Cross–Correlation Structure of Isotropic Body–Wave Fields (P, S, and P–S Mixtures)

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Abstract

We derive closed-form expressions for the normalized spatial cross–correlations of displacement components in a homogeneous isotropic medium for three scenarios: (i) pure P-wave (longitudinal) plane-wave fields, (ii) pure S-wave (transverse) plane-wave fields with random polarization in the transverse plane, and (iii) an incoherent mixture of P and S fields. The results are expressed in terms of spherical Bessel functions j_0 and j_2 and are directly applicable for validating numerical simulations of isotropic body-wave ambient fields.

1 Definitions and assumptions

Let $\mathbf{u}(\mathbf{x}, \omega)$ be the frequency-domain displacement field at angular frequency ω with implicit time dependence $e^{-i\omega t}$. Consider two receivers separated by

$$\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1, \qquad d = \|\mathbf{r}\|, \qquad \hat{\mathbf{r}} = \mathbf{r}/d. \tag{1}$$

Define the (cross-)spectral density tensor

$$C_{ij}(\mathbf{r}; \omega) = \langle u_i(\mathbf{x}_1, \omega)\, u_j^*(\mathbf{x}_2, \omega)\rangle, \tag{2}$$

where $\langle\cdot\rangle$ denotes an ensemble average (over directions, phases, and realizations).

We will focus on the *normalized* (dimensionless) correlation (often called coherence in this context):

$$\mathrm{CC}_{ij}(\mathbf{r}; \omega) \equiv \frac{C_{ij}(\mathbf{r}; \omega)}{\sqrt{C_{ii}(\mathbf{0}; \omega)\, C_{jj}(\mathbf{0}; \omega)}}. \tag{3}$$

For an isotropic homogeneous field, $C_{ii}(\mathbf{0}; \omega)$ is the same at both receivers, so the denominator reduces to $C_{ii}(\mathbf{0}; \omega)$ if $i = j$, and (3) becomes

$$\mathrm{CC}_{ii}(\mathbf{r}; \omega) = \frac{C_{ii}(\mathbf{r}; \omega)}{C_{ii}(\mathbf{0}; \omega)}. \tag{4}$$

Isotropic plane-wave field model. We represent the field as a superposition of plane waves with random phases and isotropically distributed propagation directions $\hat{\mathbf{n}}$:

$$\mathbf{u}(\mathbf{x}, \omega) = \int d\Omega_{\hat{n}}\; A(\hat{n}, \omega)\, \mathbf{p}(\hat{n}, \omega)\, e^{ik\,\hat{n}\cdot\mathbf{x}}, \tag{5}$$

where $k = \omega/v$ is the wavenumber for the relevant wave type, and $\mathbf{p}$ is the polarization vector. For P waves, $\mathbf{p} = \hat{n}$; for S waves, $\mathbf{p} \perp \hat{n}$.

We assume (i) different directions are uncorrelated, (ii) random phases remove cross-terms, and (iii) amplitudes are direction-independent in expectation. Under these assumptions, the correlation tensor reduces (up to an overall factor) to an angular average of the form

$$C_{ij}(\mathbf{r}) \propto \int \frac{d\Omega_{\hat{n}}}{4\pi}\, Q_{ij}(\hat{n})\, e^{ik\,\hat{n}\cdot\mathbf{r}}, \tag{6}$$

where Q_{ij} depends on wave type and polarization statistics.

2 Key angular identities

By isotropy we may align coordinates so that $\hat{\mathbf{r}}$ is the polar axis. Then

$$\hat{n}\cdot\hat{r} = \mu = \cos\theta, \qquad \hat{n}\cdot\mathbf{r} = d\mu, \qquad e^{ik\hat{n}\cdot\mathbf{r}} = e^{ikd\mu}, \tag{7}$$

and $d\Omega = \sin\theta\, d\theta\, d\phi$.

2.1 Plane-wave expansion and orthogonality

Use the plane-wave expansion in Legendre polynomials:

$$e^{ix\mu} = \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell} j_\ell(x)\, P_\ell(\mu), \tag{8}$$

where j_ℓ are spherical Bessel functions and P_ℓ are Legendre polynomials.

Using orthogonality,

$$\int_{-1}^{1} P_\ell(\mu)P_m(\mu)\, d\mu = \frac{2}{2\ell+1}\delta_{\ell m}, \tag{9}$$

one obtains the identity

$$\int_{-1}^{1} P_\ell(\mu)\, e^{ix\mu}\, d\mu = 2i^{\ell} j_\ell(x). \tag{10}$$

2.2 The needed scalar integrals

Define $x \equiv kd$.

Zeroth-order integral.

$$I_0(x) \equiv \int \frac{d\Omega}{4\pi}\, e^{ix\mu} = \frac{1}{2}\int_{-1}^{1} e^{ix\mu}\, d\mu = j_0(x). \tag{11}$$

Second-moment integral. We need $\langle \mu^2 e^{ix\mu}\rangle$. Use

$$\mu^2 = \frac{1}{3} + \frac{2}{3}P_2(\mu), \qquad P_2(\mu) = \frac{1}{2}(3\mu^2 - 1). \tag{12}$$

Then

$$\int_{-1}^{1} \mu^2 e^{ix\mu}\, d\mu = \frac{1}{3}\int_{-1}^{1} e^{ix\mu}\, d\mu + \frac{2}{3}\int_{-1}^{1} P_2(\mu)e^{ix\mu}\, d\mu \tag{13}$$

$$= \frac{1}{3}\cdot 2j_0(x) + \frac{2}{3}\cdot 2i^2 j_2(x) = \frac{2}{3}j_0(x) - \frac{4}{3}j_2(x). \tag{14}$$

Averaging over solid angle gives

$$\int \frac{d\Omega}{4\pi}\,\mu^2 e^{ix\mu} = \frac{1}{2}\int_{-1}^{1} \mu^2 e^{ix\mu}\,d\mu = \frac{1}{3}j_0(x) - \frac{2}{3}j_2(x). \tag{15}$$

Complementary integral for $1-\mu^2$. Since $1-\mu^2 = \sin^2\theta$,

$$\int \frac{d\Omega}{4\pi}(1-\mu^2)e^{ix\mu} = I_0(x) - (15) = j_0(x) - \left(\frac{1}{3}j_0(x) - \frac{2}{3}j_2(x)\right) \tag{16}$$

$$= \frac{2}{3}j_0(x) + \frac{2}{3}j_2(x). \tag{17}$$

3 Pure P-wave field

For a P wave, polarization is longitudinal:

$$\mathbf{p} = \hat{\mathbf{n}}, \qquad Q_{ij}^{(P)}(\hat{n}) = \hat{n}_i\hat{n}_j. \tag{18}$$

Thus

$$C_{ij}^{(P)}(\mathbf{r}) \propto \int \frac{d\Omega}{4\pi}\,\hat{n}_i\hat{n}_j\,e^{ix\mu}. \tag{19}$$

3.1 Tensor reduction by symmetry

By isotropy about $\hat{r}$, the integral must be of the form

$$\int \frac{d\Omega}{4\pi}\,\hat{n}_i\hat{n}_j\,e^{ix\mu} = A(x)\delta_{ij} + B(x)\hat{r}_i\hat{r}_j. \tag{20}$$

Determine A, B by contractions.

Trace. Using $\hat{n}_i\hat{n}_i = 1$,

$$3A(x) + B(x) = \int \frac{d\Omega}{4\pi}\,e^{ix\mu} = j_0(x). \tag{21}$$

Projection along $\hat{r}$.

$$A(x) + B(x) = \hat{r}_i\hat{r}_j \int \frac{d\Omega}{4\pi}\,\hat{n}_i\hat{n}_j\,e^{ix\mu} = \int \frac{d\Omega}{4\pi}\,(\hat{n}\cdot\hat{r})^2 e^{ix\mu} = \int \frac{d\Omega}{4\pi}\,\mu^2 e^{ix\mu}. \tag{22}$$

Using (15),

$$A(x) + B(x) = \frac{1}{3}j_0(x) - \frac{2}{3}j_2(x). \tag{23}$$

Solving yields

$$A(x) = \frac{1}{3}\big(j_0(x) + j_2(x)\big), \qquad B(x) = -j_2(x), \tag{24}$$

so the P-wave tensor is

$$C_{ij}^{(P)}(\mathbf{r}) \propto \frac{1}{3}\big(j_0 + j_2\big)\delta_{ij} - j_2\,\hat{r}_i\hat{r}_j. \tag{25}$$

3.2 Component form and normalization

Choose $\hat{r} = \hat{x} = (1, 0, 0)$ (receivers separated along x). Then

$$C_{xx}^{(P)}(d) \propto A + B = \frac{1}{3}\big(j_0 - 2j_2\big), \tag{26}$$

$$C_{yy}^{(P)}(d) \propto A = \frac{1}{3}\big(j_0 + j_2\big), \tag{27}$$

$$C_{zz}^{(P)}(d) \propto A = \frac{1}{3}\big(j_0 + j_2\big), \tag{28}$$

where $j_\ell \equiv j_\ell(x)$ and $x = kd$.

At zero separation $d \to 0$, $j_0(0) = 1$, $j_2(0) = 0$, hence

$$C_{xx}^{(P)}(0) = C_{yy}^{(P)}(0) = C_{zz}^{(P)}(0) \propto \frac{1}{3}. \tag{29}$$

Using the normalized definition (4) gives the familiar pure-P results:

$$\boxed{\mathrm{CC}_{xx}^{(P)}(d) = j_0(x) - 2j_2(x), \qquad \mathrm{CC}_{yy}^{(P)}(d) = \mathrm{CC}_{zz}^{(P)}(d) = j_0(x) + j_2(x).} \tag{30}$$

4 Pure S-wave field with random transverse polarization

For S waves, $\mathbf{p} \perp \hat{n}$ and polarization is random in the plane transverse to $\hat{n}$. We will show (Appendix A) that

$$\big\langle p_i p_j \big\rangle_\perp = \frac{1}{2}\Big(\delta_{ij} - \hat{n}_i \hat{n}_j\Big). \tag{31}$$

Thus in (6),

$$Q_{ij}^{(S)}(\hat{n}) = \frac{1}{2}\Big(\delta_{ij} - \hat{n}_i \hat{n}_j\Big), \tag{32}$$

and

$$C_{ij}^{(S)}(\mathbf{r}) \propto \frac{1}{2}\left[\delta_{ij} \int \frac{d\Omega}{4\pi} e^{ix\mu} - \int \frac{d\Omega}{4\pi} \hat{n}_i \hat{n}_j e^{ix\mu}\right]. \tag{33}$$

Using $I_0 = j_0$ and the P-wave tensor integral (25),

$$C_{ij}^{(S)}(\mathbf{r}) \propto \frac{1}{2}\left[\delta_{ij} j_0 - \left(\frac{1}{3}(j_0 + j_2)\delta_{ij} - j_2 \hat{r}_i \hat{r}_j\right)\right] \tag{34}$$

$$= \left(\frac{1}{3} j_0 - \frac{1}{6} j_2\right) \delta_{ij} + \frac{1}{2} j_2\, \hat{r}_i \hat{r}_j. \tag{35}$$

4.1 Component form and normalization

With $\hat{r} = \hat{x}$,

$$C_{xx}^{(S)}(d) \propto \left(\frac{1}{3} j_0 - \frac{1}{6} j_2\right) + \frac{1}{2} j_2 = \frac{1}{3}\big(j_0 + j_2\big), \tag{36}$$

$$C_{yy}^{(S)}(d) \propto \frac{1}{3} j_0 - \frac{1}{6} j_2 = \frac{1}{3}\left(j_0 - \frac{1}{2} j_2\right), \tag{37}$$

$$C_{zz}^{(S)}(d) \propto \frac{1}{3}\left(j_0 - \frac{1}{2} j_2\right). \tag{38}$$

At $d \to 0$, again $C_{ii}^{(S)}(0) \propto 1/3$, hence the normalized S-wave results are:

$$\boxed{\mathrm{CC}_{xx}^{(S)}(d) = j_0(x) + j_2(x), \qquad \mathrm{CC}_{yy}^{(S)}(d) = \mathrm{CC}_{zz}^{(S)}(d) = j_0(x) - \frac{1}{2} j_2(x).} \tag{39}$$

5 Incoherent mixture of P and S fields

Assume the total field is a sum of statistically independent P and S contributions:

$$\mathbf{u} = \mathbf{u}^{(P)} + \mathbf{u}^{(S)}, \qquad \langle u_i^{(P)} u_j^{(S)*} \rangle = 0. \tag{40}$$

Then the total correlation tensor is a power-weighted sum:

$$C_{ij}^{(mix)}(d) = E_P\, C_{ij}^{(P)}(d) + E_S\, C_{ij}^{(S)}(d), \tag{41}$$

where E_P and E_S are proportional to the mean squared amplitudes (energy/variance) of the respective parts. Define the mixture ratio

$$p \equiv \frac{E_S}{E_P}. \tag{42}$$

5.1 Normalization of the mixture

For each of the pure fields we found $C_{ii}(0) \propto 1/3$. Therefore

$$C_{ii}^{(mix)}(0) = E_P \frac{1}{3} + E_S \frac{1}{3} = \frac{E_P + E_S}{3}. \tag{43}$$

Hence the normalized component correlations are simply the weighted average of normalized pure-field correlations:

$$\mathrm{CC}_{ii}^{(mix)}(d) = \frac{E_P\, \mathrm{CC}_{ii}^{(P)}(d) + E_S\, \mathrm{CC}_{ii}^{(S)}(d)}{E_P + E_S} = \frac{\mathrm{CC}_{ii}^{(P)}(d) + p\, \mathrm{CC}_{ii}^{(S)}(d)}{1 + p}. \tag{44}$$

5.2 Final mixture formulas (baseline along x)

Let $x_P = k_P d = (\omega/v_P) d$ and $x_S = k_S d = (\omega/v_S) d$, and denote $j_\ell^{(P)} \equiv j_\ell(x_P)$ and $j_\ell^{(S)} \equiv j_\ell(x_S)$.

Using the pure-field results in (44):

$$\boxed{\mathrm{CC}_{xx}^{(mix)}(d) = \frac{\big(j_0^{(P)} - 2 j_2^{(P)}\big) + p\big(j_0^{(S)} + j_2^{(S)}\big)}{1 + p}} \tag{45}$$

$$\boxed{\mathrm{CC}_{yy}^{(mix)}(d) = \mathrm{CC}_{zz}^{(mix)}(d) = \frac{\big(j_0^{(P)} + j_2^{(P)}\big) + p\big(j_0^{(S)} - \frac{1}{2} j_2^{(S)}\big)}{1 + p}} \tag{46}$$

Special cases.

- Pure P ($p = 0$): $\mathrm{CC}_{xx} = j_0^{(P)} - 2 j_2^{(P)}$, $\mathrm{CC}_{yy} = \mathrm{CC}_{zz} = j_0^{(P)} + j_2^{(P)}$.
- Pure S ($p \to \infty$): $\mathrm{CC}_{xx} = j_0^{(S)} + j_2^{(S)}$, $\mathrm{CC}_{yy} = \mathrm{CC}_{zz} = j_0^{(S)} - \frac{1}{2} j_2^{(S)}$.

- Equal energies ($p = 1$) gives the mixture used in many numerical tests when P and S source counts and amplitude statistics match.
- Equipartition in the classical elastic sense corresponds to a specific energy partition across wave types and polarizations; in one common idealized limit the directional anisotropy cancels and $\mathrm{CC}_{xx} \approx \mathrm{CC}_{yy} \approx \mathrm{CC}_{zz} \approx j_0$.

6 Mapping to arbitrary receiver orientation

The preceding component formulas assumed $\hat{r} = \hat{x}$. For an arbitrary receiver separation direction $\hat{r}$, the tensor forms (25) and (35) may be used directly. Alternatively, for any separation direction, the *parallel* component is the component along $\hat{r}$ and the two *perpendicular* components lie in the orthogonal plane.

A Polarization average for S waves

For a given propagation direction $\hat{n}$, an S-wave polarization vector $\mathbf{p}$ lies in the plane orthogonal to $\hat{n}$:

$$\mathbf{p} \cdot \hat{n} = 0, \qquad \|\mathbf{p}\| = 1.$$

Assume $\mathbf{p}$ is uniformly distributed over that transverse plane (random "mix angle").

Let P_{ij} be the projector onto the plane orthogonal to $\hat{n}$:

$$P_{ij} = \delta_{ij} - \hat{n}_i \hat{n}_j. \tag{47}$$

This projector satisfies $P_{ij}\hat{n}_j = 0$ and $\mathrm{tr}(P) = 2$ (two transverse degrees of freedom).

Uniformity in the transverse plane implies the second moment $\langle p_i p_j \rangle_\perp$ must be proportional to P_{ij}:

$$\langle p_i p_j \rangle_\perp = \alpha \, P_{ij}. \tag{48}$$

Determine α by taking the trace:

$$\langle p_i p_i \rangle_\perp = 1 = \alpha \, \mathrm{tr}(P) = \alpha \cdot 2, \tag{49}$$

so $\alpha = 1/2$. Therefore,

$$\boxed{\langle p_i p_j \rangle_\perp = \frac{1}{2} \left(\delta_{ij} - \hat{n}_i \hat{n}_j \right).} \tag{50}$$

This is the polarization tensor used in the main text for isotropic S-wave fields with random transverse polarizations.

B Useful spherical Bessel functions

The first two spherical Bessel functions appearing in the correlation structure are

$$j_0(x) = \frac{\sin x}{x}, \tag{51}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x} \right) \sin x - \frac{3}{x^2} \cos x. \tag{52}$$

They satisfy $j_0(0) = 1$ and $j_2(0) = 0$, ensuring $\mathrm{CC}_{ii}(0) = 1$ after normalization.