MINIBAR: The search for minimal effective field theory bases

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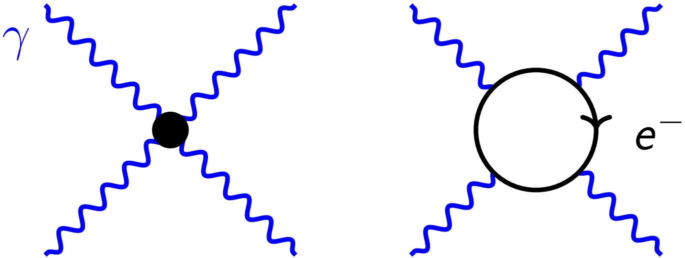
Effective Field Theories

Euler-Heisenberg Theory
QED at low elergies $E << m_e$

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{a}{m_e^4}\left(F^{\mu\nu}F_{\mu\nu}\right)^2 + \frac{b}{m_e^4}F^{\mu\nu}F_{\nu\sigma}F^{\sigma\rho}F_{\rho\mu} + \mathcal{O}\left(\frac{F^6}{m_e^8}\right)$$



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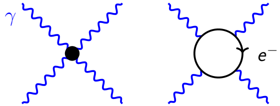
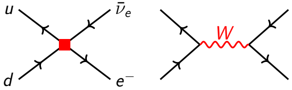
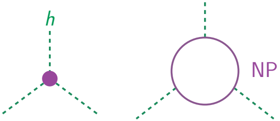


Fermi Theory
Four-fermion at $E << m_{Z,W}$



(Broken EW gauge group)
Fingerprints of H at $E < m_H$

Standard Model EFT
Integrating NP $E << m_{\text{NP}}$



Effective Lagrangians

High energy physics encoded in the coefficients of local effective operators

$$\mathcal{L}_{\mathrm{EFT}} = \sum_{i}^{\infty} C_i \, \mathcal{O}_i$$

Lagrangian operators determined by

- Field content
- Symmetries

Effective Lagrangians

High energy physics encoded in the coefficients of local effective operators

$$\mathcal{L}_{\mathrm{EFT}} = \sum_i^{\infty} C_i\, \mathcal{O}_i = \mathcal{L}_{d\leq 4} + \underbrace{\frac{1}{\Lambda}\sum_i C_i \mathcal{O}_i}_{\text{dim 5}} + \underbrace{\frac{1}{\Lambda^2}\sum_i C_i \mathcal{O}_i}_{\text{dim 6}} + \mathcal{O}\left(\frac{1}{\Lambda^3}\right)$$

Lagrangian operators determined by

- Field content
- Symmetries
- Power counting scheme

Effective Lagrangians

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Lagrangian operators determined by

- Field content
- Symmetries
- Power counting scheme

Brute-force method

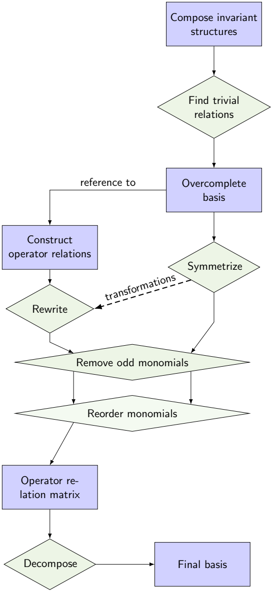
1. Write all operators
2. Remove linearly dependent
3. Remove symmetry breaking

The MINIBAR approach

https://github.com/jruizvid/MINIBAR

MINIBAR

Tools to calculate MINImal operator BAses Rapidly with MATHEMATICA



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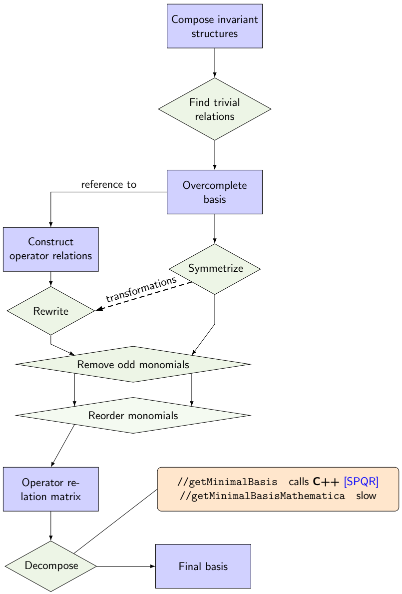

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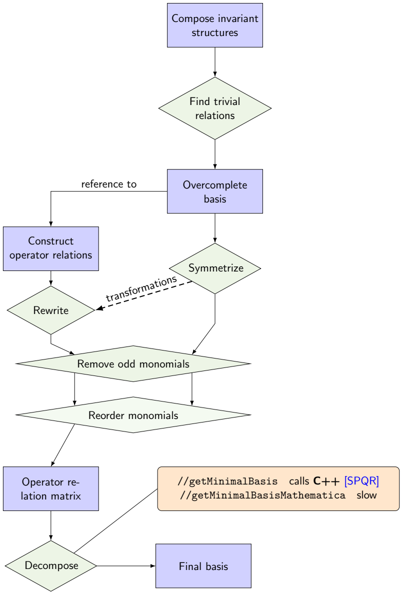

Other public tools: [Rosetta] [Sym2Int] [DEFT] [BasisGen] [GrIP] [ABC4EFT] [MassiveEFT-Operators] [Matchete] [AutoEFT] [opbasis] https://github.com/jruizvid/MINIBAR

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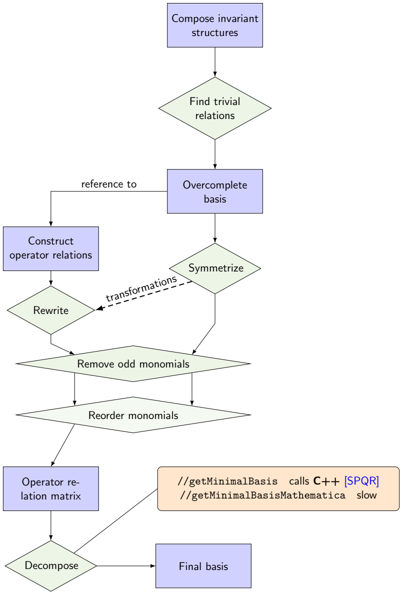

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[MINIBAR] ChPT; flexible notation

Scope of MINIBAR

Effective Lagrangians

- **ChPT** $\mathcal{O}(p^2 - p^8)$ 2310.20547
- **ChPT+Gravity** $\mathcal{O}(p^2 - p^6)$
- ChPT+QED $\mathcal{O}(p^2 - p^6)$
- ChPT+EW $\mathcal{O}(p^2 - p^6)$

- ChPT with baryons?
- ChPT with ALPS?
- EWET?

- **SMEFT**

- LEFT / WET?

Rest of the talk

- Three **Lagrangians**
- A tiny bit of code

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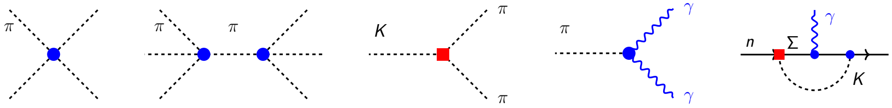
The shelves of the MINIBAR

Chiral Perturbation Theory



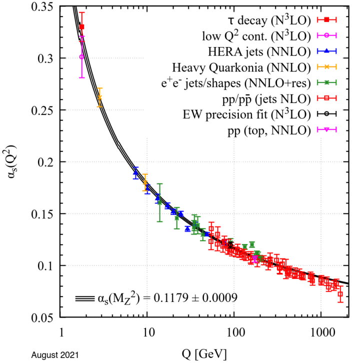
in collaboration with J. Bijnens and N. Hermansson-Truedsson

Chiral Perturbation Theory (ChPT)

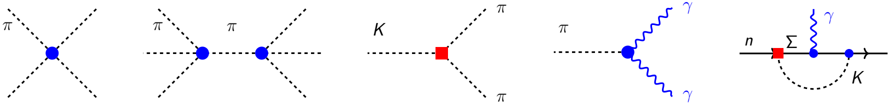
$$\mathcal{L}_{\rm QCD} = i\bar{q}(\not{D} - m_q)q - \frac{1}{4}\, G^a_{\mu\nu} G^{\mu\nu}_a$$

- Effective theory of Quantum Chromodynamics at low energies
- Exact same symmetries
  $SU(3)_C$, $U(1)^*$, $SU(3)_L \times SU(3)_R^{\ddagger}$

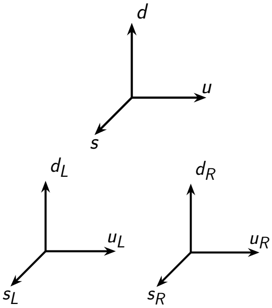
* Broken $U(1)_A$ generates **anomalous sector** (here)
‡ provided $m_q \to 0$

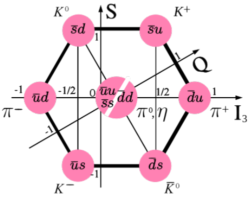


Bound states as asymptotic states



From QCD symmetries to mesons





$$\mathcal{L}_{\rm QCD} = i\bar{q}(\not{D} - m_q)q - \frac{1}{4}\, G^a_{\mu\nu} G_a^{\mu\nu}$$

- all possible rotations in {u,d,s} coordinates

$$U(3)$$

- assuming $m_q = 0$, independent rotations $i\bar{q}_R \not{D} q_R + i\bar{q}_L \not{D} q_L$

$$U(3)_L \times U(3)_R$$

- factoring out global phase changes

$$= \underbrace{SU(3)_L \times SU(3)_R}_{\text{chiral group}} \times U(1)_{L+R} \times \underbrace{U(1)_{L-R}}_{\text{anomaly}}$$

- Spontaneous chiral symmetry breaking

$$G \equiv SU(3)_L \times SU(3)_R \longrightarrow SU(3)_V$$

- Resulting 8 pseudo-Goldstone bosons: **light mesons**

$$\pi^0, \pi^\pm, K^\pm, K^0, \bar{K}^0, \eta_8$$

Chiral Perturbation Theory (ChPT)

- 8 *massless* Goldstone bosons. Best parametrized through matrix U

$$U = e^{i\frac{\Phi}{f}} \xrightarrow{G} g_R U g_L^\dagger \quad , \quad \mathbf{\Phi} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}$$

- External background fields introduce $m_q \neq 0$ and help relation to quarks

$$\mathcal{L} = \mathcal{L}_{QCD}^{m_q=0} + \overline{q}\gamma^\mu (v_\mu + a_\mu\gamma_5) q - \overline{q}(s - ip\gamma_5) q$$

- Other *standard* parametrizations

$\Phi(x)$ s p ℓ_α r_α

U χ $F_R^{\alpha\beta}$ $F_L^{\alpha\beta}$

u_α $\chi_\pm$ $f_\pm^{\alpha\beta}$

- Most general symmetry-preserving $\mathcal{L}$ at lowest order

$$\mathcal{L}_2 = \frac{f^2}{4}\langle D_\mu U^\dagger D^\mu U + \underbrace{U^\dagger\chi + \chi^\dagger U}_{\text{external fields}}\rangle$$

- Known at $\mathcal{O}(p^2)$, $\mathcal{O}(p^4)$, $\mathcal{O}(p^6)$.

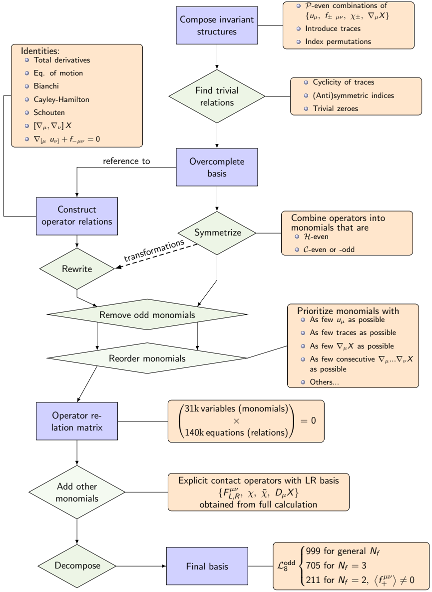
[p^4: Gasser, Leutwyler; Annals Phys. 158 (1984) 142]

[Scherer et al, hep-ph/9408346] [Bijnens et al, hep-ph/9902437]

$$\mathcal{L}_{\text{NNNLO}} = ?$$

- Intrinsic parity → two Lagrangians
  - Even sector:
    e.g. $Tr(u^\mu u_\mu u^\nu u_\nu)$
  - **Anomalous sector**:
    e.g. $Tr(u^\mu u^\nu u^\rho u^\sigma)\varepsilon_{\mu\nu\rho\sigma}$

- Contact operators with different parametrization



Bases

$\Phi(x)$ s p ℓ_α r_α

U χ $F_R^{\alpha\beta}$ $F_L^{\alpha\beta}$

u_α $\chi_\pm$ $f_\pm^{\alpha\beta}$

Matrix	Definition	Transformation
U	$\exp\left(i\frac{\Phi(x)}{f}\right)$	$g_R U g_L^\dagger$
χ	$2B_0(s+ip)$	$g_R \chi g_L^\dagger$
$F^L_{\alpha\beta}$	$\partial_\alpha \ell_\beta - \partial_\beta \ell_\alpha - i[\ell_\alpha, \ell_\beta]$	$g_L F^L_{\alpha\beta} g_L^\dagger$
$F^R_{\alpha\beta}$	$\partial_\alpha r_\beta - \partial_\beta r_\alpha - i[r_\alpha, r_\beta]$	$g_R F^R_{\alpha\beta} g_R^\dagger$
u_α	$i\left\{u^\dagger\left(\partial_\alpha - ir_\alpha\right)u - u\left(\partial_\alpha - i\ell_\alpha\right)u^\dagger\right\}$	$h u_\alpha h^\dagger$
$\chi_\pm$	$u^\dagger \chi u^\dagger \pm u \chi^\dagger u$	$h \chi_\pm h^\dagger$
$f_\pm^{\alpha\beta}$	$u F_L^{\alpha\beta} u^\dagger \pm u^\dagger F_R^{\alpha\beta} u$	$h f_\pm^{\alpha\beta} h^\dagger$

* $U = u^2$

- Meson octet fields in $\Phi(x)$

$$\Phi(x) = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}$$

- U parametrization transforms linearly under the chiral group $(g_L, g_R) \in SU(2)_L \otimes SU(2)_R$
- u_α chiral transformation only with $h \in SU(2)_V$

Constructing $\mathcal{L}^{\mathbf{ano}}_{\mathbf{p^8}}$: all conceivable terms

1. Do all possible combinations of **building blocks** at order p^8

$$u_\alpha \quad \chi_\pm \quad f_\pm^{\alpha\beta} \ ; \quad \nabla^\mu X$$

2. Impose Lorentz invariance, odd intrinsic parity, chiral symmetry, parity
3. Distribute them over flavour traces

Algorithm

1. Combinations of building blocks + derivatives
2. Do permutations* and freeze order
3. Add indices and their permutations:
$$\underbrace{\left\langle f_+^{\alpha\beta} \nabla^\gamma u^\delta \right\rangle}_{\text{4 ind. with } \varepsilon_{\alpha\beta\gamma\delta}} \underbrace{\left\langle f_+^{\mu\nu} u_\mu u_\nu \right\rangle}_{\text{(2 ind. pairs)}} \varepsilon_{\alpha\beta\gamma\delta} \quad \Rightarrow \textbf{4-8 indices}$$
4. Group in ⟨flavour traces⟩

*needed permutations

Summary

Possible terms: $\sim$ 244k

Constructing $\mathcal{L}^{\mathbf{ano}}_{\mathbf{p^8}}$: vanishing terms and trivial relations

- Directly vanishing terms from **trivial identities** $\langle u \rangle = 0$; $\langle f^{\pm}_{\mu\nu} \rangle = 0$
- One-to-one relations due to **index (anti)symmetries**
- Vanishing terms due to (Symmetric)×(Antisymmetric)=0

$$e.g.\ \varepsilon^{\alpha\beta\gamma\delta} A_\alpha A_\beta = 0$$

Algorithm

1. Directly vanishing (244k → 74k)
   $\langle u \rangle = 0$; $Tr[f^{\pm}_{\mu\nu}] = 0$; $f^{\pm}_{jj} = 0$

2a. Calculate all shapes of each term and cross-compare to find duplicates

2b. Cycle traces and order indices with a canonical criterion. Remove duplicates

3. (Symmetric)×(Antisymmetric)=0

Summary

Possible terms: $\sim$ 244k
After trivial relations: $\sim$ 31k

Constructing $\mathcal{L}^{\mathbf{ano}}_{\mathbf{p^8}}$: operator relations (I)

- The terms that survived (31k) are still related through different identities
- These can connect up to ~ 20 terms simultaneously
- Determine all relations systematically from each of **these identities:**

Total derivatives

- Total derivatives in the Lagrangian $\rightarrow$ surface terms in the action

$$S = \int \mathcal{L}\, d^4 x$$

- 12k relations

Algorithm

1. Generate all possible total derivatives and store relations

$$\frac{d}{dx^\mu}(ABC...) = 0 \quad \rightarrow \quad \frac{\partial A}{\partial x^\mu} BC... + A \frac{\partial B}{\partial x^\mu} C... + ... = 0$$

Constructing $\mathcal{L}^{\mathbf{ano}}_{\mathbf{p^8}}$: operator relations (II)

Definition of $(f_- \leftrightarrow \nabla u)$

$$\left.\begin{array}{c} u_\mu = i\Big[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - i\ell_\mu)u^\dagger\Big] \\ \nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X] \\ \Gamma_\mu = \frac{1}{2}\Big[u^\dagger\left(\partial_\mu - ir_\mu\right)u + u\left(\partial_\mu - i\ell_\mu\right)u^\dagger\Big] \\ f_\pm^{\mu\nu} = uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu}u \end{array}\right\} \to \nabla_\mu u_\nu - \nabla_\nu u_\mu + f_{-\mu\nu} = 0$$

- 12k relations

Algorithm

1. Find instances of $\nabla_\nu u_\mu$ or $f_{-\mu\nu}$ in the terms *e.g.*
$$\left\langle \chi_+ \mathbf{f}_{-\alpha\beta}\right\rangle\left\langle f_{-\mu\gamma}\nabla_\mu u_\delta\right\rangle$$
2. Each yields a relation
$$\begin{array}{c}\left\langle \chi_+ \mathbf{f}_{-\alpha\beta}\right\rangle\left\langle f_{-\mu\gamma}\nabla_\mu u_\delta\right\rangle + \\ \left\langle \chi_+ \boldsymbol{\nabla}_{\alpha}\mathbf{u}_{\beta}\right\rangle\left\langle f_{-\mu\gamma}\nabla_\mu u_\delta\right\rangle - \\ \left\langle \chi_+ \boldsymbol{\nabla}_{\beta}\mathbf{u}_{\alpha}\right\rangle\left\langle f_{-\mu\gamma}\nabla_\mu u_\delta\right\rangle = 0\end{array}$$

Constructing $\mathcal{L}^{\text{ano}}_{p^8}$: operator relations (III)

Commutation of covariant derivatives

- The commutation of derivatives

$$[\nabla_\mu, \nabla_\nu] X = [\Gamma_{\mu\nu}, X]$$

- is given by the field-strength tensor $\Gamma_{\mu\nu}$

$$\Gamma_{\mu\nu} = \frac{1}{4}[u_\mu, u_\nu] - \frac{i}{2} f_{+\mu\nu}$$

- 9k relations

Bianchi identities

- Introduced in differential geometry. Holds for the field-strength tensor $\Gamma_{\mu\nu}$ as

$$\nabla_\mu \Gamma_{\nu\rho} + \nabla_\nu \Gamma_{\rho\mu} + \nabla_\rho \Gamma_{\mu\nu} = 0$$

- 2k relations

Constructing $\mathcal{L}^{\text{ano}}_{\text{p}^8}$: operator relations (IV)

Schouten identity

- Two indices in 1D $\mu, \nu = \{1\}$ cannot be antisym. $(A^\mu A^\nu \neq -A^\nu A^\mu)$
- In 4D $\mu_i = \{1, 2, 3, 4\}$, we cannot have 5 antisymmetric indices
- In each term, permute 5 indices to build antisymmetric combinations

$$0 = A_\nu \epsilon_{\alpha\beta\gamma\delta} - A_\alpha \epsilon_{\nu\beta\gamma\delta} - A_\beta \epsilon_{\alpha\nu\gamma\delta} - A_\gamma \epsilon_{\alpha\beta\nu\delta} - A_\delta \epsilon_{\alpha\beta\gamma\nu}.$$

- 27k relations

Equations of motion / field redefinitions

$$\nabla^\mu u_\mu - \frac{i}{2}\left(\chi_- - \frac{1}{N_f}\langle\chi_-\rangle\right) = 0$$

- EoM following from LO Lagrangian $\mathcal{L}_0 = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle$
- Equivalent from most general field redefinition [App.A, hep-ph/9902437]
- 2k relations

Constructing $\mathcal{L}^{\text{ano}}_{p^8}$: operator relations (V)

Cayley-Hamilton trace relations

- The determinant of the flavour matrix A is (for $N_F = 2$)

$$\det A \; 1_{N_F} = A^2 - \langle A\rangle A \quad \rightarrow \quad \det A \; \underbrace{\langle 1_{N_F}\rangle}_{N_F} = \langle A^2\rangle - \langle A\rangle^2$$

- Cayley-Hamilton relations ($N_F = 2$)

$$0 = \underbrace{A^2 - \langle A\rangle A}_{\det A} - \underbrace{\frac{1}{N_F}\left(\left\langle A^2\right\rangle - \langle A\rangle^2\right)}_{\det A}$$

- Introduce in our terms by developing $A = B + C$ ($N_F = 2$); and analogously for the identities in $N_F = 3,\ 4,\ 5, \ldots$

- 67k relations

Summary

Possible terms: $\sim$ 244k
After trivial relations: $\sim$ 31k
Relations:

- Total derivatives $\sim$ 12k
- Definition ($f_- \leftrightarrow \nabla u$) $\sim$ 12k
- Equations of motion $\sim$ 2k
- Bianchi $\sim$ 2k
- Commuting derivatives $\sim$ 9k
- Schouten $\sim$ 27k
- Trace relations $\sim$ 67k ($N_F = 2$)

Constructing $\mathcal{L}^{\text{ano}}_{p^8}$: ensure H and C symmetries

	P	C	h.c.
u_μ	$-\varepsilon(\mu)u_\mu$	u_μ^T	u_μ^T
$\chi_\pm$	$\pm\chi_\pm$	$\chi_\pm^T$	$\pm\chi_\pm^T$
$f_{\pm\mu\nu}$	$\pm\varepsilon(\mu)\varepsilon(\nu)f_{\pm\mu\nu}$	$\mp f_{\pm\mu\nu}^T$	$f_{\pm\mu\nu}^T$

- Find linear combinations of operators that are
  - Hermitian: $\tilde{\mathcal{O}}_i^\dagger = \tilde{\mathcal{O}}_i$
  - Eigenstates of C: $\tilde{\mathcal{O}}_i \xrightarrow{C} \pm\tilde{\mathcal{O}}_i$
- New basis $\{\tilde{\mathcal{O}}_i\}$ with same number of operators (31k)
- C-even identified (14k)

Algorithm

1. Transform operators

$$\mathcal{O}_i \xrightarrow{C,\ H} \lambda_\pm^C\, \lambda_\pm^{\text{h.c.}}\, \mathcal{O}_j$$

2a. Ready eigenstates $\mathcal{O}_j = \mathcal{O}_i$
Correct H-odd by adding i

$$\mathcal{O}_i = i\tilde{\mathcal{O}}_i$$

2b. Modified operators ($j \neq i$)
Depending on $(\lambda_\pm^C, \lambda_\pm^{\text{h.c.}})$

$$(+,+):\quad \mathcal{O}_i = \frac{\tilde{\mathcal{O}}_i + i\tilde{\mathcal{O}}_j}{2}\,,\quad \mathcal{O}_j = \frac{\tilde{\mathcal{O}}_i - i\tilde{\mathcal{O}}_j}{2}\,,$$

$$(-,+):\quad \mathcal{O}_i = \frac{i\tilde{\mathcal{O}}_i + \tilde{\mathcal{O}}_j}{2}\,,\quad \mathcal{O}_j = \frac{-i\tilde{\mathcal{O}}_i + \tilde{\mathcal{O}}_j}{2}\,,$$

$$(+,-):\quad \mathcal{O}_i = \frac{i\tilde{\mathcal{O}}_i + \tilde{\mathcal{O}}_j}{2}\,,\quad \mathcal{O}_j = \frac{i\tilde{\mathcal{O}}_i - \tilde{\mathcal{O}}_j}{2}\,,$$

$$(-,-):\quad \mathcal{O}_i = \frac{\tilde{\mathcal{O}}_i + i\tilde{\mathcal{O}}_j}{2}\,,\quad \mathcal{O}_j = \frac{-\tilde{\mathcal{O}}_i + i\tilde{\mathcal{O}}_j}{2}\,.$$

Constructing $\mathcal{L}_{p^8}^{\text{ano}}$: find complete minimal basis

- Relations between terms form a system of linear equations $[A\vec{O} = 0]$
- Many linearly dependent: **Gaussian elimination**
- System not rank-complete: get maximal set of independent variables

Algorithm

1. Build relation matrix A (31k $\times$ 200k)

$$[A\vec{O} = 0]$$

2. Decomposition into unitary (Q) and right-upper-triangular matrix (R)

$$[QR\vec{O} = 0] \rightarrow [R\vec{O} = 0]$$

3. Nonzero terms in diag(R): contains a set of linearly-independent terms

- Bases obtained for general N_F and $N_F = 2$ (u, d), $N_F = 3$ (u, d, s)

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- Schouten $\sim$ 27k
- Trace relations $\sim$ 67k ($N_F = 2$)

Linearly-independent terms (general N_F): **999**

Contact terms explicitly

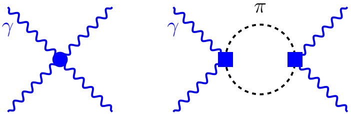
- Operators that only involve **external fields**
- **Counter terms** to renormalize loops with vertices of lower p order
- Basis to make them explicit

$$U \quad \chi \quad F_R^{\alpha\beta} \quad F_L^{\alpha\beta}$$

- Additional building block

$$\tilde{\chi} \equiv \left(\det(\chi)\chi^{-1}\right)^{\dagger} \longrightarrow g_R \tilde{\chi} g_L^{\dagger} .$$

- **Repeat calculation** with this basis



Algorithm

1. Build all possible terms
2. **Check chiral invariance**
3. Reduce with trivial relations
4. Obtain operator relations from same identities except
   - Equations of Motion
   - Definition ($f_- \leftrightarrow \nabla u$)
5. Build relation matrix
6. Gaussian elimination and final basis

Final number of terms

	N_f		$N_f = 3$		$N_f = 2$		$N_f = 2$ $\langle f_{+\alpha\beta} \rangle = 0$*	
	Total	Contact	Total	Contact	Total	Contact	Total	Contact
Full	999	0	705	0	211	5	92	2
No $\chi_\pm$	565	0	369	0	77	2	0	0
No $f_\pm^{\mu\nu}$	79	0	45	0	2	0	2	0
Only u_μ	36	-	16	-	0	-	0	-

* For the anomalous sector with $N_F = 2$, the physical case needs the inclusion of a (non chirally-invariant) singlet vector source, *i.e.* $\langle f_{+\alpha\beta} \rangle \neq 0$. Not considered in last column to allow comparison with Hilbert series [2212.07901]

ChPT Lagrangian – summary

[Bijnens, Hermansson-Truedsson, JRV, 2310.20547]

- Obtained the **anomalous ChPT Lagrangian at $\mathbf{O(p^8)}$**
- Completing $O(p^8)$
- **Agreement with Hilbert series** method on number of operators at $N_F \to \infty$ [2212.07901]
- Additionally provide **explicit basis**

	I.P.	N_f	$N_f = 3$	$N_f = 2$
p^2	even	2	2	2
p^4	even	13	12	10
p^6	even	115	94	56
	odd	24	23	13 $[5]_{\langle f_+ \rangle = 0}$
p^8	even	1862	1254	475
	odd	**999**	**705**	**211** $\mathbf{[92]}_{\langle f_+ \rangle = 0}$

- `MINIBAR` code published with the article

https://github.com/jruizvid/MINIBAR

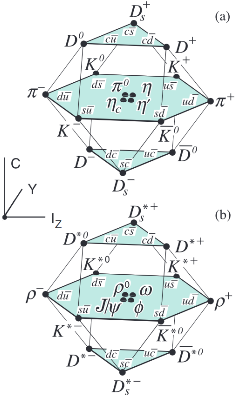
Beyond the pseudoscalar octet

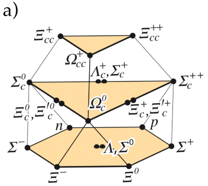
Challenges of ChPT

Reviews: [Pich, hep-ph/9806303] [Scherer, A primer of ChPT]

- **Adding resonances**
  - For E above the first resonance $m_\rho \approx 770\mathrm{MeV}$, other mesons play an important role

    $V(1^{--})$, $A(1^{++})$, $S(0^{++})$ and $P(0^{-+})$
  - Systematic frameworks developed. Many new interactions and **new free parameters**
- **Adding baryons**
  - New allowed interactions modify **power counting** and add many extra terms
- **Adding heavy quarks**
  - Chiral *perturbation* theory relies on small masses $q = \{u, d, s\}$. Formalism with $SU(N_F = 4)$
  - Combination with Heavy Quark Effective Theory





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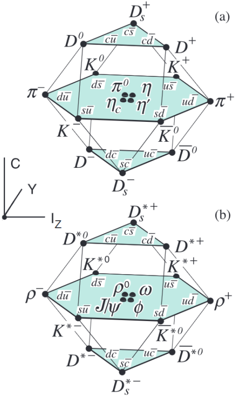
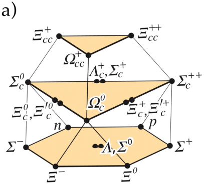
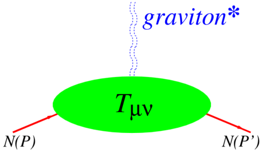
- **Adding gravitons**
  - Calculation of Gravitational Form Factors
  - Phenomenologically accessible through GPDs





2nd Lagrangian

ChPT + Gravity



in collaboration with J. Bijnens

Introduction

$$S = S_{\rm grav} + S_{\rm matter} = \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}_{\rm matter}$$

- Least action principle with $\delta g_{\mu\nu} \rightarrow$ Einstein Field equations

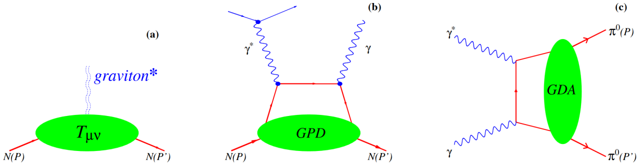
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \quad , \quad \text{where} \quad T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} \mathcal{L}_{\rm matter}}{\delta g^{\mu\nu}}$$

- $T_{\mu\nu}$ encodes fundamental information of the particles
  $\rightarrow$ mass $\rightarrow$ spin $\rightarrow$ **"mechanical properties"**
- **Form factors of conserved currents** at zero momentum transfer

em: $\partial_\mu J^\mu_{\rm em} = 0$	$\langle N' \vert J^\mu_{\bf em} \vert N \rangle$	$\longrightarrow$	Q	$= 1.602176487(40) \times 10^{-19}\text{C}$
			μ	$= 2.792847356(23) \mu_N$
weak: PCAC	$\langle N' \vert J^\mu_{\bf weak} \vert N \rangle$	$\longrightarrow$	g_A	$= 1.2694(28)$
			g_p	$= 8.06(55)$
gravity: $\partial_\mu T^{\mu\nu}_{\bf grav} = 0$	$\langle N' \vert T^{\mu\nu}_{\bf grav} \vert N \rangle$	$\longrightarrow$	m	$= 938.272013(23)\,\text{MeV}/c^2$
			J	$= \frac{1}{2}$
			$\boldsymbol{D}$	$=$ **?**

[Polyakov, Schweitzer; 1805.06596]

Phenomenology of gravitational form factors



- Experimental effects
  - In **strong gravitational field**: direct coupling to gravitons
  - **Nucleon GFF:** *Deeply virtual compton scattering / threshold J/ψ photoproduction* access GFF through generalized parton distributions (GPD): *two spin-1 photons $\sim$ spin-2 graviton*
  - **Pion GFF:** Hard exclusive production in $\gamma\gamma \to \pi^0\pi^0$ through generalized distribution amplitudes (GDA)

- Revived interest in recent years [2207.05212] [GlueX, 1905.10811]

Energy momentum tensor for mesons in ChPT

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} \mathcal{L}_{\mathrm{matter}}}{\delta g^{\mu\nu}}$$

- Build all terms in the Lagrangian compliant with symmetries
- Include couplings of matter fields to curvature
- Practically: ChPT Lagrangian with **extra building blocks**

Riemann curvature tensor

$$R^{\beta}{}_{\mu\nu\alpha} = \partial_\mu \Gamma^{\beta}{}_{\nu\alpha} - \partial_\nu \Gamma^{\beta}{}_{\mu\alpha} + \Gamma^{\beta}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\alpha} - \Gamma^{\beta}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\alpha}$$

$$\Gamma^{\beta}{}_{\mu\nu} = \frac{1}{2} g^{\beta\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right)$$

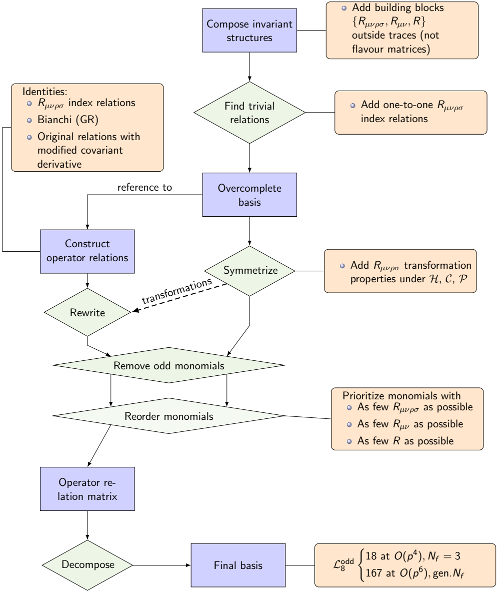
Ricci tensor

$$R_{\mu\nu} = R^{\gamma}{}_{\mu\gamma\nu}$$

Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}$$

ChPT + Gravity: calculation with MINIBAR



Covariant derivatives

ChPT	*General covariance (GR)*
$\hat{\nabla}_\mu X = \partial_\mu X + [\Gamma_\mu, X]$	$\tilde{\nabla}_\mu X^\alpha = \partial_\mu X^\alpha + \Gamma^\alpha_{\ \gamma\mu} X^\gamma$
$\Gamma_\mu = \frac{1}{2}\left[u^\dagger\left(\partial_\mu - i r_\mu\right)u + u\left(\partial_\mu - i\ell_\mu\right)u^\dagger\right]$	$\Gamma^\alpha_{\ \mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu})$

- $\hat{\nabla}$ and $\tilde{\nabla}$ combined into one covariant derivative ∇
- Adapt identities for operator relations

Commutation of covariant derivatives

$$\left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right] X = [\Gamma_{\mu\nu}, X]$$

$$\left[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu\right] X^{\alpha\beta\dots} = R^\alpha_{\ \gamma\mu\nu} X^{\gamma\beta} + R^\beta_{\ \gamma\mu\nu} X^{\alpha\gamma} + \dots$$

$$\Gamma_{\mu\nu} = \frac{1}{4}\left[u_\mu, u_\nu\right] - \frac{i}{2} f_{+\mu\nu}$$

Riemann indices

Index permutations

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$$

$$R_{\mu\nu\underline{\alpha\beta}} = -R_{\mu\nu\underline{\beta\alpha}} \qquad R_{\underline{\mu\nu}\alpha\beta} = -R_{\underline{\nu\mu}\alpha\beta}$$

1st Bianchi identity

$$R_{\mu\underline{\alpha\beta\gamma}} + R_{\mu\underline{\beta\gamma\alpha}} + R_{\mu\underline{\gamma\alpha\beta}} = 0$$

2nd Bianchi identity

$$R_{\mu\nu\underline{\alpha\beta;\gamma}} + R_{\mu\nu\underline{\beta\gamma;\alpha}} + R_{\mu\nu\underline{\gamma\alpha;\beta}} = 0$$

*Notation: $R_{\mu\nu\alpha\beta;\gamma} = \nabla_\gamma R_{\mu\nu\alpha\beta}$

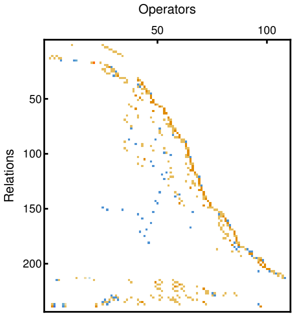
Lagrangian at $\mathcal{O}(p^4)$

- First calculation $\mathcal{O}(p^4)$
  [Donoghue, Leutwyler, Z.Phys.C 52 (1991) 343]
  $\rightarrow$ 18 terms for $N_F = 3$

	Pure ChPT	Mixed	Pure GR
1	$\mathrm{TR}[u_{j1}, u_{j1}, u_{j2}, u_{j2}]$	$\mathrm{TR}[R_{j1,j2}] \times \mathrm{TR}[u_{j1}, u_{j2}]$	$\mathrm{TR}[R_{j1,j2,j3,j4}]^2$
2	$\mathrm{TR}[u_{j1}, u_{j1}] \times \mathrm{TR}[u_{j2}, u_{j2}]$	$\mathrm{TR}[R] \times \mathrm{TR}[u_{j1}, u_{j1}]$	$\mathrm{TR}[R_{j1,j2}]^2$
3	$\mathrm{TR}[u_{j1}, u_{j2}]^2$	$\mathrm{TR}[R] \times \mathrm{TR}[\chi^+]$	$\mathrm{TR}[R]^2$
4	$\mathrm{TR}[\chi^+, u_{j1}, u_{j1}]$		
5	$\mathrm{TR}[\chi^+] \times \mathrm{TR}[u_{j1}, u_{j1}]$		
6	$\mathrm{TR}[\chi^+, \chi^+]$		
7	$\mathrm{TR}[\chi^-, \chi^-]$		
8	$\mathrm{TR}[\chi^+]^2$		
9	$\mathrm{TR}[\chi^-]^2$		
10	$i\,\mathrm{TR}[f^+{}_{j1,j2}, u_{j1}, u_{j2}]$		
11	$\mathrm{TR}[f^+{}_{j1,j2}, f^+{}_{j1,j2}]$		
12	$\mathrm{TR}[f^-{}_{j1,j2}, f^-{}_{j1,j2}]$		

- From $\mathcal{L}$ to $T^{\mu\nu}$ to $\langle\pi|T^{\mu\nu}|\pi\rangle$
- Renormalization of 1-loop diagrams

- Lagrangian reproduced with `MINIBAR`



ChPT + Gravity $\mathcal{O}(p^4)$

Lagrangian at $\mathcal{O}(p^6)$

Summary

Possible terms: $\sim$ 690k

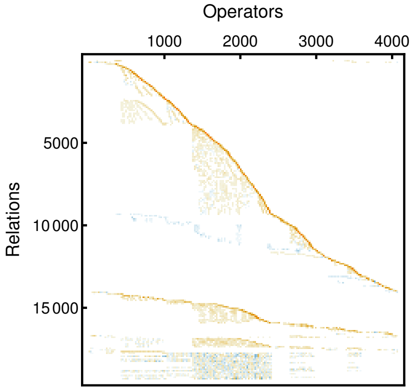
After trivial relations: $\sim$ 4.2k

Relations:

- Bianchi 1 $\sim$ 4.1k
- Bianchi 2 $\sim$ 600
- Commuting derivatives $\sim$ 1k
- Index $R^{\mu\nu\rho\sigma}$ $\sim$ 10k
- Definition R and $R^{\mu\nu}$ $\sim$ 1k
- Total derivatives $\sim$ 1k
- Definition $(f_- \leftrightarrow \nabla u)$ $\sim$ 200
- Equations of motion $\sim$ 80
- Bianchi ChPT $\sim$ 20
- Schouten $\sim$ 1.5k
- Trace relations $\sim$ 380 $(N_F = 2)$

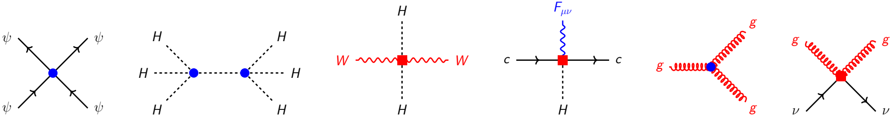
Linearly-independent terms (general N_F): **167**

PRELIMINARY



ChPT + Gravity $\mathcal{O}(p^6)$

3rd Lagrangian
Standard Model Effective Field Theory
[WIP]



with Bachelor students at Maastricht

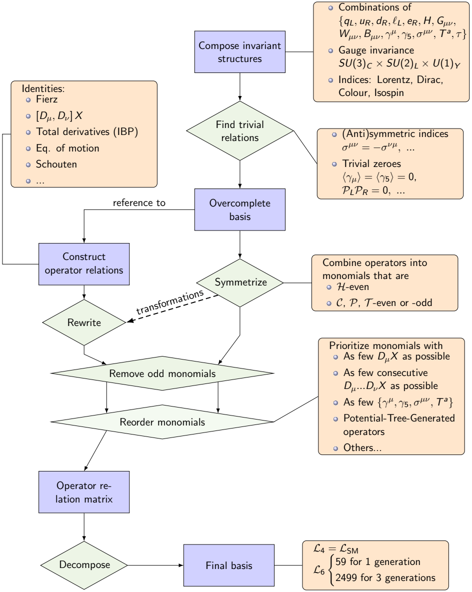
Jakub Gorzelany & Lâm Trinh

SMEFT

- EFT to search for New Physics above the SM
- Full particle content of the SM
- Full Gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$

[Warsaw basis]

help of EOMs in Ref. [3] has been a partial success. It is really amazing that no author of almost 600 papers that quoted Ref. [3] over 24 years has ever decided to rederive the operator basis from the outset to check its correctness. As the current work shows, the exercise has been straightforward enough for an M. Sc. thesis [29, 30]. It has required no extra experience with respect to what was standard already in the 1980's.



Representations and indices

$$q_L \quad u_R \quad d_R \quad \ell_L \quad e_R \quad H \quad G_{\mu\nu} \quad W_{\mu\nu} \quad B_{\mu\nu} \quad \gamma^\mu \quad \gamma_5 \quad \sigma^{\mu\nu} \quad T^a \quad \tau$$

Dirac Spinors

$$\psi = \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \\ \psi_{R1} \\ \psi_{R2} \end{pmatrix}$$

SU(3) Colour

$$q = \begin{pmatrix} q_{\rm red} \\ q_{\rm blue} \\ q_{\rm green} \end{pmatrix}$$

$\mathrm{SU}(2)_L$ electroweak

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

Lorentz

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Generations

$$q = \begin{pmatrix} q_{ud} \\ q_{cs} \\ q_{tb} \end{pmatrix}$$

`MINIBAR` v0: only Lorentz indices; v0.6: flexible index types

Operator relations

Trivial zeroes

$$\mathrm{Tr}(\gamma^\mu) = 0, \quad \mathrm{Tr}(\gamma^5) = 0$$

$$\mathrm{Tr}(\text{odd number of } \gamma) = 0$$

$$\mathrm{Tr}(\sigma^i) = 0, \quad \mathrm{Tr}(T^A) = 0$$

One-to-one relations

$$(\gamma^5)^2 = \mathbf{1}, \quad (\gamma^5)^T = \gamma^5, \quad \{\gamma^5, \gamma^\mu\} = 0$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\,\epsilon^{\mu\nu\rho\sigma}$$

$$\gamma^\mu\gamma_\mu = 4, \quad \gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}$$

New identities

Fierz:

$$(\bar{\psi}_1\psi_2)(\bar{\psi}_3\psi_4) = \frac{1}{4}(\bar{\psi}_1\psi_4)(\bar{\psi}_3\psi_2) + \frac{1}{4}(\bar{\psi}_1\gamma^\mu\psi_4)(\bar{\psi}_3\gamma_\mu\psi_2) + \frac{1}{8}(\bar{\psi}_1\sigma^{\mu\nu}\psi_4)(\bar{\psi}_3\sigma_{\mu\nu}\psi_2) - \frac{1}{4}(\bar{\psi}_1\gamma^\mu\gamma_5\psi_4)(\bar{\psi}_3\gamma_\mu\gamma_5\psi_2) + \frac{1}{4}(\bar{\psi}_1\gamma_5\psi_4)(\bar{\psi}_3\gamma_5\psi_2)$$

Commutation: $[D_\mu, D_\nu] \sim F_{\mu\nu}$, **EoM:** $\gamma^\mu\partial_\mu\psi = m\psi$, ...

The shelves of the MINIBAR

https://github.com/jruizvid/MINIBAR

Compose invariant terms

applyDerivativesAndPermutations applyDistributed cyclicSort cyclicSortOLD generateIndexOrderings getCombinations indPerm insertIndexPlaceHolders insertIndices insertOneIndexCombination nbrOfIndices partitionSchemesForXElem partitionSchemesForXElemInYGroups

Trivial relations

addSigns assesJPair completeRuleIndices cyclicSortBare deleteDuplicatesSign deleteDuplicatesSignBy deleteZeroesBy flipSigns getDictionaryOfShapes getTraceScore ignoreSigns landRulesOnZero makeIdentifyTerms onSplittedList onSplittedListParallel onTerms onTermsStrict orderFields processSublist processSublistAndSave pullSign rulesToFunction simplifySign sortTracesBare sortTracesSimple termScore

Operator relations

antisymmetricCombinations checkRule cleanTraces Dx findLinearlyDependent findSameOperators firstHead getCayleyHamilton getRelationsFromRule getRuleCayleyHamilton getSchouten4m getSchoutenJ getTotalDerivatives groupTR insertIndicesForTD justOpes realQ relWithIntegers removeDollarSign removeOverallFactor removeOverallFactorReal removePatterns rmOverallReal rotateTraces separateArgsTR sortGroupsCH subsIndices testRelationRankUnder

Symmetry transformations and reordering

getEvenTermsUnder getInverseTransformation getInverseTransformationNaive getTransformationRules getTransformationRulesFromListOrder hasField hasNotHas moveUp nDer nFields nSeqDer nTraces remove reorderOperators reverseTR scorePreferredOperators separateEvenOddUnder signsUnder simpleConjugate symmetrize testTransformation

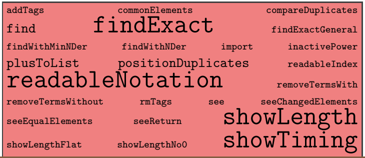
Reduction with Gaussian elimination

additionalOpesFor basesDiff correctDim getMinimalBasis getMinimalBasisAndRelations getMissingOperatorsUsing getRank getRankMathematica getRankMathematicaQuick getRedundantOperatorsSetsUsing getRedundantOperatorsUsing getRelationMatrix inspectRelationMatrix isCompleteAndMinimalWrt removeTags summary testEquivalenceBases toOpeTagsIn toSquare

Helpers to debug

addTags commonElements compareDuplicates find findExact findExactGeneral findWithMinNDer findWithNDer import inactivePower plusToList positionDuplicates readableIndex readableNotation removeTermsWith removeTermsWithout rmTags see seeChangedElements seeEqualElements seeReturn showLength showLengthFlat showLengthNoO showTiming

MINIBAR shelves: **Helpers to debug**



Helpers to debug

- **Notation**

  `DD[field, nDer][indices]`

  - inspect terms with `readableNotation`, `see`
  - find terms easily with `find`, `findExact`, `findWithNDer` …

- Useful to debug **large lists**

  `seeEqualElements`, `seeChangedElements`, `positionDuplicates`, `addTags`, `rmTags`, …

In[91]:=
```
TR[DD[chim,0][],DD[fp,1][j[1],j[1],j[2]],DD[u,0][j[2]]] //readableNotation
```

Out[91]= $\mathrm{TR}[\chi^-, \nabla_{j1}\cdot f^+_{j1,j2}, u_{j2}]$

In[92]:=
```
miniBasisNf2//findExact[u,u,chip]//showLength//showTiming//see;
```

4 terms

0.000551 s

$\mathrm{TR}[\nabla_{j1}\cdot\chi^+, \nabla_{j2}\cdot u_{j2}, u_{j1}] + \mathrm{TR}[\nabla_{j1}\cdot\chi^+, u_{j1}, \nabla_{j2}\cdot u_{j2}]$

» $\mathrm{TR}[\chi^+, \nabla_{j1}\cdot u_{j2}, \nabla_{j1}\cdot u_{j2}]$

$\mathrm{TR}[\chi^+, \nabla_{j1}\cdot u_{j1}, \nabla_{j2}\cdot u_{j2}]$

$i\,(\mathrm{TR}[\chi^+, u_{j1}, u_{j2}, f^+_{j1,j2}] + \mathrm{TR}[\chi^+, f^+_{j1,j2}, u_{j1}, u_{j2}])$

In[93]:=
```
test//showLength;
seeChangedElements[%,%/.{chip→chim}]
```

56 terms

14 changes

#	Position	list1	list2
1	6	$\mathrm{TR}[\nabla_{j1}\cdot\chi^+]^2$	$\mathrm{TR}[\nabla_{j1}\cdot\chi^-]^2$
2	7	$\mathrm{TR}[\chi^+, u_{j1}]^2$	$\mathrm{TR}[\chi^-, u_{j1}]^2$
3	8	$\mathrm{TR}[\chi^+]\times\mathrm{TR}[\chi^+, \chi^+]$	$\mathrm{TR}[\chi^-]\times\mathrm{TR}[\chi^-, \chi^-]$
4	11	$\mathrm{TR}[\chi^+, \nabla_{j1}\cdot u_{j2}, \nabla_{j1}\cdot u_{j2}]$	$\mathrm{TR}[\chi^-, \nabla_{j1}\cdot u_{j2}, \nabla_{j1}\cdot u_{j2}]$
5	12	$\mathrm{TR}[\chi^+, \nabla_{j1}\cdot u_{j1}, \nabla_{j2}\cdot u_{j2}]$	$\mathrm{TR}[\chi^-, \nabla_{j1}\cdot u_{j1}, \nabla_{j2}\cdot u_{j2}]$
6	16	$\mathrm{TR}[\chi^+, u_{j1}, u_{j2}, u_{j1}, u_{j2}]$	$\mathrm{TR}[\chi^-, u_{j1}, u_{j2}, u_{j1}, u_{j2}]$
7	17	$\mathrm{TR}[\chi^+, u_{j1}, u_{j1}, u_{j2}, u_{j2}]$	$\mathrm{TR}[\chi^-, u_{j1}, u_{j1}, u_{j2}, u_{j2}]$
8	18	$\mathrm{TR}[\nabla_{j1}\cdot\chi^+, \nabla_{j1}\cdot\chi^+]$	$\mathrm{TR}[\nabla_{j1}\cdot\chi^-, \nabla_{j1}\cdot\chi^-]$
9	22	$\mathrm{TR}[\chi^+, u_{j1}, \chi^+, u_{j1}]$	$\mathrm{TR}[\chi^-, u_{j1}, \chi^-, u_{j1}]$
10	23	$\mathrm{TR}[\chi^+, \chi^+, u_{j1}, u_{j1}]$	$\mathrm{TR}[\chi^-, \chi^-, u_{j1}, u_{j1}]$
11	26	$\mathrm{TR}[\chi^+, \chi^+, \chi^+]$	$\mathrm{TR}[\chi^-, \chi^-, \chi^-]$
12	27	$\mathrm{TR}[\chi^-, \chi^-, \chi^+]$	$\mathrm{TR}[\chi^-, \chi^-, \chi^-]$
13	54	$\mathrm{TR}[\chi^+, f^+_{j1,j2}, f^+_{j1,j2}]$	$\mathrm{TR}[\chi^-, f^+_{j1,j2}, f^+_{j1,j2}]$
14	55	$\mathrm{TR}[\chi^+, f^-_{j1,j2}, f^-_{j1,j2}]$	$\mathrm{TR}[\chi^-, f^-_{j1,j2}, f^-_{j1,j2}]$

*Screenshots from MINIBAR/TUTORIAL.nb

MINIBAR shelves: **Compose terms**

applyDerivativesAndPermutations
applyDistributed
cyclicSort
cyclicSortOLD
generateIndexOrderings
getCombinations
indPerm
insertIndexPlaceHolders
insertIndices
insertOneIndexCombination
nbrOfIndices
partitionSchemesForXElem
partitionSchemesForXElemInYGroups

Compose invariant terms

- Construct operators with MINIBAR notation
- User must define the **building blocks** and **type of indices**
- Only Lorentz* indices are supported
  - jIndices contracted in pairs
  - mIndices with external Levi-Civita

*Or other. But just one category (currently)

In[54]:=
```
mIndices={};
jIndices={j[1],j[2],j[3]};

BB= { A    in^1   dim^2,
      B    in^2   dim^2,
      der  in^1   dim^1};

comb = getCombinations[BB,{2,3,4}]//Flatten
```

$\{\text{A der dim}^3\ \text{in}^2,\ \ldots 31\text{ terms } \ldots,\ \text{A B}^3\ \text{dim}^8\ \text{in}^7,\ \text{B}^4\ \text{dim}^8\ \text{in}^8\}$

Select combinations with 2 or 4 indices, and dimension 4

In[58]:=
```
comb //Cases[_ in^(2|4)] //Cases[_ dim^(4)];
%/.{in→1,dim→1}//DeleteCases[der^_]
```

Out[59]= $\{A^2,\ B^2,\ \text{B der}^2\}$

Find field orderings, apply derivatives, wrap fields in traces, insert indices

In[60]:=
```
%//Map[applyDerivativesAndPermutations]//introduceTraces;
possibleTerms=%//ins[illegible]xPlaceHolders //insertIndices //see;
```

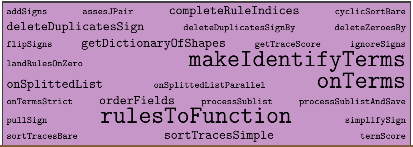
$\text{TR}[A_{j1},\ A_{j1}]$

$\text{TR}[A_{j1}]^2$

» $\text{TR}[B_{j1,j2},\ B_{j1,j2}]$	$\text{TR}[B_{j1,j2},\ B_{j2,j1}]$	$\text{TR}[B_{j1,j1},\ B_{j2,j2}]$
$\text{TR}[B_{j1,j2}]^2$	$\text{TR}[B_{j1,j2}]\times\text{TR}[B_{j2,j1}]$	$\text{TR}[B_{j1,j1}]\times\text{TR}[B_{j2,j2}]$
$\text{TR}[\nabla_{j1}.\nabla_{j2}.B_{j1,j2}]$	$\text{TR}[\nabla_{j1}.\nabla_{j2}.B_{j2,j1}]$	$\text{TR}[\nabla_{j1}.\nabla_{j1}.B_{j2,j2}]$

Initial list of **possibleTerms** obtained

MINIBAR shelves: **Trivial relations**



Trivial relations

- `identifyTerms` with `ope[i]` tags

Identify terms

In[343]:=
```
identifyTerms = makeIdentifyTerms[lag];
```

In[350]:=
```
3*I*lag[[2;;18;;4]];
{%,%//identifyTerms}//Transpose//see;
```

3 i TR[χ^+] × TR[χ^+, χ^+]	3 i ope[2]
3 i TR[$f^+_{j1,j2}$, $f^+_{j1,j3}$, $f^+_{j2,j3}$]	3 i ope[6]
» 3 i TR[χ^-]^2 TR[χ^+]	3 i ope[10]
3 i TR[χ^-] × TR[$f^-_{j1,j2}$, $f^+_{j1,j2}$]	3 i ope[14]
3 i TR[$\nabla_{j1}.\nabla_{j1}.\chi^+$] × TR[$\chi^+$]	3 i ope[18]

- User defines how to `generateAlternativeShapes` through *"trivial"* relations
- Then one can `DeleteDuplicates` and `getDictionaryOfShapes`

Apply trivial relations

In[357]:=
```
possibleTerms//showLength;
basis = possibleTerms//Map[generateAlternativeShapes] //Map[Sort] //
      Map[First] //DeleteDuplicates //showLength;
```

4490 terms

1445 terms

Prepare to reshape terms

In[359]:=
```
rulesTrivial = basis//Map[generateAlternativeShapes]//Map[getDictionaryOfShapes];
%[[9]]//see;
```

TR[χ^+, $f^+_{j1,j1}$, $f^+_{j2,j2}$] → TR[χ^+, $f^+_{j1,j1}$, $f^+_{j2,j2}$]
TR[χ^+, $f^+_{j1,j1}$, $f^+_{j2,j2}$] → TR[χ^+, $f^+_{j1,j1}$, $f^+_{j2,j2}$]
TR[χ^+, $f^+_{j2,j2}$, $f^+_{j1,j1}$] → TR[χ^+, $f^+_{j1,j1}$, $f^+_{j2,j2}$]
» TR[$f^+_{j1,j1}$, $f^+_{j2,j2}$, χ^+] → TR[χ^+, $f^+_{j1,j1}$, $f^+_{j2,j2}$]
TR[$f^+_{j2,j2}$, $f^+_{j1,j1}$, χ^+] → TR[χ^+, $f^+_{j1,j1}$, $f^+_{j2,j2}$]
TR[$f^+_{j2,j2}$, χ^+, $f^+_{j1,j1}$] → TR[χ^+, $f^+_{j1,j1}$, $f^+_{j2,j2}$]
TR[$f^+_{j1,j1}$, χ^+, $f^+_{j2,j2}$] → TR[χ^+, $f^+_{j1,j1}$, $f^+_{j2,j2}$]

- Substitution accelerated 50x using **hash tables** with `rulesToFunction`

In[362]:=
```
rulesTrivialF = rulesTrivial //showLength //rulesToFunction;
```

30 843 terms

In[363]:=
```
possibleTerms/.rulesTrivial//showTiming;
possibleTerms//rulesTrivialF//showTiming;
```

2.00479 s

0.038142 s

MINIBAR shelves: **Operator relations**

antisymmetricCombinations checkRule cleanTraces Dx findLinearlyDependent findSameOperators firstHead getCayleyHamilton getRelationsFromRule getRuleCayleyHamilton getSchouten4m getSchoutenJ getTotalDerivatives groupTR insertIndicesForTD justOpes realQ relWithIntegers removeDollarSign removeOverallFactor removeOverallFactorReal removePatterns rmOverallReal rotateTraces separateArgsTR sortGroupsCH subsIndices testRelationRankUnder

Operator relations

- **Common relations**: Cayley-Hamilton, Schouten, total derivatives.

Cayley-Hamilton trace relations for NxN matrices (N = 2, … , 27)

In[36]:=
```
TR[A,A,B,B]//getCayleyHamilton[2]
```

3 relations found

$\{\mathrm{TR}[A]^2\,\mathrm{TR}[B,B] - \mathrm{TR}[A,A]\times\mathrm{TR}[B,B] - 2\,\mathrm{TR}[A]\times\mathrm{TR}[A,B,B] + 2\,\mathrm{TR}[A,A,B,B],\ \ldots\}$

Schouten relation for D+1 antisymmetric indices

In[34]:=
```
test2//see//getSchoutenJ;
%[[1]]//see;
```

» $\mathrm{TR}[A_{j1,j2}, B_{j1,j2}]\times\mathrm{TR}[B_{j3,j4}, B_{j3,j4}, C_{j5,j5}]$

32 relations found

» $\mathrm{TR}[A_{j4,j5}, B_{j1,j2}]\times\mathrm{TR}[B_{j1,j2}, B_{j3,j4}, C_{j3,j5}] - \mathrm{TR}[A_{j5,j4}, B_{j1,j2}]\times\mathrm{TR}[B_{j1,j2}, B_{j3,j4}, C_{j3,j5}] - \mathrm{TR}[A_{j3,j5}, B_{j1,j2}]\times\mathrm{TR}[B_{j1,j2}, B_{j3,j4}, C_{j4,j5}] + \mathrm{TR}[A_{j5,j3}, B_{j1,j2}]\times\mathrm{TR}[B_{j1,j2}, B_{j3,j4}, C_{j4,j5}] + \ldots$ 120 terms

- **Specific identities** are defined in a Rule, used by getRelationsFromRule. It allows **flexible notation**

User-defined relations

In[39]:=
```
rule1 = DD[B,n_?(#≥2&)][x_,y_,ind__] :>
        DD[B,n][x,y,ind] + DD[B,n][y,x,ind];
% // checkRule;
```

» $\nabla_x\cdot\nabla_y\cdot B_{ind} \to \nabla_x\cdot\nabla_y\cdot B_{ind} + \nabla_y\cdot\nabla_x\cdot B_{ind}$

In[41]:=
```
rel1 = getRelationsFromRule[possibleTerms,rule1]//see;
```

3 relations found

$\mathrm{TR}[\nabla_{j1}\cdot\nabla_{j2}\cdot B_{j1,j2}] + \mathrm{TR}[\nabla_{j2}\cdot\nabla_{j1}\cdot B_{j1,j2}]$

» $\mathrm{TR}[\nabla_{j1}\cdot\nabla_{j2}\cdot B_{j2,j1}] + \mathrm{TR}[\nabla_{j2}\cdot\nabla_{j1}\cdot B_{j2,j1}]$

$2\,\mathrm{TR}[\nabla_{j1}\cdot\nabla_{j1}\cdot B_{j2,j2}]$

MINIBAR shelves: **Symmetries and reordering**

getEvenTermsUnder getInverseTransformation getInverseTransformationNaive
getTransformationRules getTransformationRulesFromListOrder hasField hasNotHas moveUp nDer
nFields nSeqDer nTraces remove reorderOperators reverseTR scorePreferredOperators
separateEvenOddUnder signsUnder simpleConjugate symmetrize testTransformation

Symmetry transformations and reordering

- Find new **symmetric basis** with symmetrize

In[49]:=
```
Clear[H,P]
H[expr_]:=expr /.DD[B,n_][ind__]:>-DD[B,n][ind] //cleanTraces//reshapeTerms;
P[expr_]:=expr /.DD[B,n_][ind___,a_,b_]:>(-1)^n DD[B,n][ind,b,a] //cleanTraces,
```

In[52]:=
```
symbasis = possibleTerms // symmetrize[H,P];
{%,%//identifyTerms}//Transpose//see;
```

$TR[A_{j1}, A_{j1}]$	ope[1]
$TR[A_{j1}]^2$	ope[2]
$TR[B_{j1,j2}, B_{j1,j2}]$	ope[3]
$TR[B_{j1,j2}, B_{j2,j1}]$	ope[4]
$TR[B_{j1,j1}, B_{j2,j2}]$	ope[5]
$TR[B_{j1,j2}]^2$	ope[6]
$TR[B_{j1,j2}] \times TR[B_{j2,j1}]$	ope[7]
$TR[B_{j1,j1}] \times TR[B_{j2,j2}]$	ope[8]
$i\,(TR[\nabla_{j1}.\nabla_{j2}.B_{j1,j2}] + TR[\nabla_{j1}.\nabla_{j2}.B_{j2,j1}])$	i ope[9] + i ope[10]
$i\,TR[\nabla_{j1}.\nabla_{j1}.B_{j2,j2}]$	i ope[11]
$i\,(TR[\nabla_{j1}.\nabla_{j2}.B_{j1,j2}] - TR[\nabla_{j1}.\nabla_{j2}.B_{j2,j1}])$	i ope[9] − i ope[10]

- scorePreferredOperators to determine which should be kept in the final basis.

In[92]:=
```
Clear[scorePreferredOperators]
scorePreferredOperators[term_]:=
hasNotHas[term,A,B]  * 100 +
nFields[term,B]      * 10  +
nDer[term]           * 1   +
nTraces[term]        * 0.1

prefbasis = basis//reorderOperators[scorePreferredOperators] //identifyTerms;
```

- Change bases:
  raw ↔ symmetric ↔ preferred

In[85]:=
```
symToRawOpes = symbasis //identifyTerms //getTransformationRules;
rawToSymOpes = symToRawOpes //getInverseTransformation;
```

In[95]:=
```
symToPrefOpes = prefbasis //getTransformationRulesFromListOrder;
prefToSymOpes = symToPrefOpes //getInverseTransformationNaive;
```

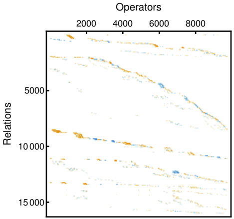
MINIBAR shelves: **Final reduction**

additionalOpesFor basesDiff correctDim
getMinimalBasis
getMinimalBasisAndRelations getMissingOperatorsUsing
getRank getRankMathematica
getRankMathematicaQuick getRedundantOperatorsSetsUsing
getRedundantOperatorsUsing
getRelationMatrix inspectRelationMatrix
isCompleteAndMinimalWrt removeTags summary
testEquivalenceBases toOpeTagsIn
toSquare

Reduction with Gaussian elimination

- Put relations into a `SparseArray` with `getRelationMatrix`

In[67]:=
```
{rel1,rel2} // Flatten // reshapeTerms // identifyTerms
relMatrix = % // rawToSymOpesF // symToPrefOpesF // getRelationMatrix
```



Relation matrix of ChPT Odd IP Lagrangian

10k variables,
16k equations

[Factorization with SuiteSparseQR]

- **Interface to C++** to solve the system
  - `getRank`: nbr of indep. operators
  - `getMinimalBasis`: basis

In[69]:=
```
relMatrix //correctDim[Length[basis]] // getMinimalBasis
```

```
Factorising ...

CHOLMOD sparse:  R:  11-by-11, nz 4, up/lo.  OK
CHOLMOD sparse:  D:  11-by-11, nz 3, up/lo.  OK
Rank:          3

Minimal basis obtained:  8 operators altogether
```

Out[69]= {ope[1], ope[3], ope[4], ope[5], ope[6], ope[7], ope[8], ope[9]}

These relations were found (=0):

$TR[\nabla_{j1}.\nabla_{j2}.B_{j1,j2}] + TR[\nabla_{j2}.\nabla_{j1}.B_{j1,j2}]$
$TR[\nabla_{j1}.\nabla_{j2}.B_{j2,j1}] + TR[\nabla_{j2}.\nabla_{j1}.B_{j2,j1}]$
$2\,TR[\nabla_{j1}.\nabla_{j1}.B_{j2,j2}]$
$-TR[A_{j1}, A_{j1}] + TR[B_{j1,j2}, B_{j2,j1}]$

Minimal and symmetric basis:

$TR[A_{j1}]^2$
$TR[B_{j1,j1}] \times TR[B_{j2,j2}]$
$TR[B_{j1,j2}] \times TR[B_{j2,j1}]$
$TR[B_{j1,j2}]^2$
$TR[B_{j1,j2}, B_{j2,j1}]$
$TR[B_{j1,j2}, B_{j1,j2}]$
$TR[B_{j1,j1}, B_{j2,j2}]$
$i\,TR[\nabla_{j1}.\nabla_{j2}.B_{j1,j2}] - i\,TR[\nabla_{j1}.\nabla_{j2}.B_{j2,j1}]$

- Slow alternatives: `getRankMathematica`, `getMinimalBasisMathematica`

Stocked shelves of MINIBAR

applyDerivativesAndPermutations applyDistributed cyclicSort cyclicSortOLD generateIndexOrderings getCombinations indPerm insertIndexPlaceHolders insertIndices insertOneIndexCombination nbrOfIndices partitionSchemesForXElem partitionSchemesForXElemInYGroups

Compose invariant terms

addSigns assesJPair completeRuleIndices cyclicSortBare deleteDuplicatesSign deleteDuplicatesSignBy deleteZeroesBy flipSigns getDictionaryOfShapes getTraceScore ignoreSigns landRulesOnZero makeIdentifyTerms onSplittedList onSplittedListParallel onTerms onTermsStrict orderFields processSublist processSublistAndSave pullSign rulesToFunction simplifySign sortTracesBare sortTracesSimple termScore

Trivial relations

antisymmetricCombinations checkRule cleanTraces Dx findLinearlyDependent findSameOperators firstHead getCayleyHamilton getRelationsFromRule getRuleCayleyHamilton getSchouten4m getSchoutenJ getTotalDerivatives groupTR insertIndicesForTD justOpes realQ relWithIntegers removeDollarSign removeOverallFactor removeOverallFactorReal removePatterns rmOverallReal rotateTraces separateArgsTR sortGroupsCH subsIndices testRelationRankUnder

Operator relations

getEvenTermsUnder getInverseTransformation getInverseTransformationNaive getTransformationRules getTransformationRulesFromListOrder hasField hasNotHas moveUp nDer nFields nSeqDer nTraces remove reorderOperators reverseTR scorePreferredOperators separateEvenOddUnder signsUnder simpleConjugate symmetrize testTransformation

Symmetry transformations and reordering

additionalOpesFor basesDiff correctDim getMinimalBasis getMinimalBasisAndRelations getMissingOperatorsUsing getRank getRankMathematica getRankMathematicaQuick getRedundantOperatorsSetsUsing getRedundantOperatorsUsing getRelationMatrix inspectRelationMatrix isCompleteAndMinimalWrt removeTags summary testEquivalenceBases toOpeTagsIn toSquare

Reduction with Gaussian elimination

addTags commonElements compareDuplicates find findExact findExactGeneral findWithMinNDer findWithNDer import inactivePower plusToList positionDuplicates readableIndex readableNotation removeTermsWith removeTermsWithout rmTags see seeChangedElements seeEqualElements seeReturn showLength showLengthFlat showLengthNoO showTiming

Helpers to debug

- 136 functions, all documented
- Time- and memory-efficient algorithms
- Flexible notation

- MINIBAR v0.6
- Mathematica v13.0
- SuiteSparse v7.0.1

Conclusions

- Software MINIBAR to construct Effective Lagrangians
  + flexible notation
  + tools for large analytical calculations in Mathematica

  https://github.com/jruizvid/MINIBAR

- Shown Lagrangians
  - **Anomalous ChPT** at $O(p^8)$ $\mathcal{L}^{\mathbf{ano}}_{\mathbf{p^8}}$
  - **ChPT** with **gravity** at $O(p^6)$ $\mathcal{L}_{\mathrm{ChPT+Gravity}}$
  - **Standard Model Effective Field Theory** [WIP]

- Other examples already finished or coming

- Manual in preparation

Thank you for the attention!

Backup

Power counting and low energy constants

Matrix	Definition	Transformation
U	$\exp\left(i\frac{\Phi(x)}{f}\right)$	$g_R U g_L^\dagger$
χ	$2B_0(s+ip)$	$g_R \chi g_L^\dagger$
$F^L_{\alpha\beta}$	$\partial_\alpha \ell_\beta - \partial_\beta \ell_\alpha - i[\ell_\alpha, \ell_\beta]$	$g_L F^L_{\alpha\beta} g_L^\dagger$
$F^R_{\alpha\beta}$	$\partial_\alpha r_\beta - \partial_\beta r_\alpha - i[r_\alpha, r_\beta]$	$g_R F^R_{\alpha\beta} g_R^\dagger$

Power counting

- The expansion parameter for this effective theory is the **momentum transfer p** of the process (divided by an hadronic scale, typically m_π)
- Chiral + Lorentz symmetry in Goldstones → even powers $\mathbf{p^{2n}}$

→ $\mathcal{O}(p^2)$

$$\mathcal{L}_2 = \frac{f^2}{4}\langle D_\mu U^\dagger D^\mu U \, + \, U^\dagger \chi \, + \, \chi^\dagger U \rangle$$

→ $\mathcal{O}(p^4)$ [p^4: Gasser, Leutwyler; Annals Phys. 158 (1984) 142]

$$\begin{aligned}\mathcal{L}_4 \quad &= L_1 \langle D_\mu U^\dagger D^\mu U\rangle^2 \, + \, L_2 \langle D_\mu U^\dagger D_\nu U\rangle \langle D^\mu U^\dagger D^\nu U\rangle + L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle \\ &+ L_4 \langle D_\mu U^\dagger D^\mu U\rangle \langle U^\dagger \chi + \chi^\dagger U\rangle + L_5 \langle D_\mu U^\dagger D^\mu U \left(U^\dagger \chi + \chi^\dagger U\right)\rangle \, + \, L_6 \langle U^\dagger \chi + \chi^\dagger U\rangle^2 \\ &+ L_7 \langle U^\dagger \chi - \chi^\dagger U\rangle^2 \, + \, L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle - i L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger \, + \, F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \\ &+ L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle + H_1 \langle F_{R\mu\nu} F_R^{\mu\nu} \, + \, F_{L\mu\nu} F_L^{\mu\nu}\rangle \, + \, H_2 \langle \chi^\dagger \chi\rangle \, .\end{aligned} \tag{1}$$

Higher-order Lagrangians

Motivations

- Broaden knowledge of ChPT; more accurate predictions *eventually*
- Explore mathematical structure of scattering **amplitudes**

  [Kampf et al, 1611.03137] [Kampf, 2109.11574]
- Comparison to other theoretical techniques:
  - **Hilbert series**: extracts the dimensions of the operator space for different irreducible representations. Based only on symmetries [Bijnens et al, 2212.07901] [Graf et al, 2009.01239]
  - **Adler zero** self-consistency condition on the amplitudes + group-theory considerations + zero Gram determinant = constraints to find minimal basis [Low et al, 2209.00198]

Calculations

$\rightarrow \mathcal{O}(p^6)$ [Scherer et al, hep-ph/9408346] [Bijnens et al, hep-ph/9902437]

- Growing number of terms (and LECs): 115 / 94 / 56 for $N_F = N_F$ / 3 / 2
- Calculated mostly by hand

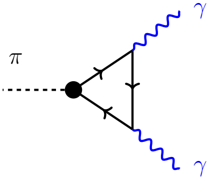
$\rightarrow \mathcal{O}(p^8)$ [Bijnens, Hermansson-Truedsson, Wang, 1810.06834]

- Unmanageable by hand: 1862 / 1254 / 475
- **Automated analytical calculation** with FORM [FORM, math-ph/0010025]

$\Rightarrow$ Another part of the interaction Lagrangian: **anomalous sector** $\Leftarrow$

Chiral anomaly and intrinsic parity

- Chiral symmetry is broken in QCD
  - *Explicitly*: through quark mass term
  - *Quantum effects*: **axial anomaly** in triangle diagrams



- Anomaly reproduced in the ChPT Lagrangian in vertices with odd number of pseudoscalar mesons or axial vectors $\rightarrow$ **odd intrinsic parity (IP)**

Intrinsic Parity (IP)

Transforms the fields, not the space coordinates

$$Y^{\mu\nu} \xrightarrow{\mathcal{P}} \delta(\mu)\delta(\nu) Y_{\mathcal{P}}^{\mu\nu}$$

$$Y^{\mu\nu} \xrightarrow{IP} Y_{\mathcal{P}}^{\mu\nu}$$

$$\delta(0) = 1,\ \delta(1,2,3) = -1$$

- Even IP: $X_{\mathcal{P}} = X$

$$X^{\mu\nu}_{\mu\nu} \xrightarrow{\mathcal{P}} \underbrace{\delta(\mu)^2\delta(\nu)^2}_{1} X_{\mathcal{P}}^{\mu\nu}{}_{\mu\nu} = X_{\mathcal{P}}^{\mu\nu}{}_{\mu\nu}$$

- Odd IP: $X_{\mathcal{P}} = -X$

$$X^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} \xrightarrow{\mathcal{P}} \underbrace{\delta(\mu)\delta(\nu)\delta(\rho)\delta(\sigma)}_{-1} X_{\mathcal{P}}^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma}$$

Anomalous Lagrangian

[Wess, Zumino, Phys.Lett.B 37 (1971) 95-97]

→ Anomalous $\mathcal{O}(p^4)$ [Witten, Nucl.Phys.B 223 (1983) 422-432]

- In ChPT, the structure of the anomaly is encoded in the WZW action of $\mathcal{O}(p^4)$.
- No new parameters. No hadronization problems. Highly predictive

break chiral symmetry **odd intrinsic parity**

→ Anomalous $\mathcal{O}(p^6)$ [Scherer et al, hep-ph/0110261] [Bijnens et al, hep-ph/0110400]

- The EFT must accomodate any possible interaction of the fundamental Lagrangian
- Systematic analytical computation J. Weber; MSc thesis, Mainz 2008

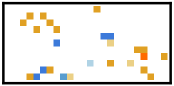
chirally symmetric **odd intrinsic parity**

→ Anomalous $\mathcal{O}(p^8)$ [Bijnens, Hermansson-Truedsson, JRV, 2310.20547] **[Today]**

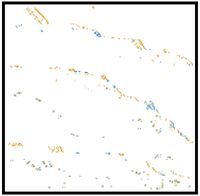
- Developed independently with FORM and MATHEMATICA

chirally symmetric **odd intrinsic parity**

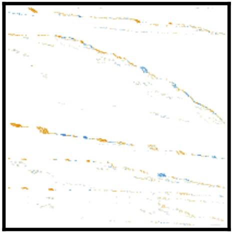
Example calculations with `MINIBAR`



Even IP $\mathcal{O}(p^4)$
13 terms – $<$1s

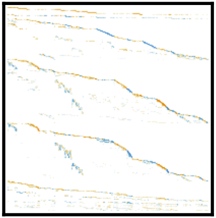


Even IP $\mathcal{O}(p^6)$
115 terms – 2s



Even IP $\mathcal{O}(p^8)$
1862 terms – 15min

Odd IP $\mathcal{O}(p^6)$
24 terms – 1s



Odd IP $\mathcal{O}(p^8)$
999 terms – 12min

Computation of form factors

1. Lagrangian `MINIBAR`

$$\mathcal{L}_{\text{ChPT+Gravity}}$$

2. Energy momentum tensor `xAct`

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left(\sqrt{-g}\ \mathcal{L} \right)$$

3. Form factors FEYNCALC [WIP]

$$\langle \pi | T^{\mu\nu} | \pi \rangle = 2P^{\mu}P^{\nu}\mathbf{A} + \frac{1}{2}(Q^{\mu}Q^{\nu} - g^{\mu\nu}Q^2)\mathbf{D}$$

$$\text{where } P_\mu = p_\mu + p'_\mu, \;\; Q_\mu = p_\mu - p'_\mu$$