Gravitational Waves from General Relativity

Lecture 1

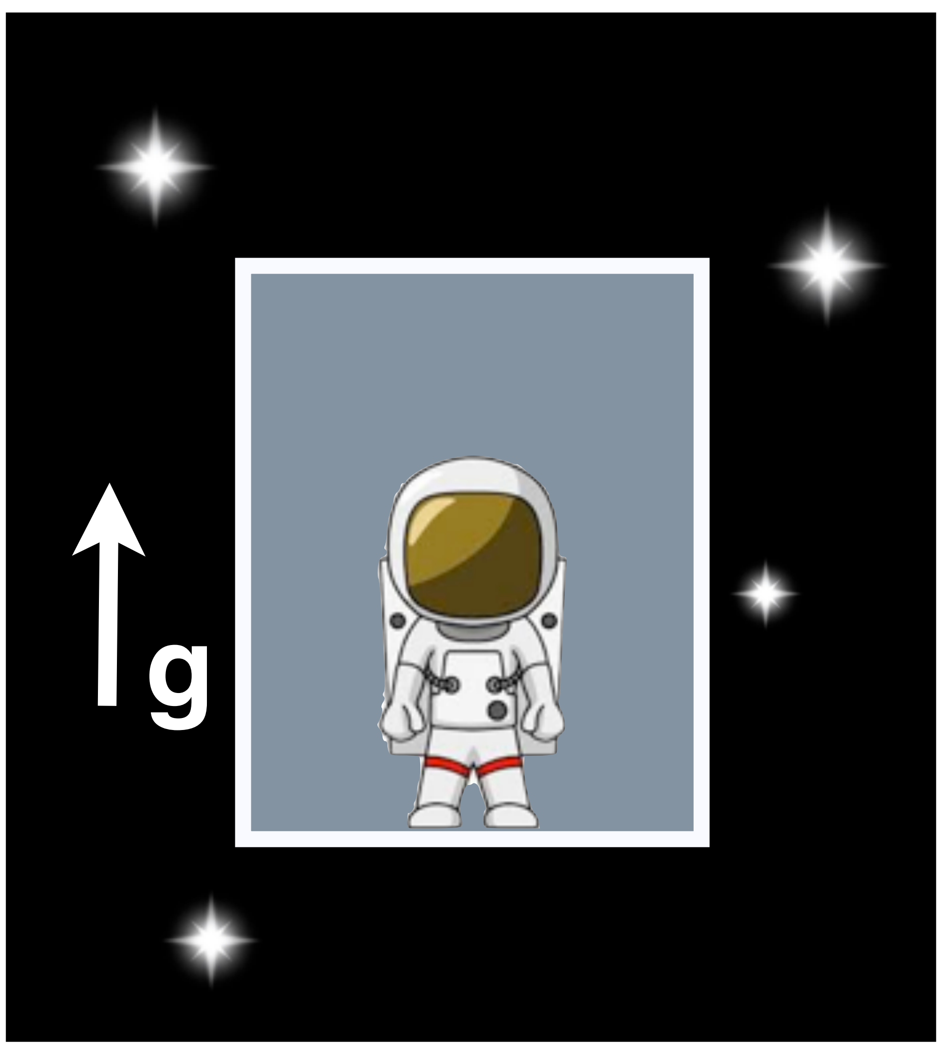
Agenda

1. Motivate the Einstein Equations.
2. Motivate how one might solve these equations.
3. Explain the meaning of the solutions.

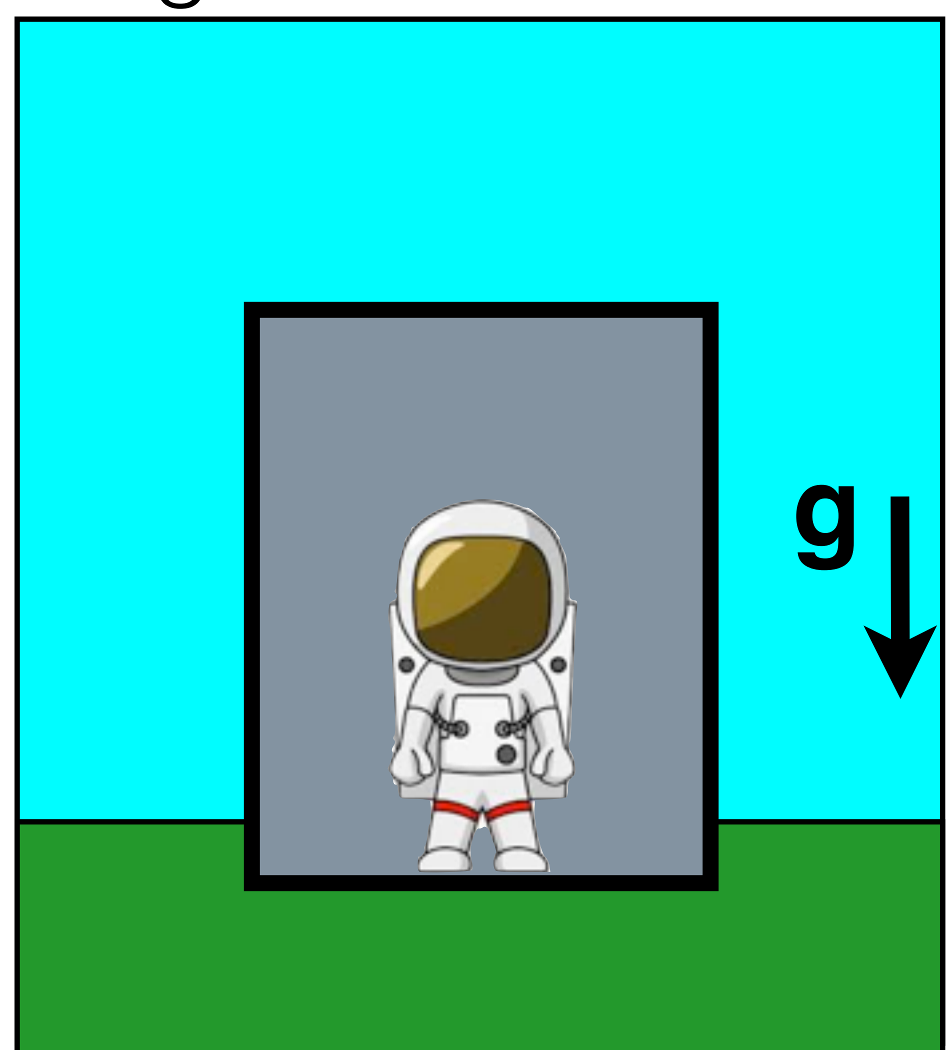
Principle of Equivalence

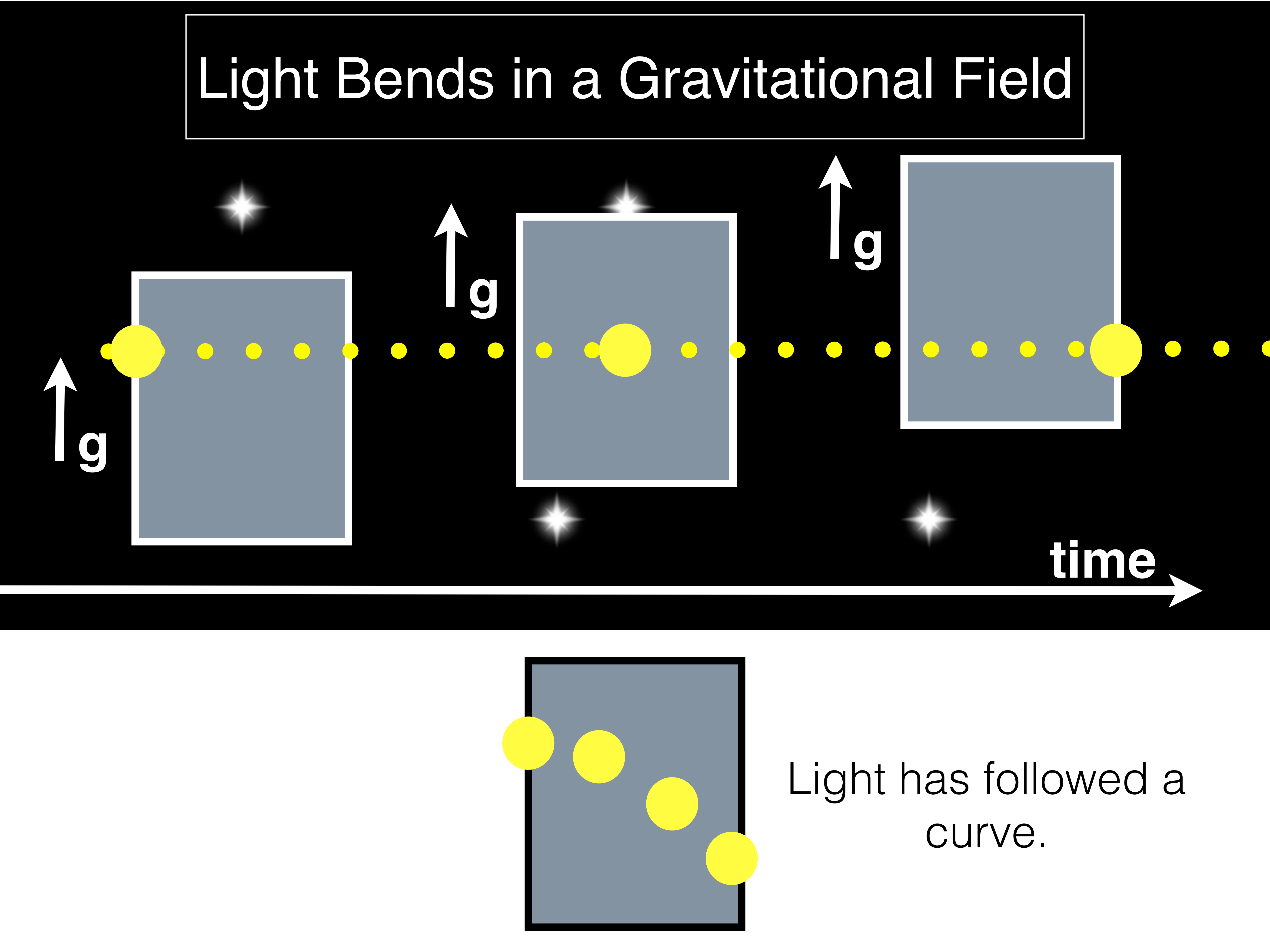
There is no experiment you can do that will distinguish between the following two experiments.

Accelerating at **g**



Stationary but subject to gravitational force



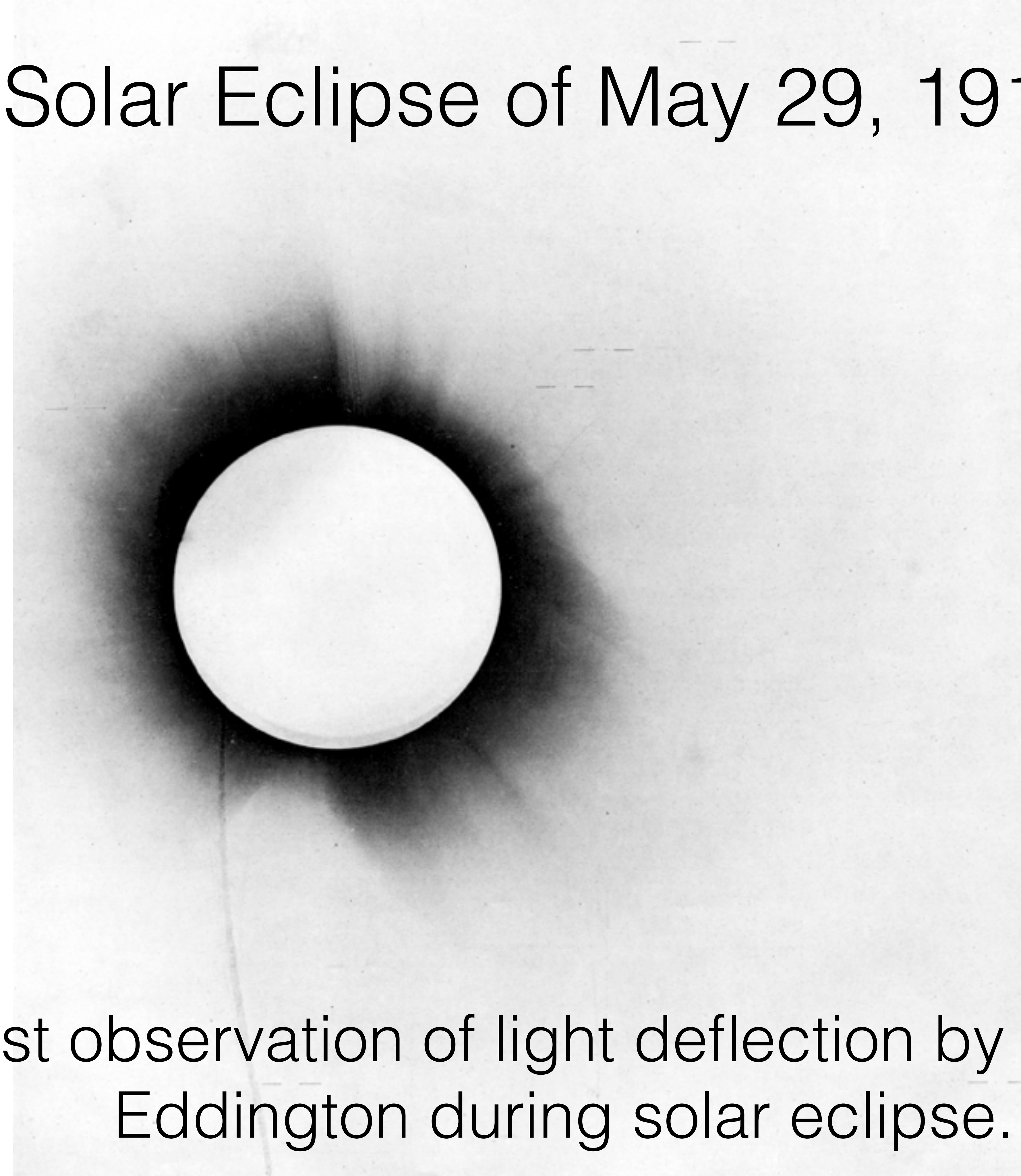

Light Bends in a Gravitational Field
g
g
g
g
time
Light has followed a curve.

Light appears to curve when you are accelerating through space with acceleration **g**.

By principle of equivalence, accelerating with acceleration **g** is equivalent to being stationary subject to acceleration **g**.

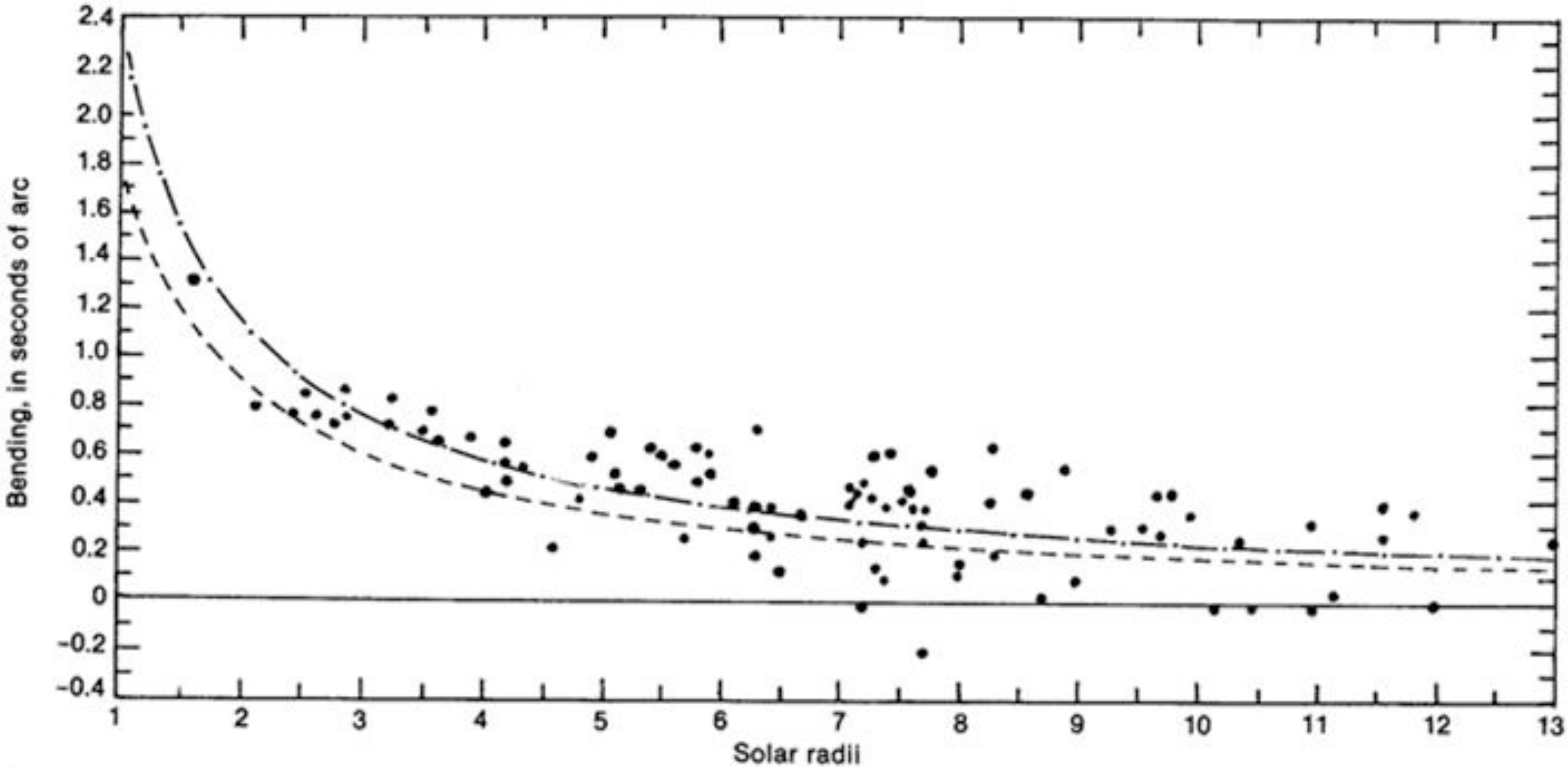
Then, light should also appear to curve in a gravitational field.

Solar Eclipse of May 29, 1919



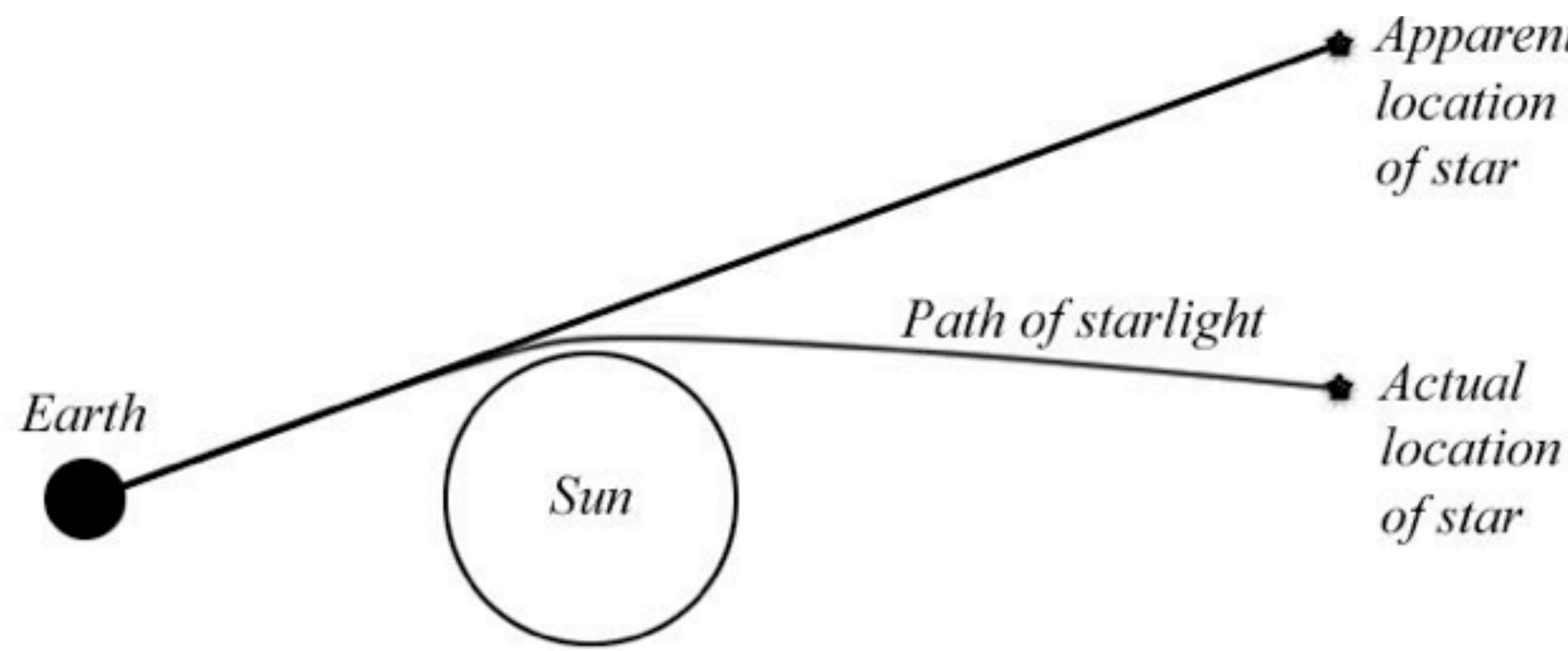
First observation of light deflection by Arthur Eddington during solar eclipse.

Later Eclipse Measurements



Results from later eclipse experiments in 1922 and 1929. Dashed line is Einstein's prediction. Dot dashed is least-squared fit of actual data.

Why do these measurements imply spacetime is curved?

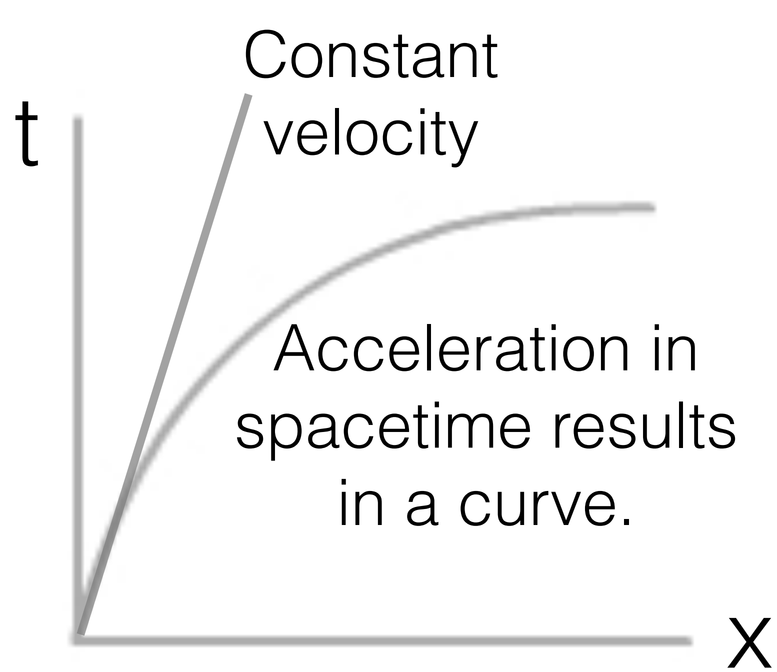


Newton's law of gravitation

$$F = \frac{GMm}{r^2}$$

$$m_{\text{photon}} = 0$$

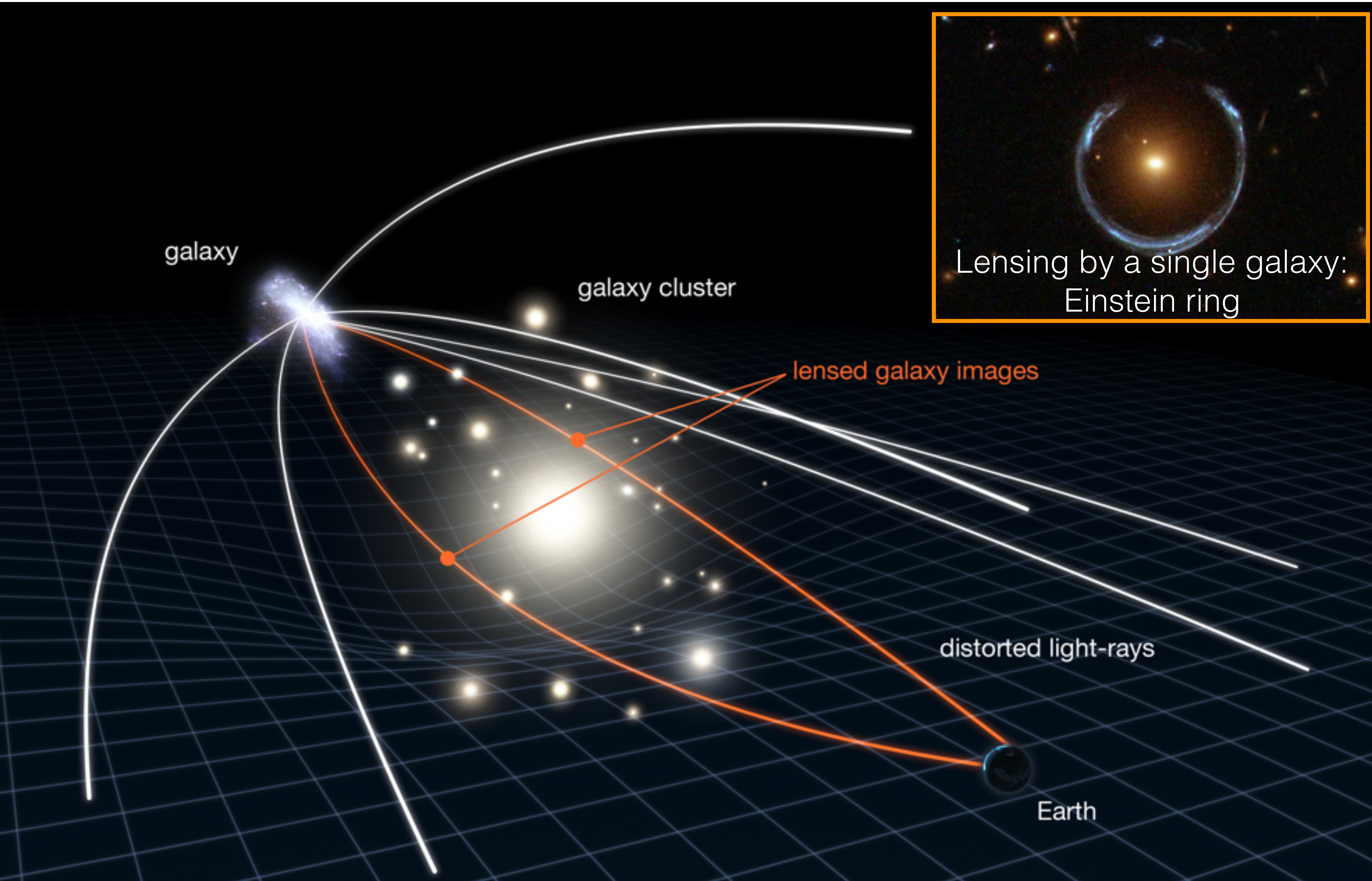
This form of Newton's law doesn't work for light!



"Gravity" causes acceleration.

Therefore, spacetime must be curved if it is creating an acceleration.

Gravitational Lensing



Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu}$$

μ, ν space time coordinates
Ricci curvature tensor
Metric tensor
Curvature scalar
Cosmological Constant
Stress-Energy-Momentum Tensor

16 equations, only 10 are unique.

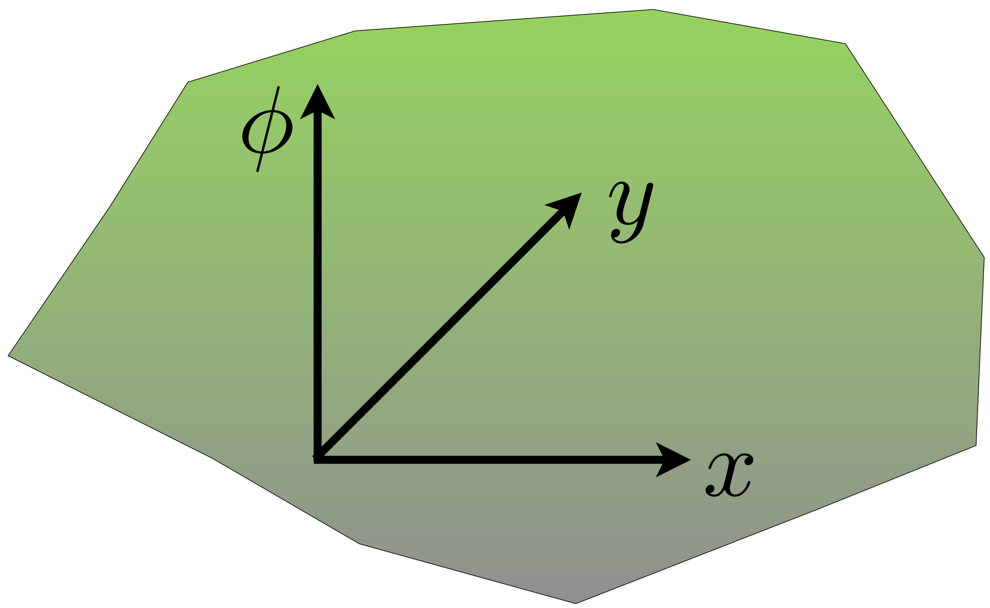
Left hand - curvature of spacetime
Right hand - mass and energy

"Mass tells spacetime how to curve and curved spacetime tells mass how to move." - Wheeler

1. Some Background

A Generic Field Equation

Consider field with height ϕ



Expression for change in height of field along path s

$$d\phi_s = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

More generically,

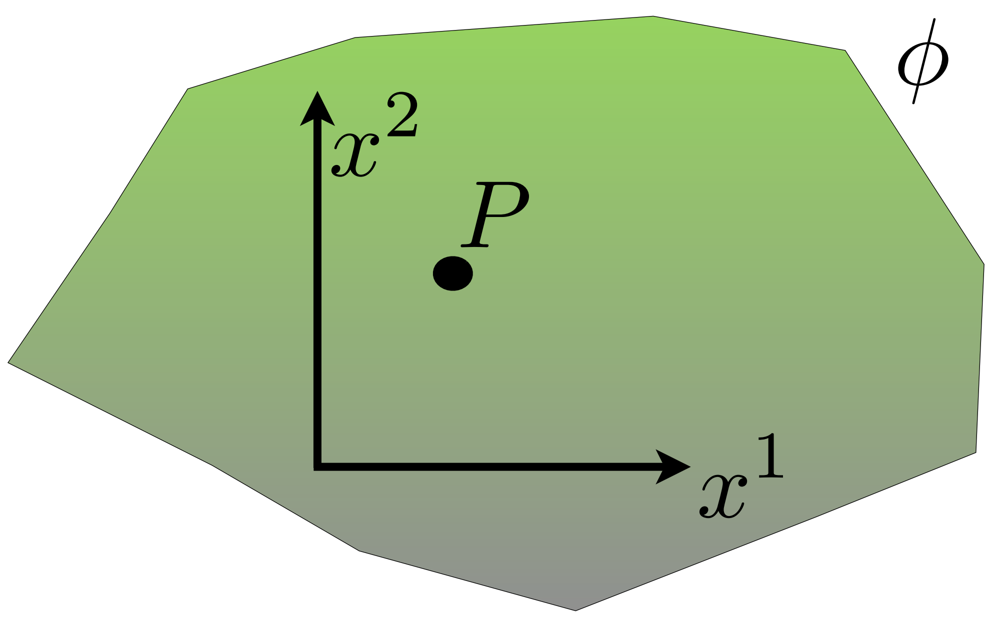
$$t \to x^0$$
$$x \to x^1$$
$$y \to x^2$$
$$z \to x^3$$
$$\ldots$$

$$d\phi = \frac{\partial \phi}{\partial x^1} dx^1 + \frac{\partial \phi}{\partial x^2} dx^2 + \ldots$$

$$= \sum_n \frac{\partial \phi}{\partial x^n} dx^n$$

Relativity

Rules must be independent of the frame of reference used to make observation.



If we know all gradients of field in x-frame of reference, how can we find all gradients in the y-frame of reference? Use chain rule...

Relativity

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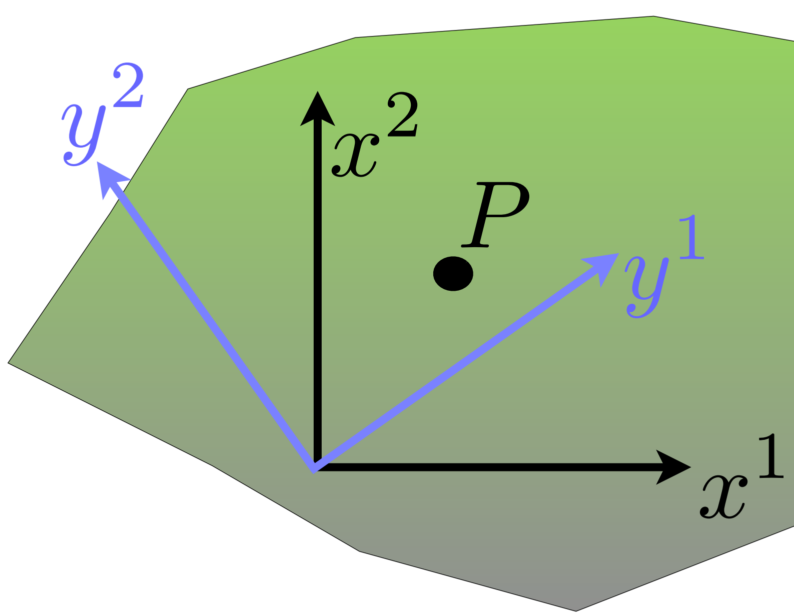

$$\frac{\partial \phi}{\partial x^n} \xrightarrow{\quad ? \quad} \frac{\partial \phi}{\partial y^n}$$

If we know all gradients of field in x-frame of reference, how can we find all gradients in the y-frame of reference? Use chain rule...

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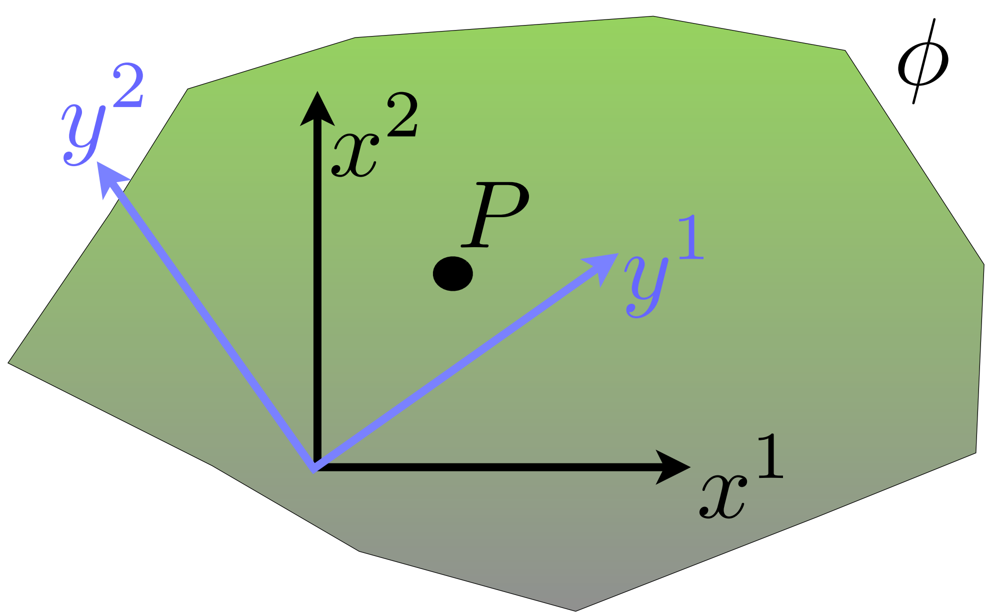


$$\frac{\partial \phi}{\partial x^n} \xrightarrow{?} \frac{\partial \phi}{\partial y^n}$$

$$\frac{\partial \phi}{\partial y^1} = \frac{\partial \phi}{\partial x^1}\frac{\partial x^1}{\partial y^1} + \frac{\partial \phi}{\partial x^2}\frac{\partial x^2}{\partial y^1}$$

$$\frac{\partial \phi}{\partial y^n} = \sum_m \frac{\partial \phi}{\partial x^m}\frac{\partial x^m}{\partial y^n}$$

If we know all gradients of field in x-frame of reference, how can we find all gradients in the y-frame of reference? Use chain rule...

What is a tensor?

Scalar - tensor rank 0, magnitude, ex: Temperature
Vector - tensor rank 1, magnitude and direction, ex: Force
Tensor - combination of vectors where there is a fixed relationship, independent of coordinate system; ex: Dot product, work

$$T^{mn} = A^m B^n$$

Contravariant transformation - x-frame to y-frame

$$T_y^{mn} = A_y^m B_y^n = \sum_{rs} \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} A_x^r B_x^s = \sum_{rs} \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} T_x^{rs}$$

Covariant transformation - x-frame to y-frame

$$T_{mn}(y) = \sum_{rs} \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} T_{rs}(x)$$

Derivative of a Tensor

$$W_{mn}(x) = V_{mn}(x) \xrightarrow{\quad ? \quad} W_{mn}(y) = V_{mn}(y)$$

If two tensors are equal in the x-frame, are they equal in other frames? Yes, they are always equal.

However, derivatives of tensors do not necessarily transform between frames of reference.

$$T_{mn}(x) = \frac{\partial V_m}{\partial x^n}(x) \xrightarrow{\quad ? \quad} T_{mn}(y) \neq \frac{\partial V_m}{\partial y^n}(y)$$

Derivative of a tensor cont'd

What goes wrong? You end up with the derivative of a product:

$$\frac{\partial V_m}{\partial y^n}(y) = \frac{\partial x^r}{\partial y^m}\frac{\partial V_r}{\partial y^n}(x) + \frac{\partial}{\partial y^n}\frac{\partial x^r}{\partial y^m}V_r(x)$$

Motivates definition of new type of derivative: covariant derivative of vector

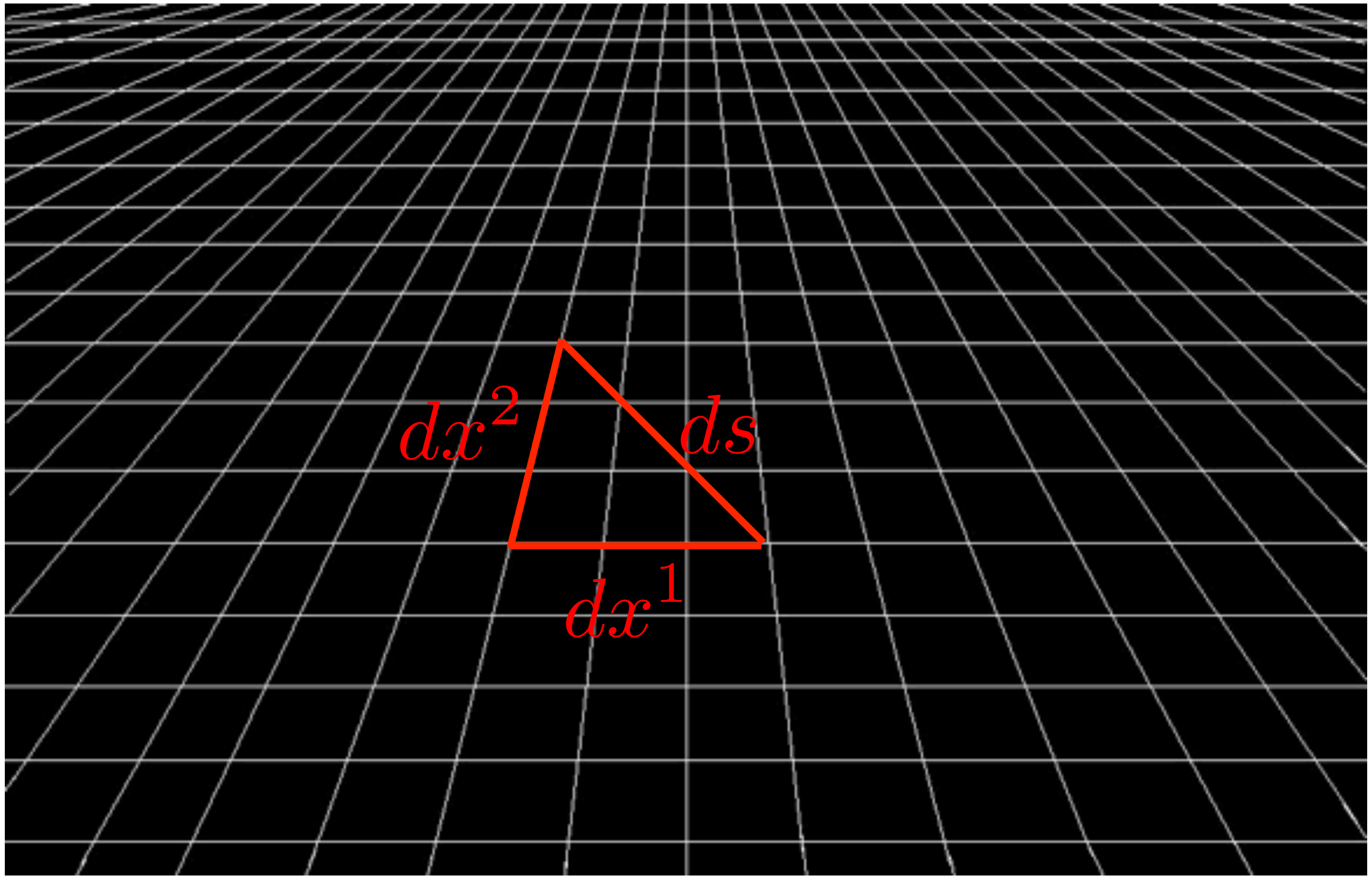
$$T_{mn}(y) = \frac{\partial V_m}{\partial y^n}(y) + \Gamma^r_{nm}V_r(x) \ \ = \nabla_n V_m$$

Extra term with Christoffel symbol Γ^r_{nm}

2. Metric Tensor

The Metric of Flat Space

Pythagorean Theorem in Euclidean Space

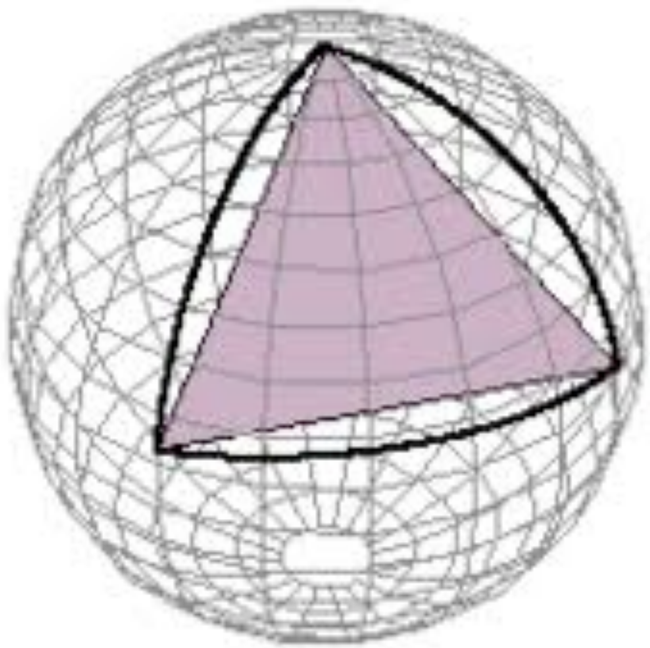


$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2$$
$$= \sum_{mn} dx^m dx^n \delta_{mn}$$
$$= \delta_{mn} \sum_{mn} dx^m dx^n$$

Kronecker delta is metric tensor in flat space.

$$g_{mn} = \delta_{mn}$$

Curved Metric Tensor



Consider a triangle on the surface of sphere. How do we make a correction to the measurement of ds?

$$ds^2 = \delta_{mn} \sum \frac{\partial x^m}{\partial y^r} \frac{\partial x^n}{\partial y^s} dy^r dy^s$$
$$= g_{mn} dy^r dy^s$$

The metric tensor makes a correction to the Pythagorean Theorem with a right triangle on curved space rather than flat space.

2. Curvature Tensor

Covariant Derivative of Tensor

$$\nabla_p T_{mn} = \frac{\partial T_{mn}}{\partial y^p} + \Gamma^r_{pm} T_{rn} + \Gamma^r_{pn} T_{mr}$$

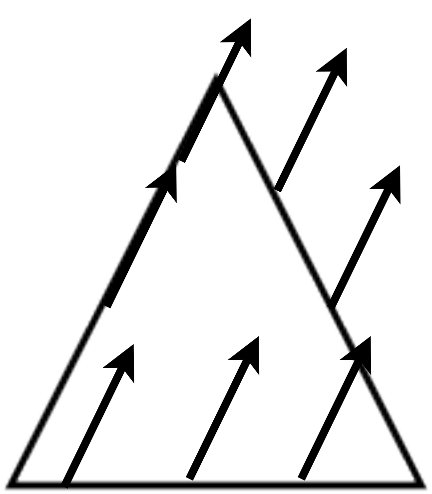
Take covariant derivative of metric tensor in flat space to find:

$$\Gamma^{\mathrm{a}}_{\mathrm{bc}}(x) = \frac{1}{2} g^{\mathrm{ad}} \left(\frac{\partial g_{\mathrm{dc}}}{\partial x^{\mathrm{b}}} + \frac{\partial g_{\mathrm{ab}}}{\partial x^{\mathrm{c}}} - \frac{\partial g_{\mathrm{bc}}}{\partial x^{\mathrm{d}}} \right)$$

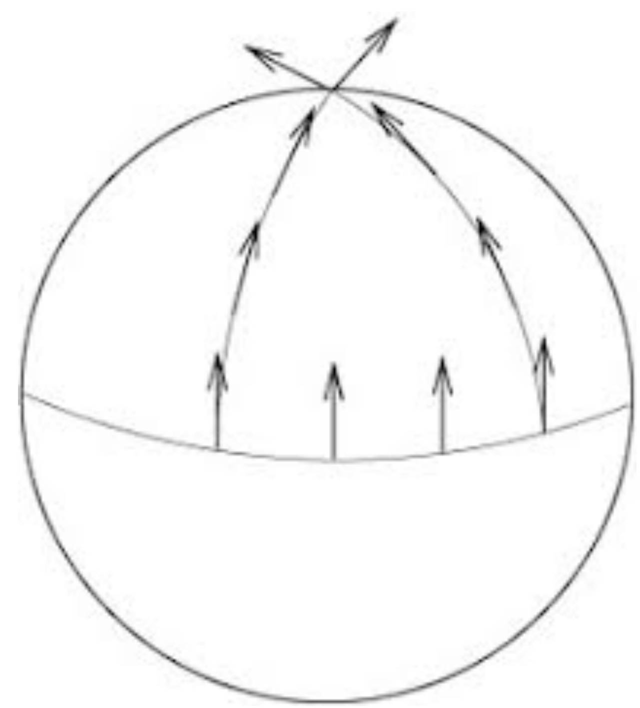
- Christoffel symbol is not a tensor; it is a correction term.
- It is equal to zero in flat space.
- It is not equal to zero in curvilinear space.
- It is a function solely of the metric tensor and derivatives of the metric tensor.
- Why do we need Christoffel symbol? It is buried in the Ricci curvature tensor.

How to detect curvature?

1. Take a vector and parallel transport it around path, keeping same length and orientation.
2. If vector is same length and direction as where it started, space is flat.
3. If direction of vector has changed, space is curved.



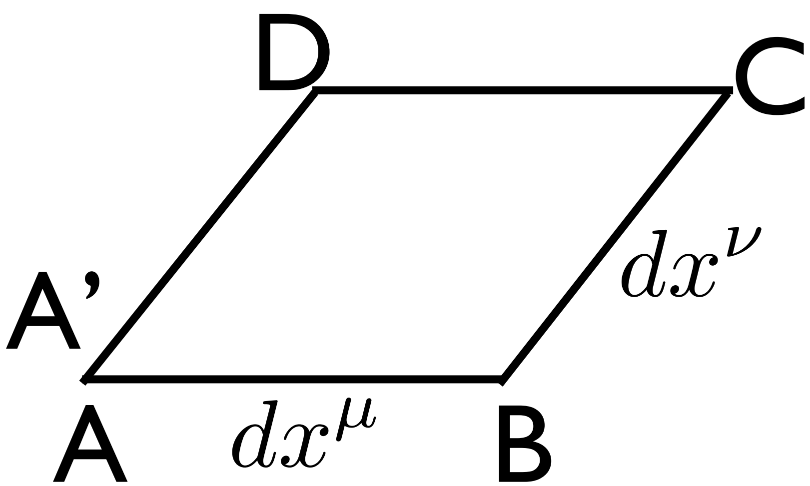
Flat space



Curved space

Ricci Tensor

Definition of commutator: $\left[\frac{\partial}{\partial x}, f(x)\right] = \frac{\partial f(x)}{\partial x}$



$$V_{\mathrm{A}} - V_{\mathrm{A}'} = dV$$

$$\begin{aligned} dV &= dx^\mu dx^\nu V \left(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu\right) \\ &= dx^\mu dx^\nu V \left[\nabla_\nu, \nabla_\mu\right] \end{aligned}$$

Commutator composed of Christoffel symbols and their derivatives.

Normally called Riemann tensor. For our purposes, we regard it as Ricci tensor $R_{\mu\nu}$

$$dV = dx^\mu dx^\nu V R_{\mu\nu}$$

3. Stress-Energy-Momentum Tensor

Meaning of Christoffel Symbols

We can define a tangent vector, rate of change of distance we travel with respect to proper time

$$\frac{dx^\mu}{d\tau}$$

The derivative of that tangent vector must be zero in order to find minimum distance between two points = path for **geodesic**.

But we need the covariant derivative!

$$\nabla \frac{dx^\mu}{d\tau} = \frac{\partial}{\partial \tau} \frac{\partial x^\mu}{\partial \tau} + \Gamma \left(\ldots \right) = 0$$

Meaning of Christoffel Symbols cont'd

$$\nabla \frac{dx^\mu}{d\tau} = \frac{\partial}{\partial \tau}\frac{\partial x^\mu}{\partial \tau} + \Gamma\,(\ldots) = 0$$

$$\frac{\partial^2 x^\mu}{\partial \tau^2} = -\Gamma$$

Looks like acceleration that equals $-\Gamma$

$$a = F/m$$

$$\Gamma \equiv F$$

Christoffel symbol has a kind of equivalence to force in Newton's laws.

Meaning of Christoffel Symbols cont'd

$$\Gamma^{\mathrm{a}}_{\mathrm{bc}}(x) = \frac{1}{2} g^{\mathrm{ad}} \left(\frac{\partial g_{\mathrm{dc}}}{\partial x^{\mathrm{b}}} + \frac{\partial g_{\mathrm{ab}}}{\partial x^{\mathrm{c}}} - \frac{\partial g_{\mathrm{bc}}}{\partial x^{\mathrm{d}}} \right)$$

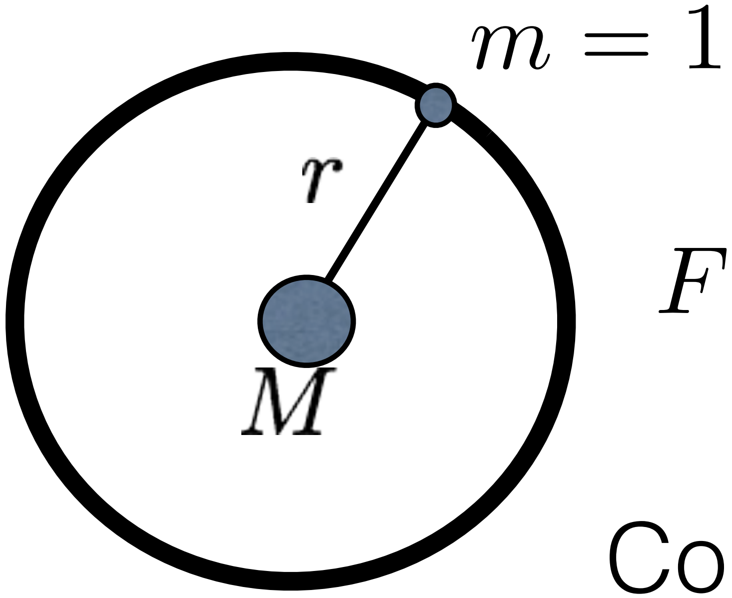
Consider low gravity, low speed, ordinary flat space. Then, GR must reduce to Newtonian gravity.

Only term with any significance is derivative of time component g_{00}. All other derivatives go to zero and **g** goes to 1.

$$\Gamma = \frac{1}{2} \frac{\partial g_{00}}{\partial x} \equiv F$$

$$F = -\frac{\partial \phi}{\partial x} \qquad F = -\nabla \phi \quad \longrightarrow \quad g_{00} = 2\phi + C$$

Motivating Einstein Equations



$$F = -\frac{GMm}{r^2} \longrightarrow F = -\frac{GM}{r^2}$$

Consider force capability across whole sphere

$$\int F \cdot \mathrm{d}A = \int -\frac{GM}{r^2} \cdot \mathrm{d}A = -\frac{GM}{r^2} 4\pi r^2 = -4\pi GM$$

Divergence theorem

$$\int_{\text{area}} F \cdot \mathrm{d}A = \int_{\text{vol}} \nabla \cdot F \mathrm{d}V$$

$$\rho = \frac{M}{V} \qquad M = \int \rho dV$$

$$-4\pi G \int \rho \mathrm{d}V = \int \nabla \cdot F \mathrm{d}V$$

$$-4\pi G\rho = \nabla \cdot F = \nabla \cdot (-\nabla \phi)$$

$$\nabla^2 \phi = 4\pi G\rho$$

Motivating Einstein Equations

$$\nabla^2 \left(\frac{1}{2} g_{00} \right) = 4\pi G \rho$$

$$\nabla^2 g_{00} = 8\pi G \rho$$

But this is not a tensor equation and for general relativity, we need tensor equations.

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Now we have tensors on both left and right hand side. **Einstein tensor** is on the left. Instead of mass density on the right, we have a **stress-energy-momentum tensor** with all mass-energy-stress-pressure terms that you can have.

Stress-Energy-Momentum Tensor

Momentum 4-vector

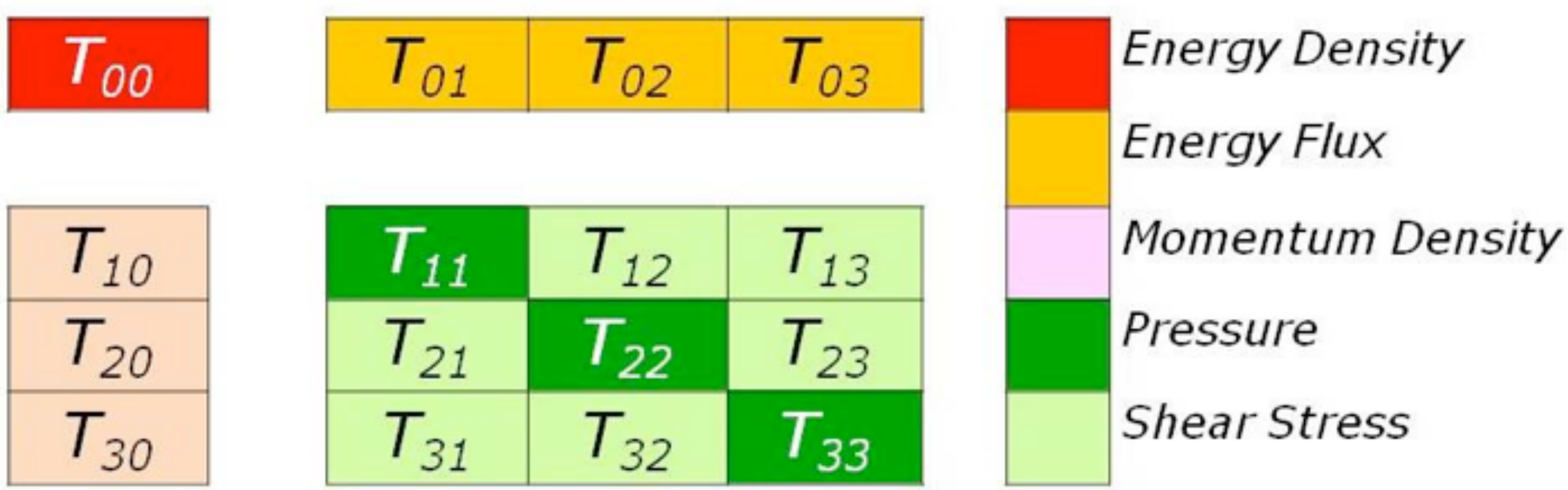
$$\vec{p} = m\left(\frac{x_0}{\tau}, \frac{x_1}{\tau}, \frac{x_2}{\tau}, \frac{x_3}{\tau}\right)$$

But we need a tensor!

$$\frac{E}{V} = \frac{W}{V} = \frac{F \times L}{L^3} = \frac{F}{L^2} = \frac{F}{A}$$

$T_{\mu\nu}$ has 0 to 3 indices.

- 00 - time component / energy part.
- Along top - energy flow
- Along side - momentum density
- 9 middle components - momentum flux-stress energy part



4. Motivating Einstein’s Equations

Motivating Einstein Equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \xrightarrow{?} \quad R_{\mu\nu} = 8\pi G T_{\mu\nu}$$

We need the spacetime curvature term on the left. Einstein thought it should be the Ricci curvature tensor. But there is a problem.

Due to energy conservation:

$$\nabla T_{\mu\nu} = 0$$

But the derivative of Ricci tensor does not equal zero:

$$\nabla R_{\mu\nu} = \frac{1}{2}\nabla g_{\mu\nu} R$$

Motivating Einstein Equations

Thus, the equation could have the form:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Einstein thought he forgot something because it is also true that

$$\nabla g_{\mu\nu} = 0$$

Then we can add the metric tensor term with a constant:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Λ is the cosmological constant for space in math terms. It is often left out except for major cosmological scales.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

5. Solving Einstein’s Equations

Methods

Solving Einstein's equations is difficult. They're non-linear. In fact, the equations of motion are impossible to solve unless there is some symmetry present.

In the absence of symmetry, there are two methods:

1. Numerical relativity
2. Approximation techniques

For the approximation technique, we consider a metric very close to flat space with a small perturbation. And we consider only first order perturbations.

Linearized Gravity

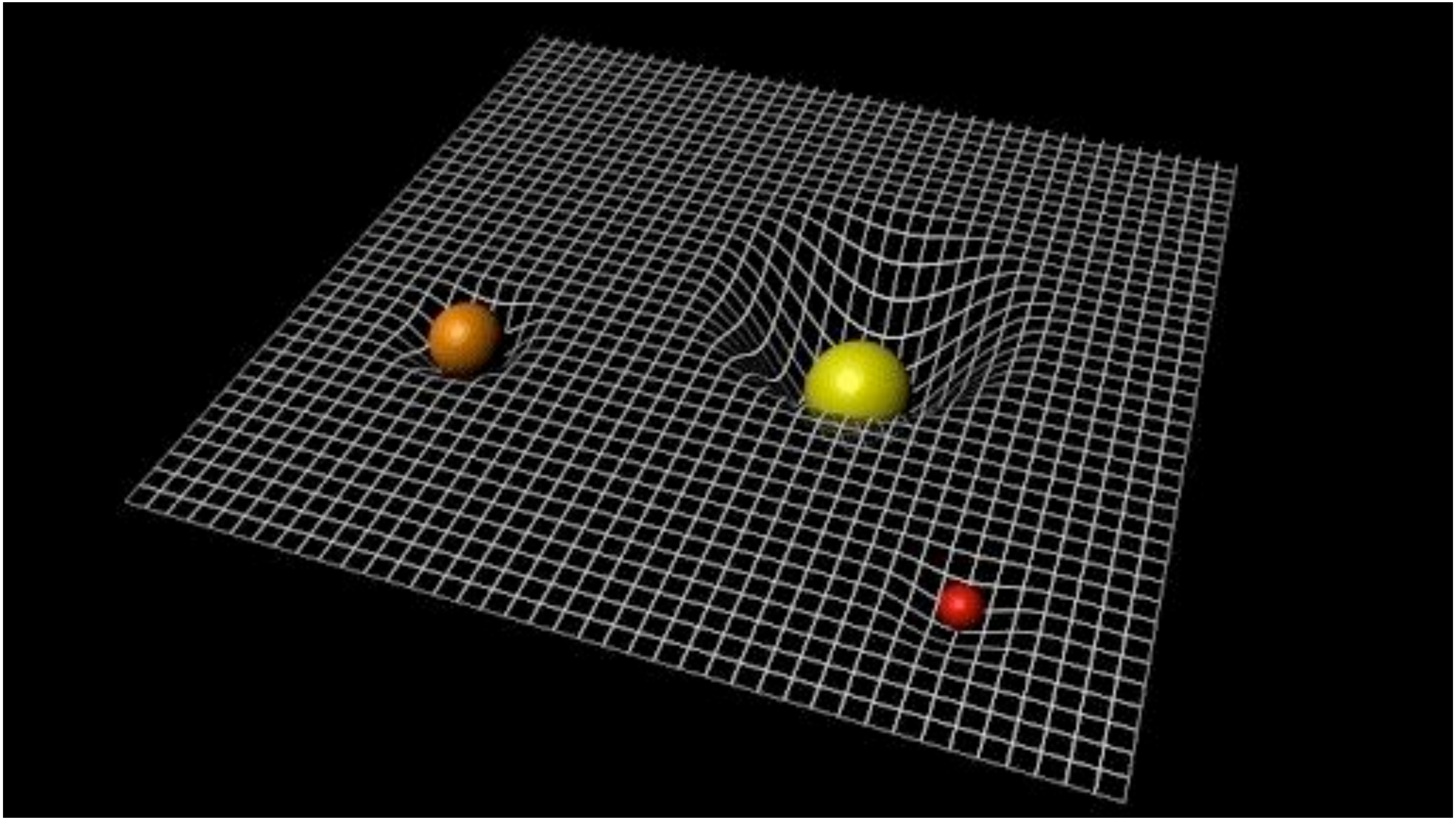
Consider the Minkowski metric - a combination of three dimensional Euclidean space and time into four dimensions.

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$$\eta_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Consider a small perturbation $h_{\mu\nu}$ on flat space:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$



Equation Outside Source

Here, we consider the Einstein equations with no matter, i.e. no energy-momentum tensor

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Take trace of both sides to find: $R = 0$

Then the Einstein equation reduces to:

$$R_{\mu\nu} = 0$$

We can take the metric of flat space, calculate $R_{\mu\nu}$, set it equal to zero to get an equation for $h_{\mu\nu}$

Equation Outside Source

Recall, $R \sim \partial\Gamma + \Gamma\Gamma$

And that we are working with,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$|h_{\mu\nu}| \ll 1$$

Then, Christoffel symbols are of the form:

$$\Gamma \sim \frac{1}{2} g^{-1} \partial g \sim (\eta - h)\, \partial h$$

Equation Outside Source

We can discard small terms, quadratic in h

$$\Gamma \sim \eta \partial h \sim \partial h$$

$$R \sim \partial^2 h + \partial h \partial h \sim \partial^2 h$$

All we have left are second derivatives with respect to position and time

So equation of motion looks like

$$R \sim \partial^2 h \sim 0$$

Wave equation!
But there are 16 components here (10 of which are independent).

Gravitational Waves in Vacuum

Consider the d'Alembertian operator,

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Then, we can write Einstein's equations as

$$-\Box \bar{h}_{\alpha\beta} = \frac{16\pi G}{c^4} T_{\alpha\beta} \xrightarrow{\text{in vacuum}} -\Box \bar{h}_{\alpha\beta} = 0$$

Solution to vacuum wave equation:

$$\bar{h}_{\alpha\beta} = |a_{\alpha\beta}| \cos\left(k_\mu x^\mu\right)$$

- k_μ is wave four vector
- perturbation travels at speed of light
- monochromatic plane wave
- direction of propagation of wave is orthogonal to its amplitude components, i.e., transverse wave

Transverse, Traceless Gauge

Transverse: GWs produce forces only perpendicular to their propagation direction
Traceless: Forces are shearing, i.e. do not cause overall expansion/contraction

If direction of propagation is z-axis,

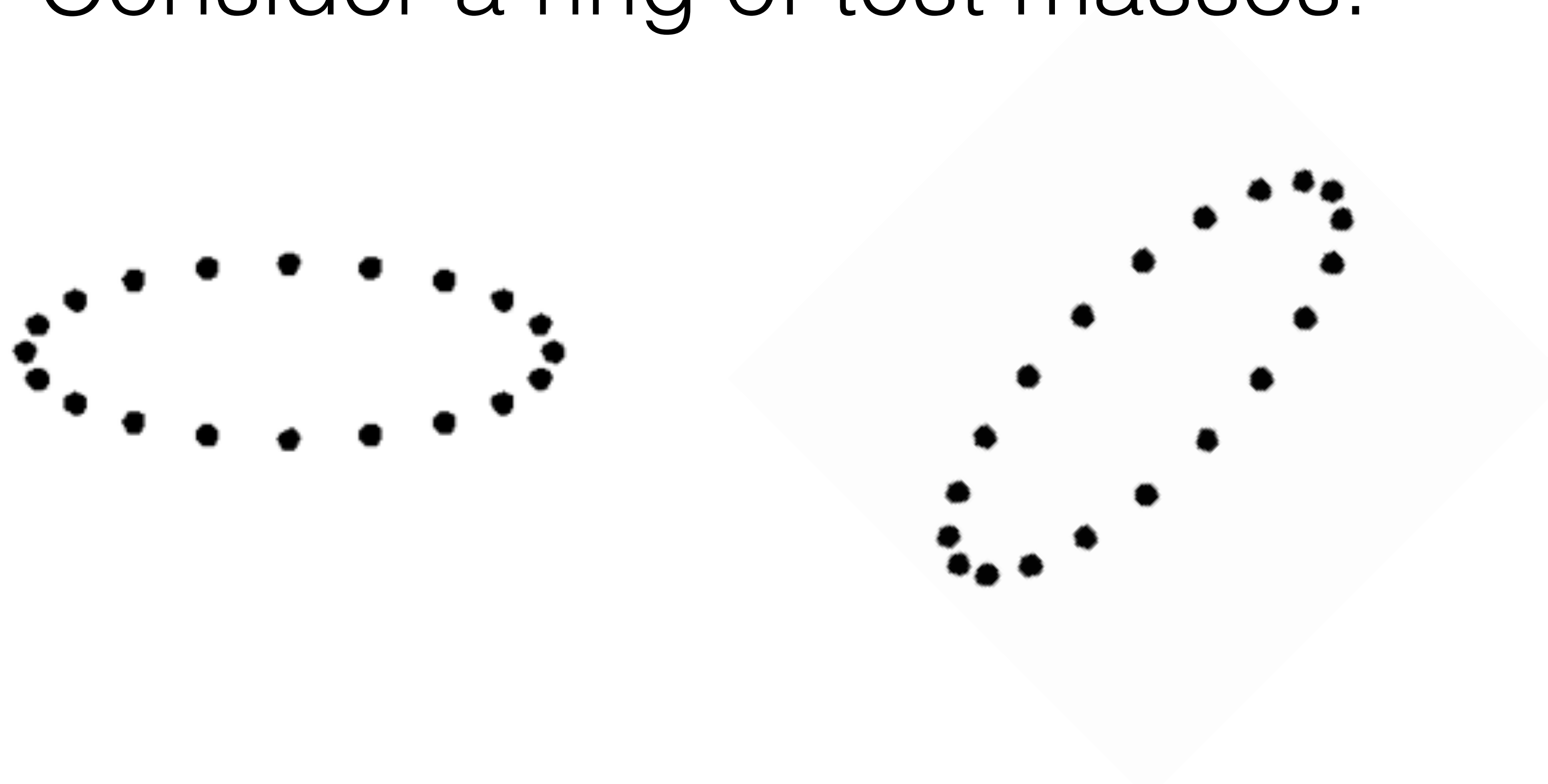
$$h_{\alpha\beta}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos\left[\omega\left(t - z/c\right)\right]$$

- h_+ and $h_\times$ are amplitudes of plus and cross polarizations of gravitational wave
- h changes as wave propagates
- h changes sign at 1/2 cycle of wave

Two Polarizations

Any solution is linear superposition of h_+ and $h_\times$.

Consider a ring of test masses:



Gravitational waves can be measured with strain gauge. If the wave was not moving, it would not really be measurable. But its oscillating behavior with time really induces strain.


Scale of Effect Vastly Exaggerated

Equation With Source

Inhomogenous wave equation with known solution:

$$\Box\bar{h}^{\alpha\beta} = -\frac{16\pi G}{c^4}T^{\alpha\beta} + O\left(h^2\right) = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

Solution to field equation using Green's function:

$$\bar{h}^{\alpha\beta}\left(t,\vec{x}\right) = \frac{4G}{c^4}\int\frac{\tau^{\alpha\beta}\left(t-\left\|\vec{x}-\vec{x}'\right\|/c,\vec{x}'\right)}{\left\|\vec{x}-\vec{x}'\right\|}d^3\vec{x}'$$

Far-field, slow motion approximation:

$$\bar{h}^{ij}\left(t,\vec{x}\right) \approx \frac{2G}{c^4 r}\frac{\partial^2}{\partial t^2}\int x'^i x'^j \tau^{00}\left(t-r/c,\vec{x}'\right)d^3\vec{x}'$$

Equation With Source

Definition of quadrupole tensor, or second mass moment:

$$I^{ij}\left(t\right) = \int x^i x^j \tau^{00}\left(t, \vec{x}\right) d^3\vec{x}$$

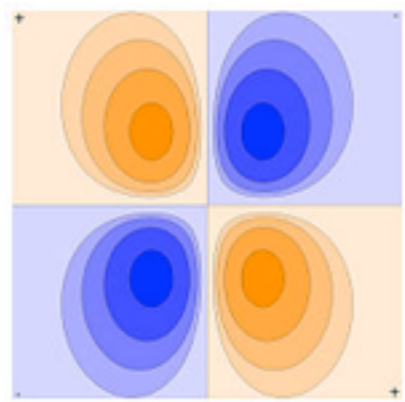
Perturbation is generated by second derivatives of quadrupole tensor:

$$h_{ij}^{TT}(t) \approx \frac{2G}{c^4 r}\frac{\partial^2}{\partial t^2} I_{ij}^{TT}\left(t - r/c\right)$$

Multipole Expansion

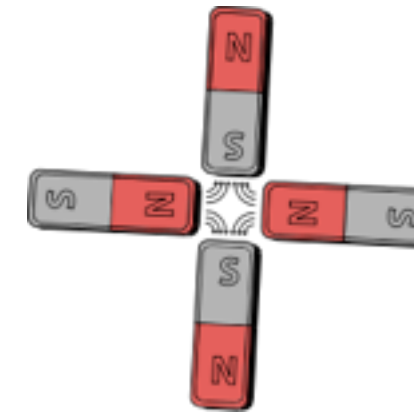
Electric:

- has monopoles, dipoles, quadrupoles



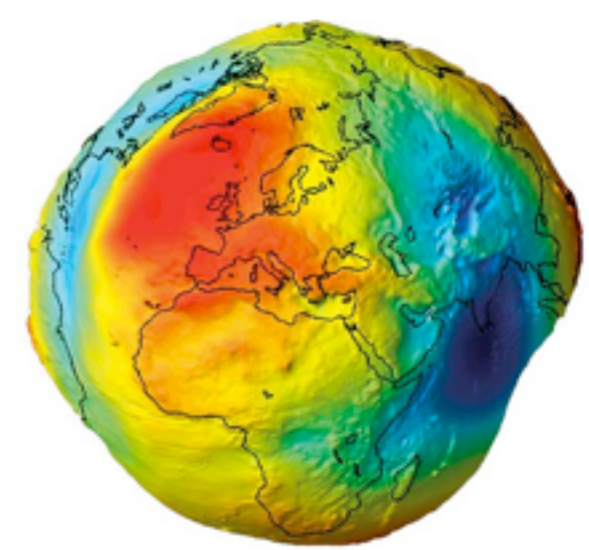
Magnetic:

- no monopoles
- have dipoles and quadrupoles



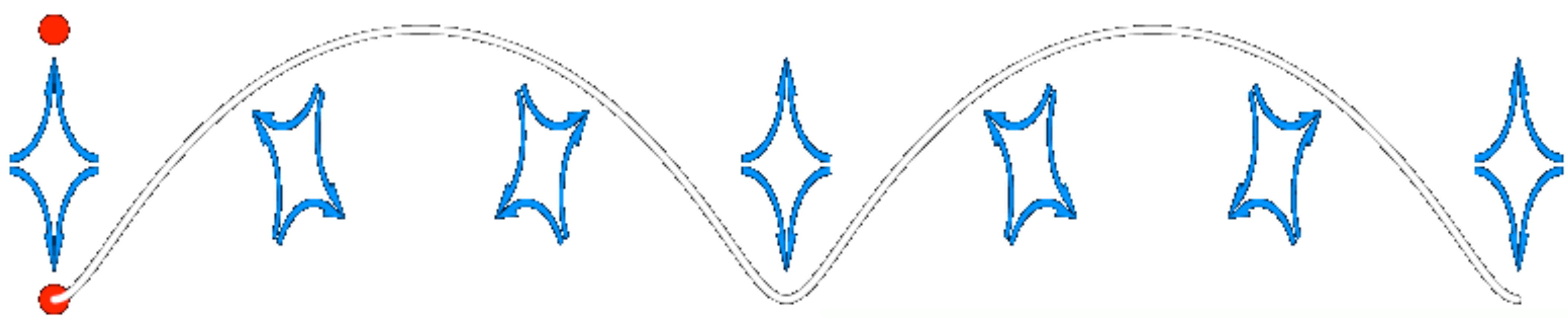
Gravitational:

- Monopole term depends on M (conserved)
- Dipole term depends on center of mass and first derivative P (conserved)
- Only quadrupole and higher moments radiate

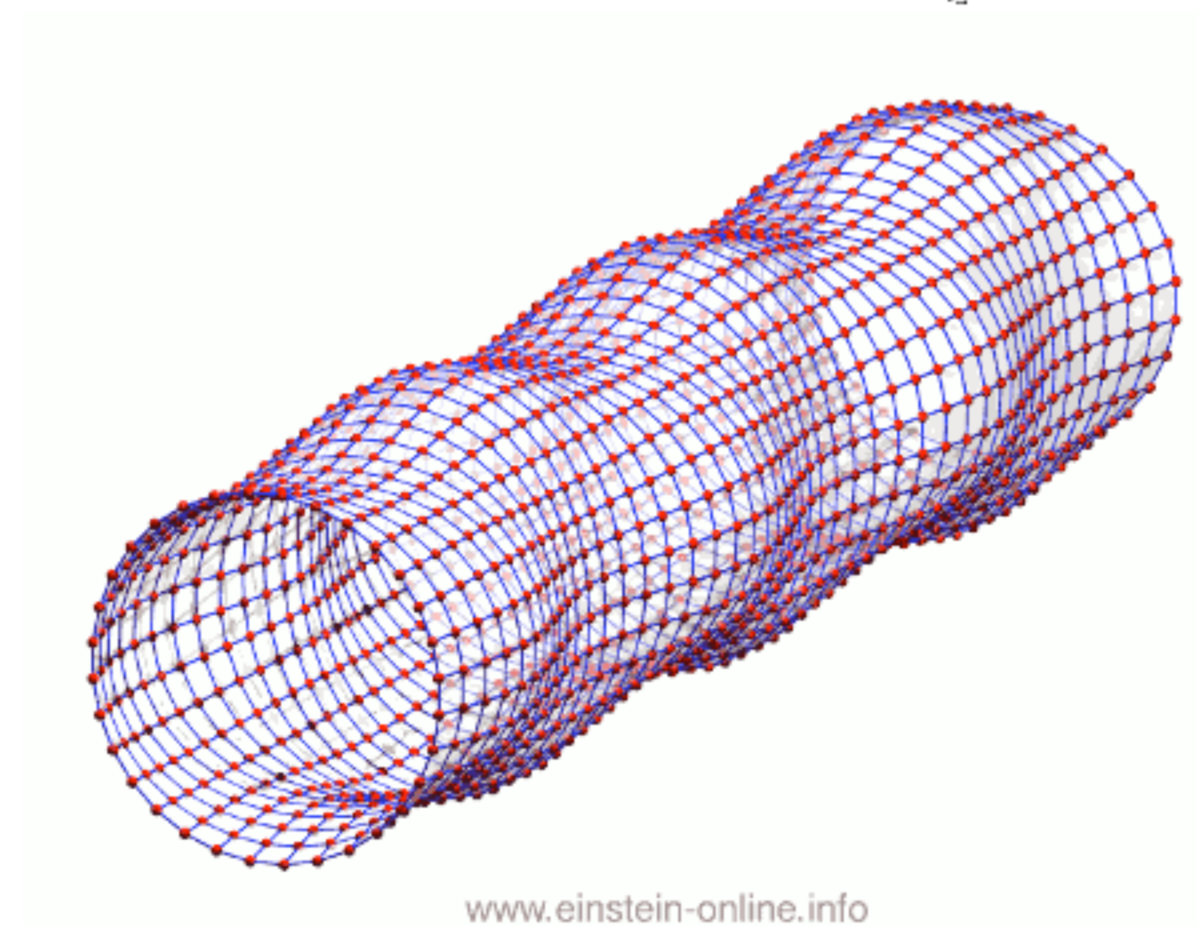


Quadrupole Radiation

When gravitational tidal field of source changes with time, those changes propagate out from the source at speed c.



A gravitational wave passing through a 3d cylinder of particles.



Astrophysical Sources of Gravitational Radiation