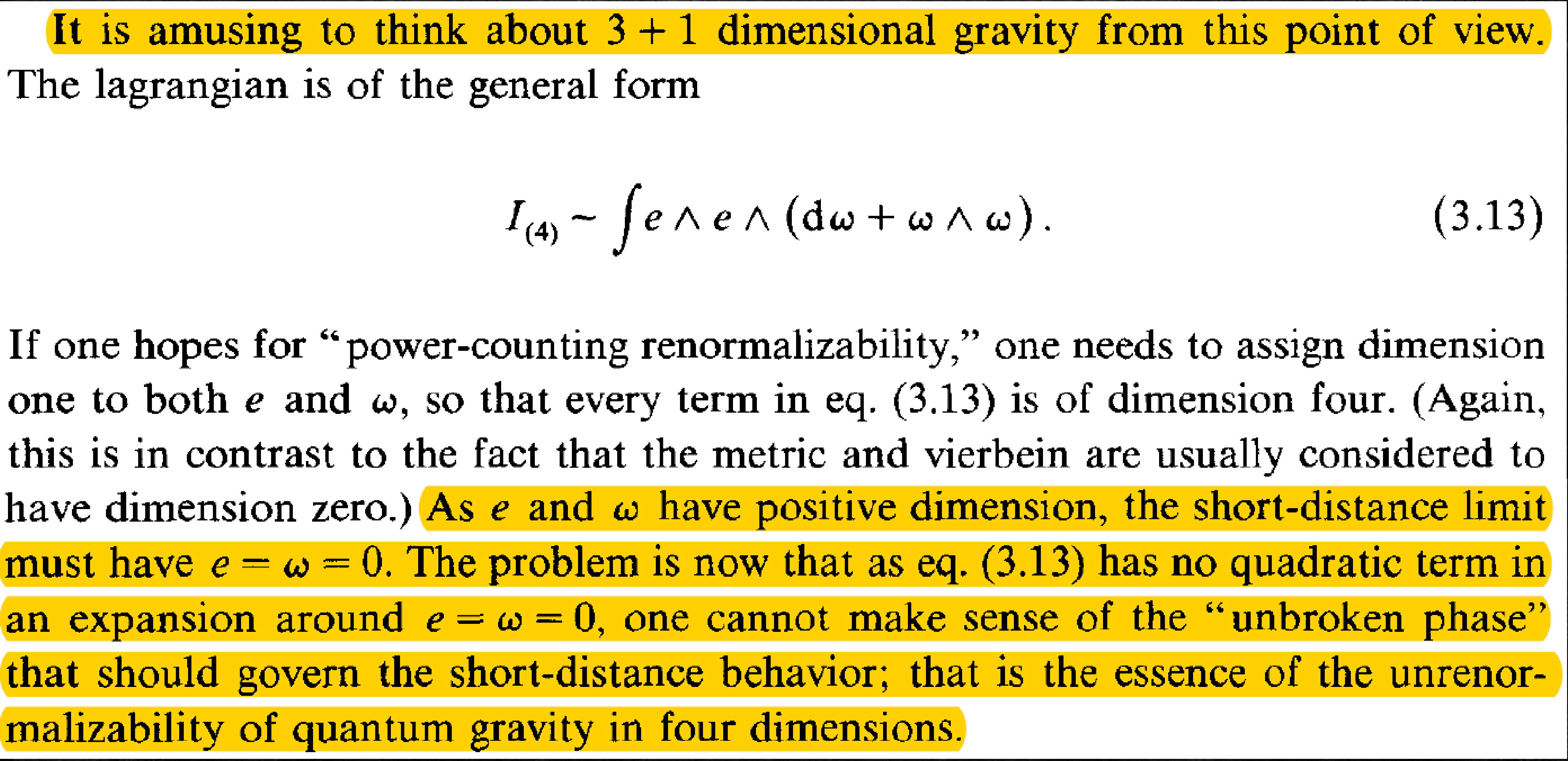


✗

What can we calculate (predict) ?

Next step !!

E. Witten, Nucl. Phys. B311 (1988), 46

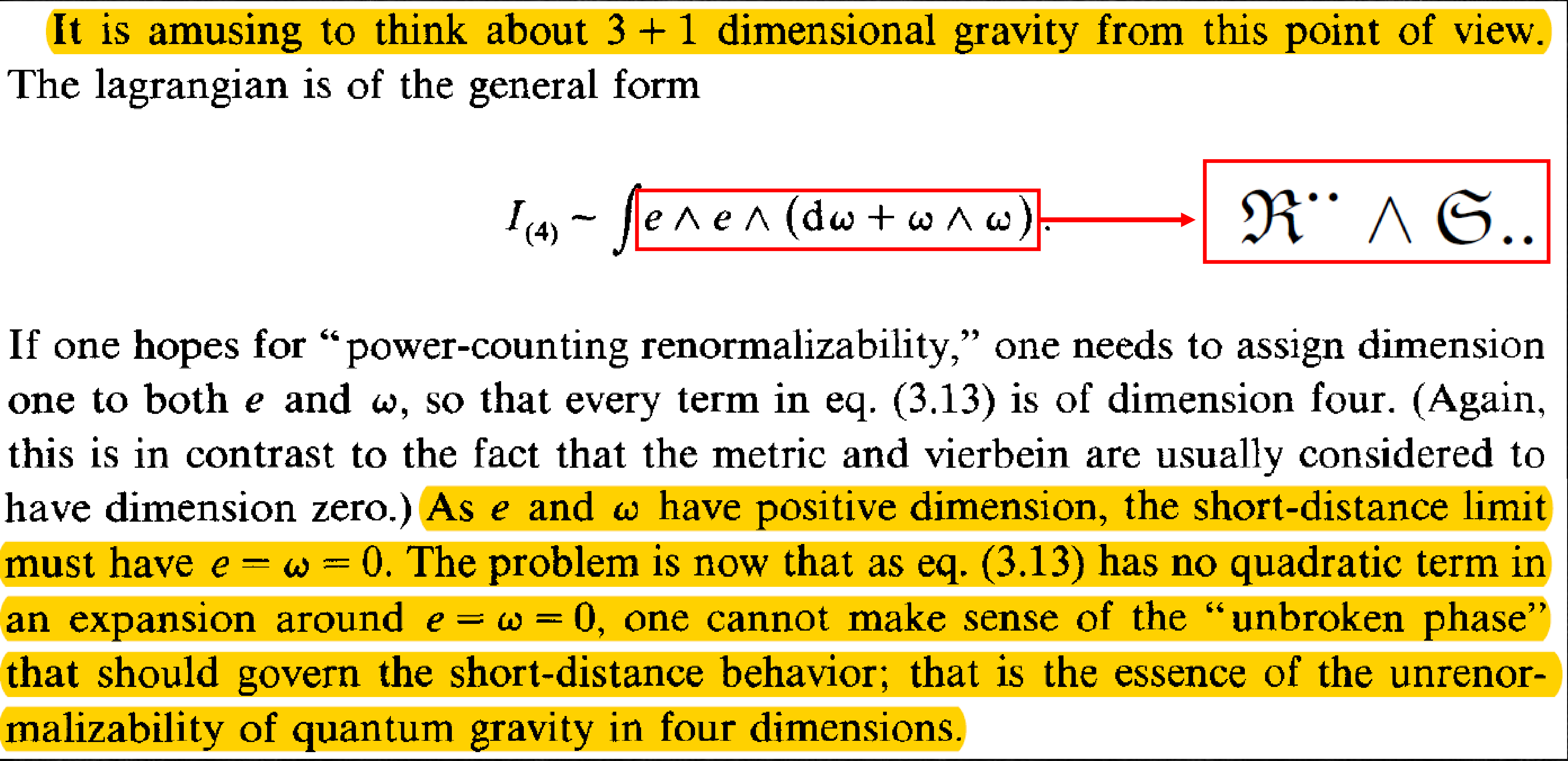


It is amusing to think about 3 + 1 dimensional gravity from this point of view. The lagrangian is of the general form

$$I_{(4)} \sim \int e \wedge e \wedge (\mathrm{d}\omega + \omega \wedge \omega)\,. \qquad (3.13)$$

If one hopes for "power-counting renormalizability," one needs to assign dimension one to both e and ω, so that every term in eq. (3.13) is of dimension four. (Again, this is in contrast to the fact that the metric and vierbein are usually considered to have dimension zero.) As e and ω have positive dimension, the short-distance limit must have $e = \omega = 0$. The problem is now that as eq. (3.13) has no quadratic term in an expansion around $e = \omega = 0$, one cannot make sense of the "unbroken phase" that should govern the short-distance behavior; that is the essence of the unrenormalizability of quantum gravity in four dimensions.

E. Witten, Nucl. Phys. B311 (1988), 46



It is amusing to think about 3 + 1 dimensional gravity from this point of view. The lagrangian is of the general form

$$I_{(4)} \sim \int \boxed{e \wedge e \wedge (\mathrm{d}\omega + \omega \wedge \omega)}. \longrightarrow \boxed{\mathfrak{R}^{\cdot\cdot} \wedge \mathfrak{S}_{\cdot\cdot}}$$

If one hopes for "power-counting renormalizability," one needs to assign dimension one to both e and ω, so that every term in eq. (3.13) is of dimension four. (Again, this is in contrast to the fact that the metric and vierbein are usually considered to have dimension zero.) As e and ω have positive dimension, the short-distance limit must have $e = \omega = 0$. The problem is now that as eq. (3.13) has no quadratic term in an expansion around $e = \omega = 0$, one cannot make sense of the "unbroken phase" that should govern the short-distance behavior; that is the essence of the unrenormalizability of quantum gravity in four dimensions.

Summary

Summary

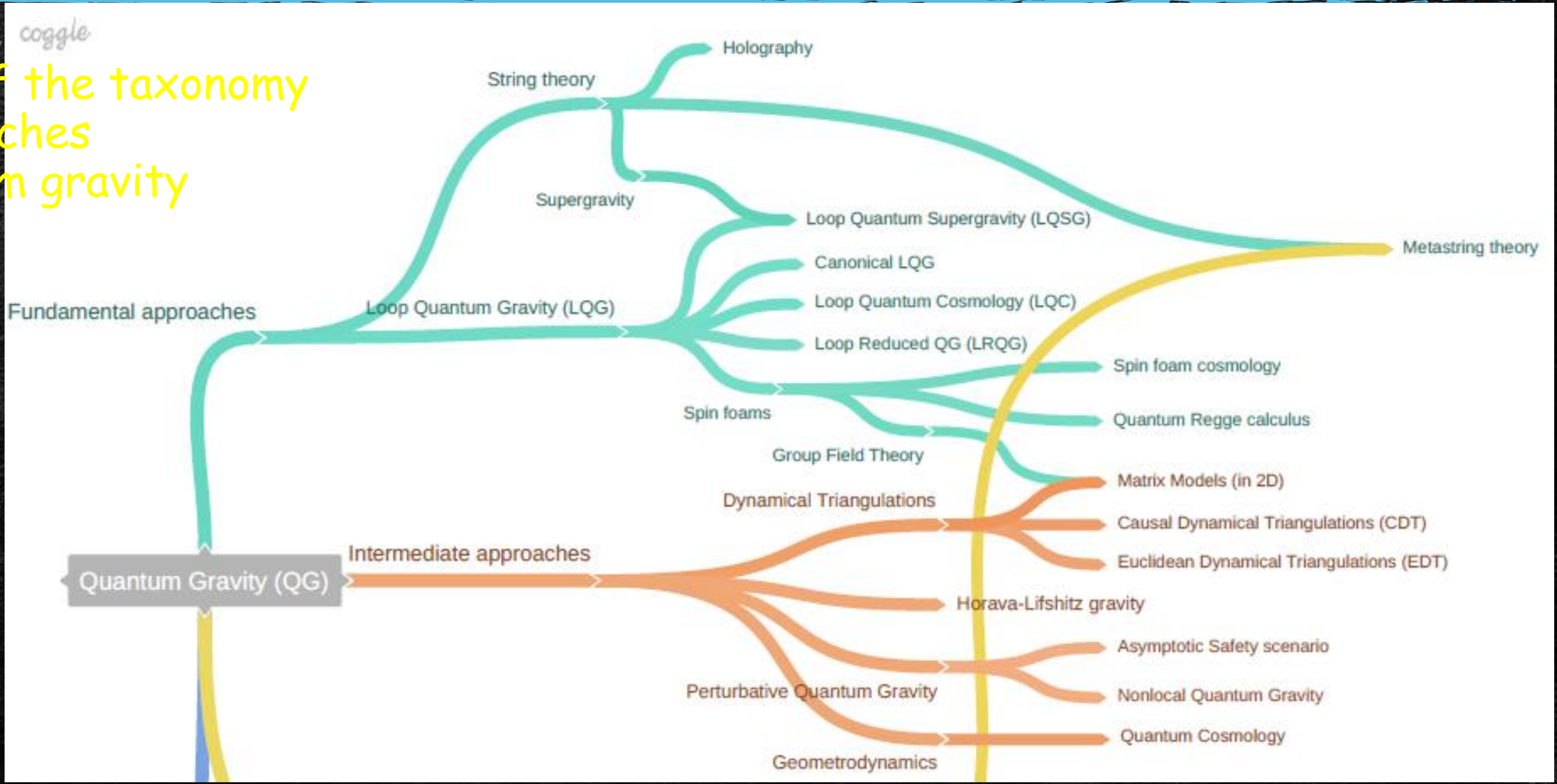
- Covariant & non-perturbative quantization of general relativity.
  - ✓ Closed set of Lagrangian, BRS-transformations, BRS-charges
  - ✓ Unitarity is ensured by K-O theorem.
- A co-translation invariance is found in the four dimensional Einstein-Hilbert action.
  - ✓ New topological invariance.
- Applications for cosmology ⟶ Next step !!

Back pocket

Introduction

J. Mielczarek, T. Trzesniewski
arXive:1708.07445

Outline of the taxonomy
of approaches
to quantum gravity

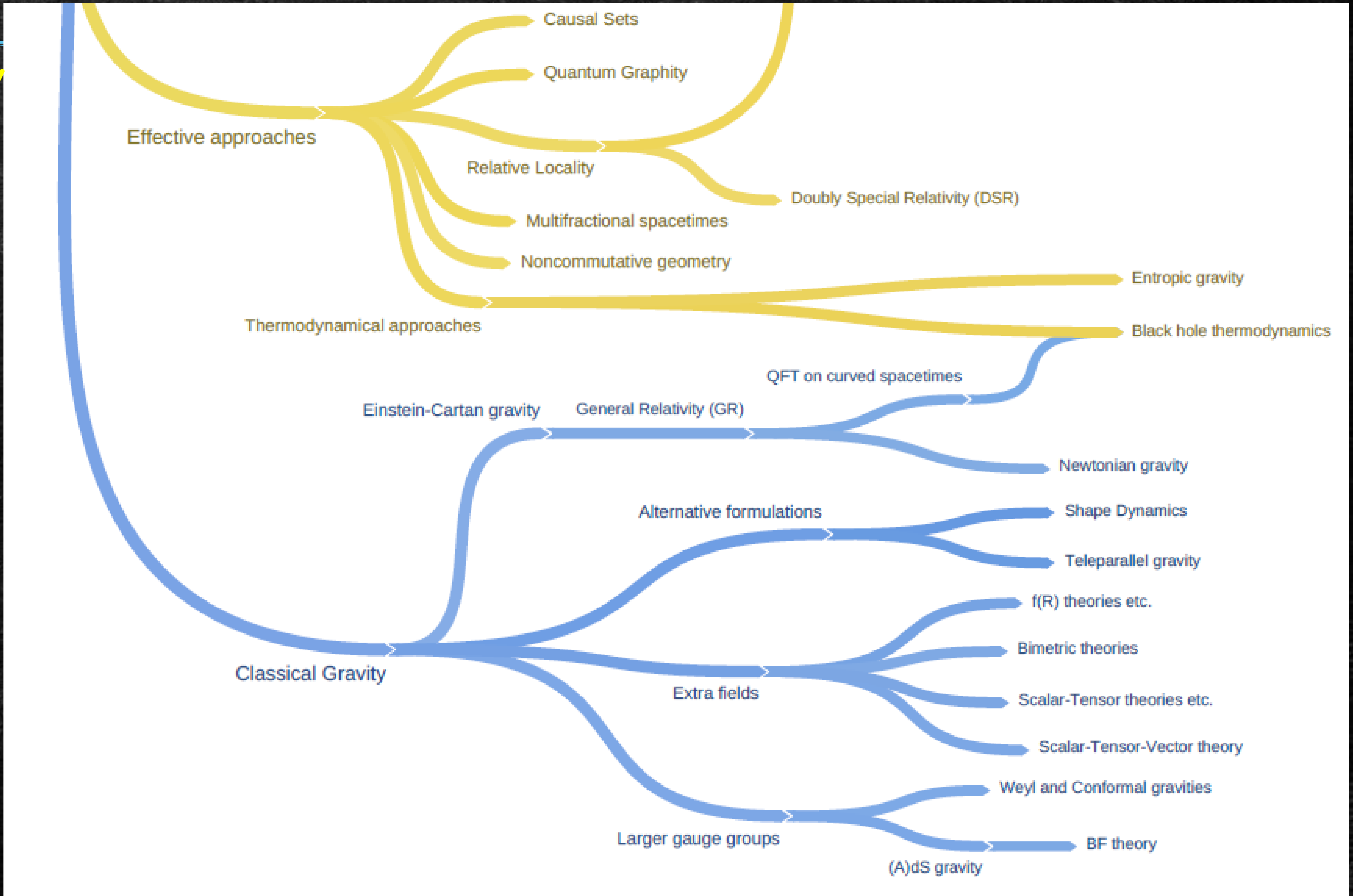


Quantum Gravity (QG)
Fundamental approaches
String theory
Holography
Supergravity
Loop Quantum Supergravity (LQSG)
Loop Quantum Gravity (LQG)
Canonical LQG
Loop Quantum Cosmology (LQC)
Loop Reduced QG (LRQG)
Spin foams
Spin foam cosmology
Quantum Regge calculus
Group Field Theory
Matrix Models (in 2D)
Metastring theory
Intermediate approaches
Dynamical Triangulations
Causal Dynamical Triangulations (CDT)
Euclidean Dynamical Triangulations (EDT)
Horava-Lifshitz gravity
Perturbative Quantum Gravity
Asymptotic Safety scenario
Nonlocal Quantum Gravity
Geometrodynamics
Quantum Cosmology

Introduction

J. Mielczarek, T. Trzesniewski
arXive:1708.07445

Outline of the taxonomy
of approaches
to quantum gravity



What is a topological quantum field theory?

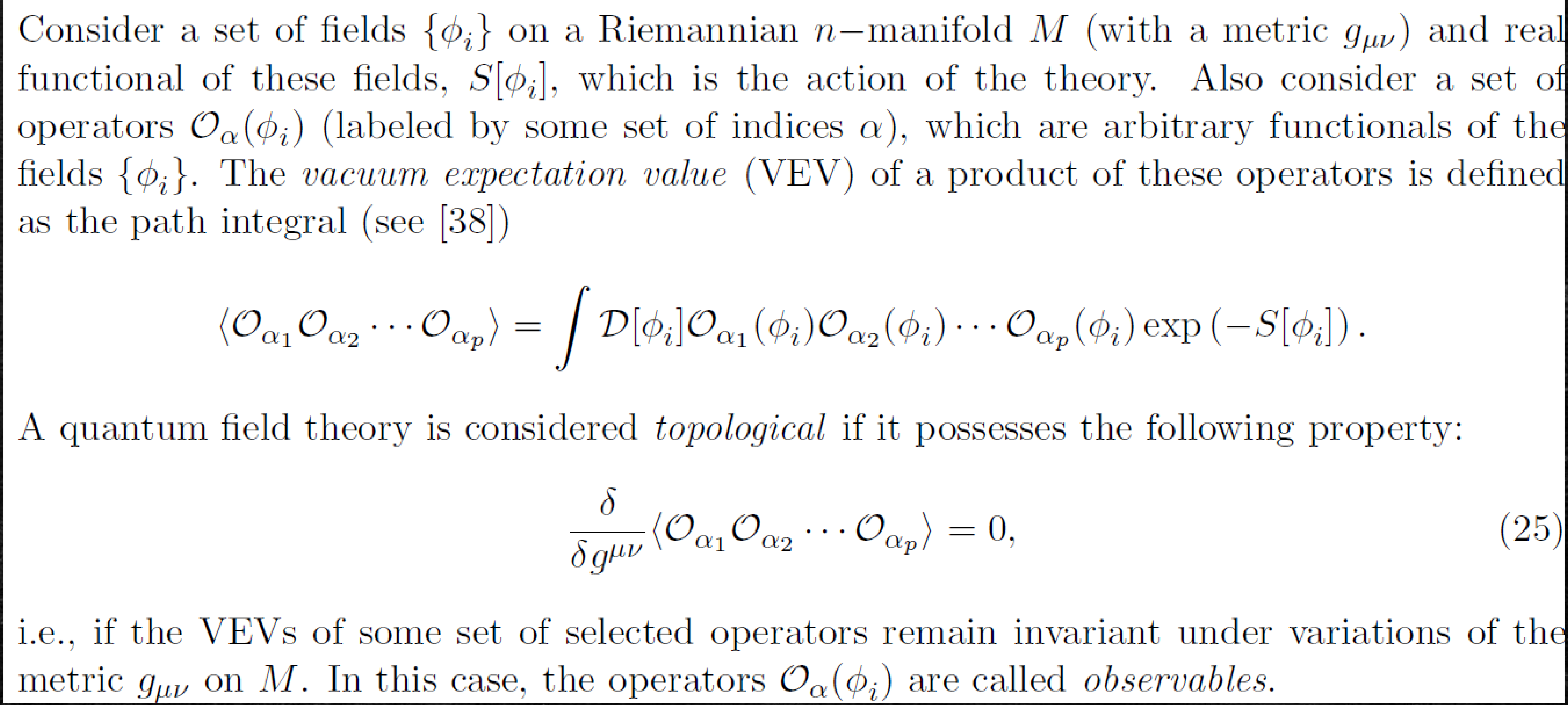
What is a topological quantum field theory?

Field theory which has topological invariants

≠

Topological Field Theory

What is a topological quantum field theory?



Consider a set of fields $\{\phi_i\}$ on a Riemannian $n-$manifold M (with a metric $g_{\mu\nu}$) and real functional of these fields, $S[\phi_i]$, which is the action of the theory. Also consider a set of operators $\mathcal{O}_\alpha(\phi_i)$ (labeled by some set of indices α), which are arbitrary functionals of the fields $\{\phi_i\}$. The *vacuum expectation value* (VEV) of a product of these operators is defined as the path integral (see [38])

$$\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\cdots\mathcal{O}_{\alpha_p}\rangle = \int \mathcal{D}[\phi_i]\mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)\exp\left(-S[\phi_i]\right).$$

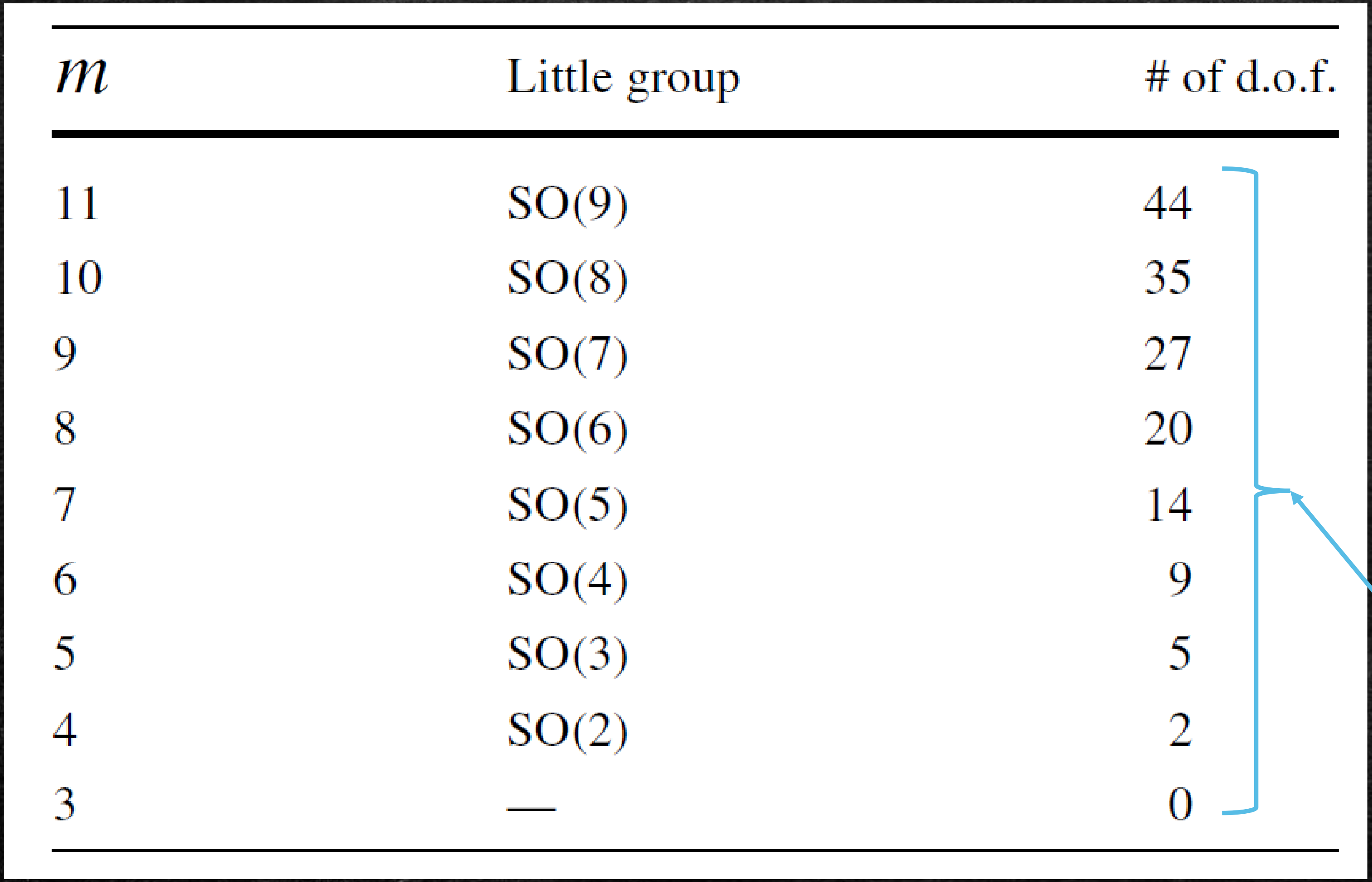
A quantum field theory is considered *topological* if it possesses the following property:

$$\frac{\delta}{\delta g^{\mu\nu}}\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\cdots\mathcal{O}_{\alpha_p}\rangle = 0, \qquad (25)$$

i.e., if the VEVs of some set of selected operators remain invariant under variations of the metric $g_{\mu\nu}$ on M. In this case, the operators $\mathcal{O}_\alpha(\phi_i)$ are called *observables*.

V.G. Ivancevic & T.T. Icancevic, arXive:0810.0344v5

Degrees of freedom of the gravitation in diverse dimension



m	Little group	# of d.o.f.
11	SO(9)	44
10	SO(8)	35
9	SO(7)	27
8	SO(6)	20
7	SO(5)	14
6	SO(4)	9
5	SO(3)	5
4	SO(2)	2
3	—	0

$$G_{time} = SO(m)$$

$$G_{space} = SO(1, m-1)$$

$$\big\downarrow$$

Little group

$$SO(m-2)$$

$$\frac{m(m-3)}{2}$$

Frè, P.: Gravity, a Geometrical Course: Volume 1: Development of the Theory and Basic Physical Applications. Gravity, a Geometrical Course. Springer Netherlands (2012)

Given the simplicity of the above arguments, why is it usually felt that $2+1$ dimensional gravity is unrenormalizable? $2+1$ dimensional gravity appears unrenormalizable if the connection ω is eliminated and the theory is written entirely in terms of the metric $g_{ij}=e_{ia}e_j{}^a$. In terms of g_{ij}, the Einstein–Hilbert action is non-polynomial – in sharp contrast to the simple polynomial structure of eq. (3.10). Also, to write the Einstein–Hilbert action in terms of g_{ij}, one must introduce the inverse metric g^{ij}. In particular, when general relativity is written in terms of the metric (rather than the independent vierbein and spin connection), it is impossible to see the unbroken phase $g=0$★. However, it is quite clear, in the renormalizable perturbation expansion sketched above, that as $e_i{}^a$ is a field of positive dimension, the short-distance behavior will involve the behavior in the unbroken phase $e=0$. Thus, the fact that one has renormalizability in the description with e and ω is closely related to the fact that in this description one can see the unbroken phase $e=\omega=0$. The usual attempt to quantize $2+1$ dimensional gravity in terms of g_{ij} is somewhat analogous to discussing a spontaneously broken gauge theory in a "unitary gauge" in which the underlying symmetry is not manifest. The attempt to quantize gauge theories in unitary gauge is notoriously treacherous.

WKB-approximation

Canonical Quantization

WKB-approximation

Operators:

$$\begin{cases} \widehat{\mathfrak{w}}^{\bullet\bullet} & = i\hbar G \frac{\delta}{\delta \mathfrak{S}_{\bullet\bullet}}, \\ \widehat{\mathfrak{S}}_{\bullet\bullet} & = \mathfrak{S}_{\bullet\bullet}, \end{cases}$$

Ordering:

“$\widehat{\mathfrak{w}}$-right and $\widehat{\mathfrak{S}}$-left”

Schrödinger equation:

$$\frac{(\hbar G)^2}{2} \eta_{**} \mathfrak{S}_{..} \wedge \frac{\delta}{\delta \mathfrak{S}_{.*}} \wedge \frac{\delta}{\delta \mathfrak{S}_{*.}} |\Psi\rangle \quad = \quad E|\Psi\rangle$$

Solution:

$$|\Psi\rangle \quad = \quad \exp\left(\frac{i}{\hbar G} S(\mathfrak{S})\right)$$

Canonical Quantization

WKB-approximation

Schrödinger equation:

$$\frac{1}{4}\mathfrak{S}_{..} \wedge \left(i\hbar G\ S''^{\cdot\cdot} - S'^{\cdot}_{\ *} \wedge S'^{*\cdot}\right)|\Psi\rangle \quad = \quad E|\Psi\rangle$$

$$\begin{aligned} S'^{ab} &= \frac{\delta S}{\delta \mathfrak{S}_{ab}}, \\ S''^{ab} &= \eta_{c_1 c_2} \frac{\delta}{\delta \mathfrak{S}_{ac_1}} \frac{\delta S}{\delta \mathfrak{S}_{c_2 b}}, \end{aligned}$$

Taylor expansion:

$$S \quad = \quad S_0 - i\hbar G\ S_1 + (-i\hbar G)^2 S_2 + \cdot\cdot$$

Canonical Quantization

WKB-approximation

Order-by-order comparison:

$$\begin{cases} \text{0th}: & -\mathfrak{S}_{\cdot\cdot}\wedge(S_0)'^{\cdot}{}_{*}\wedge(S_0)'^{*\cdot}/2 = E, \\ \text{1st}: & \mathfrak{S}_{\cdot\cdot}\wedge\left((S_0)''^{\cdot\cdot}+2(S_0)'^{\cdot}{}_{*}\wedge(S_1)'^{*\cdot}\right) = 0, \\ \text{2nd}: & \cdots \end{cases}$$

0th order Solution:

$$S_0(\mathfrak{S}) = -\frac{1}{2}\mathfrak{w}^{\cdot\cdot}\wedge\mathfrak{S}_{\cdot\cdot}$$

Classical solution

1st order Solution:

$$\frac{1}{\hbar G}\mathfrak{S}_{\cdot\cdot}\wedge(S_1(\mathfrak{S}))^{\cdot\cdot} = -\frac{1}{2}\log\left(\frac{\lambda}{(\hbar G)^2}\mathfrak{v}\right)$$

First quantum correction

Canonical Quantization

WKB-approximation

Schwarzschild black hole

$$|\Psi\rangle \quad = \quad A\sin\left[4\pi\left(2\frac{r}{l_p} - 3\frac{M}{m_p}\right) + \Phi\right]$$

Heisenberg picture.

I appreciate the kind hospitality of all members of the theory group of Nikhef, particularly Prof. J. Vermaseren and Prof. E. Laenen. A major part of this study has been conducted during my stay at Nikhef in 2016. In addition, I would like to thank Dr. Y. Sugiyama for his continuous encouragement and fruitful discussion.

A Proof of nilpotency

Coordinate vector

The coordinate vectors are one of the most fundamental vectors on $T\mathscr{M}$. The nilpotent can be confirmed as

$$\begin{aligned}\delta_{\mathrm{BRS}}\left[\delta_{\mathrm{BRS}}\left[x^{\mu}\right]\right] &= \delta_{\mathrm{BRS}}\left[\chi^{\mu}\right] = \delta_{\mathrm{BRS}}\left[g^{\mu\nu}\chi_{\nu}\right],\\ &= g^{\mu\rho}\left(\partial_{\rho}\chi^{\nu}\right)\chi_{\nu} + g^{\rho\nu}\left(\partial_{\rho}\chi^{\mu}\right)\chi_{\nu} - g^{\mu\nu}g_{\nu\rho}\left(\partial^{\rho}\chi^{\sigma}+\partial^{\sigma}\chi^{\rho}\right)\chi_{\sigma},\\ &= \left(\partial^{\mu}\chi^{\nu}\right)\chi_{\nu} + \left(\partial^{\nu}\chi^{\mu}\right)\chi_{\nu} - \left(\partial^{\mu}\chi^{\sigma}\right)\chi_{\sigma} - \left(\partial^{\sigma}\chi^{\mu}\right)\chi_{\sigma} = 0.\end{aligned}$$

Metric tensor

Starting from the BRS-transformation of the metric tensor (46),

$$\begin{aligned}\delta_{\mathrm{BRS}}\left[\delta_{\mathrm{BRS}}\left[g_{\mu\nu}\right]\right] &= \delta_{\mathrm{BRS}}\left[-g_{\mu\rho}\partial_{\nu}\chi^{\rho}\right] + \delta_{\mathrm{BRS}}\left[\mu \leftrightarrow \nu\right],\\ &= \left\{-\delta_{\mathrm{BRS}}\left[g_{\mu\rho}\right]\partial_{\nu}\chi^{\rho} + g_{\mu\rho}\left(\partial_{\nu}\delta_{\mathrm{BRS}}\left[x^{\sigma}\right]\right)\partial_{\sigma}\chi^{\rho}\right\} + \left\{\mu \leftrightarrow \nu\right\},\\ &= \left\{g_{\mu\sigma}\left(\partial_{\rho}\chi^{\sigma}\right)\partial_{\nu}\chi^{\rho} + g_{\mu\rho}\left(\partial_{\nu}\chi^{\sigma}\right)\partial_{\sigma}\chi^{\rho}\right\} + \left\{\mu \leftrightarrow \nu\right\} = 0,\end{aligned}$$

where anti-commutativity of the ghost filed is used.

Ghost field

Since the ghost field has two parts, nilpotent is checked separately. First for the χ_{μ},

$$\delta_{\mathrm{BRS}}\left[\delta_{\mathrm{BRS}}\left[\chi_{\mu}\right]\right] = \delta_{\mathrm{BRS}}\left[\delta_{\mathrm{BRS}}\left[g_{\mu\nu}\chi^{\nu}\right]\right] = \delta_{\mathrm{BRS}}\left[\delta_{\mathrm{BRS}}\left[g_{\mu\nu}\right]\right]\chi^{\nu} = 0,$$

where $\delta_{\mathrm{BRS}}[\chi^{\mu}] = 0$ and nilpotent of the metric tensor are used. Direct calculation from (45) gives the same result, too. The second part becomes

$$\delta_{\mathrm{BRS}}\left[\delta_{\mathrm{BRS}}\left[\chi^{a}{}_{b}\right]\right] = \delta_{\mathrm{BRS}}\left[\chi^{a}{}_{c}\chi^{c}{}_{b}\right] = \chi^{a}{}_{c_2}\chi^{c_2}{}_{c_1}\chi^{c_1}{}_{b} - \chi^{a}{}_{c_1}\chi^{c_1}{}_{c_2}\chi^{c_2}{}_{b} = 0,$$

due to anti-commutativity of the ghost field.

A tensor $\partial_{\mu}\chi^{\nu}$ is also nilpotent as

$$\delta_{\mathrm{BRS}}\left[\delta_{\mathrm{BRS}}\left[\partial_{\mu}\chi^{\nu}\right]\right] = -\delta_{\mathrm{BRS}}\left[\partial_{\mu}\chi^{\rho}\partial_{\rho}\chi^{\nu}\right] = -\partial_{\mu}\chi^{\rho_1}\partial_{\rho_1}\chi^{\rho_2}\partial_{\rho_2}\chi^{\nu} + \partial_{\mu}\chi^{\rho_1}\partial_{\rho_1}\chi^{\rho_2}\partial_{\rho_2}\chi^{\nu} = 0.$$

Vierbein form

$$\delta_{\mathrm{BRS}}\left[\delta_{\mathrm{BRS}}\left[\mathfrak{e}^{a}\right]\right] = \delta_{\mathrm{BRS}}\left[\mathfrak{e}^{b}\chi^{a}{}_{b}\right] = \mathfrak{e}^{b_1}\chi^{b_2}{}_{b_1}\chi^{a}{}_{b_2} + \mathfrak{e}^{b_2}\chi^{a}{}_{b_1}\chi^{b_1}{}_{b_2} = 0,$$

due to anti-commutativity of the ghost field.

Spin form

One can trace the same calculation as a case of the vierbein form due to $\delta_{\mathrm{BRS}}\left[d\chi^{ab}\right] = 0$. Detailed calculations are omitted here.

Volume form

The volume form is global scalar and their BRS-transformation is expected to vanish, which can be confirmed as

$$\delta_{\mathrm{BRS}}\left[\mathfrak{v}\right] = \frac{1}{4!}\epsilon_{....}\delta_{\mathrm{BRS}}\left[\mathfrak{e}^{\cdot}\wedge\mathfrak{e}^{\cdot}\wedge\mathfrak{e}^{\cdot}\wedge\mathfrak{e}^{\cdot}\right] = \frac{1}{3!}\epsilon_{a_1...}\delta_{\mathrm{BRS}}\left[\chi^{a_1}{}_{a_2}\mathfrak{e}^{a_2}\wedge\mathfrak{e}^{\cdot}\wedge\mathfrak{e}^{\cdot}\wedge\mathfrak{e}^{\cdot}\right] = 0,$$

due to $\mathfrak{e}^{\cdot}\wedge\mathfrak{e}^{\cdot}\wedge\mathfrak{e}^{\cdot}\wedge\mathfrak{e}^{\cdot} \propto \epsilon^{\cdot\cdot\cdot\cdot}$ and $\chi^{a_1}{}_{a_2} = 0$ when $a_1 = a_2$.

Surface form

The BRS-transformation of the surface form is given by

$$\delta_{\mathrm{BRS}}\left[\mathfrak{S}_{ab}\right] = \frac{1}{2}\epsilon_{abc_1c_2}\delta_{\mathrm{BRS}}\left[\mathfrak{e}^{c_1}\wedge\mathfrak{e}^{c_2}\right] = \epsilon_{abc_1c_2}\chi^{c_1}_{\ c_3}\mathfrak{e}^{c_3}\wedge\mathfrak{e}^{c_2}\ \left(=\ \mathfrak{c}^{\cdot}\wedge\bar{\mathfrak{e}}_{ab\cdot}\right).$$

Applying the BRS-transformation on it again, one can get

$$\begin{aligned}\delta_{\mathrm{BRS}}\left[\delta_{\mathrm{BRS}}\left[\mathfrak{S}_{ab}\right]\right] &= \epsilon_{abc_1c_2}\delta_{\mathrm{BRS}}\left[\chi^{c_1}_{\ c_3}\mathfrak{e}^{c_3}\wedge\mathfrak{e}^{c_2}\right],\\ &= \epsilon_{abc_1c_2}\left\{\chi^{c_1}_{\ c_4}\chi^{c_4}_{\ c_3}\mathfrak{e}^{c_3}\wedge\mathfrak{e}^{c_2} - \chi^{c_1}_{\ c_3}\chi^{c_3}_{\ c_4}\mathfrak{e}^{c_4}\wedge\mathfrak{e}^{c_2} - \chi^{c_1}_{\ c_3}\chi^{c_2}_{\ c_4}\mathfrak{e}^{c_3}\wedge\mathfrak{e}^{c_4}\right\} = 0,\end{aligned}$$

because first term is the same as the second term and the third term is symmetric with c_1 and c_2 exchange.

ghost forms

The BRS-transformation of $\mathfrak{c}^{a}$ is given by

$$\begin{aligned}\delta_{\mathrm{BRS}}\left[\mathfrak{c}^{a}\right] &= \delta_{\mathrm{BRS}}\left[\chi^{a}_{\ b}\,\mathcal{E}^{b}_{\mu}\,dx^{\mu}\right],\\ &= \chi^{a}_{\ b_1}\chi^{b_1}_{\ b_2}\mathcal{E}^{b_2}_{\mu}dx^{\mu} - \chi^{a}_{\ b_1}\,\mathcal{E}^{b_2}_{\mu}\chi^{b_2}_{\ b_1}dx^{\mu} + \chi^{a}_{\ b}\,\left(\partial_{\mu}\chi^{\nu}\right)\mathcal{E}^{b}_{\nu}\,dx^{\mu} - \chi^{a}_{\ b}\,\mathcal{E}^{b}_{\mu}d\chi^{\mu} = 0,\end{aligned}$$

Other forms

Nilpotent of other forms are trivial and the proof is omitted here.

Gravitational Lagrangian

The quantum Lagrangian must be the BRS-null. The gauge-fixing and Fadeef-Popov terms, $\mathfrak{L}_{GF} + \mathfrak{L}_{FP}$, will be constructed to satisfy the BRS-null condition. Therefore, the proof for the gravitational Lagrangian is given here by

$$\delta_{\mathrm{BRS}}\left[\mathfrak{L}_{G}\right] = \frac{1}{2}\delta_{\mathrm{BRS}}\left[\left(d\mathfrak{w}^{\cdot\cdot} + \mathfrak{w}^{\cdot}{}_{*}\wedge\mathfrak{w}^{*\cdot}\right)\wedge\mathfrak{S}_{\cdot\cdot} - \frac{\Lambda}{3!}\mathfrak{v}\right].$$

The BRS-transformation for the volume form is vanished by itself. For the derivative term,

$$\begin{aligned}\delta_{\mathrm{BRS}}\left[d\mathfrak{w}^{\cdot\cdot}\wedge\mathfrak{S}_{\cdot\cdot}\right] &= \epsilon_{abc_2c_3}\chi^{b}{}_{c_1}d\mathfrak{w}^{ac_1}\wedge\mathfrak{e}^{c_2}\wedge\mathfrak{e}^{c_3} + \epsilon_{abc_2c_3}\mathfrak{w}^{ac_1}\wedge d\chi^{b}{}_{c_1}\wedge\mathfrak{e}^{c_2}\wedge\mathfrak{e}^{c_3}\\ &\quad +\epsilon_{abc_1c_2}\chi^{c_1}_{\ c_3}\ d\mathfrak{w}^{ab}\wedge\mathfrak{e}^{c_3}\wedge\mathfrak{e}^{c_2}\\ &= 2\ \mathfrak{w}^{ac_1}\wedge d\chi^{b}{}_{c_1}\wedge\mathfrak{S}_{ab},\end{aligned}$$

where first- and third-terms are cancelled each other. Remnant term is transformed as

$$\begin{aligned}\delta_{\mathrm{BRS}}\left[\mathfrak{w}^{\cdot}{}_{*}\wedge\mathfrak{w}^{*\cdot}\wedge\mathfrak{S}_{\cdot\cdot}\right] &= \epsilon_{abc_2c_3}\chi^{c_2}_{\ c_4}\mathfrak{w}^{ac_1}\wedge\mathfrak{w}_{c_1}{}^{b}\wedge\mathfrak{e}^{c_4}\wedge\mathfrak{e}^{c_3} + \epsilon_{abc_3c_4}\chi^{c_1}_{\ c_2}\mathfrak{w}^{ac_1}\wedge\mathfrak{w}_{c_2}{}^{b}\wedge\mathfrak{e}^{c_3}\wedge\mathfrak{e}^{c_4}\\ &\quad +\epsilon_{abc_3c_4}\chi^{b}{}_{c_2}\mathfrak{w}^{ac_1}\wedge\mathfrak{w}_{c_1}{}^{c2}\wedge\mathfrak{e}^{c_3}\wedge\mathfrak{e}^{c_4} - 2\ \mathfrak{w}^{ac_1}\wedge d\chi^{b}{}_{c_1}\wedge\mathfrak{S}_{ab},\\ &= -2\ \mathfrak{w}^{ac_1}\wedge d\chi^{b}{}_{c_1}\wedge\mathfrak{S}_{ab}.\end{aligned}$$

In the r.h.s of the first line, the second term is zero as itself, and first- and third-terms are cancelled each other. Therefore one can confirmed $\delta_{\mathrm{BRS}}\left[\mathfrak{L}_{G}\right] = 0$ after summing up all terms.

If we use a following remake, we can give simpler proofs for above forms.

Remark

If both of two fields, α and β, are nilpotent, $\alpha\beta$ is also nilpotent.

Proof:

If a field X is nilpotent, signatures of the Leibniz rule satisfy $\epsilon_{X} = -\epsilon_{\delta X}$ due to $\delta_{\mathrm{BRS}}[\delta_{\mathrm{BRS}}[X]] = 0$ and (45), where ϵ_{X} ($\epsilon_{\delta X}$) is a signature of X ($\delta_{\mathrm{BRS}}[X]$), respectively. Therefore

$$\delta_{\mathrm{BRS}}\left[\delta_{\mathrm{BRS}}\left[\alpha\beta\right]\right] = \epsilon_{\alpha}\delta_{\mathrm{BRS}}\left[\alpha\right]\delta_{\mathrm{BRS}}\left[\beta\right] + \epsilon_{\delta\alpha}\delta_{\mathrm{BRS}}\left[a\right]\delta_{\mathrm{BRS}}\left[\beta\right] = 0.$$

B Equations of motion

From the classical Lagrangian form, the torsion-less condition and the Einstein equation are obtained as the equations of motion by requiring a stationary condition for a variation of the action. The same procedure can extract Euler-Lagrange equations from the quantum Lagrangian. Here the quantum Lagrangian is summarized as

$$\mathfrak{L}_{QG} = \mathfrak{L}_G + \mathfrak{L}_{GF} + \mathfrak{L}_{FP}, \tag{49}$$

$$\left\{\begin{array}{lcll} \mathfrak{L}_G &=& \frac{1}{2}\left(\mathfrak{R}^{\cdot\cdot} - \frac{\Lambda}{3!}\overline{\mathfrak{S}}^{\cdot\cdot}\right) \wedge \mathfrak{S}_{\cdot\cdot}, & (11) \\ \mathfrak{L}_{GF} &=& -\frac{1}{2}\left(d\mathfrak{b}^{\cdot\cdot} + \alpha\mathfrak{b}^{\cdot}{}_{*} \wedge \mathfrak{b}^{*\cdot}\right) \wedge \mathfrak{S}_{\cdot\cdot}, & (50) \\ \mathfrak{L}_{FP} &=& -\frac{i}{2}\left(d\tilde{\mathfrak{c}}^{\cdot\cdot} + \alpha\tilde{\mathfrak{c}}^{\cdot}{}_{*} \wedge \mathfrak{b}^{*\cdot}\right) \wedge \mathfrak{c}^{\star} \wedge \overline{\mathfrak{e}}_{\star\cdot\cdot}, & (51) \end{array}\right.$$

Euler-Lagrange equations are obtained as follows:

$\boxed{\delta_{\mathfrak{w}}}$

$$\mathfrak{T}^a \;=\; d\mathfrak{e}^a + \mathfrak{w}^a{}_{\cdot} \wedge \mathfrak{e}^{\cdot} \;=\; 0.$$

This is the torsion-less condition as the the same as a case of the classical Lagrangian.

$\boxed{\delta_{\mathfrak{e}}}$

$$\frac{1}{2}\overline{(\mathfrak{R}^{\cdot\cdot} \wedge \mathfrak{e}^{\cdot})}_a - \Lambda\mathfrak{V}_a - \frac{1}{2}\left(d\mathfrak{b}^{\cdot\cdot} + \alpha\mathfrak{b}^{\cdot}{}_{*} \wedge \mathfrak{b}^{*\cdot}\right) \wedge \overline{\mathfrak{e}}_{a\cdot\cdot} - \frac{i}{2}\left(d\tilde{\mathfrak{c}}^{\cdot\cdot} + \alpha\tilde{\mathfrak{c}}^{\cdot}{}_{*} \wedge \mathfrak{b}^{*\cdot}\right) \wedge \overline{\mathfrak{c}}_{a\cdot\cdot} \;=\; 0, \tag{72}$$

where $\mathfrak{V}_a = \epsilon_{ab_1b_2b_3}\mathfrak{e}^{b_1} \wedge \mathfrak{e}^{b_2} \wedge \mathfrak{e}^{b_3}/3!$. The first two terms give the Einstein equation without matter fields. Third- and fourth-terms newly appeared from the gauge-fixing and Feddeev–Popov Lagrangian forms.

$\boxed{\delta_{\mathfrak{b}}}$

$$d\mathfrak{S}_{ab} - \alpha\left(\mathfrak{b}^{\cdot}_{a} \wedge \mathfrak{S}_{\cdot b} - i\tilde{\mathfrak{c}}^{\cdot}_{a} \wedge \mathfrak{c}^{\cdot} \wedge \overline{\mathfrak{e}}_{\cdot\cdot b}\right) \;=\; 0.$$

We note that $\mathfrak{b}$ and $\delta_{\mathfrak{b}}$ are anti-commute each other and the variation operator is applied from the left. When the Landau-gauge is used, the de Donder gauge-fixing condition $d\mathfrak{S}_{ab} = 0$ is obtained.

$\boxed{\delta_{\mathfrak{c}}}$

$$\left(d\tilde{\mathfrak{c}}^{\cdot\cdot} + \alpha\tilde{\mathfrak{c}}^{\cdot}{}_{*} \wedge \mathfrak{b}^{*\cdot}\right) \wedge \overline{\mathfrak{e}}_{a\cdot\cdot} \;=\; 0, \tag{73}$$

where the anti-commutation among $\mathfrak{c}$, $\delta\mathfrak{c}$ and $\mathfrak{b}$ is used.

$\boxed{\delta_{\tilde{\mathfrak{c}}}}$

$$\epsilon_{ab\cdot\cdot}\left(d\left(\mathfrak{c}^{\cdot} \wedge \mathfrak{e}^{\cdot}\right) - \alpha\mathfrak{b}^{\cdot}{}_{*} \wedge \mathfrak{c}^{*} \wedge \mathfrak{e}^{\cdot}\right) \;=\; \epsilon_{ab\cdot\cdot}\left(d\mathfrak{c}^{\cdot} - \alpha\mathfrak{b}^{\cdot}{}_{*} \wedge \mathfrak{c}^{*}\right) \wedge \mathfrak{e}^{\cdot} \;=\; 0,$$

where the de Donder condition is used.

The BRS-transformation may give another set of equations, which must be consistent with above equations:

$$\begin{aligned} \delta_{\mathrm{BRS}}\left[\mathfrak{T}^a\right] &= \chi^a{}_{\cdot}\, d\mathfrak{e}^{\cdot} + d\chi^a{}_{\cdot} \wedge \mathfrak{e}^{\cdot} + \chi^a{}_{\cdot}\, \mathfrak{w}_{*}{}^{\cdot} \wedge \mathfrak{e}^{*} + \chi_{*}{}^{\cdot}\, \mathfrak{w}^a{}_{\cdot} \wedge \mathfrak{e}^{*} - d\chi^a{}_{\cdot} \wedge \mathfrak{e}^{\cdot} + \chi^{\cdot}{}_{*}\mathfrak{w}^a{}_{\cdot} \wedge \mathfrak{e}^{*} \\ &= \chi^a{}_{\cdot}\, \mathfrak{T}^{\cdot} \;=\; 0. \end{aligned}$$

This is consistent with the torsion-less condition. The BRS-transformation for the volume form is vanished and last two-terms are cancelled each other such as

$$\delta_{\mathrm{BRS}}\left[i\left(d\tilde{\mathfrak{c}}^{\cdot\cdot} + \alpha\tilde{\mathfrak{c}}^{\cdot}{}_{*} \wedge \mathfrak{b}^{*\cdot}\right) \wedge \overline{\mathfrak{c}}_{a\cdot\cdot}\right] \;=\; -\left(d\mathfrak{b}^{\cdot\cdot} + \alpha\mathfrak{b}^{\cdot}{}_{*} \wedge \mathfrak{b}^{*\cdot}\right) \wedge \overline{\mathfrak{e}}_{a\cdot\cdot}$$

Therefor, the BRS-transformation of (72) is given by

$$0 \;=\; \epsilon_{abc\bullet}\delta_{\mathrm{BRS}}\left[\left(d\mathfrak{w}^{ab} + \mathfrak{w}^a{}_{\cdot} \wedge \mathfrak{w}^{\cdot b}\right) \wedge \mathfrak{e}^c\right] = \epsilon_{abc\bullet}\left\{\chi^b{}_{\cdot}\left(d\mathfrak{w}^{a\cdot} + \mathfrak{w}^a{}_{*} \wedge \mathfrak{w}^{*\cdot}\right) \wedge \mathfrak{e}^c + \left(d\mathfrak{w}^{ab} + \mathfrak{w}^a{}_{\cdot} \wedge \mathfrak{w}^{\cdot b}\right) \wedge \mathfrak{c}^c\right\}.$$

This is consistent with the Einstein equation.

$$\delta_{\mathrm{BRS}}\left[d\mathfrak{e}^a - \alpha\left(\mathfrak{b}^a{}_{\cdot} \wedge \mathfrak{e}^{\cdot} - i\tilde{\mathfrak{c}}^a{}_{\cdot} \wedge \mathfrak{c}^{\cdot}\right)\right] \;=\; d\mathfrak{c}^a = 0,$$

where α-terms are cancelled each other. The BRS-transformation of (73) gives an equation of motion for $\mathfrak{b}$ in (72).

$$\epsilon_{ab\cdot\cdot}\delta_{\mathrm{BRS}}\left[d\left(\mathfrak{c}^{\cdot} \wedge \mathfrak{e}^{\cdot}\right) - \alpha\mathfrak{b}^{\cdot}{}_{*} \wedge \mathfrak{c}^{*} \wedge \mathfrak{e}^{\cdot}\right] \;=\; 0,$$

where $\epsilon_{\bullet\bullet\cdot\cdot}\mathfrak{c}^{\cdot} \wedge \mathfrak{c}^{\cdot} = 0$ is used. This is not an equation, but an identity.

C proof of (61)

The proof of (61) can be given as follows: At first, shorthand notations are introduced to omit indices in following calculations for a while such as,

$$\widehat{\mathfrak{Q}}_{\mathfrak{b}} = \frac{1}{2}\widehat{\boldsymbol{\beta}}\cdot\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right], \qquad \widehat{\mathfrak{Q}}_{\tilde{\mathfrak{c}}} = \frac{i}{4}\widehat{\boldsymbol{\chi}}\cdot\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right] - \frac{1}{4}\widehat{\boldsymbol{\beta}}\cdot\widehat{\mathfrak{Q}},$$

because the correspondence of indices is clear in them. By using these notions, the commutation relation can be represented as

$$\begin{aligned} -8i\left[\widehat{\mathfrak{Q}}_{\tilde{\mathfrak{c}}}, \widehat{\mathfrak{Q}}_{\mathfrak{b}}\right] &= \left[\widehat{\boldsymbol{\beta}}\cdot\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right], \widehat{\boldsymbol{\chi}}\cdot\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right]\right] + i\left[\widehat{\boldsymbol{\beta}}\cdot\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right], \widehat{\boldsymbol{\beta}}\cdot\widehat{\mathfrak{Q}}\right], \\ &= \widehat{\boldsymbol{\beta}}\left[\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right], \widehat{\boldsymbol{\chi}}\right]\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right] + \widehat{\boldsymbol{\chi}}\left[\widehat{\boldsymbol{\beta}}, \delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right]\right]\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right] \\ &\quad +i\,\widehat{\boldsymbol{\beta}}\left[\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right], \widehat{\boldsymbol{\beta}}\cdot\widehat{\mathfrak{Q}}\right] + i\left[\widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\beta}}\cdot\widehat{\mathfrak{Q}}\right]\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right]. \end{aligned} \tag{74}$$

where $[\widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\chi}}] = 0$ is used. The first term of (74) becomes

$$\begin{aligned} \widehat{\boldsymbol{\beta}}\left[\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right], \widehat{\boldsymbol{\chi}}\right]\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right] &= \widehat{\boldsymbol{\beta}}\left(\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right]\widehat{\boldsymbol{\chi}} - \widehat{\boldsymbol{\chi}}\,\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right]\right)\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right], \\ &= \widehat{\boldsymbol{\beta}}\left(\delta_{\mathrm{BRS}}\left[\left\{\widehat{\mathfrak{Q}}, \widehat{\boldsymbol{\chi}}\right\}\right] + i\left[\widehat{\boldsymbol{\beta}}, \widehat{\mathfrak{Q}}\right]\right)\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right], \\ &= i\widehat{\boldsymbol{\beta}}\,\left[\widehat{\boldsymbol{\beta}}, \widehat{\mathfrak{Q}}\right]\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right] = 4\widehat{\mathfrak{Q}}_{\mathfrak{b}}, \end{aligned}$$

where $\delta_{\mathrm{BRS}}\left[\left\{\widehat{\mathfrak{Q}}, \widehat{\boldsymbol{\chi}}\right\}\right] = 0$ due to (58), and (56) are used. Note that $\widehat{\boldsymbol{\chi}}$ and $\widehat{\mathfrak{Q}}$ have $\epsilon_X = -1$ in (45). The second term of (74) is zero, because

$$\left[\widehat{\boldsymbol{\beta}}, \delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right]\right] \;=\; \delta_{\mathrm{BRS}}\left[\left[\widehat{\boldsymbol{\beta}}, \widehat{\mathfrak{Q}}\right]\right] = 0,$$

due to (56). The third term is also zero as

$$i\widehat{\boldsymbol{\beta}}\left[\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right], \widehat{\boldsymbol{\beta}}\cdot\widehat{\mathfrak{Q}}\right] \;=\; i\widehat{\boldsymbol{\beta}}\widehat{\boldsymbol{\beta}}\left[\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right], \widehat{\mathfrak{Q}}\right] + i\widehat{\boldsymbol{\beta}}\,\delta_{\mathrm{BRS}}\left[\left[\widehat{\mathfrak{Q}}, \widehat{\boldsymbol{\beta}}\right]\right]\widehat{\mathfrak{Q}} = 0,$$

due to (56). A relation $\left[\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right], \widehat{\mathfrak{Q}}\right] = 0$ can be confirmed by direct calculations. The last term of (74) becomes

$$i\left[\widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\beta}}\cdot\widehat{\mathfrak{Q}}\right]\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right] \;=\; i\widehat{\boldsymbol{\beta}}\,\left[\widehat{\boldsymbol{\beta}}, \widehat{\mathfrak{Q}}\right]\delta_{\mathrm{BRS}}\left[\widehat{\mathfrak{Q}}\right] = 4\widehat{\mathfrak{Q}}_{\mathfrak{b}}.$$