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Physics & Astrophysics at the Extreme (PAX)

RECENT RESULTS AT THE 4PN ORDER

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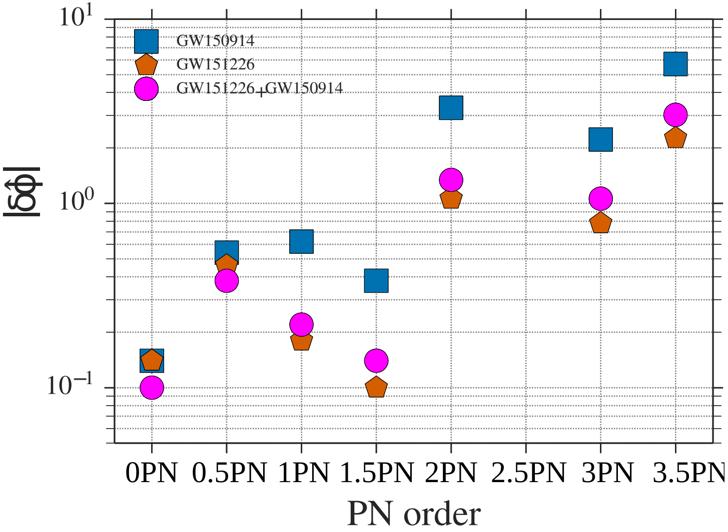
Measurement of PN parameters [LIGO/VIRGO collaboration 2016, 2017]



HIGHER-ORDER GRAVITATIONAL WAVE TAIL EFFECTS

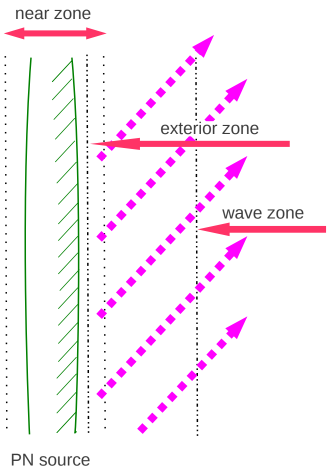
Based on a collaboration with

Tanguy Marchand & **Guillaume Faye**

[CQG **33**, 244003 (2016)]

The MPM-PN formalism [Blanchet-Damour-Iyer formalism 1980-90s]

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



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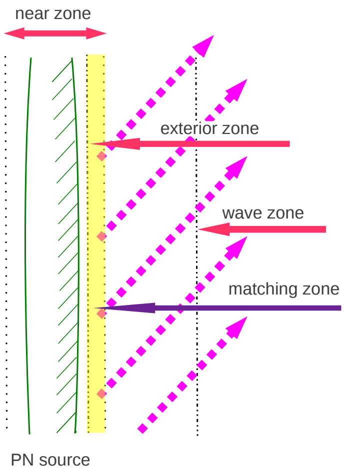


$$\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})$$

The MPM-PN formalism [Blanchet-Damour-Iyer formalism 1980-90s]

1. Radiative multipole moments observed at infinity from the source ($\mathcal{J}_+$)

$$\boxed{U_L(T-R/c)\,,\qquad V_L(T-R/c)}$$

2. Source multipole moments describe a specific matter system

$$\boxed{I_L(t)\,,\qquad J_L(t)}$$

- The relations between the radiative moments and the source moments are obtained by the MPM algorithm
- The expressions of the source moments in terms of the source parameters follow from the matching to the PN source
- The radiation reaction effects in the PN solution are also obtained

The 4.5PN radiative quadrupole moment

$$U_{ij}(t) = I^{(2)}_{ij}(t) + \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}}$$

$$+\frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty}\mathrm{d}\tau\, I^{(3)}_{a<i}I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\}$$

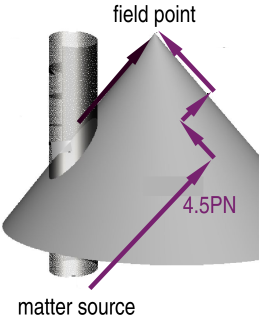
$$+\underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty}\mathrm{d}\tau\, I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}}$$

$$+\underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty}\mathrm{d}\tau\, I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}}$$

$$+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

4.5PN coefficient in the GW flux

$$\mathcal{F}^{4.5\mathrm{PN}} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ \bigg(\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\mathrm{E} - \frac{3424}{105}\ln(16x) + \left[\frac{2062241}{22176} + \frac{41}{12}\pi^2\right]\nu - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3 \bigg) \pi x^{9/2} \Bigg\}$$



- The 4.5PN tail effect represents the complete 4.5PN coefficient in the GW energy flux in the case of circular orbits
- Perfect agreement with results from BH perturbation theory in the small mass ratio limit $\nu \to 0$ [Tanaka, Tagoshi & Sasaki 1996]
- Result confirmed by factorized and resummed waveforms [Messina & Nagar 2017]
- The 4PN term in the flux is still in progress [Marchand *et al.* 2017]

THE 4PN EQUATIONS OF MOTION

Based on collaborations with

Laura Bernard, **Alejandro Bohé**, **Guillaume Faye** & **Sylvain Marsat**

[PRD **93**, 084037 (2016); PRD **95**, 044026 (2017); PRD submitted (2017)]

Tanguy Marchand, **Laura Bernard** & **Guillaume Faye**

[PRL submitted (2017)]

4PN: state-of-the-art on equations of motion

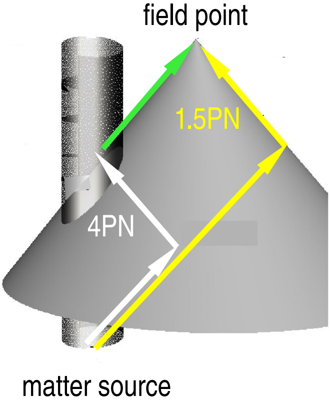
$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\cdots\right]n_{12}^i+\cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017ab]	Fokker Lagrangian
	[Foffa & Sturani 2012, 2013] (partial results)	Effective field theory

Gravitational wave tail effect at the 4PN order

[Blanchet & Damour 1988; Blanchet 1993, 1996]

- At the 4PN order there is an imprint of gravitational wave tails in the local (near-zone) dynamics of the source
- This leads to a non-local-in-time contribution in the Fokker action
- This implies the appearance of IR divergences in the Fokker action at the 4PN order



$$S_{\mathrm{F}}^{\text{tail}} = \frac{G^2 M}{5c^8} \mathop{\mathrm{Pf}}_{s_0} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I_{ij}^{(3)}(t)\, I_{ij}^{(3)}(t')$$

where the Hadamard partie finie (Pf) is parametrized by an arbitrary constant s_0

Problem of the IR divergences

1. Our initial calculation of the Fokker action was based on the Hadamard regularization to treat the IR divergences (FP procedure when $B \to 0$)
2. However computing the conserved energy and periastron advance for circular orbits we found it does not agree with GSF calculations
3. The problem was due to the HR and conjectured that a different IR regularization would give (modulo shifts)

$$L = L^{\mathsf{HR}} + \underbrace{\frac{G^4 m\, m_1^2 m_2^2}{c^8 r_{12}^4}\left(\delta_1 (n_{12} v_{12})^2 + \delta_2 v_{12}^2\right)}_{\text{two ambiguity parameters } \delta_1 \text{ and } \delta_2}$$

4. Matching with GSF results for the energy and periastron advance uniquely fixes the two ambiguity parameters and we are in complete agreement with the results from the Hamiltonian formalism [DJS]

Dimensional regularization of the IR divergences

- The Hadamard regularization of IR divergences reads

$$I^{\text{HR}}_{\mathcal{R}} = \mathop{\text{FP}}_{B=0} \int_{r>\mathcal{R}} \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B F(\mathbf{x})$$

- The corresponding dimensional regularization reads

$$I^{\text{DR}}_{\mathcal{R}} = \int_{r>\mathcal{R}} \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^{d-3}} F^{(d)}(\mathbf{x})$$

- The difference between the two regularization is of the type ($\varepsilon = d - 3$)

$$\boxed{\mathcal{D}I = \sum_q \Bigg[\underbrace{\frac{1}{(q-1)\varepsilon}}_{\text{IR pole}} - \ln\left(\frac{r_0}{\ell_0}\right)\Bigg] \int \mathrm{d}\Omega_{2+\varepsilon}\, \varphi^{(\varepsilon)}_{3,q}(\mathbf{n}) + \mathcal{O}(\varepsilon)}$$

Ambiguity-free completion of the 4PN EOM

1. The tail effect contains a UV pole which cancels the IR pole coming from the instantaneous part of the action

$$g_{00}^{\text{tail}} = -\frac{8G^2M}{5c^8}\,x^{ij}\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\sqrt{\bar{q}}\,\tau}{2\ell_0}\right)\underbrace{-\frac{1}{2\varepsilon}}_{\text{UV pole}}+\frac{41}{60}\right]I_{ij}^{(7)}(t-\tau)+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

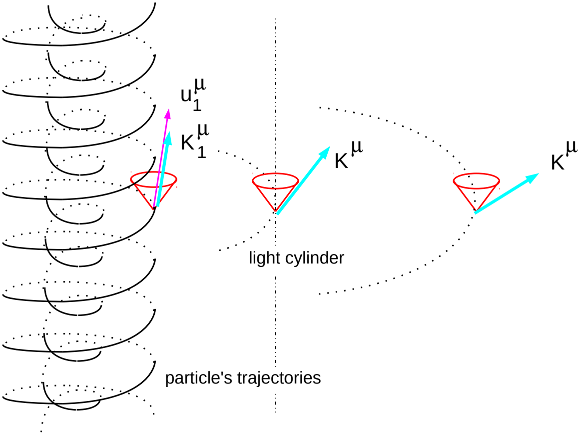
2. Adding up all contributions we obtain the conjectured form of the ambiguity terms with the correct values of the ambiguity parameters δ_1 and δ_2
3. It is likely that the EFT formalism will also succeed in deriving the full EOM without ambiguities [Porto & Rothstein 2017]
4. The lack of a consistent matching in the ADM Hamiltonian formalism [DJS] forces this formalism to be plagued by one ambiguity parameter

4PN FIRST LAW OF COMPACT BINARIES

Based on a collaboration with

Alexandre Le Tiec [CQG **34**, 164001 (2017)]

The redshift observable [Detweiler 2008; Barack & Sago 2011]

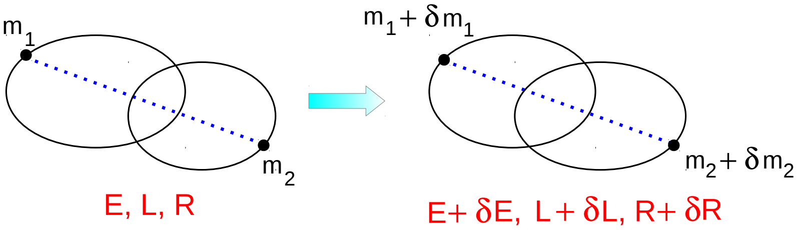


$$\boxed{K_1^\mu = z_1 u_1^\mu}$$

For eccentric orbits one must consider the averaged redshift $\langle z_1 \rangle = \frac{1}{P}\int_0^P \mathrm{d}t\, z_1(t)$

First law of compact binary mechanics

[Le Tiec, Blanchet & Whiting 2011; Blanchet, Buonanno & Le Tiec 2012; Le Tiec 2015]



$$\boxed{\delta E = \omega\,\delta L + n\,\delta R + \langle z_1\rangle\,\delta m_1 + \langle z_2\rangle\,\delta m_2}$$

- E, L : ADM energy and angular momentum
- $\boxed{R = \frac{1}{2\pi}\oint p_r \mathrm{d}r}$: radial action integral
- n, ω : radial and azimuthal frequencies

First law versus non-local dynamics

1. The basic variable computed by GSF techniques is the averaged redshift $\langle z_a \rangle$ in the test-mass limit $m_1/m_2 \to 0$
2. The first law permits to derive from $\langle z_a \rangle$ the binary's conserved energy E and periastron advance K for circular orbits
$$K = \frac{\omega}{n}$$
3. These results are then used to fix the ambiguity parameters in the 4PN equations of motion both in Hamiltonian formalism [DJS] and in Lagrangian formalism [BBBFM]
4. However the first law has been derived from a local Hamiltonian but at 4PN order the dynamics becomes non-local due to the tail term

Are we still allowed to use the first law in standard form for the non-local dynamics at the 4PN order?

The 4PN non-local-in-time dynamics

1. At 4PN order the dynamics becomes non-local due to the tail term

$$H = H_0(r, p_r, p_\varphi; m_a) + H_{\mathsf{tail}}[r, \varphi, p_r, p_\varphi; m_a]$$

with
$$\boxed{H_{\mathsf{tail}} = -\frac{m}{5} I^{(3)}_{ij}(t) \int_{-\infty}^{+\infty} \frac{\mathrm{d}t'}{|t-t'|} I^{(3)}_{ij}(t')}$$

2. Hamilton's equations involve functional derivatives

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\delta H}{\delta p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\delta H}{\delta x^i}$$

3. For the non-local dynamics H and p_φ are no longer conserved but instead

$$\begin{aligned} E &= H + \Delta H^{\mathsf{DC}} + \Delta H^{\mathsf{AC}} \\ L &= p_\varphi + \Delta p_\varphi^{\mathsf{DC}} + \Delta p_\varphi^{\mathsf{AC}} \end{aligned}$$

where H^{AC} and p_φ^{AC} (given by Fourier series) average to zero and

$$\boxed{\Delta H^{\mathsf{DC}} = -2m\,\mathcal{F}^{\mathsf{GW}} \qquad \Delta p_\varphi^{\mathsf{DC}} = -2m\,\mathcal{G}^{\mathsf{GW}}}$$

Derivation of the first law at 4PN order

1. Varying the non-local Hamiltonian and averaging we get

$$\boxed{\delta E = \omega\,\delta L + n\,\delta\mathcal{R} + \sum_a \langle z_a\rangle\,\delta m_a}$$

but where the radial action integral gets corrected at 4PN order

$$\boxed{\mathcal{R} = R + 2m\left(\mathcal{G}^{\mathsf{GW}} - \frac{\mathcal{F}^{\mathsf{GW}}}{\omega}\right) - \frac{1}{2\pi}\oint \Delta p_\varphi^{\mathsf{AC}}\,\mathrm{d}\varphi}$$

2. By performing a non-local-in-time shift of canonical variables
   - the non-local Hamiltonian can be transformed into an ordinary one [DJS]
   - the modified action integral is identical to the local one in local coordinates
3. With the first law derived at 4PN order we have fully confirmed the expressions of E^{4PN} and K^{4PN} in the test-mass limit