BND Lectures 2017

Quantum Field Theory

Exercises for the tutorial

Alberto Mariotti

1. By operating with $\gamma^\nu \partial_\nu$ on the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{1}$$

prove that the components of ψ satisfies the Klein-Gordon equation.

2. Show that the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{2}$$

is Lorentz invariant, reminding that under Lorentz transformation

$$\psi(x) \to \Lambda_{1/2}\psi(\Lambda^{-1}x) \qquad \partial_\mu \to (\Lambda^{-1})^\nu_\mu \partial_\nu \tag{3}$$

with $\Lambda_{1/2} = \exp(-\frac{i}{2} w_{\mu\nu} S^{\mu\nu})$ and using $\Lambda^{-1}_{1/2}\gamma^\mu \Lambda_{1/2} = \Lambda^\mu_\nu \gamma^\nu$.

3. Derive the equation of motion for the following Lagrangians using the Euler-Lagrangian equation

   (a) $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$

   (b) $\mathcal{L} = (\partial_\mu\Phi)^*\partial^\mu\Phi - m^2\Phi^*\Phi + \lambda(\Phi^*\Phi)^2$

   (c) $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \lambda\bar{\psi}\psi\phi$

   (d) $\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with $D_\mu = \partial_\mu + ieA_\mu$

4. Using Noether theorem compute the conserved charge of the $U(1)_V$ global symmetry of the Dirac equation

$$\psi \to e^{i\alpha}\psi \qquad \bar{\psi} \to \bar{\psi}e^{-i\alpha} \tag{4}$$

5. Consider a Lagrangian describing a free complex scalar field. Show that the current

$$j_\mu = i(\Phi \partial_\mu \Phi^* - \Phi^* \partial_\mu \Phi) \tag{5}$$

is conserved (using the equations of motion).

6. Show that the QED Lagrangian (eqn 3d) is invariant under gauge transformation ($\psi \to e^{ie\alpha(x)}\psi(x), A_\mu \to A_\mu - \partial_\mu \alpha$)

7. Consider a quantum real scalar field and its expansion in terms of creation and annihilation operators $a_p^\dagger$ and a_p. Assuming the following commutation relation

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{6}$$

compute the following commutators

$$[\phi(t, \vec{x}), \pi(t, \vec{y})] \qquad , \qquad [\phi(t, \vec{x}), \phi(t, \vec{y})] \tag{7}$$

8. Consider a quantum complex scalar field. Compute the conserved charge associated to the conserved current (5) in terms of creation and annihilation operators (be careful with normal ordering), finding

$$Q = \int \frac{d^3p}{(2\pi)^3} \left(b_p^\dagger b_p - c_p^\dagger c_p\right) \tag{8}$$

9. Verify that

$$\psi(x) = e^{-ip\cdot x} u(p) \qquad u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\ \xi \\ \sqrt{p \cdot \bar{\sigma}}\ \xi \end{pmatrix} \tag{9}$$

solves the dirac equation.

10. Prove that

$$\gamma^\mu \gamma_\mu = 4 \qquad \gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \qquad \gamma^\mu \gamma^\sigma \gamma^\rho \gamma_\mu = 4\eta^{\sigma\rho} \qquad (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \tag{10}$$

11. Show that $P_R = \frac{1}{2}(1+\gamma^5)$ and $P_L = \frac{1}{2}(1-\gamma^5)$ are projectors.

12. Consider a Dirac field. Compute the conserved charge associated to the symmetry (4) in terms of creation and annihilation operators (be careful with normal ordering and anticommutation), finding

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left[(a_p^s)^\dagger a_p^s - (b_p^s)^\dagger b_p^s\right] \tag{11}$$

13. Suppose to work in d dimensions. The action is $S = \int d^dx\mathcal{L}$. The canonical mass dimension of fields is set by the kinetic term which is the same of dimension $d = 3+1$ that we saw in the lecture. Show that the canonical mass dimension of scalar, Dirac and gauge fields are

$$[\phi] = \frac{d-2}{2} \qquad [A_\mu] = \frac{d-2}{2} \qquad [\psi] = \frac{d-1}{2} \tag{12}$$

Hence, given these results, what are the dimensions of the following operators? Which of them is marginal/relevant/irrelevant in e.g. d=2+1?

(a) $\mathcal{L} = -e\bar{\psi}\gamma^\mu\psi A_\mu$

(b) $\mathcal{L} = -y\bar{\psi}\psi\phi$

(c) $\mathcal{L} = -\lambda_n\phi^n$ with integer n

Comment with respect to the $d = 3+1$ case that we know.

14. Show that the sum of the Mandelstam variables in a 2 to 2 scattering $ab \to cd$ satisfies the following identity

$$s = (p_1+p_2)^2 \qquad t = (p_1-p_3)^2 \qquad u = (p_1-p_4)^2 \tag{13}$$

$$s + u + t = m_a^2 + m_b^2 + m_c^2 + m_d^2 \tag{14}$$

15. Consider a two body decay of an heavy particle with mass M_a, $a \to bc$. Show that the momenta of the two daughter particles is

$$p^* = \frac{1}{2M_a}\sqrt{(M_a^2-(m_b+m_c)^2)(M_a^2-(m_b-m_c)^2)} \tag{15}$$

16. Show that the two body phase space including the four momentum conservation for a 2 to 2 process in the center of mass frame

$$\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\delta^{(4)}(P_i - p_1 - p_2) \tag{16}$$

can be simplified to

$$\frac{p^*}{4\sqrt{s}}d\Omega \tag{17}$$

where $p^* = |\vec{p}_1| = |\vec{p}_2|$ is the outgoing particle momentum in the c.o.m. frame.

Hint: use the delta function relation

$$\delta(f(x)) = \left|\frac{\partial f(x)}{\partial x}\right|_{x=x_0}^{-1} \delta(x - x_0) \tag{18}$$

17. Verify that the Feynman propagator for the Dirac field is a Green function of the Dirac equation

18. Consider a theory with two real scalar fields with Lagrangian (with $m_A > 2m_B$)

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi_A\partial^\mu\phi_A + \partial_\mu\phi_B\partial^\mu\phi_B\right) - \frac{1}{2}(m_A^2\phi_A^2 + m_B^2\phi_B^2) - \frac{\mu}{2}\phi_A\phi_B^2 \tag{19}$$

Compute the decay rate of the heavy real scalar into a pair of light real scalars $\phi_A \to \phi_B\phi_B$, finding

$$\Gamma[\phi_A \to \phi_B\phi_B] = \frac{\mu^2}{32\pi m_A}\sqrt{1 - \frac{4m_B^2}{m_A^2}} \tag{20}$$

19. Consider a theory with a real scalar of mass M coupled to a fermion with mass m with interaction term

$$\mathcal{L} \supset -y\phi\bar{\psi}\psi \tag{21}$$

Assuming $M > 2m$ compute the decay rate of the scalar into a fermion-antifermion pair

$$\Gamma[\phi \to \psi\bar{\psi}] = \frac{y^2 M}{8\pi}\left(1 - \frac{4m^2}{M^2}\right)^{3/2} \tag{22}$$

20. Compute the QED cross section $e^+e^- \to \mu^+\mu^-$

21. Compute the QED cross section $e^-\mu^- \to e^-\mu^-$

22. Compute the QED Moeller scattering cross section $e^+e^- \to e^+e^-$ starting from the expression given in the lectures.