Cosmology & Dark Matter

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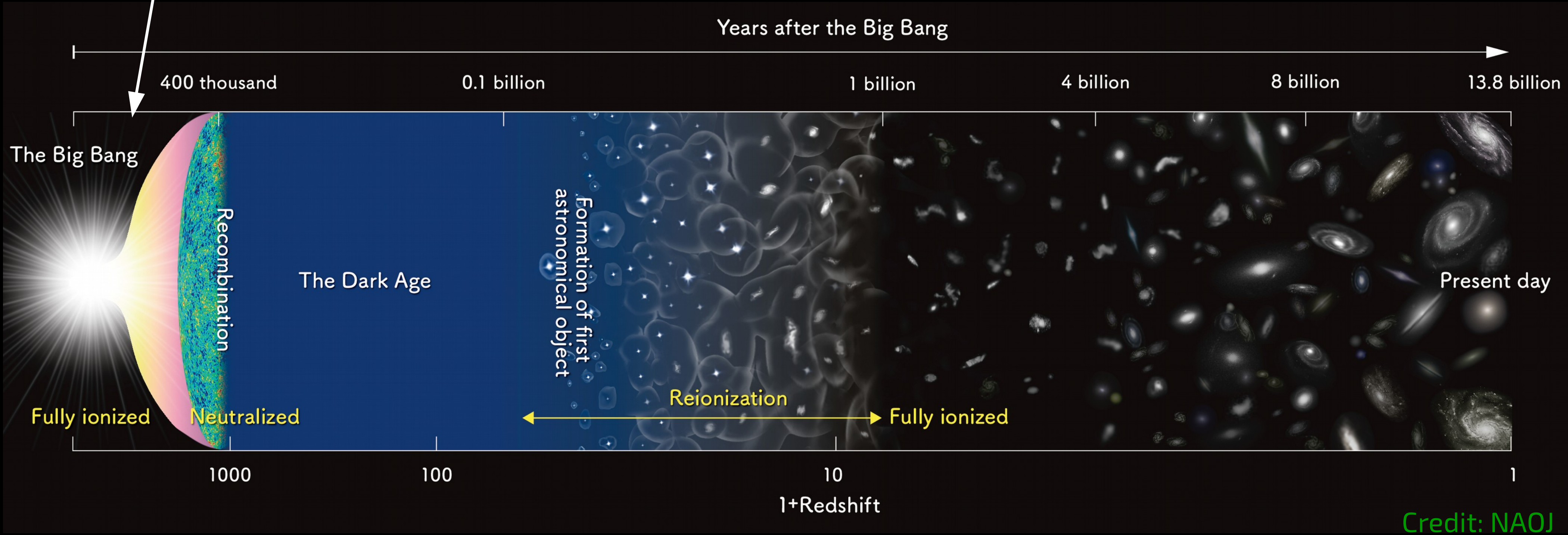
A brief history of the Universe

- 1e-43 s: Quantum gravity
- >1e-34 s: Inflation
- <1e-11 s: Baryogenesis
- 1e-11 s: Electroweak phase transition (particles acquire rest mass)
- 1e-5 s: QCD phase transition (protons & neutrons form)
- 1 s: Neutrino decoupling
- 3 min: Big-bang nucleosynthesis



Credit: NAOJ

Outline

Lecture 1 – Geometry, dynamics & thermal history of the Universe

- Geodesic equation and FRW-metric
- Einstein Equation and expansion of Universe
- Thermodynamics in expanding Universe
- Dark relics, neutrino decoupling, big-bang nucleosynthesis

Lecture 2 – Inflation and linear structure formation

- Horizon problem
- Inflation and scalar field dynamics
- Linear structure formation
- Formation of the cosmic microwave background

Outline

Lecture 3 – The non-linear Universe, dark matter and MOND

- Evidence for dark matter
- N-body simulations of CDM
- CDM problems and solutions
- Status of Modified Newtonian Dynamics (MOND)

Lecture 4 – Searches for particle dark matter

- Properties of dark matter & dark matter production
- Dark matter candidates & Searches
- Indirect searches for dark matter
  - Signal characteristics
  - Searches with cosmic rays
  - Searches with photons
  - Signal hints and challenges

“Natural units”

$$c = \hbar = k_B = 1$$

Lecture 1

Geometry, dynamics & thermal history of the Universe

Literature

Here and in the following I use material from the Baumann cosmology lectures

http://cosmology.amsterdam/education/cosmology/

&

Scott Dodelson, *Modern Cosmology*

Geometry of space

The geometry of space is encoded in the **metric**, which connects observer-dependent coordinates with physical distances.

In 3-dimensional Euclidean space, with coordinates (x, y, z), the infinitesimal length dl is connected to dx, dy and dz via:

$$\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \sum_{ij=1}^{3} \delta_{ij}\, \mathrm{d}x^i \mathrm{d}x^j$$

Example: The same metric in **spherical coordinates**, (r, θ, ϕ), reads

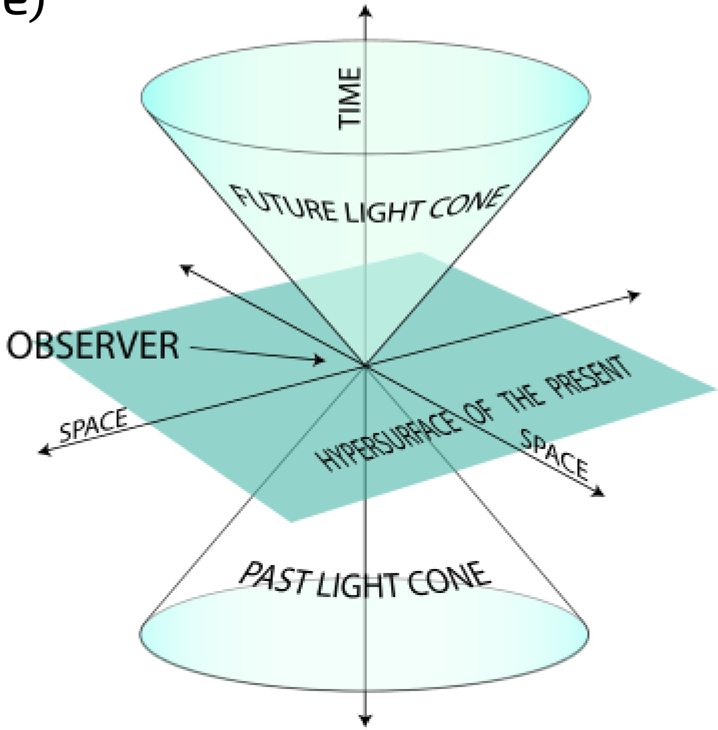
$$\mathrm{d}\ell^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta\, \mathrm{d}\phi^2 \equiv \sum_{ij=1}^{3} g_{ij}\, \mathrm{d}x^i \mathrm{d}x^j$$

Space-time geometry = Gravity

In **general relativity**, the fundamental object is the 4-dim space-time metric, which relates the observer-dependent coordinates, $X^\mu = (t, x, y, z)$, to the **invariant line element** (corresponds to proper time)

$$\mathrm{d}s^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \mathrm{d}X^\mu \mathrm{d}X^\nu \equiv g_{\mu\nu} \mathrm{d}X^\mu \mathrm{d}X^\nu$$

Einstein sum convention



Special relativity has the constant metric

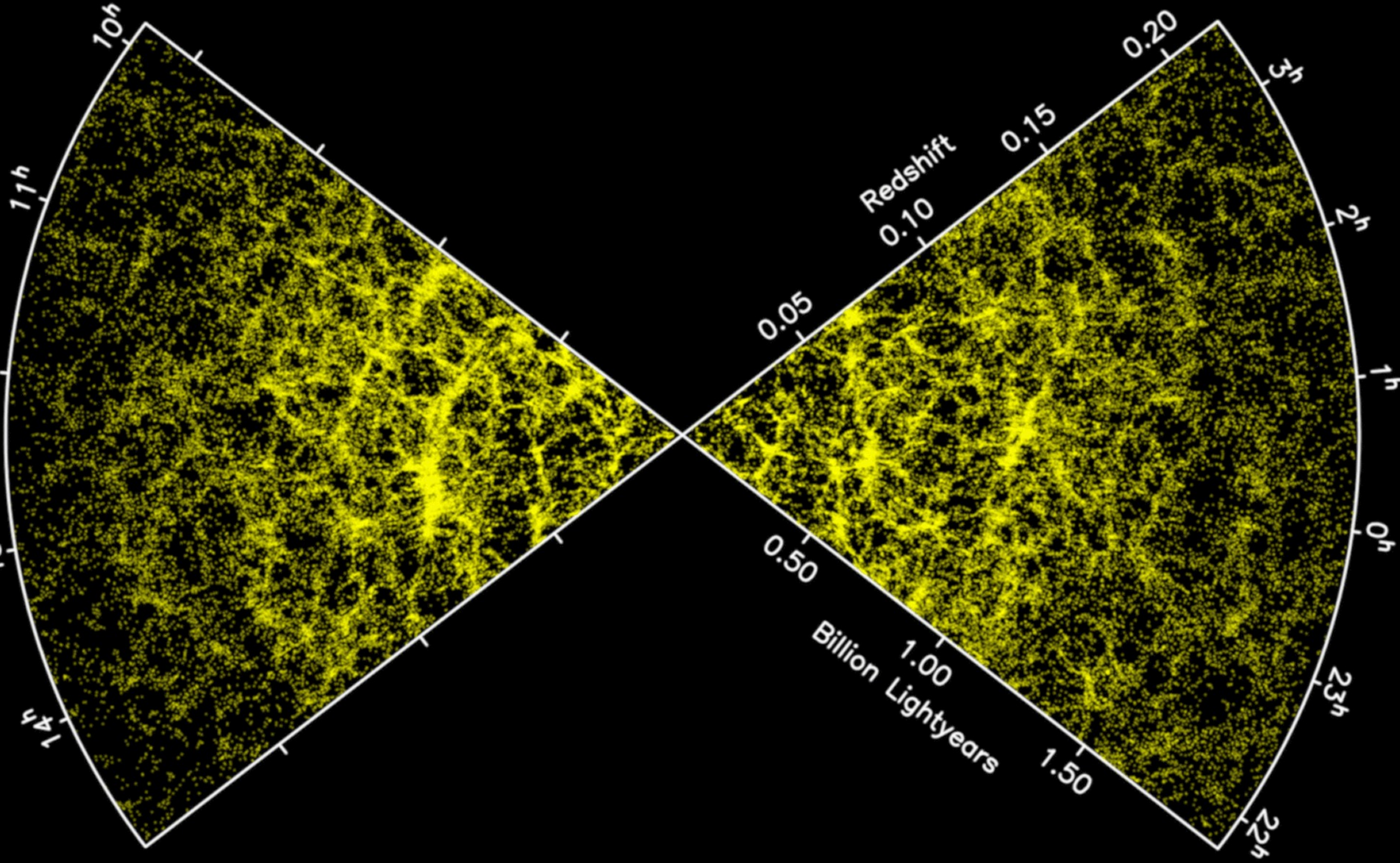
$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

The metric in **general relativity** depends on the space-time position:

$$g_{\mu\nu}(t, \boldsymbol{x})$$

Gravity = Geometry

Universe today: The 2dF Galaxy Redshift Survey



http://www.roe.ac.uk/~jap/2df/

Metric of the Universe

The Universe appears to be on large scales

- **Isotropic** = the same in all directions
- **Homogeneous** = the same at all places
- **Changing with time**

This means that the time-dependence of the metric can be factored out in the scale factor, a(t),

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \underset{\text{scale factor}}{\boxed{a^2(t)}} \times \overset{\text{symmetric 3-space}}{\boxed{\mathrm{d}\ell^2}}$$

and the line-element $d\ell$ is supposed to describe a “maximally symmetric 3-space”.

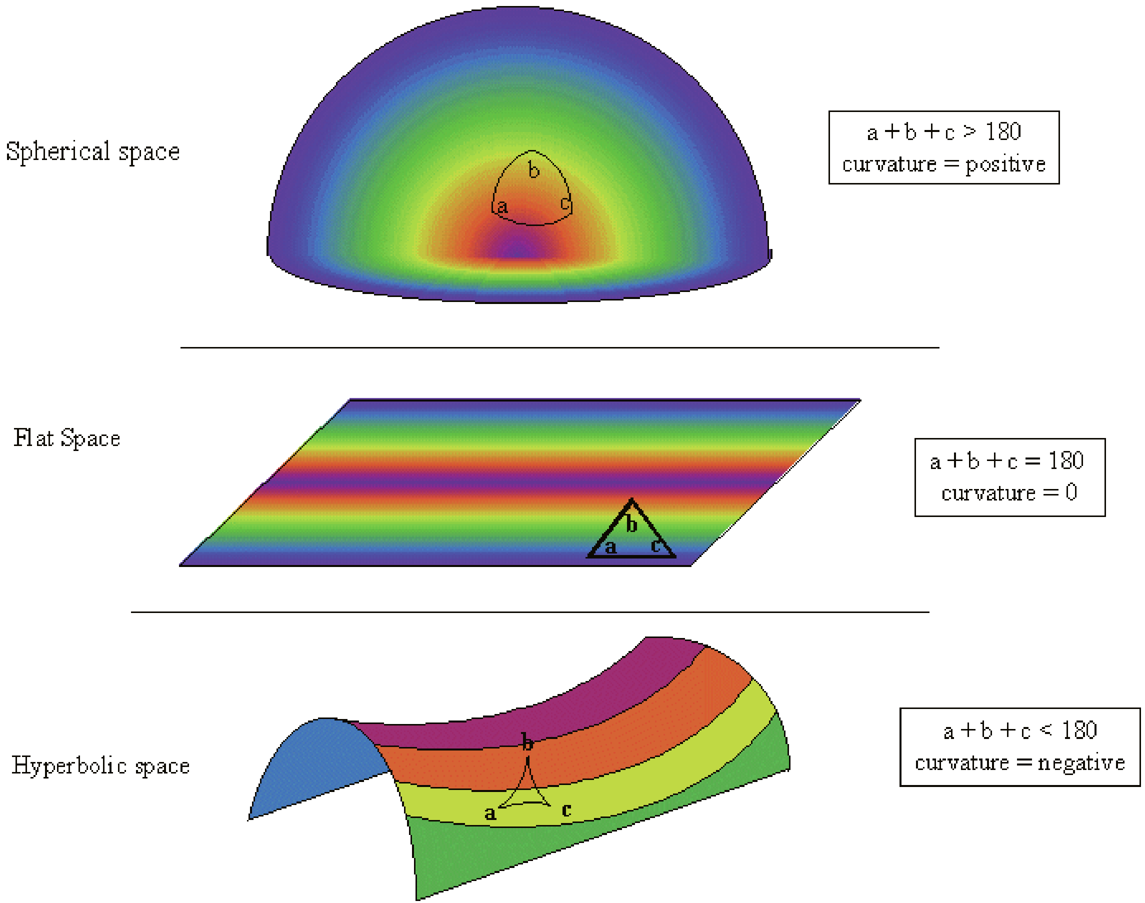
Side remark: one can use conformal time, $d\eta = dt/a(t)$, to simplify equations.

$$ds^2 = a(t)^2 \times \left[d\eta^2 - d\ell^2\right]$$

Friedmann-Robertson-Walker Metric

Three maximally symmetric 3-spaces

- **Positive curvature** (spherical, k=1)
- **Zero curvature** (flat, k=0)
- **Negative curvature** (hyperbolic, k=-1)



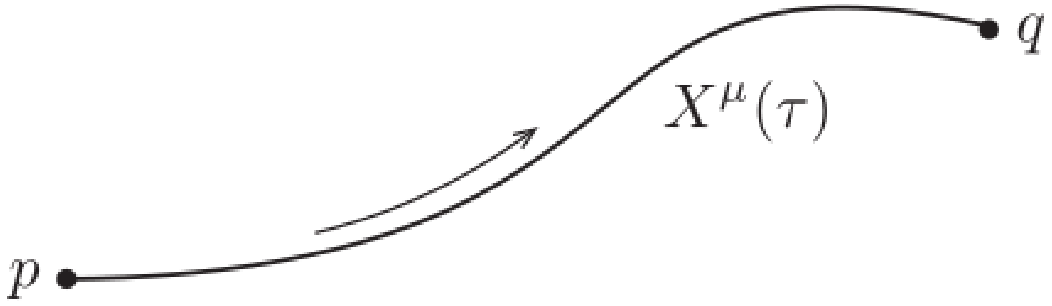
Credit: Researchgate

FRW Metric

$$\boxed{\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right]}$$

Geodesic equation

In absence of (non-gravitational) forces, particles move along **geodesics.** These can be defined as curves of least action, which extremize the proper time of a curves connecting two points.



Geodesic equation

$$\frac{d^2X^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dX^\alpha}{d\tau}\frac{dX^\beta}{d\tau}$$

Christoffel symbols (vanishing for constant metric)

$$\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2}g^{\mu\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta})$$

$$\partial_j \equiv \partial/\partial x^j$$

Dynamics and energy momentum Tensor

The geodesic equation is **invariant under a parameter rescaling**

$$\tau \rightarrow \alpha \cdot \tau \qquad \frac{d^2 X^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta} \frac{dX^\alpha}{d\tau} \frac{dX^\beta}{d\tau}$$

Definition of **proper time** by (time measured by clock along the geodesic) via the **four-velocity**

$$U^\mu \equiv \frac{dX^\mu}{d\tau} \qquad U^\mu U^\nu g_{\mu\nu} = 1$$

The **four-momentum** is defined as

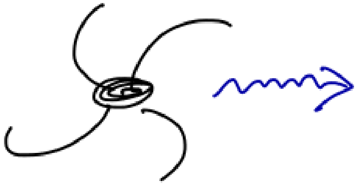
$$P^\mu \equiv m U^\mu \qquad P^\mu = (E, p_x, p_y, p_z)$$

Massless and massive particles

For a massless particle in a FRW Universe, one can show that

$$E\frac{dE}{dt} = \frac{\dot{a}}{a} g_{ij}p^i p^j \quad \Rightarrow \quad \frac{1}{E}\frac{dE}{dt} = -\frac{\dot{a}}{a} \quad \Rightarrow \quad \boxed{E(t) \propto \frac{1}{a(t)}}$$

This implies that radiation redshifts as the Universe expands



emitted at t_1 with λ_1

observed at t_0 with $\lambda_0 = \frac{a(t_0)}{a(t_1)}\lambda_1 \; > \; \lambda_1$

$$\boxed{z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1}} \quad \Rightarrow \quad \boxed{1 + z = \frac{1}{a(t_1)}} \; , \; \text{for } a(t_0) \equiv 1$$

For massive particles: $E(t) = \sqrt{m^2 + p(t)^2}$ with $p(t) \propto \frac{1}{a(t)}$

Distance ladder and expansion

Local Universe: Simple connection of redshift and Hubble parameter today (modulo peculiar motion of object relative to the Hubble flow)

Hubble's Law $z = H_0 d$

$$d = c \cdot (t_1 - t_0)$$

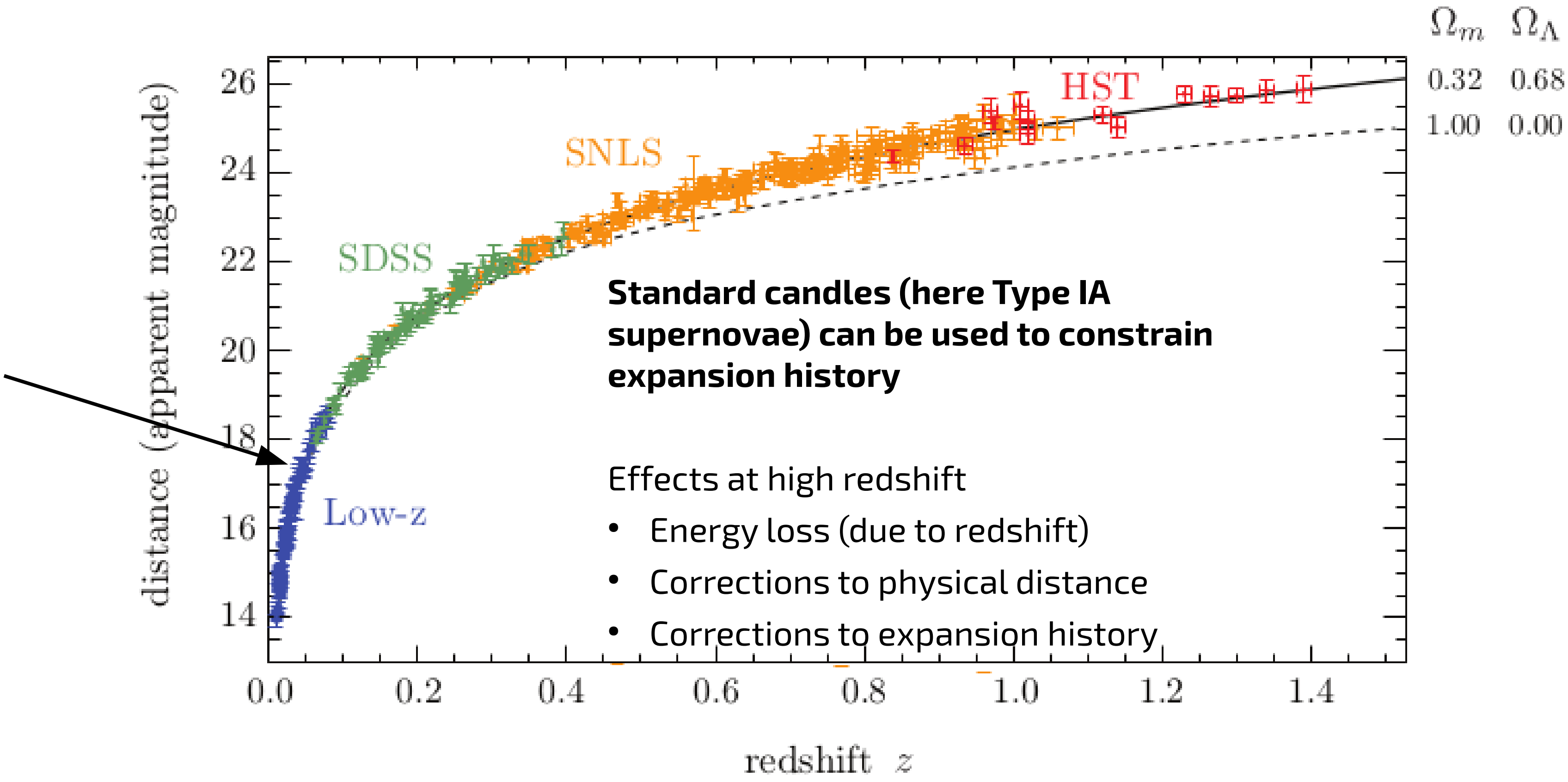
$$a(t_1) = a(t_0)\left[1 + (t_1 - t_0)\boxed{H_0} + \cdots\right]$$

Hubble constant

$$H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)}$$

$$H_0 = 100\,\boxed{h}\,\mathrm{km\,s^{-1}Mpc^{-1}}$$

$h = 0.67 \pm 0.01$



Standard candles (here Type IA supernovae) can be used to constrain expansion history

Effects at high redshift

- Energy loss (due to redshift)
- Corrections to physical distance
- Corrections to expansion history

Various cosmological distance measures

Comoving line-of-sight (LOS) distance towards object emitting at redshift z.

(here only flat Universe!)

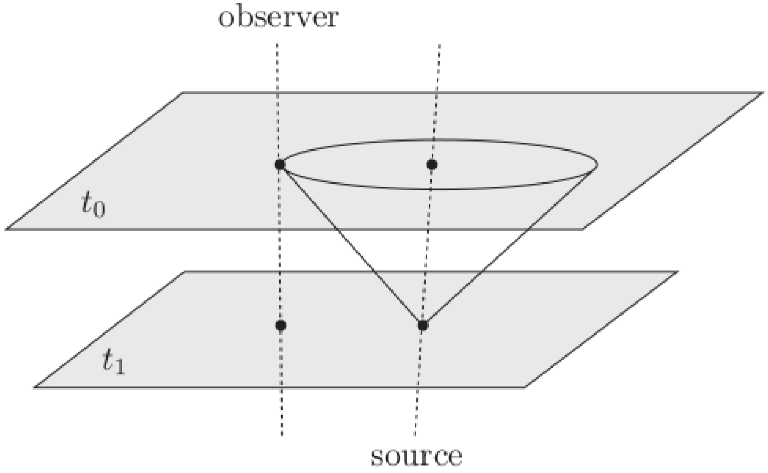
$$d_C = \int_{t(z)}^{t_0} \frac{dt}{a(t)} = \int_0^z dz' \frac{1}{H(z')}$$

Luminosity distance

$$F = \frac{L}{4\pi d_L^2}$$

(*F*: energy flux)

$$d_L = (1+z) d_C(z)$$

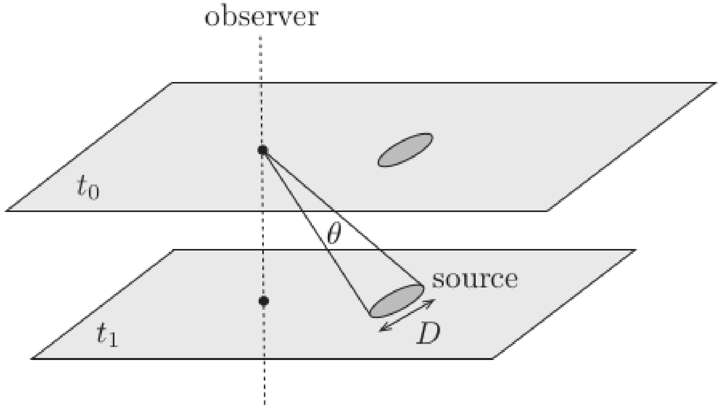


Angular distance

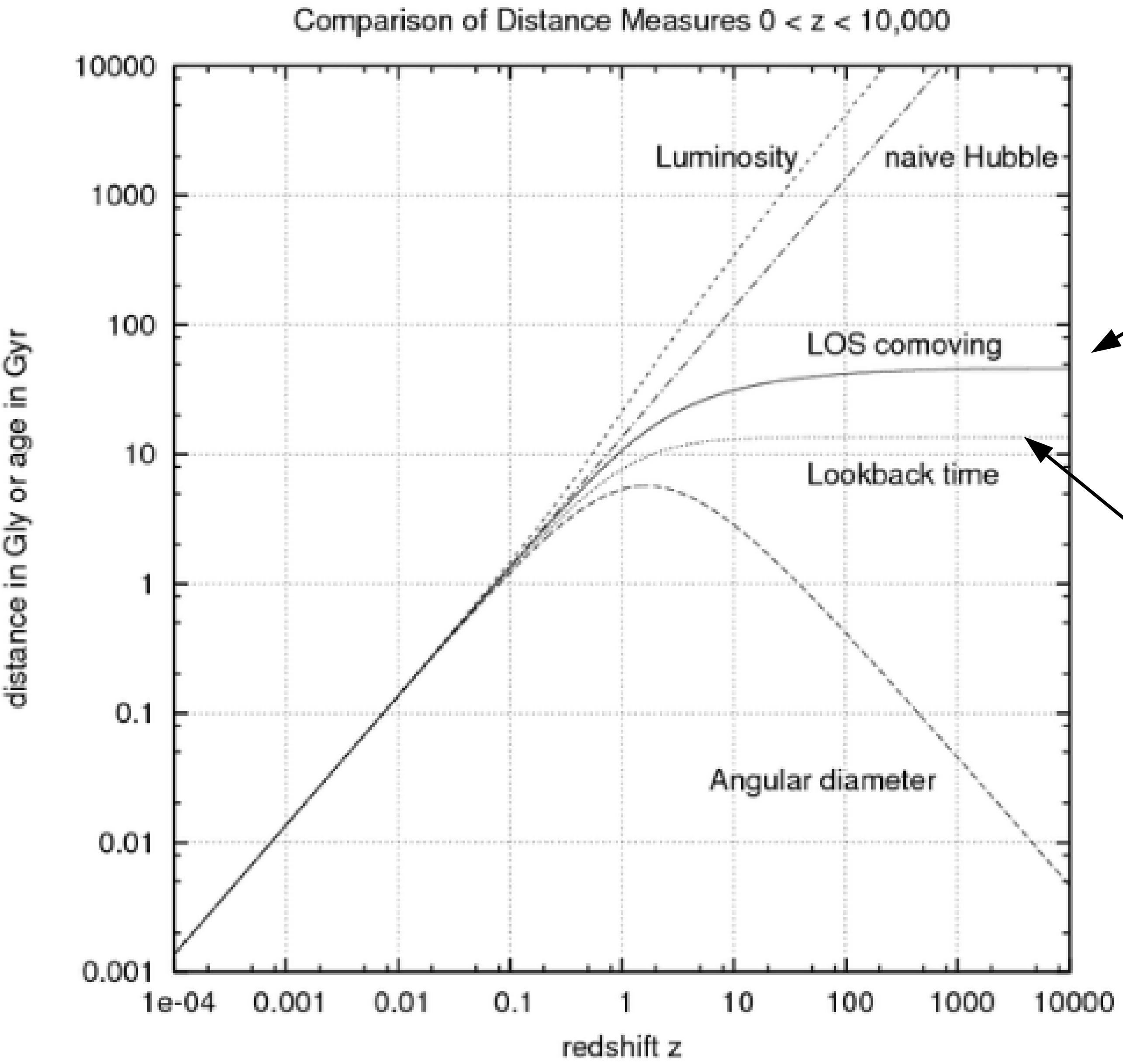
$$\theta = \frac{D}{d_A}$$

(*D*: physical size)

$$d_A = \frac{1}{1+z} d_C(z)$$



Distance measures



$\sim 47\,\mathrm{Gly} \sim 14\,\mathrm{Gpc}$

Time passed since redshift *z*

$$t_{\rm LB} = \int_{t(z)}^{t_0} dt = \int_0^z dz' \frac{1}{(1+z')H(z')}$$

$$t_{\rm LB}(z \to \infty) \sim 14\,\mathrm{Gyr}$$

Dynamics

The dynamics of the metric, of a(t), is determined by the

Einstein equations

$$\underbrace{\boxed{G_{\mu\nu}[a(t)]}}_{\text{“CURVATURE”}} = 8\pi G \underbrace{\boxed{T_{\mu\nu}}}_{\text{“MATTER”}}$$

1. Energy momentum tensor of a perfect fluid in the rest-frame

$$T^{\mu}{}_{\nu} \equiv g^{\mu\lambda}T_{\lambda\nu} = \begin{pmatrix} \rho & & & \\ & -P & & \\ & & -P & \\ & & & -P \end{pmatrix}$$

P : Pressure density

ρ : Energy density

Equation of state $w \equiv \dfrac{P}{\rho}$

Energy momentum tensor

Continuity equation

$$\dot{\rho} \;=\; -3\,\frac{\dot{a}}{a}\,(\rho + P) \qquad \text{or} \qquad \frac{d(\rho a^3)}{dt} = -P\frac{d(a^3)}{dt}$$

$$\text{“}\mathrm{d}U \;=\; -P\mathrm{d}V\text{”}$$

Contributions to the energy density at late times

$$w = 0: \qquad\qquad w = \tfrac{1}{3}: \qquad\qquad w = -1:$$

$$\overbrace{\underbrace{\text{electrons }(e)\quad \text{protons }(p)}_{\text{baryons }(b)}\quad \text{CDM }(c)}^{\text{matter }(m)}\quad \overbrace{\text{photons }(\gamma)\quad \text{neutrinos }(\nu)}^{\text{radiation }(r)}\quad \text{dark energy }(\Lambda)$$

This implies, for constant *w*,

$$\boxed{\rho \propto a^{-3(1+w)}} = \begin{cases} a^{-3} & \text{matter} \\ a^{-4} & \text{radiation} \\ a^{0} & \text{dark energy} \end{cases}$$

Friedmann equations

2. Einstein tensor

$$\overset{\text{EINSTEIN TENSOR}}{\boxed{G_{\mu\nu}}} = \underset{\text{RICCI TENSOR}}{\boxed{R_{\mu\nu}}} - \frac{1}{2} g_{\mu\nu} \underset{\text{RICCI SCALAR}}{\boxed{R}}$$

Implied equations of motion

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$\Omega_k = -\frac{k}{H_0^2 a_0^2}$$

$$H \equiv \frac{\dot{a}}{a}$$

$$\rho \equiv \underbrace{\rho_\gamma + \rho_\nu}_{\rho_r} + \underbrace{\rho_c + \rho_b}_{\rho_m} + \rho_\Lambda$$

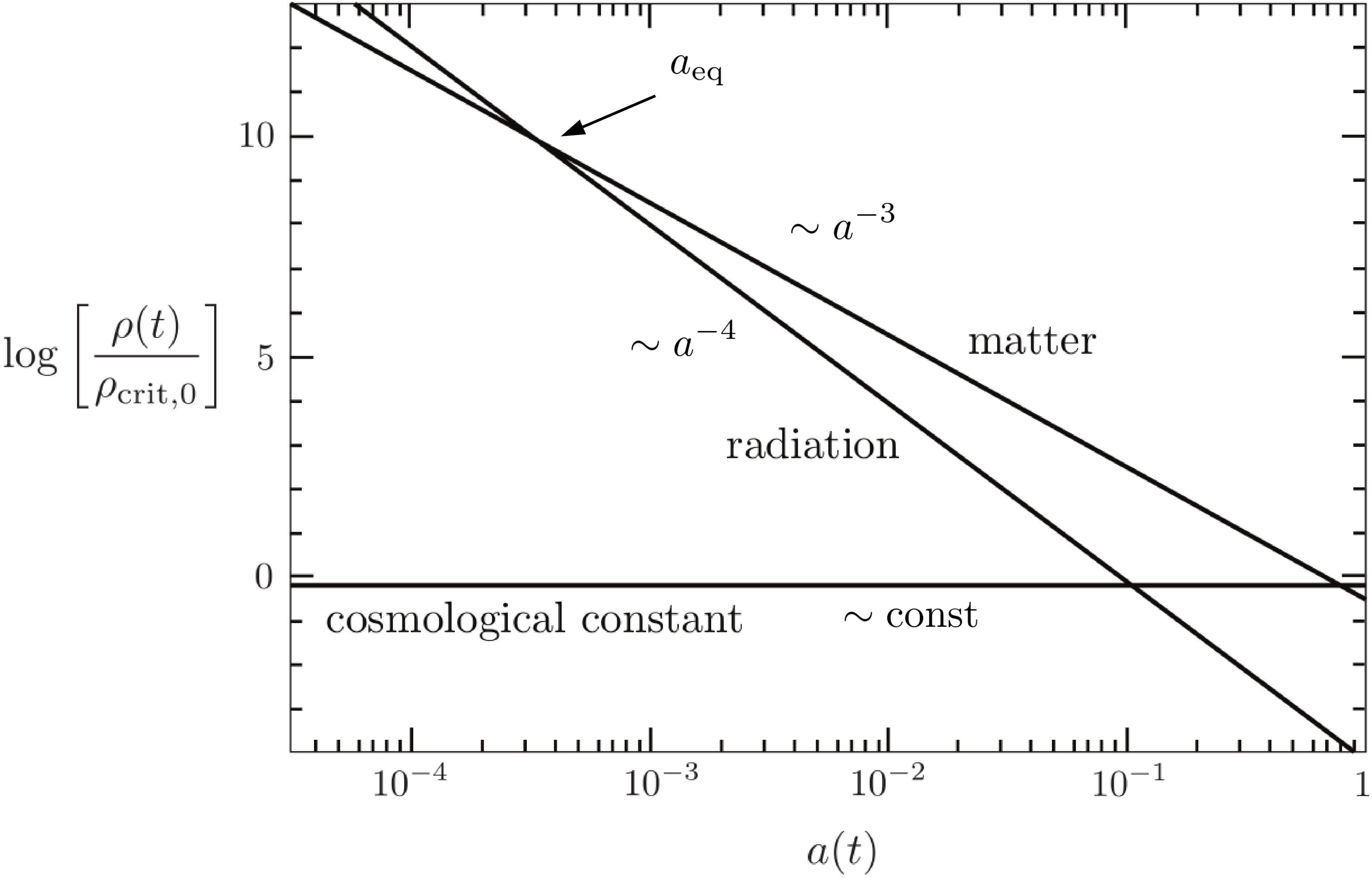
$$\rho_{\mathrm{crit},0} = \frac{3H_0^2}{8\pi G} = 1.9 \times 10^{-29}\, h^2\, \mathrm{grams\,cm}^{-3}$$

$$\boxed{\Omega_I \equiv \frac{\rho_{I,0}}{\rho_{\mathrm{crit},0}}}$$

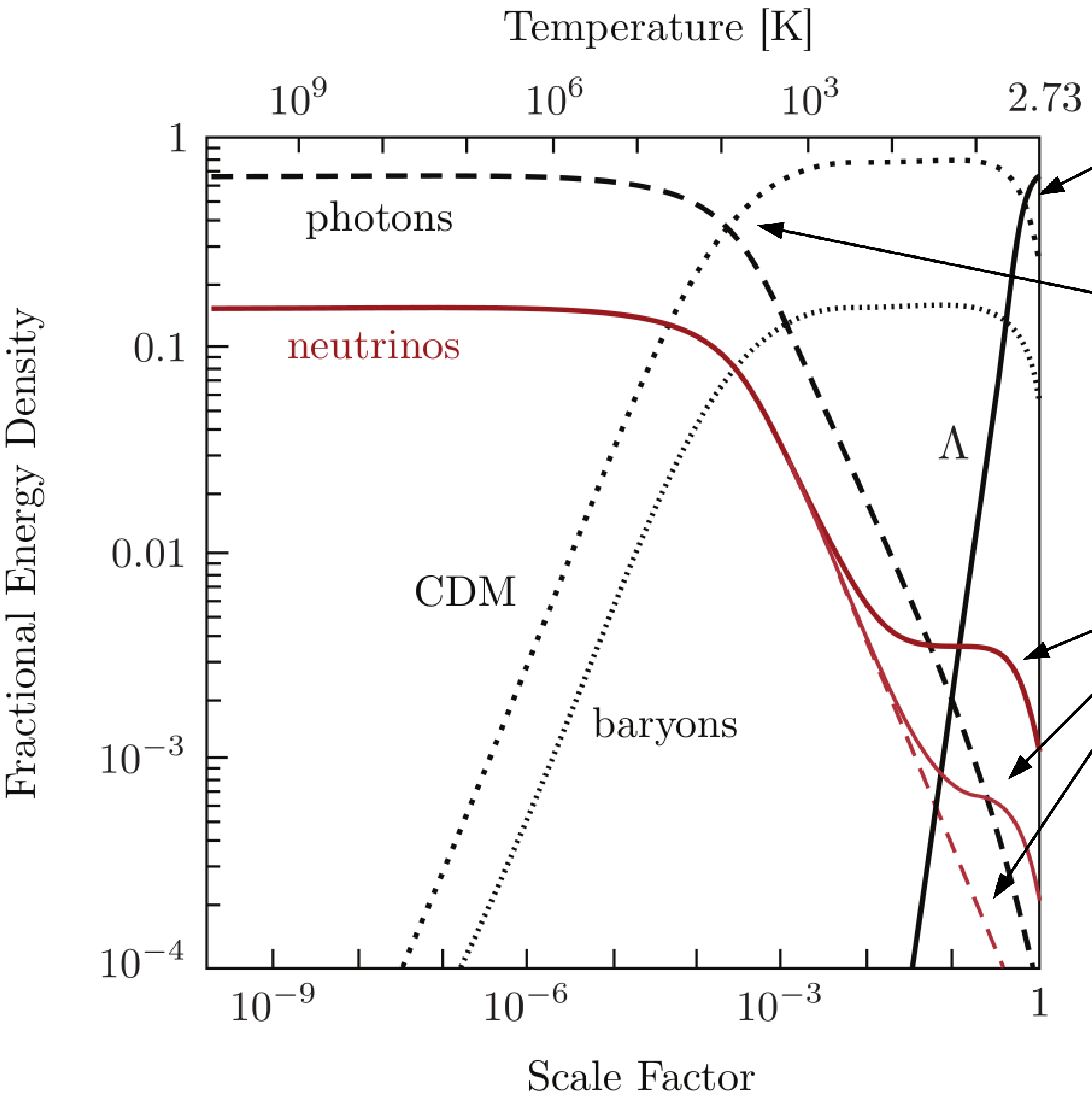
Hubble parameter as function of scale parameter

$$H^2(a) = H_0^2\left[\Omega_r\, a^{-4} + \Omega_m\, a^{-3} + \Omega_k\, a^{-2} + \Omega_\Lambda\right]$$

Energy densities scaling



Fractional energy densities scaling



Dark energy matter equality

$z \sim 0.4$

Matter radiation equality (MRE)

$z_{\rm eq} \sim 3400$

$\rho_m(z_{\rm eq}) = \rho_r(z_{\rm eq})$

different neutrino masses

Equilibrium thermodynamics

equilibrium

$>$

freeze-out

Γ $=$ H

decoupling

interaction rate *expansion rate*

$<$

non-equilibrium

$$N_{\rm int} = \int_{t_1}^{t_0} dt \Gamma(t) = \int_{z_0}^{z_1} dz \frac{\Gamma(z)}{(1+z)H(z)} \simeq \int_{\ln z_0}^{\ln z_1} d \ln z \frac{\Gamma(z)}{H(z)}$$

Overview over different epochs

Event	time t	redshift z	temperature T
Singularity	$\mathbf{0}$	∞	∞
Quantum gravity	$\sim 10^{-43}\,\mathrm{s}$	–	$\sim 10^{18}\,\mathrm{GeV}$
Inflation	$\gtrsim 10^{-34}\,\mathrm{s}$	–	–
Baryogenesis	$\lesssim 20\,\mathrm{ps}$	$> 10^{15}$	$> 100\,\mathrm{GeV}$
EW phase transition	$20\,\mathrm{ps}$	10^{15}	$100\,\mathrm{GeV}$
QCD phase transition	$20\,\mu\mathrm{s}$	10^{12}	$150\,\mathrm{MeV}$
Dark matter freeze-out	?	?	?
Neutrino decoupling	$1\,\mathrm{s}$	6×10^{9}	$1\,\mathrm{MeV}$
Electron-positron annihilation	$6\,\mathrm{s}$	2×10^{9}	$500\,\mathrm{keV}$
Big Bang nucleosynthesis	$3\,\mathrm{min}$	4×10^{8}	$100\,\mathrm{keV}$
Matter-radiation equality	$60\,\mathrm{kyr}$	3400	$0.75\,\mathrm{eV}$
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	$380\,\mathrm{kyr}$	1100	$0.26\,\mathrm{eV}$
Reionization	100–400 Myr	10–30	2.6–7.0 meV
Dark energy-matter equality	$9\,\mathrm{Gyr}$	0.4	$0.33\,\mathrm{meV}$
Present	**13.8 Gyr**	**0**	**0.24 meV**

Covered today

Equilibrium thermodynamics

For now, we focus on an **isotropic Universe**. The distribution function is independent of *x*.

$$f(\boldsymbol{x},\boldsymbol{p},t) \underset{\substack{\uparrow \\ \text{homogeneity + isotropy}}}{=} f(p,t) \equiv \boxed{f(p)}$$

Fermions, bosons, and intensive quantities

$$\boxed{f(p) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1}}$$

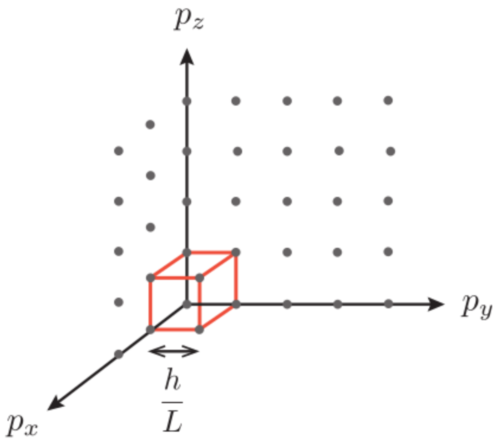
$+$ fermions

$-$ bosons

$T(t)$: temperature $(k_{\mathrm{B}} \equiv 1)$

$\mu(T)$: chemical potential

- **number density** $$\boxed{n = \frac{g}{(2\pi)^3}\int \mathrm{d}^3p\, f(p)}$$
- **energy density** $$\boxed{\rho = \frac{g}{(2\pi)^3}\int \mathrm{d}^3p\, f(p)E(p)}$$
- **pressure** $$\boxed{P = \frac{g}{(2\pi)^3}\int \mathrm{d}^3p\, f(p)\frac{p^2}{3E(p)}}$$



Relativistic and non-relativistic limits

Relativistic boson and fermion gases $T \gg m$

$$n = \frac{\zeta(3)}{\pi^2} g T^3 \begin{cases} 1 & \text{bosons} \\ \frac{3}{4} & \text{fermions} \end{cases}$$

$$\rho = \frac{\pi^2}{30} g T^4 \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases}$$

$$P = \frac{1}{3}\rho$$

Non-relativistic limit $T \ll m$

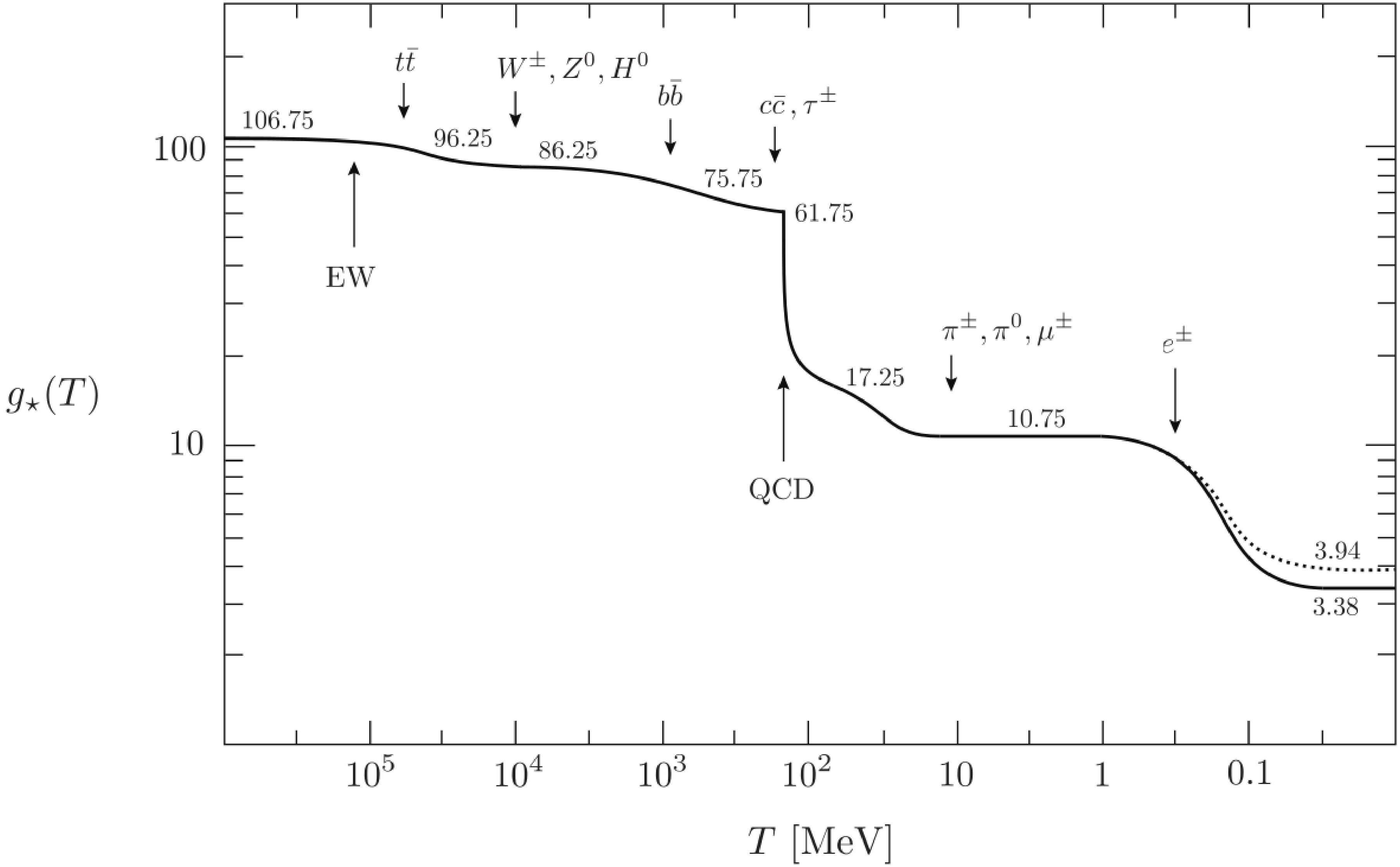
$$n = g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

$$\rho \approx mn$$

$$P = nT$$

Effective number of relativistic DOF

$$\rho_r = \sum_i \rho_i \equiv \frac{\pi^2}{30} g_\star(T)\, T^4 \qquad g_\star(T) = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=f} g_i \left(\frac{T_i}{T}\right)^4$$



Entropy

Entropy density can be defined as

One temperature

$$s \equiv \frac{\rho + P}{T}$$

Multiple components with different temperatures

$$s = \sum_i \frac{\rho_i + P_i}{T_i} \equiv \frac{2\pi^2}{45} g_{\star S}(T)\, T^3$$

Effective entropy degrees of freedom

$$g_{\star S}(T) = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=f} g_i \left(\frac{T_i}{T}\right)^3$$

Entropy is conserved in reversible adiabatic expansion

$$s \propto a^{-3} \quad \Rightarrow \quad T \propto g_{\star S}^{-1/3} a^{-1}$$

Neutrino decoupling

Neutrino freeze-out

$$\nu + e^+ \ \leftrightarrow \ \nu + e^+$$

$$\frac{\Gamma}{H} \sim \frac{G_F^2 T^5}{T^2/M_{\rm pl}} \sim \left(\frac{T}{1\,\mathrm{MeV}}\right)^3 \sim 1$$

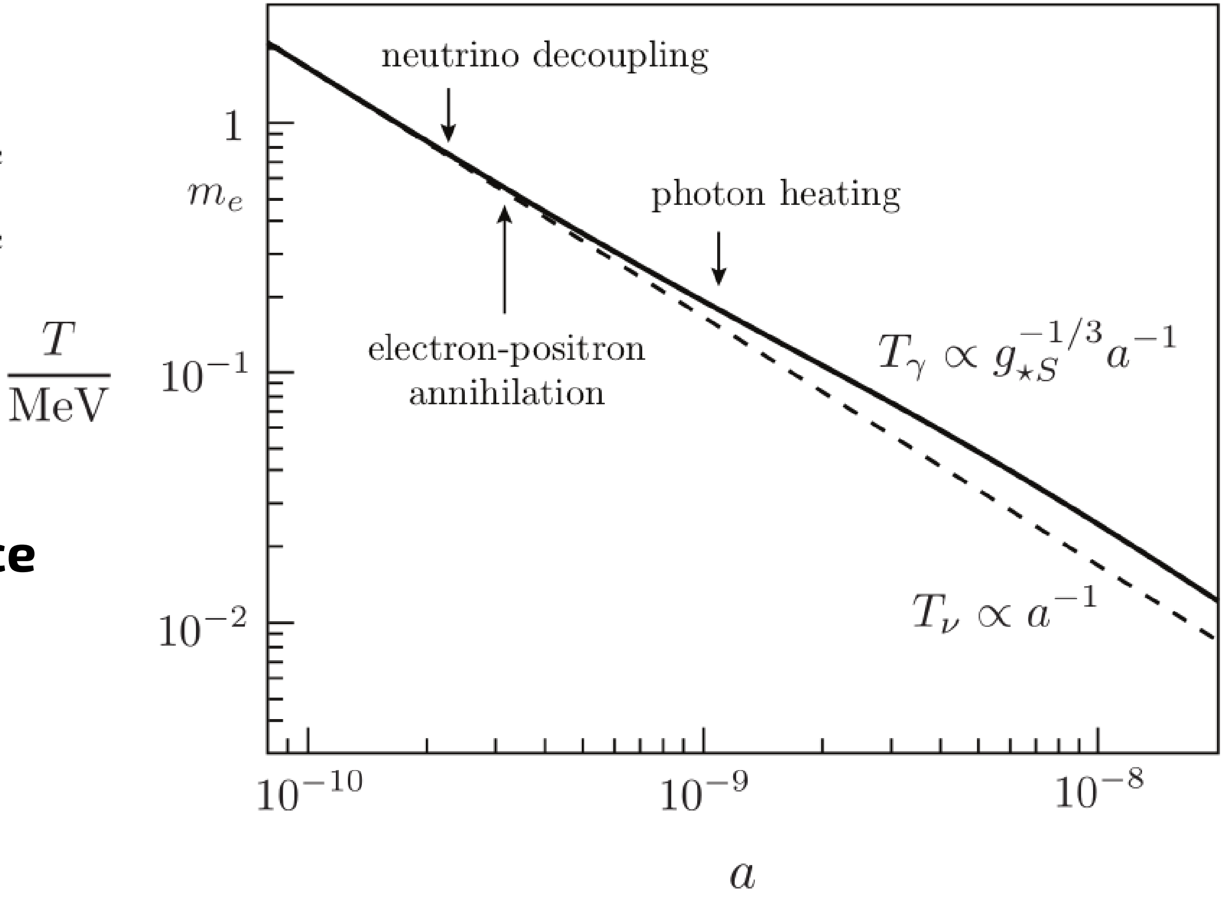
Electron positron annihilation:

$$e^+ + e^- \ \leftrightarrow \ \gamma + \gamma$$

$$T \lesssim m_e$$

Degrees of freedom

$$g_{\star S}^{th} = \begin{cases} 2 + \frac{7}{8} \times 4 = \frac{11}{2} & T \gtrsim m_e \\ 2 & T < m_e \end{cases}$$



Induced temperature difference

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

~71%

Recombination and photon decoupling

Photo dissociation of Hydrogen

$$e^- + p^+ \quad \leftrightarrow \quad \mathrm{H} + \gamma$$

$$n_i^{\mathrm{eq}} = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T} \right)$$

$$B_{\mathrm{H}} \equiv m_p + m_e - m_{\mathrm{H}} = 13.6 \text{ eV}$$

Saha equation (ionization equilibrium)

$$\boxed{\left(\frac{1 - X_e}{X_e^2} \right)_{\mathrm{eq}} = \frac{2\zeta(3)}{\pi^2} \eta \left(\frac{2\pi T}{m_e} \right)^{3/2} e^{B_{\mathrm{H}}/T}}$$

$$X_e \equiv \frac{n_e}{n_b}$$

$$\eta = \frac{n_b}{n_\gamma} \sim 5.5 \times 10^{-10}$$ baryon-to-photon ratio

$$X_e \sim 0.1 \Rightarrow \; T_{rec} \approx 0.3\,\mathrm{eV}$$

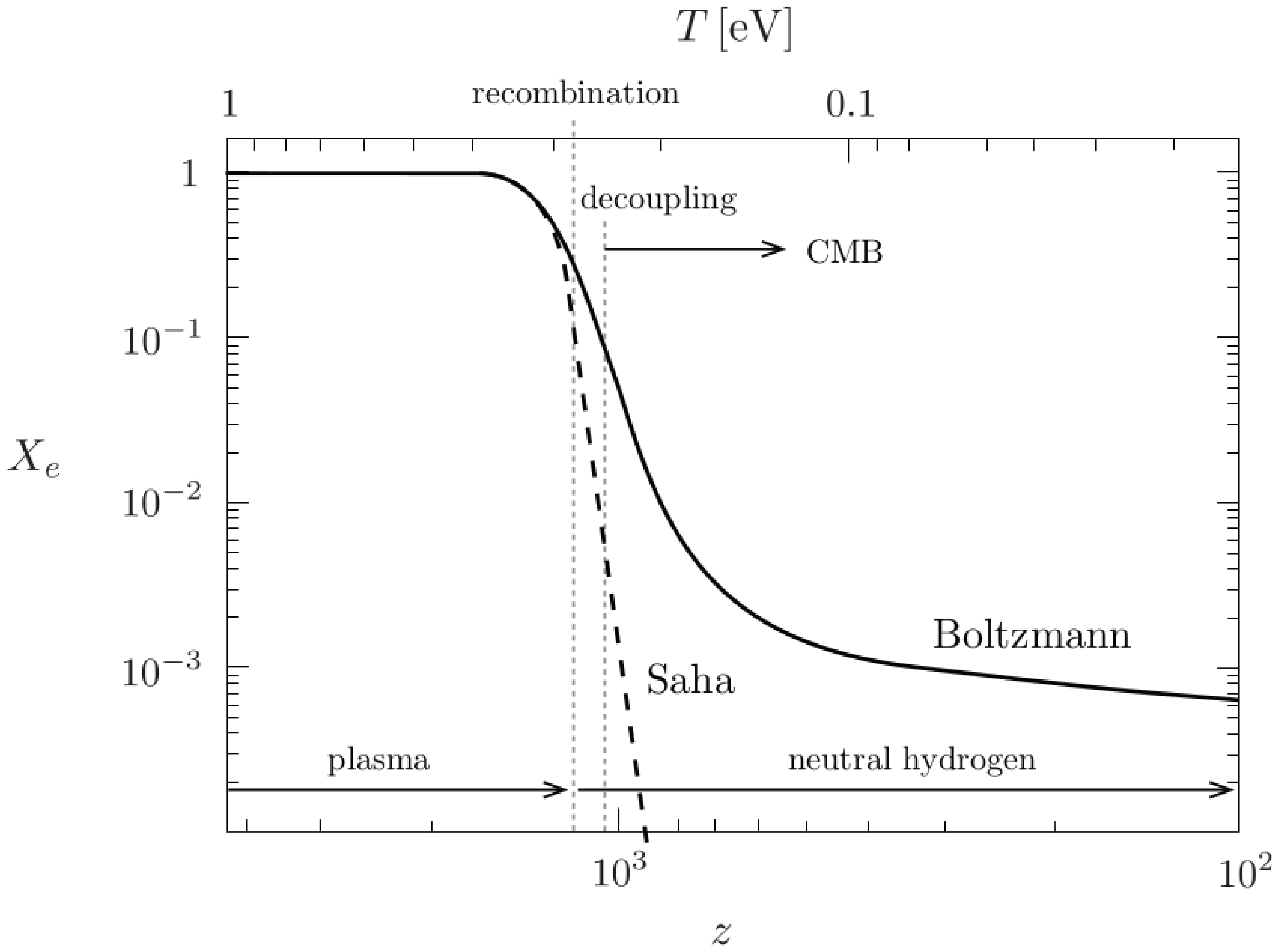
Decoupling

$$e^- + \gamma \quad \leftrightarrow \quad e^- + \gamma$$

$$\Gamma_\gamma(T_{dec}) = n_b X_e(T_{dec})\, \sigma_T$$

$$\Gamma_\gamma(T_{dec}) \sim H(T_{dec}) \qquad \Rightarrow \qquad T_{dec} \sim 0.27\,\mathrm{eV}$$

Recombination and photon decoupling



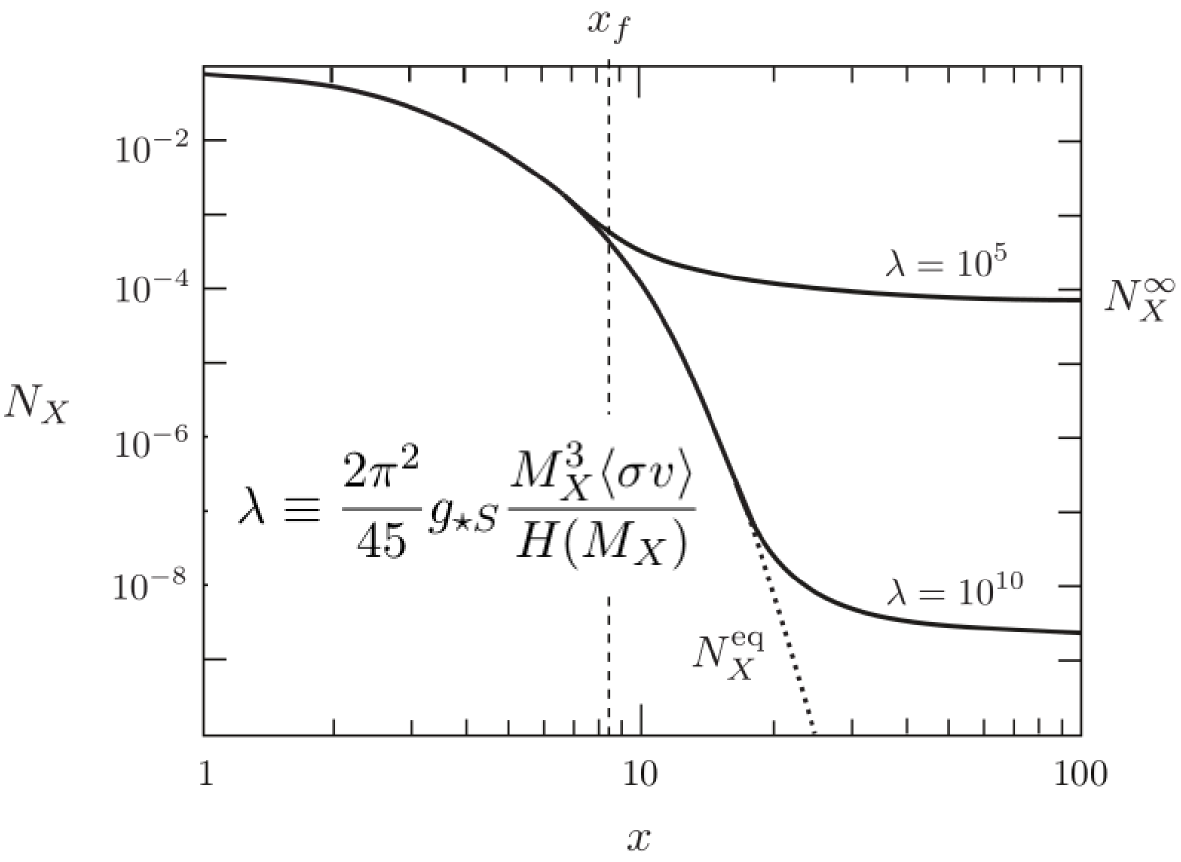
Dark matter relics

Boltzmann equation for particles in comoving volume

$$\frac{dN_X}{dt} = -s\langle\sigma v\rangle\left[N_X^2 - (N_X^{\rm eq})^2\right]$$

$$X + \bar{X} \leftrightarrow \ell + \bar{\ell}$$

$$N_X \equiv n_X/s. \qquad x \equiv \frac{M_X}{T}$$



$$\lambda \equiv \frac{2\pi^2}{45} g_{\star S} \frac{M_X^3\langle\sigma v\rangle}{H(M_X)}$$

Relic density today

$$\Omega_X = \frac{\rho_{X,0}}{\rho_{\rm crit,0}} = \frac{M_X N_X^\infty s_0}{\rho_{\rm crit,0}}$$

“WIMP miracle”

$$\Omega_X h^2 \sim 0.1 \frac{3\times 10^{-26}{\rm cm}^3{\rm s}^{-1}}{\langle\sigma v\rangle} \sim \frac{10^{-3}G_F}{\langle\sigma v\rangle}$$

BBN

Helium – and in traces other light elements – are produced during big-bang nucleosynthesis (BBN)

- Above 0.1 MeV, only neutrons and protons exist
- Primordial Helium *after* BBN mass fraction is ~25%
- Production of Helium requires the existence of deuterium

$$\mathrm{D} + p^+ \leftrightarrow {}^3\mathrm{He} + \gamma \ ,$$

$$\mathrm{D} + {}^3\mathrm{He} \leftrightarrow {}^4\mathrm{He} + p^+ \ .$$

- Binding energy of deuterium is low

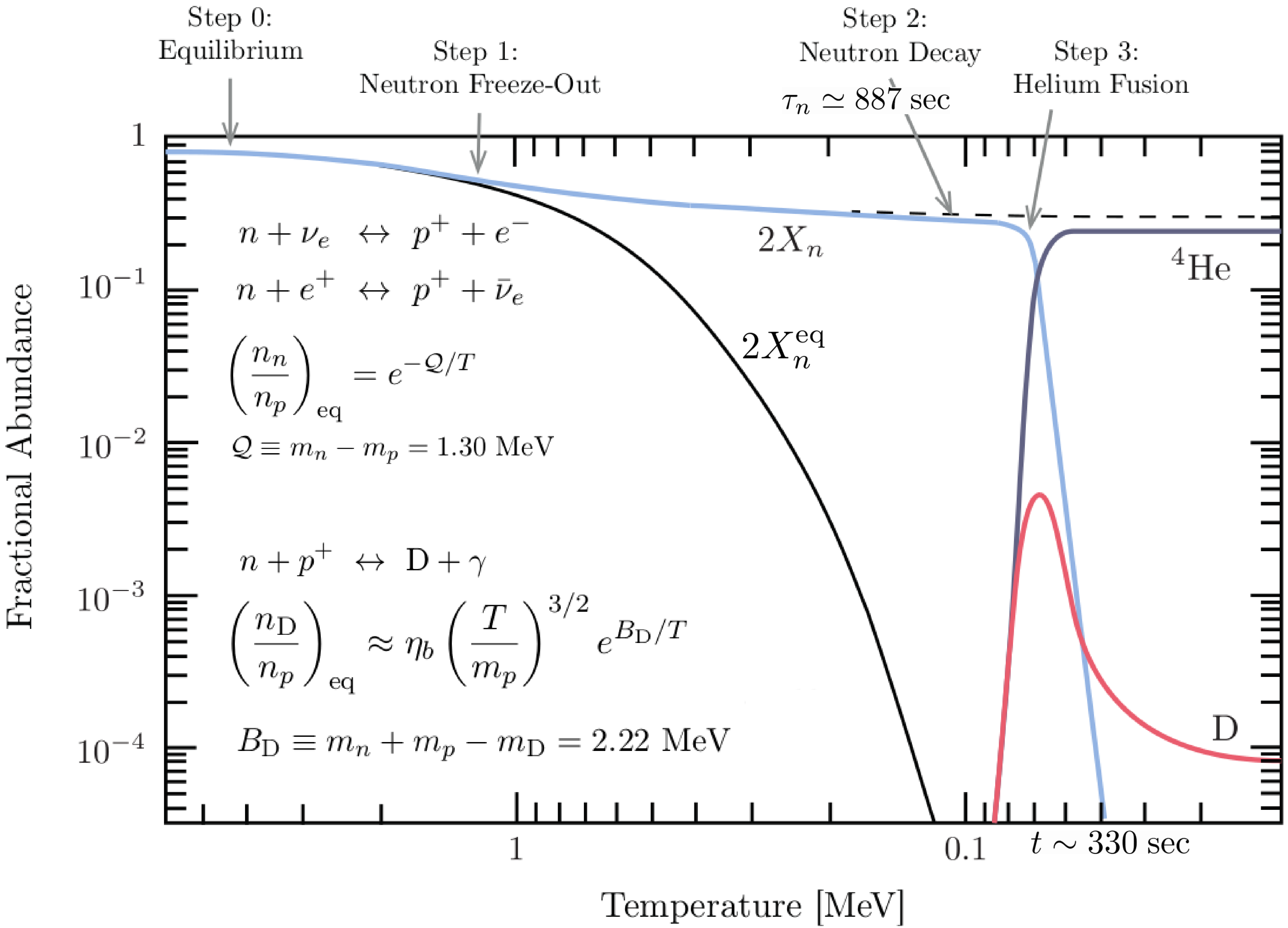
$$B_\mathrm{D} \equiv m_n + m_p - m_\mathrm{D} = 2.22 \ \mathrm{MeV}$$

  (while binding energy of Helium is 28.3 MeV!)

- Deuterium formation is delayed by photo dissociation
  → Impact on element abundances

Consider neutron fraction $X_n \equiv \dfrac{n_n}{n_n + n_p}$

BBN



$$n + \nu_e \;\leftrightarrow\; p^+ + e^-$$

$$n + e^+ \;\leftrightarrow\; p^+ + \bar{\nu}_e$$

$$\left(\frac{n_n}{n_p}\right)_{\text{eq}} = e^{-\mathcal{Q}/T}$$

$$\mathcal{Q} \equiv m_n - m_p = 1.30 \text{ MeV}$$

$$n + p^+ \;\leftrightarrow\; \text{D} + \gamma$$

$$\left(\frac{n_\text{D}}{n_p}\right)_{\text{eq}} \approx \eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{B_\text{D}/T}$$

$$B_\text{D} \equiv m_n + m_p - m_\text{D} = 2.22 \text{ MeV}$$

BBN

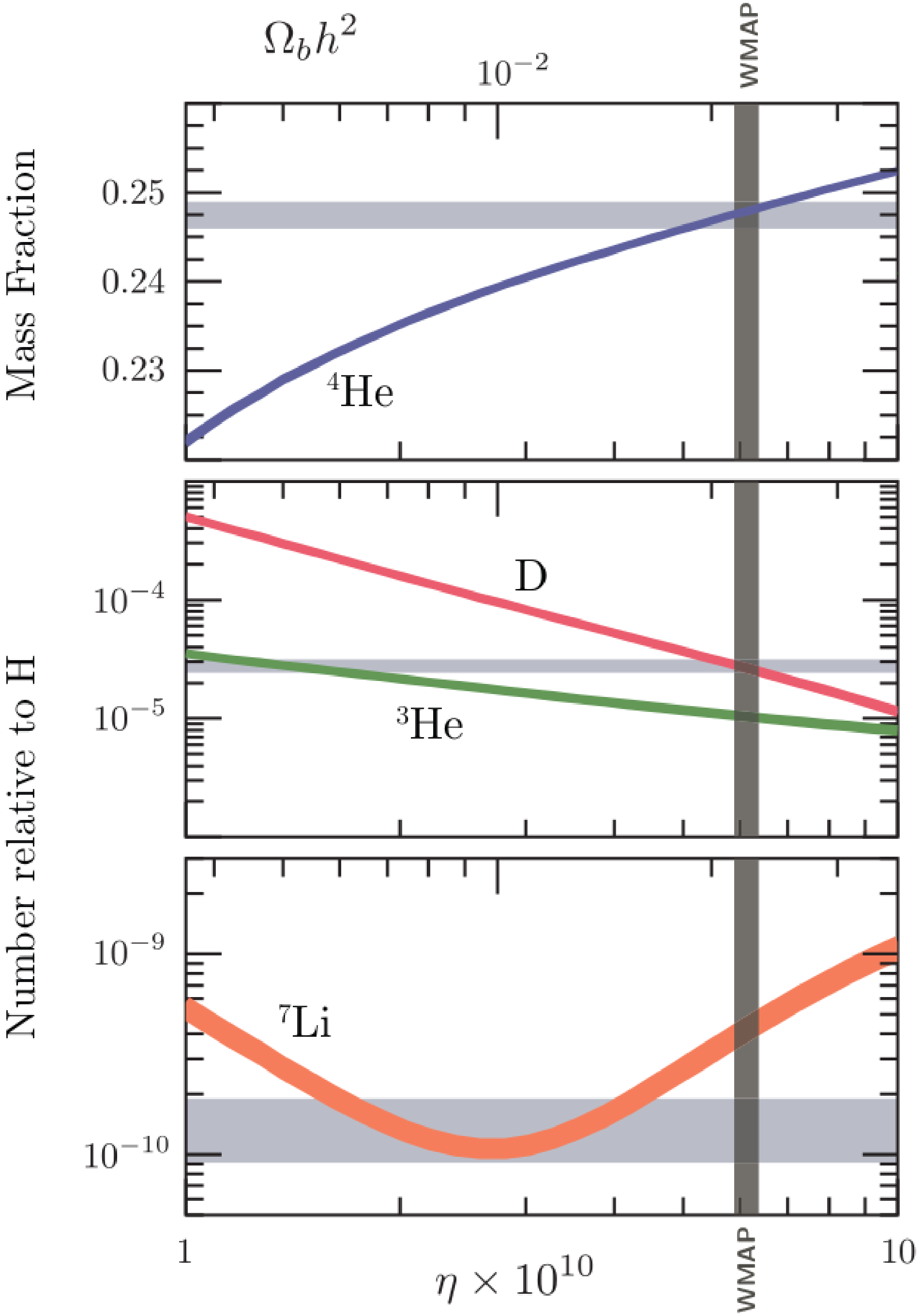
Primordial abundances of elements provide information about the baryon density in the early Universe.

A general diagnostic for new-physics searches

- Relativistic degrees of freedom

$$T_f \propto g_\star^{1/6}$$

- Changes in neutron lifetime, masses, or gravitational or electro-weak couplings would affect abundances.

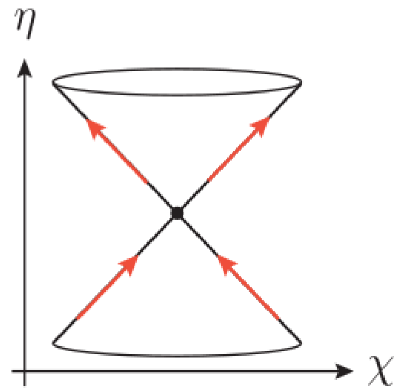


Lecture 2

Inflation and Linear Structure Formation

The particle horizon

Photon propagation



$$ds^2 = a(t)^2\left(d\eta^2 - d\chi^2\right) = 0 \quad \Rightarrow \quad \Delta\chi = \pm\Delta\eta$$

Conformal time: $d\eta = \frac{dt}{a}$ *Comoving distance*

Particle horizon

Comoving distance a photon has propagated since the beginning of time

$$\chi_{\rm p}(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{\mathrm{d}t}{a(t)} = \int_{\ln a_i}^{\ln a} \frac{\mathrm{d}\ln a}{\dot{a}} = \int_{\ln a_i}^{\ln a} \underset{\uparrow}{(aH)^{-1}}\, \mathrm{d}\ln a \qquad (\star)$$

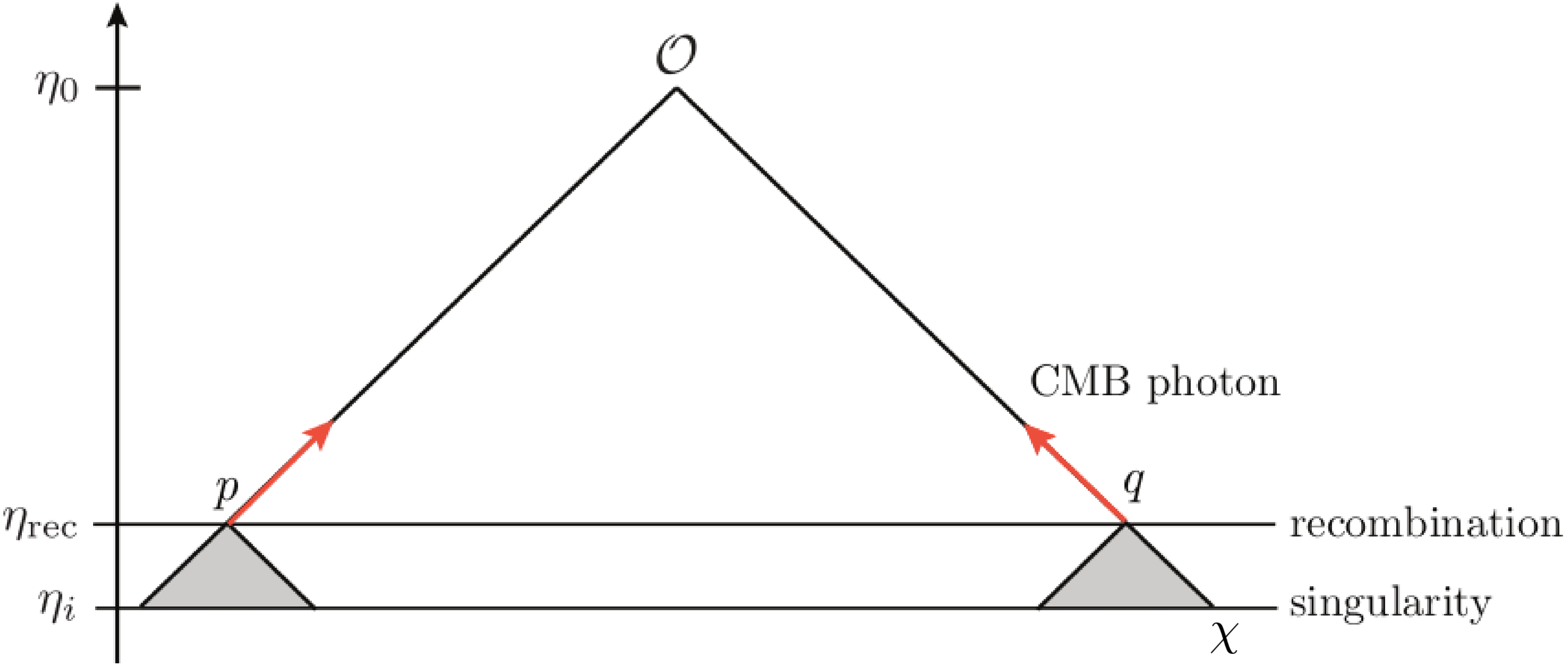
Comoving Hubble radius

Normal matter implies: $\ddot{a} < 0 \quad \Rightarrow \quad \frac{d}{dt}(aH)^{-1} > 0$

The size of particle horizon is dominated by late times.

The horizon problem

Past light cones of present-day observer

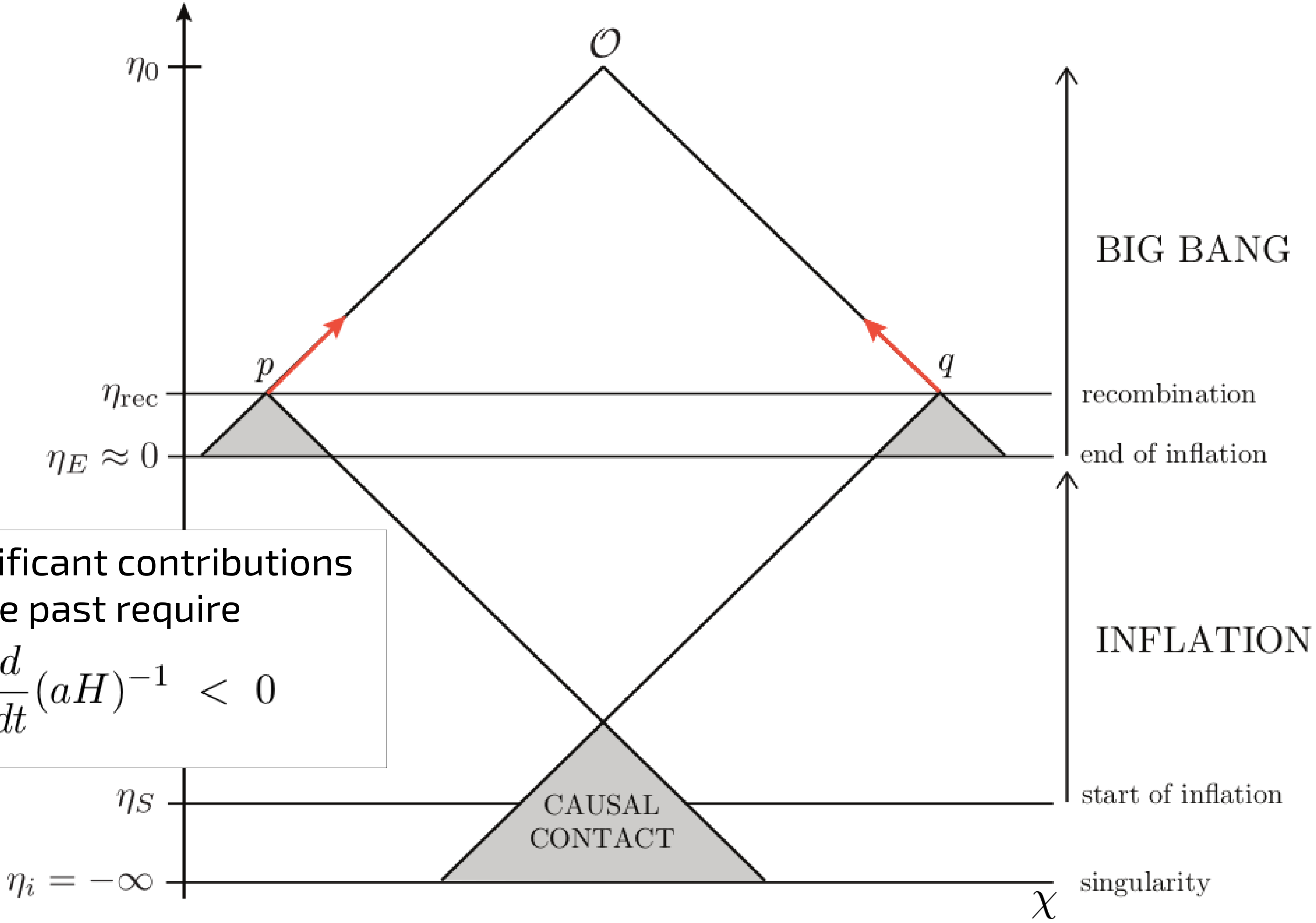


In the standard picture, points *p* and *q* of the CMB were **never in causal contact**. In fact, the CMB is made of ~1e4 causally disconnected patches.

Yet, the CMB is remarkably uniform. This is the “horizon problem”.

Inflation as solution to the horizon problem

Idea: There was more conformal time than naively thought!



Significant contributions in the past require

$$\frac{d}{dt}(aH)^{-1} \; < \; 0$$

Some characteristics of inflation

Accelerated expansion $\ddot{a} > 0$	$\frac{d(aH)^{-1}}{dt} = \frac{d}{dt}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2} < 0 \quad \Rightarrow \quad \boxed{\ddot{a} > 0}$
Slowly-varying Hubble $\varepsilon \equiv -\frac{\dot{H}}{H^2} < 1$	$\frac{d(aH)^{-1}}{dt} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = \frac{(\varepsilon - 1)}{a} < 0 \quad \Rightarrow \quad \boxed{\varepsilon < 1}$
Exponential expansion $a(t) \approx e^{Ht}$	$\varepsilon \ll 1 \;\rightarrow\; H = \frac{\dot{a}}{a} \approx const. \quad \Rightarrow \quad \boxed{a(t) = e^{Ht}}$
Negative pressure $w \equiv \frac{P}{\rho} < -\frac{1}{3}$	$\frac{\ddot{a}}{a} = -\frac{\rho + 3P}{6M_{\rm pl}^2} > 0 \quad \Rightarrow \quad \boxed{\rho + 3P < 0}$

Conditions for slow-roll inflation

1. Accelerated expansion $\ddot{a} > 0$

$$\varepsilon = -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN} \; < \; 1, \quad \text{where } \mathrm{d}N \equiv \mathrm{d}\ln a = H\,\mathrm{d}t.$$

Number of “e-folds”

2. Inflation lasts sufficiently long

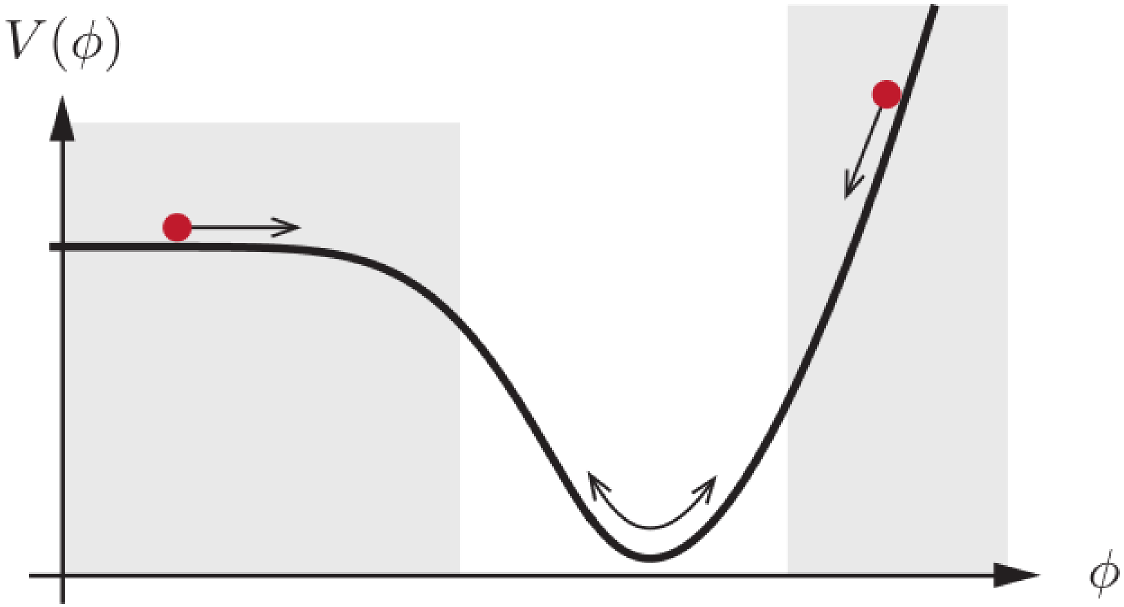
$$\eta = \left|\frac{d\ln\varepsilon}{dN}\right| = \left|\frac{\dot{\varepsilon}}{H\varepsilon}\right| \; < \; 1$$

What microphyscis leads to $\{\varepsilon, \eta\} < 1$*?*

Scalar field dynamics

Real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - V(\phi)$$



Corresponding energy and momentum densities

$$\rho_\phi \equiv T^0{}_0 = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$P_\phi \equiv -\frac{1}{3}T^i{}_i = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Equation of motion

$$\underbrace{\boxed{\ddot{\phi}}}_{\text{ACCELERATION}} + \underbrace{\boxed{3H\dot{\phi}}}_{\text{FRICTION}} = -\underbrace{\boxed{V'}}_{\text{FORCE}}$$

with $V' \equiv \frac{dV}{d\phi}$

Slow-roll inflation

Slow-roll parameters

Pragmatic parameters that describe slow-roll properties of inflaton field.

$$\epsilon_{\rm v} \equiv \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2 \quad , \qquad |\eta_{\rm v}| \equiv M_{\rm pl}^2 \frac{|V''|}{V}$$

$\{\epsilon_{\rm v}, |\eta_{\rm v}|\} \ll 1$ implies $\{\epsilon, |\eta|\} \ll 1$

Number of "e-folds"

How many factors of *e* the Universe inflated

$$N_{\rm tot} \equiv \int_{a_S}^{a_E} {\rm d}\ln a \quad \underset{\text{for single scalar field}}{\Rightarrow} \quad N_{\rm tot} = \int_{\phi_S}^{\phi_E} \frac{1}{\sqrt{2\epsilon_{\rm v}}}\frac{|{\rm d}\phi|}{M_{\rm pl}}$$

Solving horizon problem requires about 60 e-folds

Ratio of comoving Hubble radii
at end of inflation and today

$$\frac{a_0 H_0}{a_e H_e} = a_e \sim \frac{T_0}{10^{15}\ {\rm GeV}} \simeq 10^{-28} \sim \frac{1}{e^{64}}$$

Exercise

As an example, perform a slow-roll analysis of arguably the simplest model of inflation: **single-field inflation driven by a mass term**

$$V(\phi) = \frac{1}{2} m^2 \phi^2$$

1) Derive the slow roll parameters

$$\epsilon_{\rm v} \equiv \frac{M_{\rm pl}^2}{2} \left(\frac{V'}{V} \right)^2 , \qquad |\eta_{\rm v}| \equiv M_{\rm pl}^2 \frac{|V''|}{V}$$

Q: How large must the scalar field value be to satisfy the slow-roll conditions?

2) Derive the relation between the inflation field value and the number of e-folds before the end of inflation.

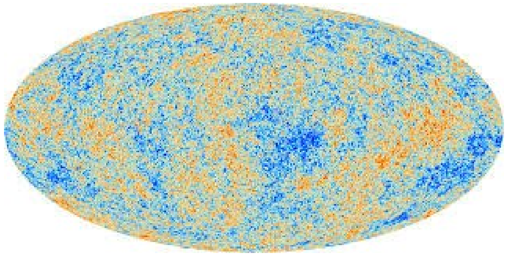
$$N_{\rm tot} = \int_{\phi_S}^{\phi_E} \frac{1}{\sqrt{2\epsilon_{\rm v}}} \frac{|{\rm d}\phi|}{M_{\rm pl}}$$

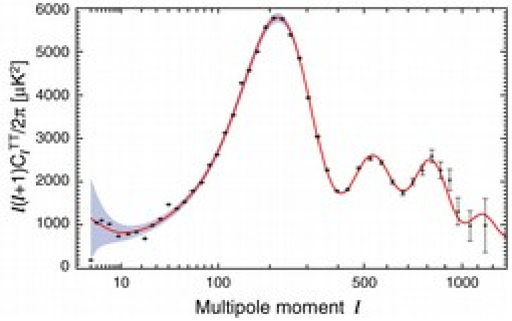
Q: What is the required minimum start value for the inflation field to obtain N~60?

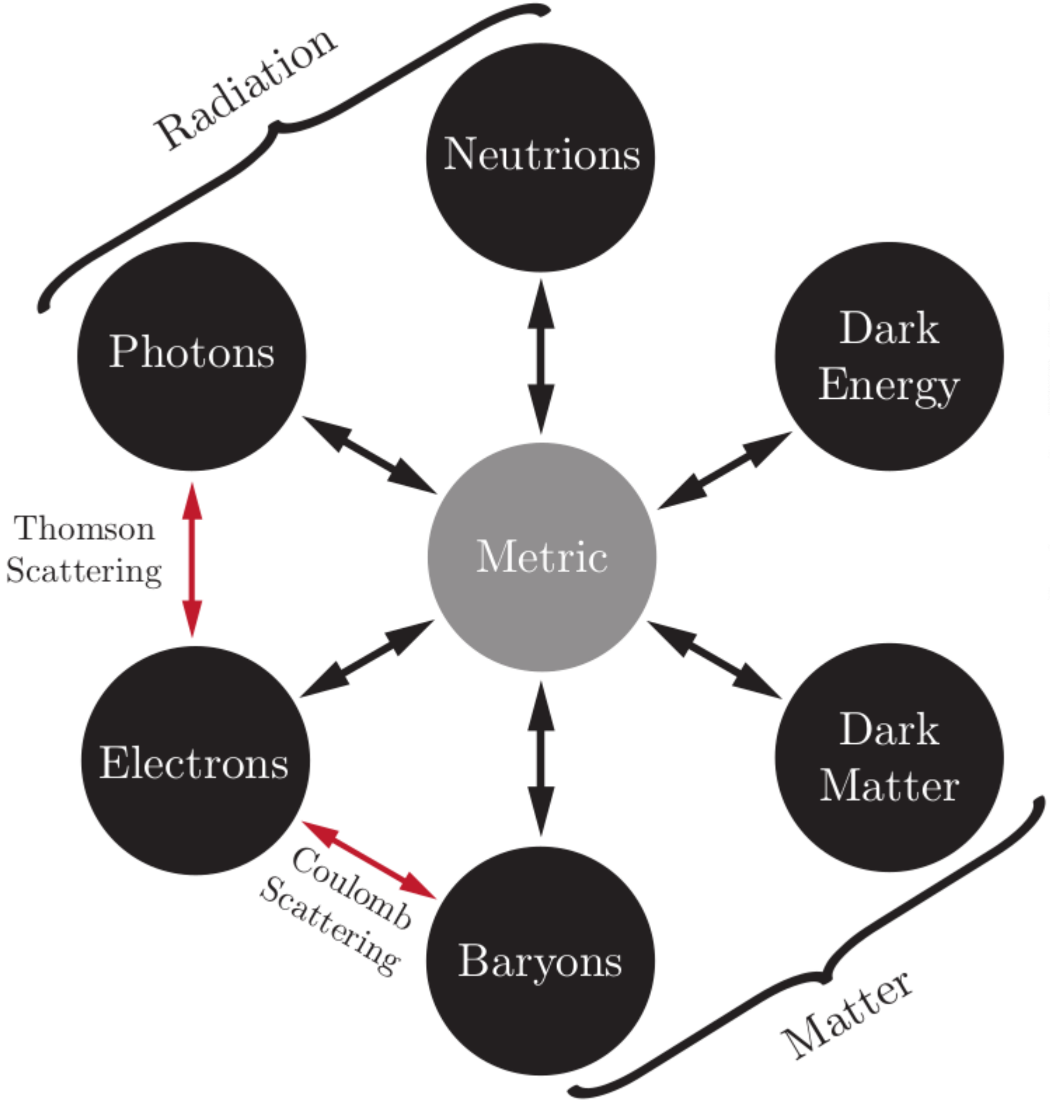
Linear structure formation and consequences

- Linear growth of initial density and metric perturbations
- Generation of the cosmic microwave background (CMB)
- Generation of large scale structures (LSS)

CMB ⇐







Radiation

Neutrinos

Photons

Dark Energy

Thomson Scattering

Metric

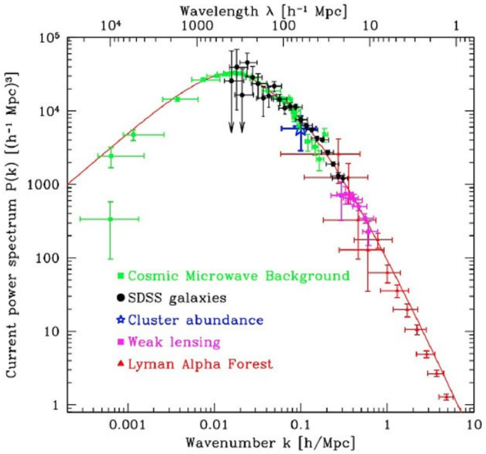
Electrons

Dark Matter

Coulomb Scattering

Baryons

Matter



⇒ LSS

End of inflation and density perturbations

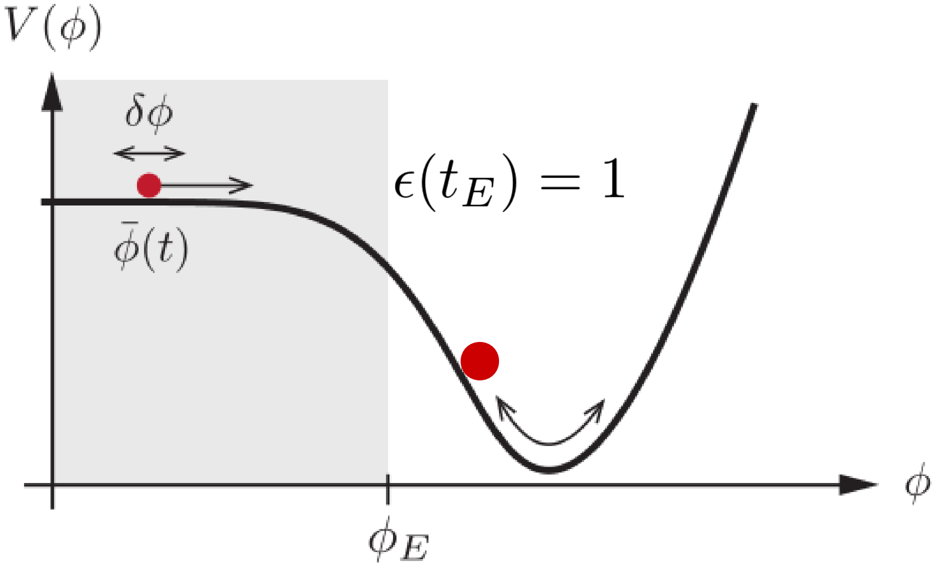
End of inflation and reheating

- Inflation ends when kinetic energy starts to dominate $\epsilon(t_E) = 1$
- Inflaton field behaves like matter

$$\langle P_\phi \rangle \approx 0 \;\; \Rightarrow \;\; \langle \rho_\phi \rangle \propto a^{-3}$$

- Inflaton field decays into SM particles, which thermalize (**reheating**)

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi$$



Adiabatic density perturbations from quantum fluctuations

Density and metric perturbations are connected

$$\delta_r \equiv \delta\rho_r/\bar{\rho}_r = -2\Phi$$

$$\delta_m \equiv \delta\rho_m/\bar{\rho}_m = -\frac{3}{2}\Phi$$

Metric perturbation power spectrum

$$P_\Phi(k) = \frac{1}{9k^3\epsilon}\frac{H^2}{M_{\rm pl}^2}\bigg|_{k=aH} \simeq A_s\left(\frac{k}{k_*}\right)^{n_s-4}$$

where $\langle \Phi(\boldsymbol{k})\Phi^*(\boldsymbol{k}')\rangle = (2\pi)^3 P_\Phi(k)\delta_D(\boldsymbol{k}-\boldsymbol{k}')$

Metric and density perturbations

Small perturbations to the mean metric and energy density.

$$g_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \boldsymbol{x})$$
$$T_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta, \boldsymbol{x})$$

$$\Rightarrow \Phi, \delta\rho_m, \delta\rho_r$$

Metric perturbations (“Newtonian gauge”)

$$\mathrm{d}s^2 = a^2(\eta)\left[(1+2\Psi)\mathrm{d}\eta^2 - (1-2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]$$

Ψ : Corresponds to Newton potential

Neglect anisotropic stress: $\Psi = \Phi$

Energy-momentum tensor perturbations

$$T^0{}_0 = \bar{\rho}(\eta) + \delta\rho\,,$$
$$T^i{}_0 = [\bar{\rho}(\eta) + \bar{P}(\eta)]\, v^i\,,$$
$$T^i{}_j = -[\bar{P}(\eta) + \delta P]\, \delta^i_j - \Pi^i{}_j\,,$$

$$\delta\rho = \delta\rho_m + \delta\rho_r$$

EoM for metric perturbations

Use Fourier transform of field

$$\Phi(\eta, \boldsymbol{x}) \equiv \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\,\Phi_{\boldsymbol{k}}(\eta)\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

EoM for metric perturbations (from $\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2\,\delta P$)

Radiation era $a \ll a_{\rm eq}$

$$\Phi''_{\boldsymbol{k}} + \frac{4}{\eta}\Phi'_{\boldsymbol{k}} + \frac{1}{3}k^2\Phi_{\boldsymbol{k}} = 0$$ (damped oscillator)

$\eta k = k/H \lesssim 1$ super-horizon modes are **frozen**

$\eta k = k/H \gtrsim 1$ sub-horizon mode **oscillate and decay**

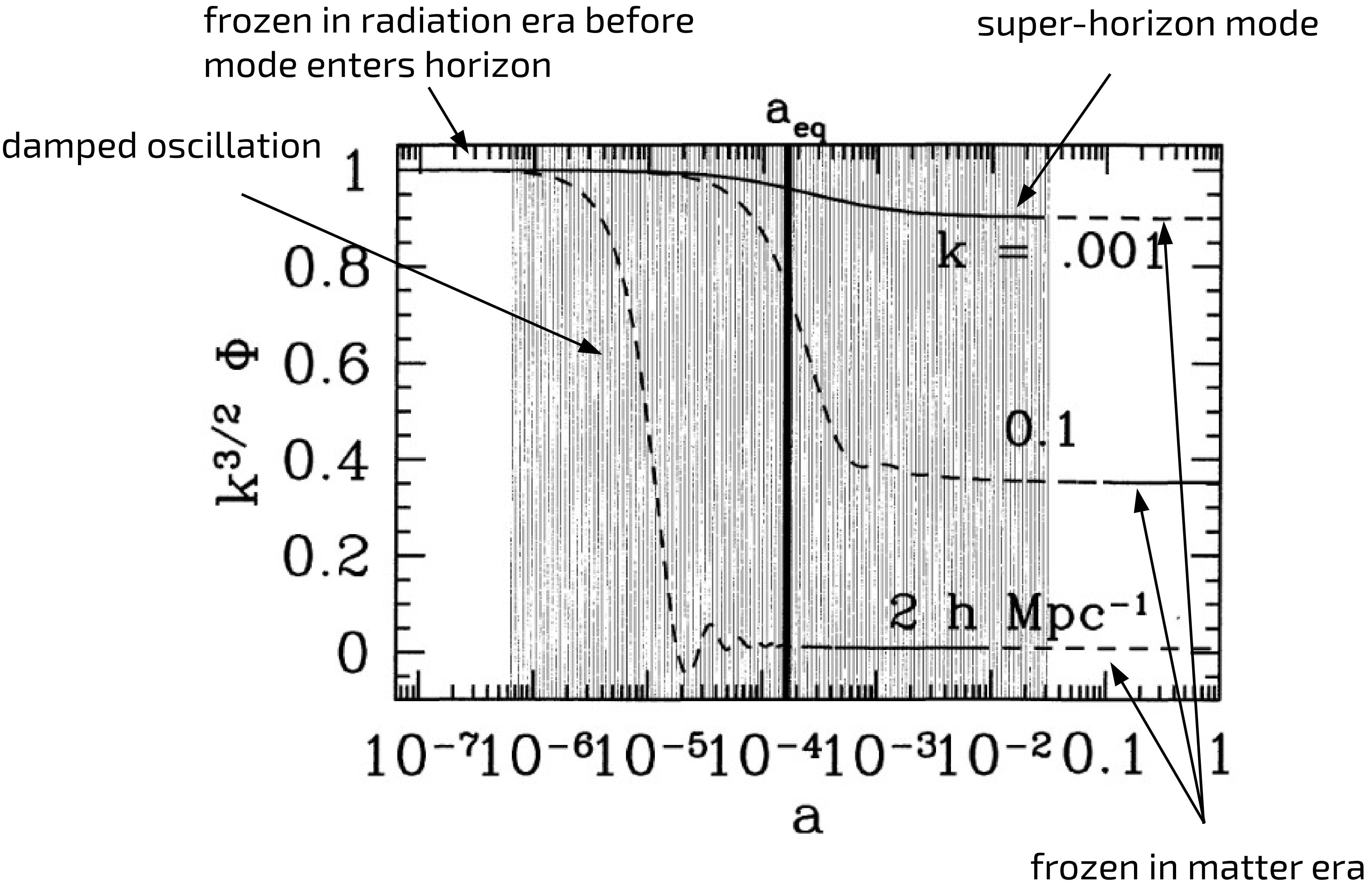
Frequency: $\omega = \frac{k}{\sqrt{3}}$ Amplitude decreases like: $\propto \frac{1}{\eta^2}$

Matter era $a \gg a_{\rm eq}$

$$\Phi'' + 3\mathcal{H}\Phi' = 0 \quad \Rightarrow \quad \Phi = const$$

all modes are **frozen**

Metric perturbations



Metric perturbations today

3. Growth function

at late times

Potential today

$$\Phi_k = \frac{9}{10}\Phi_k^p \times T(k) \times \frac{D_1(a)}{a}$$

2. Transfer function

until a~1e-2

1. Primoridal fluctuations

Small modes decay away before MRE

$$T(k) \propto \frac{1}{k^2} \qquad k > k_{\rm eq}$$

Large modes not affected

$$T(k) = 1 \qquad k < k_{\rm eq}$$

Matter perturbations

In matter era, the Poisson equation connects matter and metric perturbations

Poisson equation

$$\nabla\Phi = 4\pi G a^2 \rho_m \delta_m$$

$$\Rightarrow \qquad \delta_m = \frac{k^2 \Phi_k a}{(3/2)\Omega_m H_0^2} = \frac{3}{5}\frac{k^2}{\Omega_m H_0^2}\Phi_k^p T(k) D_1(a)$$

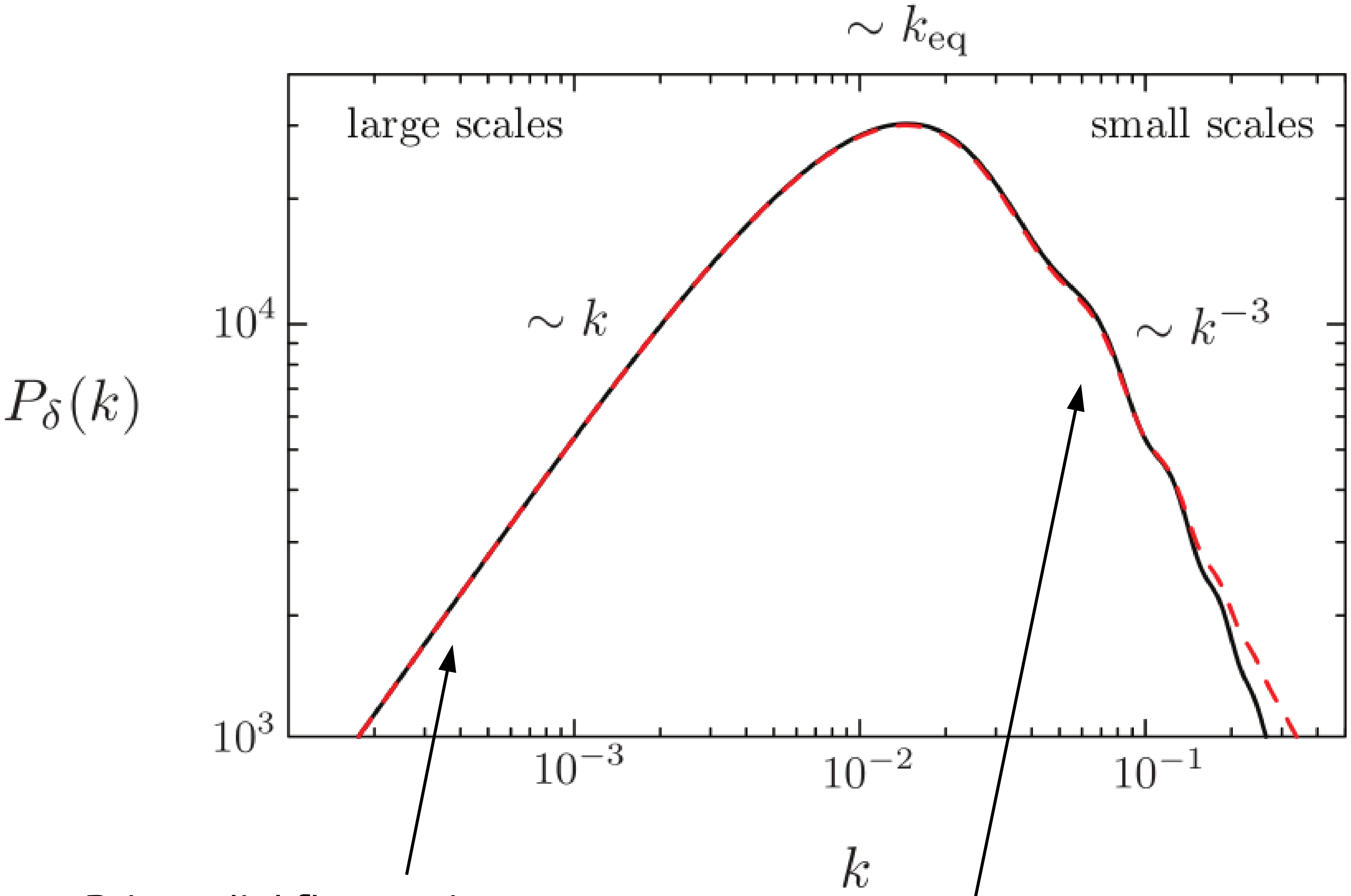
Matter density power spectrum

$$\langle \delta_m(\boldsymbol{k})\delta_m(\boldsymbol{k'})\rangle = (2\pi)^3 P_m(k)\delta_D(\boldsymbol{k}-\boldsymbol{k'})$$

$$P_\Phi^p = A_s\left(\frac{k}{k_*}\right)^{n_s-4} \qquad \Rightarrow \qquad P_m \propto k^{n_s} T(k)^2$$

(from inflation)

Matter power spectrum

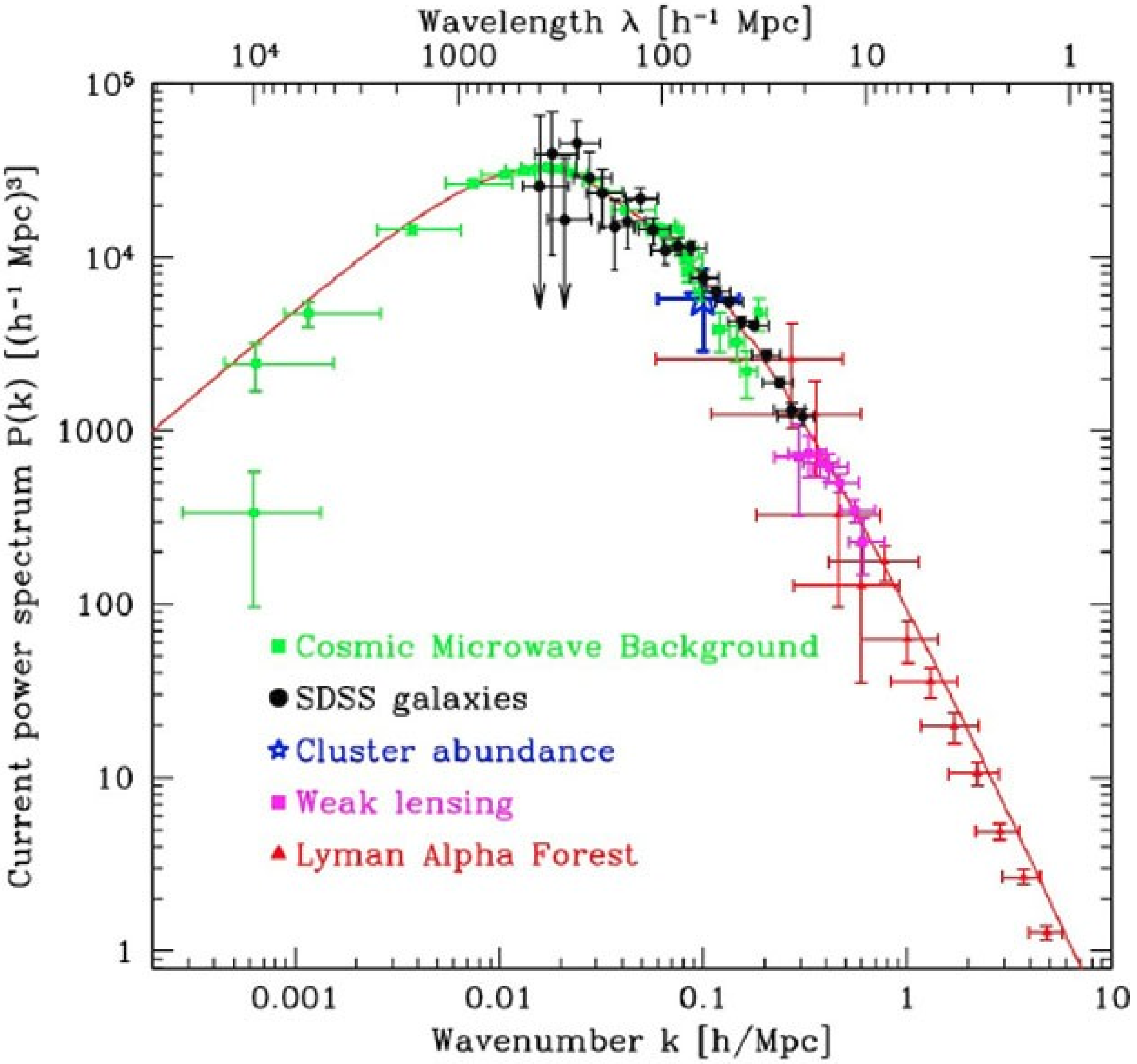


Primordial fluctuations

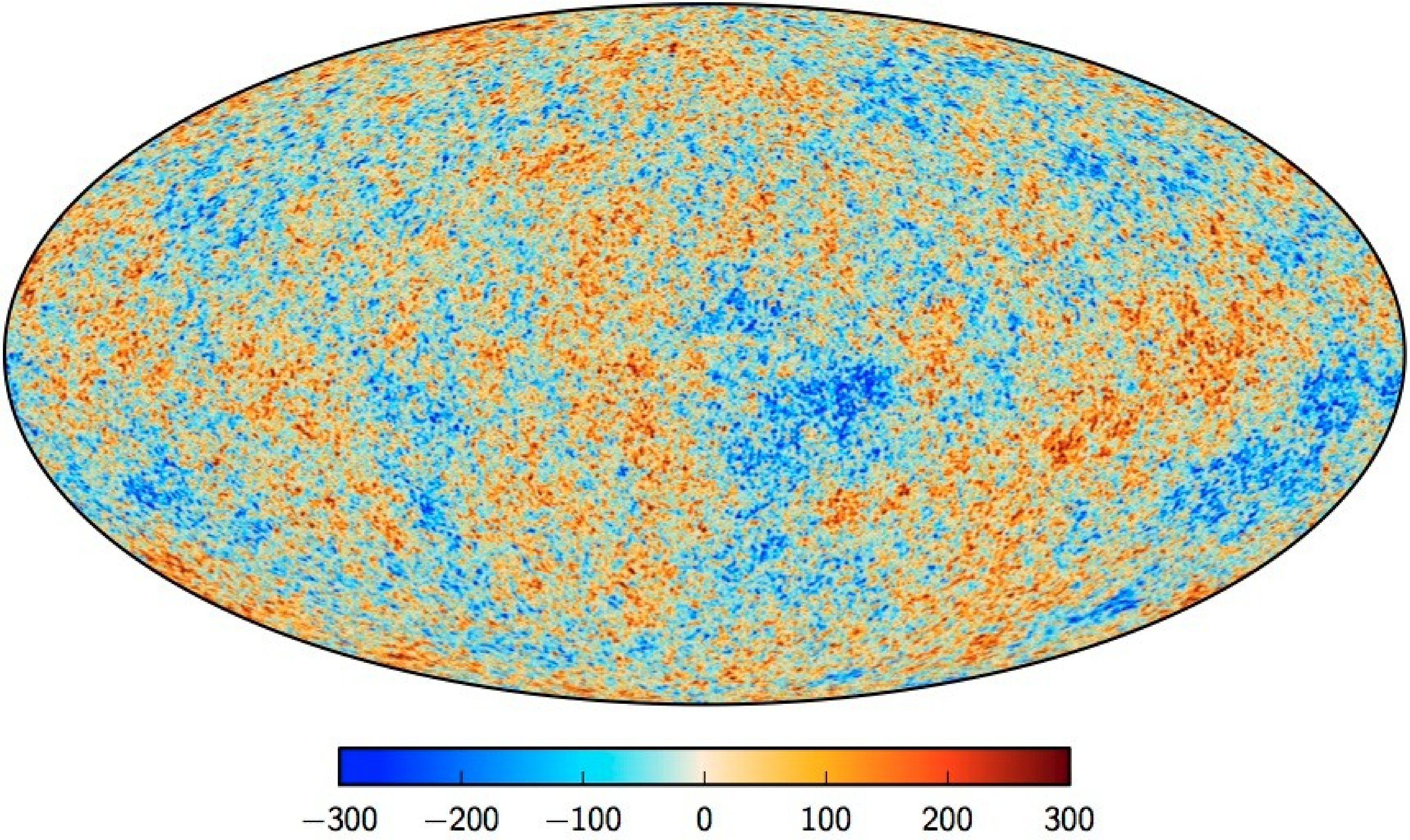
Decay (and oscillation) before MRE

Matter power spectrum

Tegmark+ 2004



The cosmic microwave background



Temperature fluctuations μK_{cmb}

Planck satellite

Radiation field perturbations

Equation of motion for the baryon-photon fluid

$$\boxed{\delta_\gamma'' + \frac{\mathcal{H}R}{1+R}\,\delta_\gamma' + \underset{\underset{\text{pressure}}{\uparrow}}{c_s^2 k^2\,\delta_\gamma} \;=\; -\underset{\underset{\text{gravity}}{\uparrow}}{\frac{4}{3}k^2\Phi} + 4\Phi'' + \frac{4R'}{1+R}\Phi'} \qquad \mathcal{H} = aH$$

Until “decoupling” at z~1100, there is a strong coupling between electrons and photons, which behave like a single fluid.

Baryon to photon ratio

$$R \equiv \frac{3}{4}\frac{\bar\rho_b}{\bar\rho_\gamma} = 0.6\left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{a}{10^{-3}}\right)$$

Sound speed

$$c_s^2 \equiv \frac{1}{3(1+R)}$$

For sub-horizon modes

before MRE $a \ll a_{\rm eq}$

- gravitational potential small

$$\delta_r'' - \frac{1}{3}\nabla^2\delta_r \approx 0$$

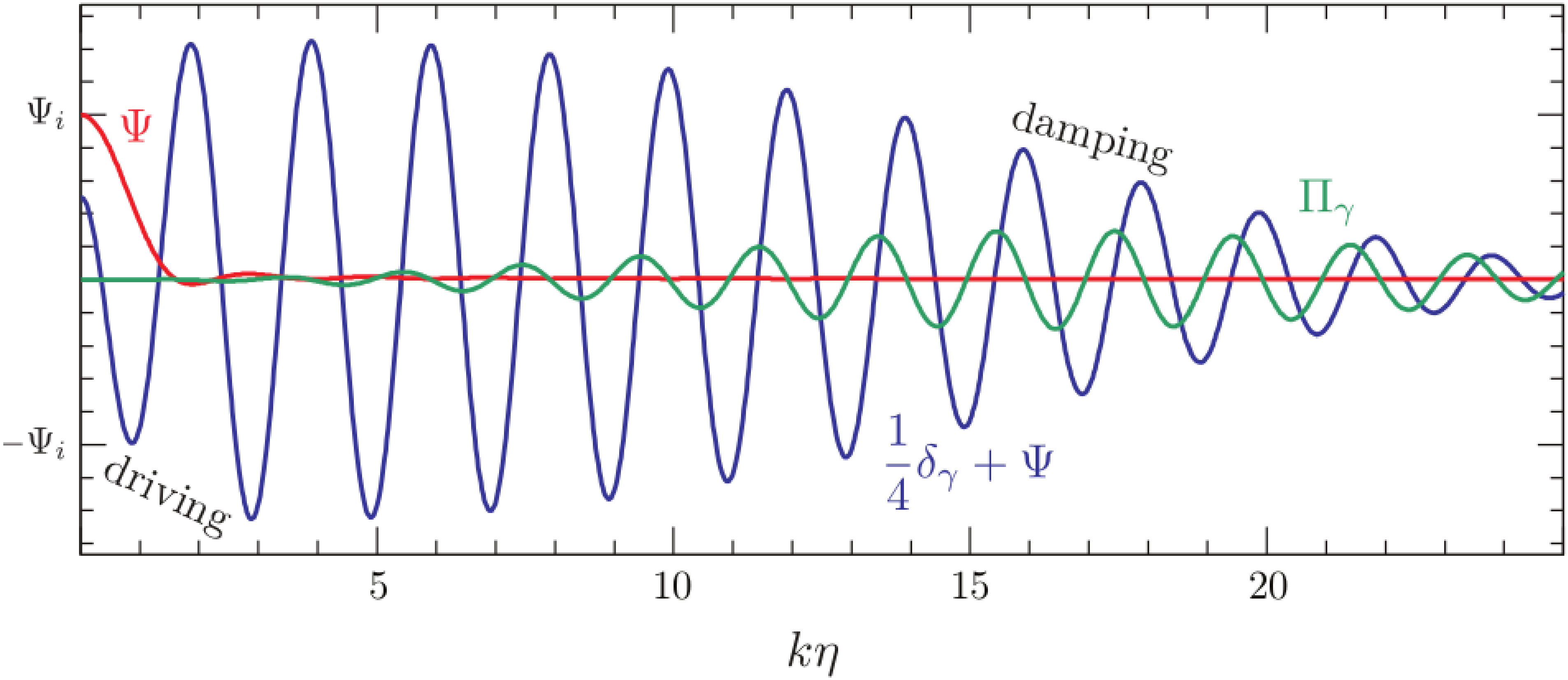
after MRE $a \gg a_{\rm eq}$

- grav. potential as external force

$$\delta_r'' - \frac{1}{3}\nabla^2\delta_r = \frac{4}{3}\nabla^2\Phi = const$$

Baryon-photon field oscillations

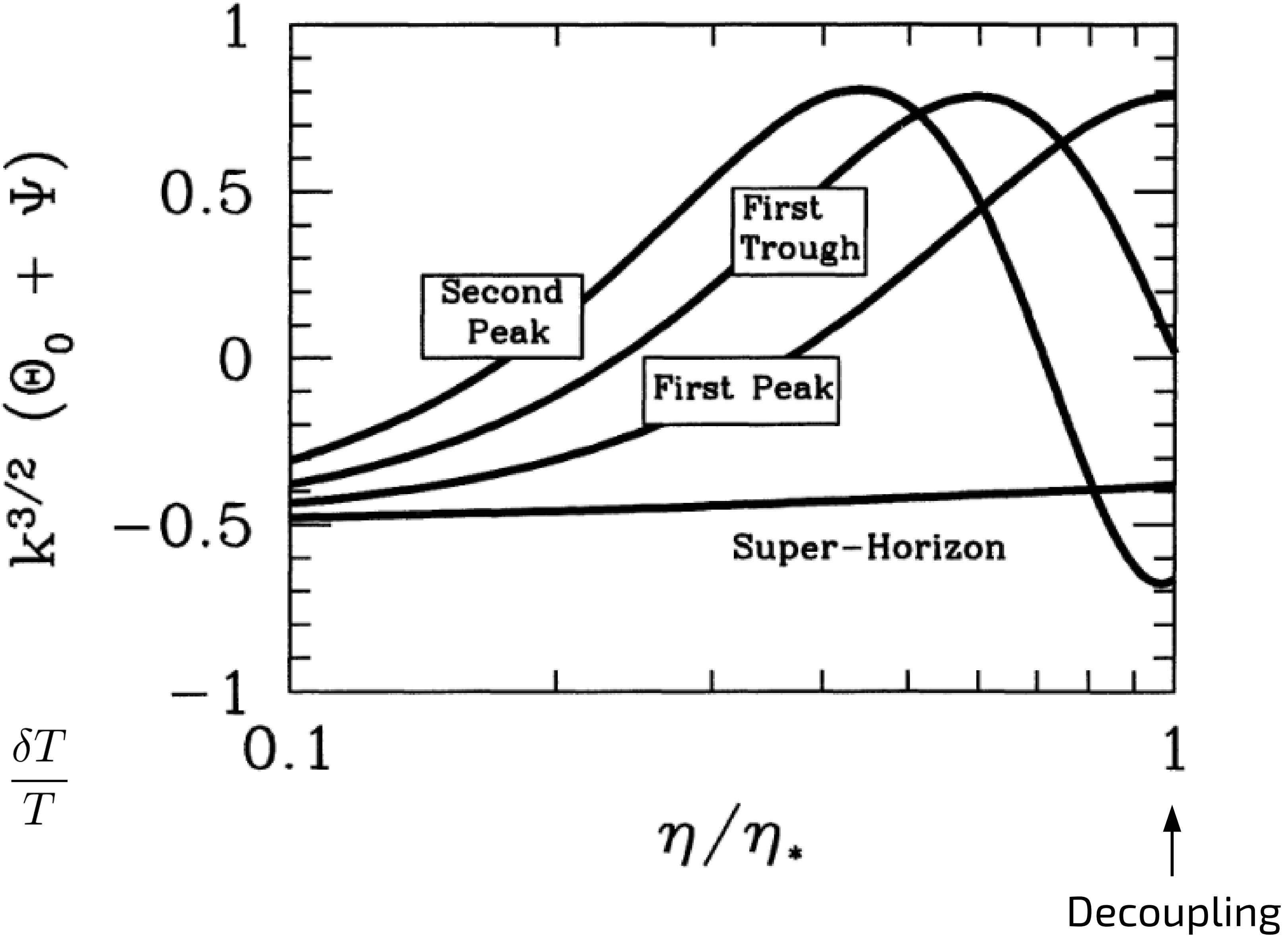
Evolution of gravitational potential and photon perturbations in radiation era



- Gravitational potential decays, and drives photon perturbations
- Photons *diffuse* over finite length scales, leading to damping and the generation of quadrupole fluctuations
- What is *actually* observable today is $\frac{1}{4}\delta_\gamma + \Psi = \frac{\delta T}{T} + \Psi$

Formation of peaks in the CMB power spectrum

The peak on the largest scales (first peak) is generated by the mode that entered the horizon so late that it could just perform ~1/4 of its oscillation.



$$\Theta_0 = \frac{\delta T}{T}$$

Observed temperature fluctuations

Momentum loss in expanding perturbed Universe

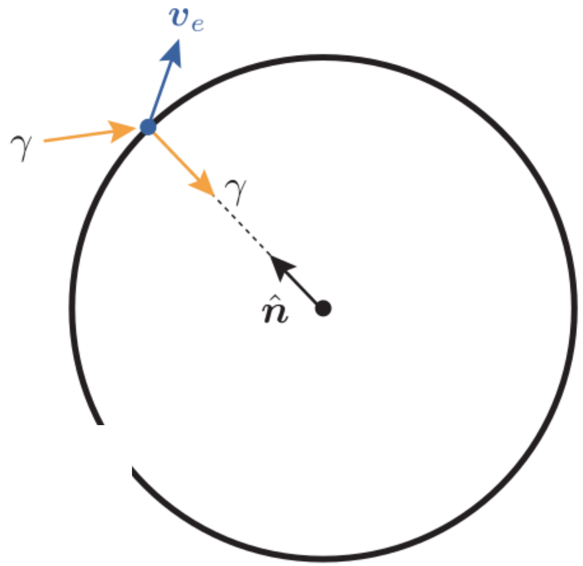
$$\boxed{\frac{1}{p}\frac{dp}{d\eta} = -\frac{1}{a}\frac{da}{d\eta} - \hat{p}^i\frac{\partial\Psi}{\partial x^i} + \frac{\partial\Phi}{\partial\eta}}$$

↑ ↑ ↑

redshift gravitational redshift

&

Doppler shift



→ Perceived temperature fluctuations

intrinsic δT gravitational redshift

↓ ↓

$$\boxed{\frac{\delta T}{\bar{T}}(\hat{\boldsymbol{n}}) = \left(\frac{1}{4}\delta_\gamma + \Psi + \hat{\boldsymbol{n}}\cdot\boldsymbol{v}_e\right)_* + \int_{\eta_*}^{\eta_0} \mathrm{d}\eta\,(\Psi' + \Phi')}$$

↑ ↑ ↑

SW Doppler ISW

Sachs-Wolfe term “Integrated Sachs-Wolfe”

At decoupling!

Definition of angular power spectrum

Angular power spectrum of the CMB:

Describes the correlations of temperature fluctuations of the CMB in different sky directions

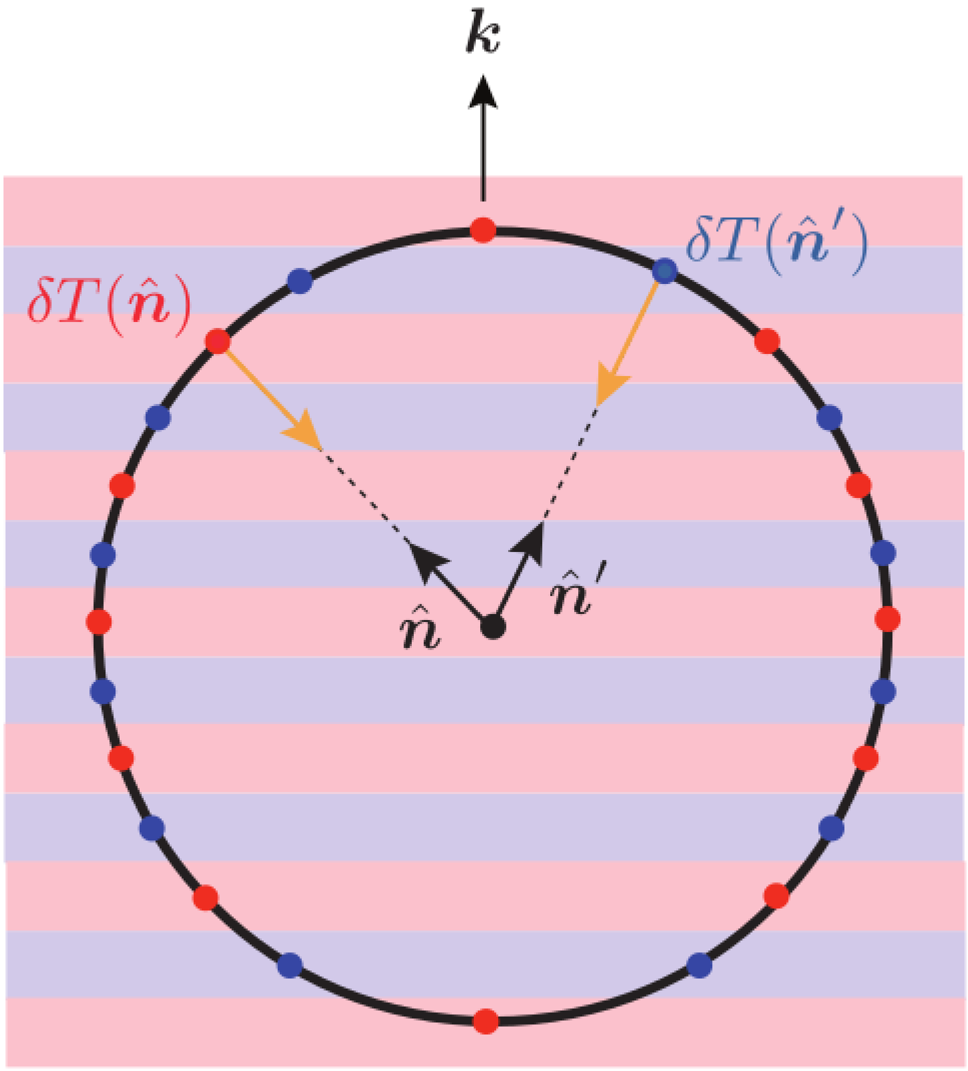
Legendre Polynomial

$$\langle \delta T(\hat{\boldsymbol{n}}) \delta T(\hat{\boldsymbol{n}}') \rangle = \sum_l \frac{2l+1}{4\pi} \underset{\uparrow}{C_l} \overset{\downarrow}{P_l}(\cos\theta)$$

Angular power spectrum

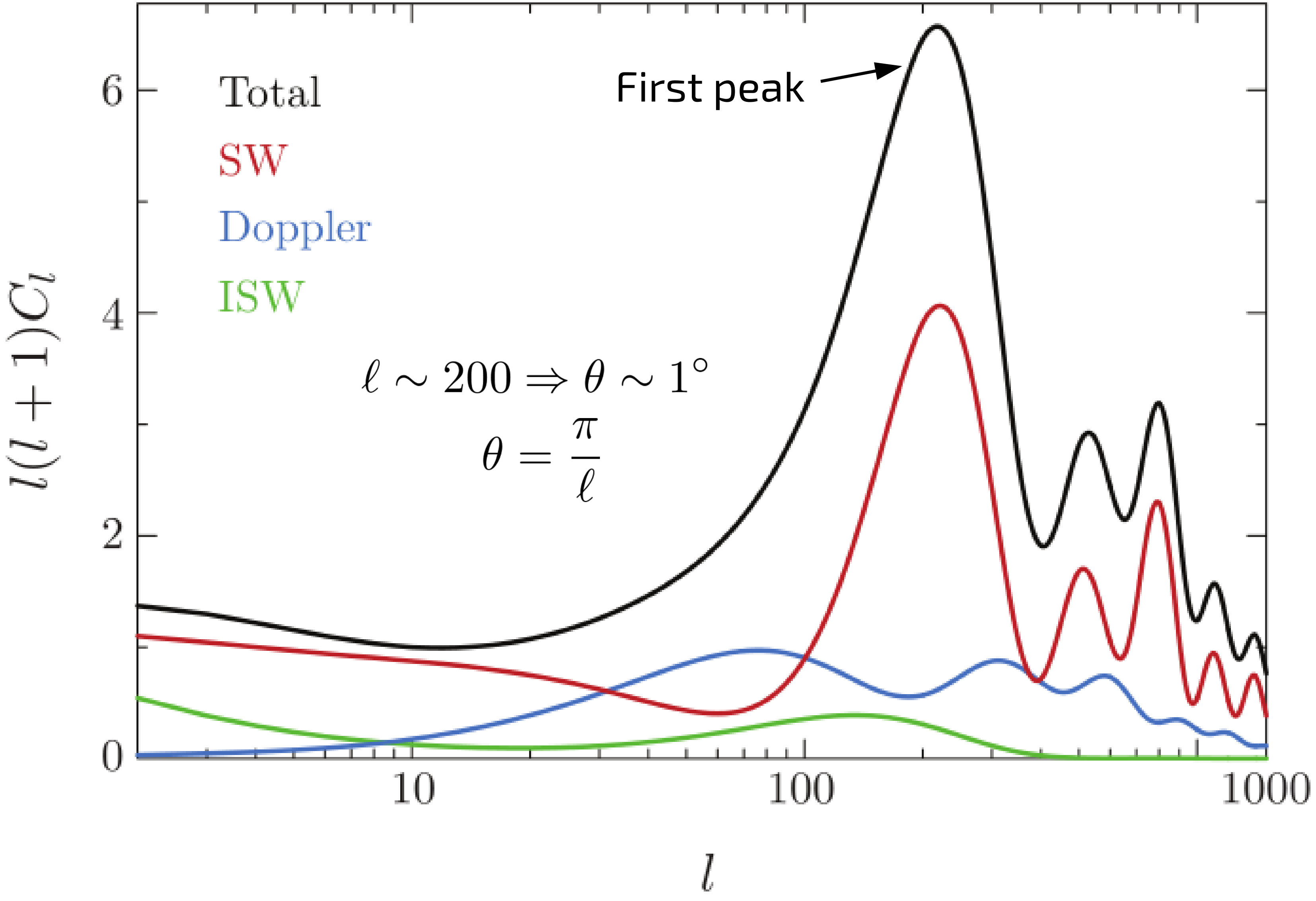
where $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}' \equiv \cos\theta$

Mapping of plane wave modes onto temperature fluctuations.

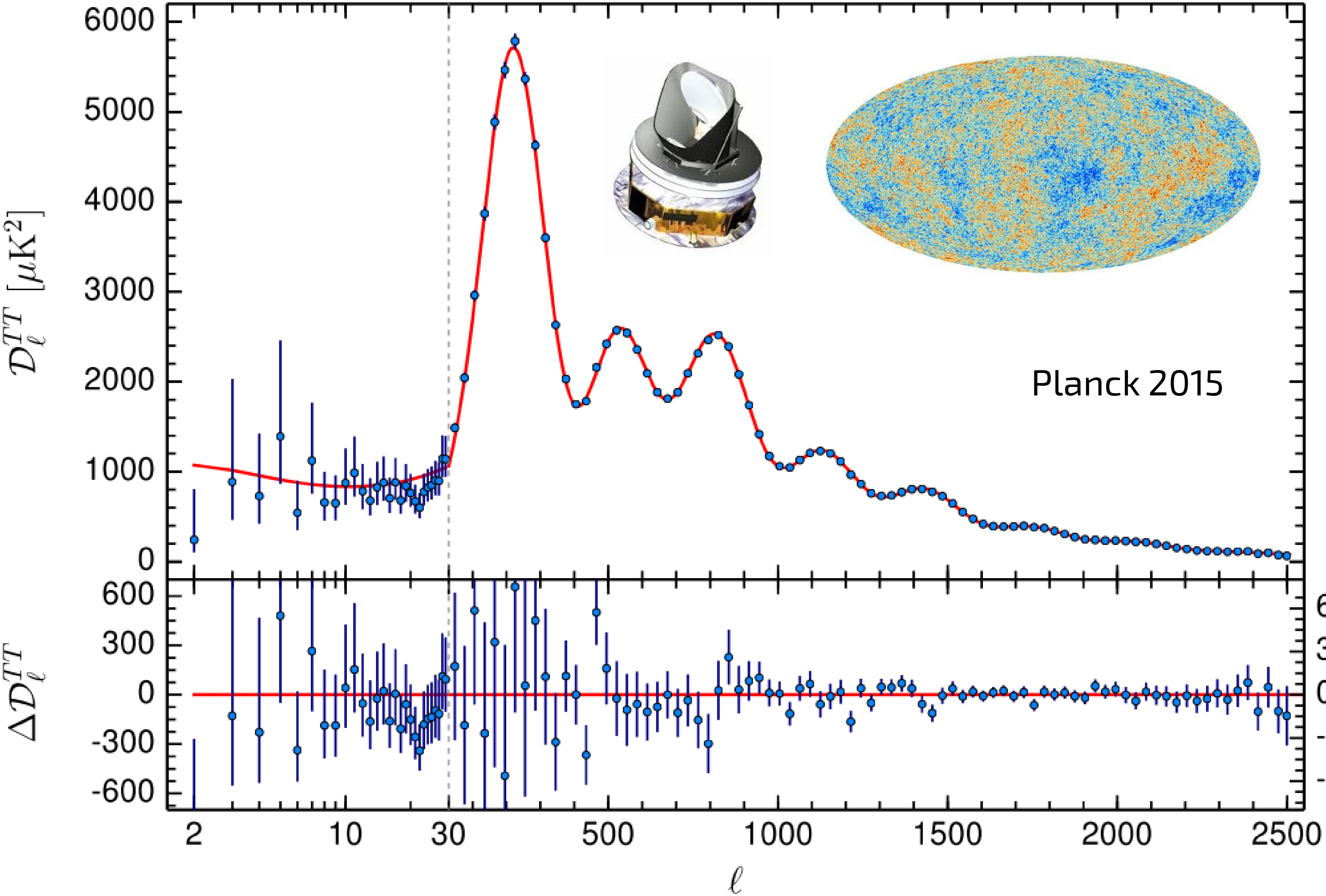


Predictions for the CMB

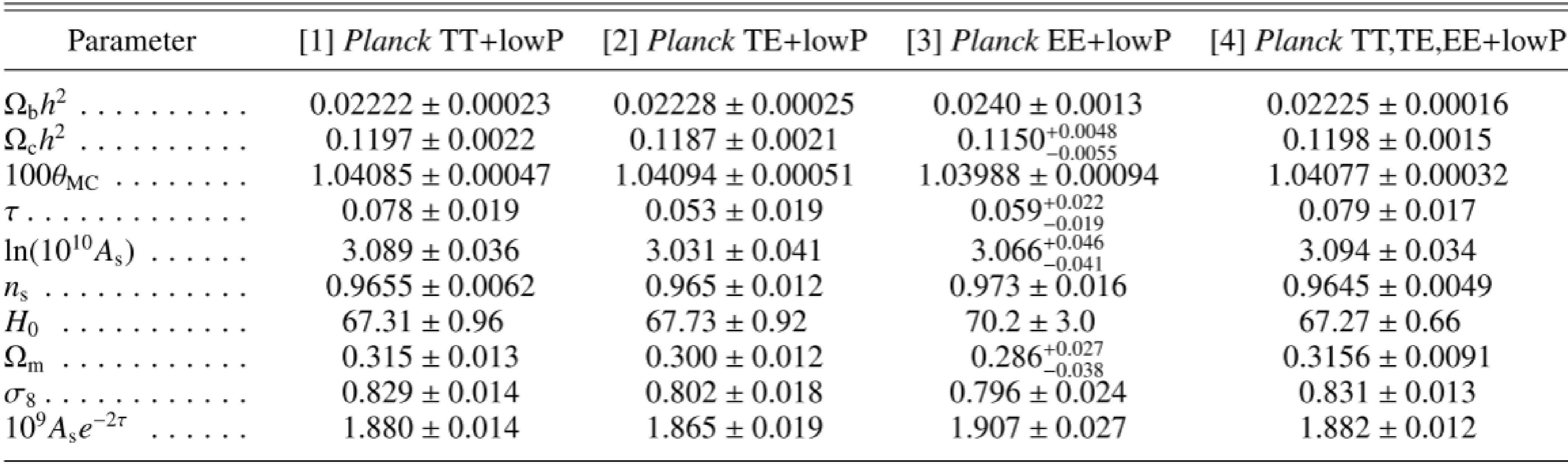
General predictions for the angular power spectrum of the temperature fluctuations in the cosmic microwave background



Results from Planck satellite



Parameters from fit to Planck data



Parameter	[1] *Planck* TT+lowP	[2] *Planck* TE+lowP	[3] *Planck* EE+lowP	[4] *Planck* TT,TE,EE+lowP
$\Omega_b h^2$	0.02222 ± 0.00023	0.02228 ± 0.00025	0.0240 ± 0.0013	0.02225 ± 0.00016
$\Omega_c h^2$	0.1197 ± 0.0022	0.1187 ± 0.0021	$0.1150^{+0.0048}_{-0.0055}$	0.1198 ± 0.0015
$100\theta_{MC}$	1.04085 ± 0.00047	1.04094 ± 0.00051	1.03988 ± 0.00094	1.04077 ± 0.00032
τ	0.078 ± 0.019	0.053 ± 0.019	$0.059^{+0.022}_{-0.019}$	0.079 ± 0.017
$\ln(10^{10} A_s)$	3.089 ± 0.036	3.031 ± 0.041	$3.066^{+0.046}_{-0.041}$	3.094 ± 0.034
n_s	0.9655 ± 0.0062	0.965 ± 0.012	0.973 ± 0.016	0.9645 ± 0.0049
H_0	67.31 ± 0.96	67.73 ± 0.92	70.2 ± 3.0	67.27 ± 0.66
Ω_m	0.315 ± 0.013	0.300 ± 0.012	$0.286^{+0.027}_{-0.038}$	0.3156 ± 0.0091
σ_8	0.829 ± 0.014	0.802 ± 0.018	0.796 ± 0.024	0.831 ± 0.013
$10^9 A_s e^{-2\tau}$	1.880 ± 0.014	1.865 ± 0.019	1.907 ± 0.027	1.882 ± 0.012

Planck 2015 results