Electroweak interactions - Exercices

1 Groups, spinors and gauge transformations

1. For Lie Groups, a group element arbitrarily close to the identity can be expanded as:

$$U = I + i\alpha^a T^a \tag{1}$$

where α^a are real parameters and T^a are the generators. The latter form a Lie algebra that is defined though the commutation relations:

$$[T^a, T^b] = if^{abc}T^c \tag{2}$$

where f^{abc} are the structure constants and are real.

   (a) For U a group element of SU(N) and SO(N) determine the properties of the generators (hermitian, etc) and, for $N \times N$ transformations (corresponding to the fundamental representation) determine the number of the group generators.

   (b) Check that for SU(2), the generators $T^a = \sigma^a/2$, where σ^a are the Pauli Matrices, satisfy the commutation relations above with $f^{abc} = \epsilon^{abc}$.

   (c) Check that for SU(2), the representation of dimension 2 satisfy $-T^{a*} = UT^aU^\dagger$ with $U = i\sigma_2$. This representation is said to be pseudo-real.[1]

   (d) Consider an SU(2) lepton doublet,

$$L_e = (\nu_{eL}\, e_L)^T . \tag{3}$$

   How does this doublet transforms under an SU(2) transformation described by (1) with $\alpha^3 = \alpha$, and $\alpha^1 = \alpha^2 = 0$.

2. We are now going to look at the Lorentz transformations acting on spinors.

   (a) The 4 dimensional matrices $S^{\mu\nu} = i/4[\gamma^\mu, \gamma^\nu]$ provide a representation of the Lorentz algebra. Give the form of S^{0i} and S^{ij} in terms of the Pauli matrices using the Weyl representation of the Dirac matrices γ^μ.

   (b) Considering that a Dirac Spinor transform as $\Psi \to \exp(-i\omega_{\mu\nu}S^{\mu\nu})\Psi$, where $\omega_{\mu\nu}$ is ar antisymmetric tensor, check that the left and right components of the spinor $\Psi = (\psi_L\, \psi_R)^T$

      - transform in the same way when considering an infinitesimal rotation of angle $\omega_{12} = -\omega_{12} = \theta$ in the xy plane

[1] For a given representation R described with the generators T^a_R, the complex conjugate representation $\bar{R}$ with $T^a_{\bar{R}} = -T^{a*}_R$ is also a representation as it satisfy (2). A real representation satisfy $T^a_{\bar{R}} = T^a_R$ (or there is a unitary transformation V such that $\tilde{T}^a_R = V^{-1}T^a_RV$ satisfy $\tilde{T}^a_{\bar{R}} = \tilde{T}^a_R$). You can check that the Adjoint representations of SU(N) or the fundamental representations of SO(N) are real representations.

- transform differently under an infinitesimal boost of rapidity $\omega_{01} = -\omega_{10} = \beta$ in the x-direction.

  This is directly related to the fact that the Dirac representation of the Lorentz group is reducible.

(c) Check that $i\sigma^2\chi_L^*$ transforms as ψ_R using the last item of Exercise 1

(d) Check that $\chi_L^T i\sigma^2\psi_L$ is a scalar under Lorentz transformations, conclude that $\bar{\Psi}\Psi$ with $\Psi = (\psi_L\,\psi_R)^T$ is a Lorentz scalar.

3. Considering the covariant derivative

$$D_\mu = \partial_\mu - igA_\mu\,, \tag{4}$$

the field strength can be defined as $[D_\mu, D_\nu] = -igF_{\mu\nu}$.

(a) For an abelian gauge group check that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(b) For a non abelian gauge group check that $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$ with $F_{\mu\nu} = F^a_{\mu\nu}T^a$ and $A_\mu = A^a_\mu T^a$.

4. The kinetic term for and $SU(2)_L\times U(1)_Y$ lepton doublet $L_e = (L_{e1}\,L_{e2})^T$ with $L_{e1} = \nu_{eL}$ and $L_{e2} = e_L$ reads $\bar{L}_e \not{D} L_e$ with $D_\mu = \partial_\mu - ig'Y_{eL}B_\mu - igW_\mu$ and $W_\mu = W^a_\mu\frac{\sigma^a}{2}$. In the compact expression of this kinetic term, write explicitly the sum over SU(2), Lorentz and spinor indices.

2 Processes

In order to fix the notations, here we follow the notations of [1] describing a Dirac spinor as

$$\psi\,(x,t) = \sum_s \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} \left(a_{k,s} u\,(k,s)\, e^{-ikx} + b^\dagger_{k,s} v\,(k,s)\, e^{ikx} \right)\,. \tag{5}$$

The sum runs over spin values s and $a^\dagger_{k,s}$ $\left(b^\dagger_{k,s}\right)$ creates a particle (antiparticle) of momentum $\vec{k}$ and spin s. The spinors u and v obey the Dirac equations

$$\begin{aligned} (\not{k} - m)u\,(k,s) &= 0 \\ (\not{k} + m)v\,(k,s) &= 0 \end{aligned} \tag{6}$$

with the sums:

$$\sum_s u\,(k,s)\,\bar{u}\,(k,s) = \not{k} + m, \quad \sum_s v\,(k,s)\,\bar{v}\,(k,s) = \not{k} - m \tag{7}$$

For a vector, we use:

$$V^\mu\,(x,t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} \left(a_{k,\lambda}\epsilon^\mu_{k,\lambda} e^{-ikx} + b^\dagger_{k,s}\epsilon^{\mu *}_{k,\lambda} e^{ikx} \right) \tag{8}$$

where the sum runs over the polarizations λ and the $\epsilon^\mu_{k,\lambda}$ are the polarization vectors that satisfy:

$$\epsilon^\mu_{k,\lambda}\epsilon_{k,\lambda'\,\mu} = -\delta_{\lambda\lambda'} \quad \text{and} \quad k_\mu\epsilon^\mu_{k,\lambda} = 0\,. \tag{9}$$

Also, in order to calculate decay widths, you can make use of the following expression for the decay width [2]:

$$d\Gamma(X \to ab) = \frac{1}{32\pi^2}|\mathcal{M}|^2 \frac{|\vec{p}_a|^2}{m_X^2} d\Omega \tag{10}$$

which is valid in the restframe of the particle X. $|\mathcal{M}|^2$ refers to the transition matrix squared summed (averaged) over final (initial) state polarization and spins and $\vec{p}_a$ and $d\Omega = d\phi_a d\cos\theta_a$ are the momentum and the solid angle of particle a respectively.

2.1 Decay of a gauge boson and number of ν_L families

1. Compute the decay width for the process:

$$W^- \to e^- \bar{\nu}_e \tag{11}$$

neglecting electron and the neutrino masses. For that purpose, use the lagrangian for gauge boson-lepton interactions that were obtained in the lectures. You can also use of the following tools:

   (a) the sum over the polarization vectors of the on-shell W boson of 4-momentum k is given by $\sum_\lambda \epsilon^\mu_{k,\lambda}\epsilon^{\nu\,*}_{k,\lambda} = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m_W^2}$

   (b) tr(any odd nb. of γ's)=0

   (c) $\text{tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = 4\,\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\alpha\nu}g^{\mu\beta}\right)$

   (d) $\text{tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^5\right) = -4i\,\epsilon^{\alpha\beta\mu\nu}$

   (e) $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$ in Dirac and Weyl representations.

2. Compute the decay width for the process:

$$Z \to \bar{\nu}\nu\,. \tag{12}$$

You can use the similitudes between the Lagrangians driving (11) and (12) for a rapid evaluation.
This is of interest because the total decay width of the Z boson can be obtained analyzing the total cross-section for e^+e^- annihilation at the Z pole obtained at LEPI and SLAC. Compare the obtained decay width with the invisible decay width of Z obtained experimentally (see e.g. PDG [2]) and deduce the number of families of active neutrinos in Nature.

2.2 Decay of the top quark and Goldstone bosons

1. Compute the decay width for the process:

$$t \to W^+ b \tag{13}$$

using the lagrangian for gauge boson-quarks interactions that were obtained in the lectures and assuming that the CKM matrix element $|V_{tb}| \simeq 1$ and neglecting the mass of the bottom quark.

2. Compute the decay width for the process:

$$t \to \phi^+ b\,, \tag{14}$$

where ϕ^+ is the charged Goldstone boson appearing in the decomposition of the Standard Model (SM) scalar doublet (use the top-SM scalar yukawa interactions). Compare to the result obtained in the previous exercise and discuss.

References

[1] Michael E. Peskin and Daniel V. Schroeder. An Introduction to quantum field theory. 1995.

[2] Beringer et al. (Particle Data Group). Review of particle physics. *Phys. Rev. D*, 86:010001, Jul 2012.