QCD at 5 Loops & the R^* operation

Franz Herzog

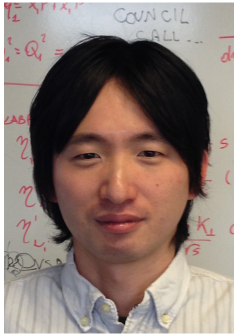
VU UNIVERSITY AMSTERDAM

Nikhef

The HEPGAME Team

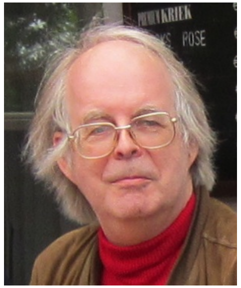
Giulio Falcioni

Ben Rujil



Takahiro Ueda

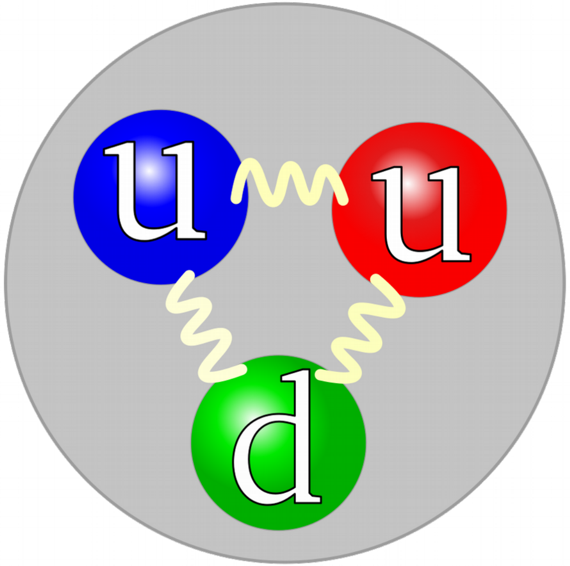
Franz Herzog



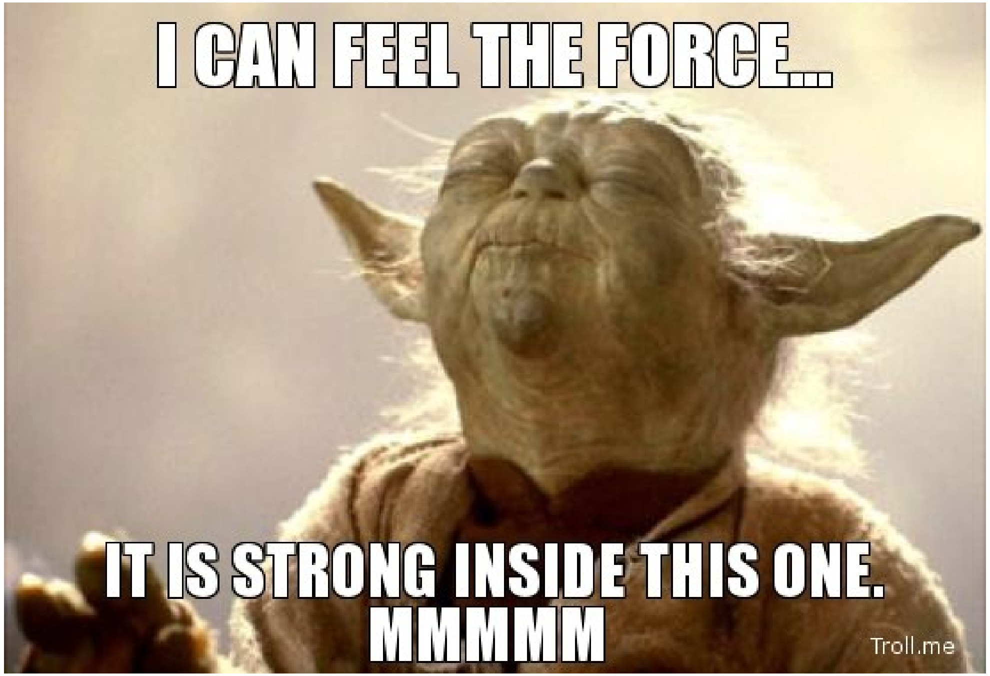
Jos Vermaseren

Andreas Vogt

QCD: The strongest force in the universe

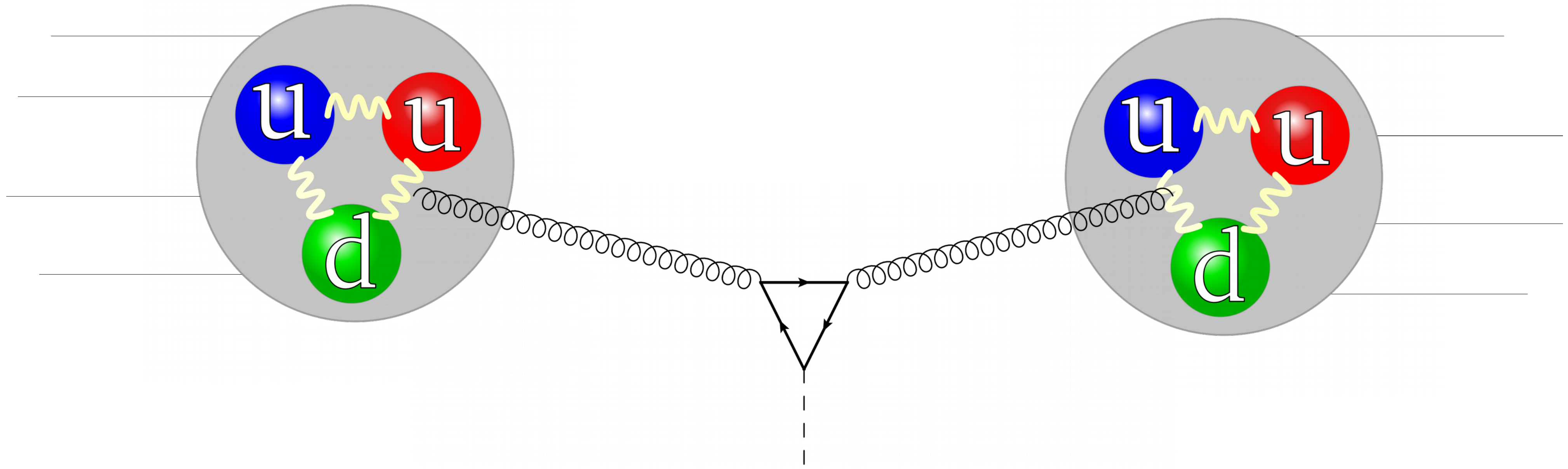


Gluons bind quarks into protons



Proton collisions

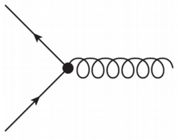
QCD is **CRUCIAL** for collider physics

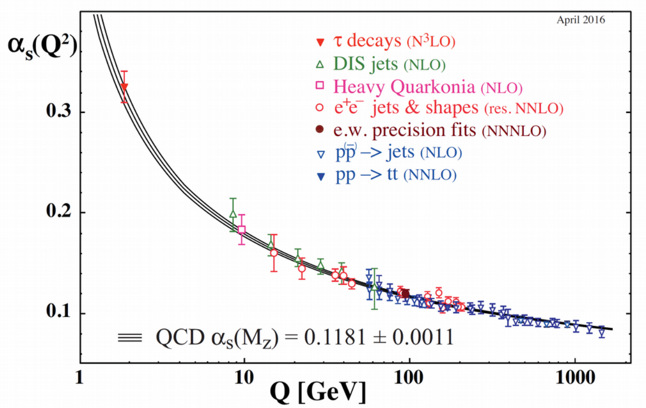


Higgs boson

QCD is asymptotically **free**

- The strong force becomes **weak** at high energy

$\alpha_s(\mu) \sim$  < 1



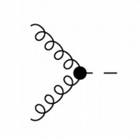
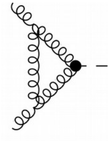
The Nobel Prize in Physics 2004

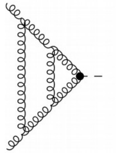
"for the discovery of asymptotic freedom in the theory of the strong interaction"

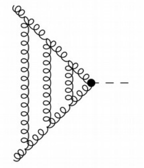
David J. Gross　H. David Politzer　Frank Wilczek

QCD is asymptotically **free**

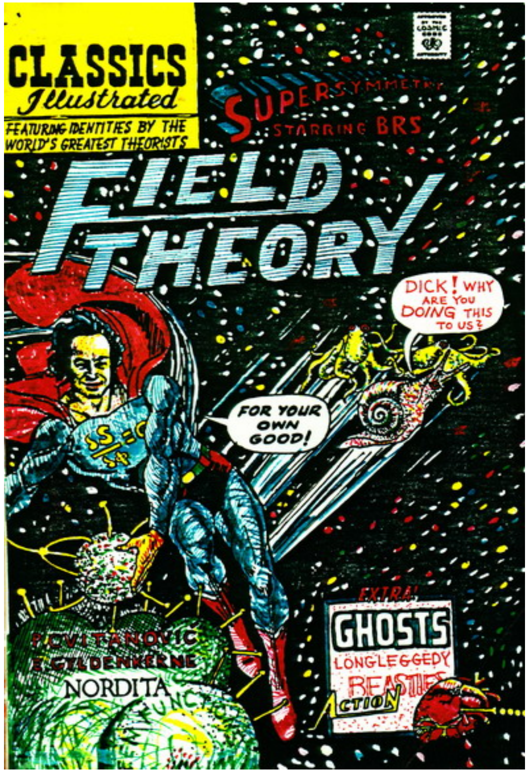
- weak coupling → **Feynman Diagrams**

$$\mathcal{A}_{gg\to H} =$$  $$+ \alpha_s \Big($$  $$+ \ldots \Big)$$

$$+ \alpha_S^2 \Big($$  $$+ \ldots \Big)$$

$$+ \alpha_S^3 \Big($$  $$+ \ldots \Big)$$

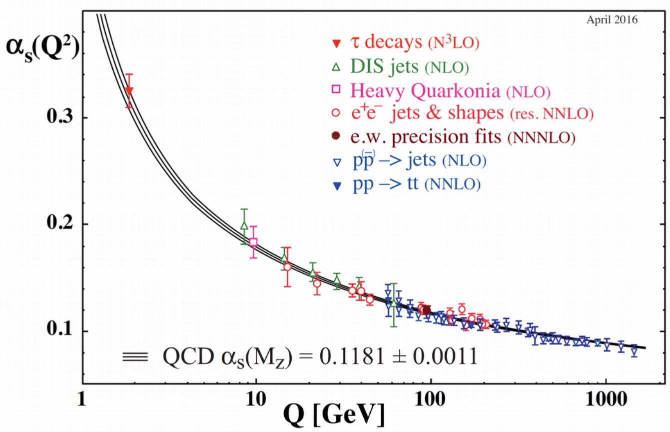
$$+ \mathcal{O}(\alpha_S^4)$$



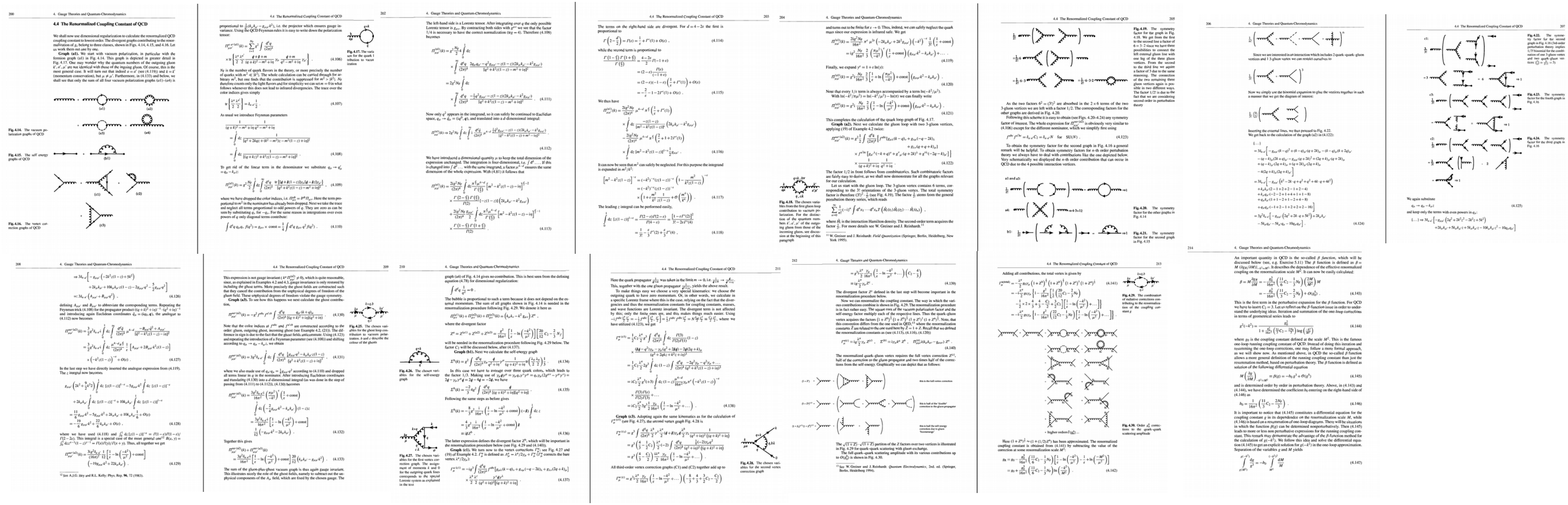
Q: How did they prove asymptotic freedom?

A: They computed a sign!

$$\beta(Q) = Q^2 \frac{d\alpha_s(Q)}{dQ^2} = -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 - \beta_2 \alpha_s^4 - \beta_3 \alpha_s^5 - \beta_4 \alpha_s^6 - ..$$



The 1-loop beta function is now textbook material



15 pages in book *Quantum Chromo Dynamics* by Greiner, Schramm and Stein

Complexity at higher loops

# Loops	1	2	3	4	5
# Diagrams	7	66	1 219	29 310	831 942
# terms of worst diagrams	9	81	729	6 561	59 049
Dimension of integrals	2	5	8	11	14

History of the QCD beta function

- β_0:
  - 1965 [Vanyashin, Terentev]; 1970 [Khriplovich];1972 [T'Hooft];
  - 1973 [Gross, Wilczek; Politzer]

History of the QCD beta function

- β_0:
  - 1965 [Vanyashin, Terentev]; 1970 [Khriplovich];1972 [T'Hooft];
  - 1973 [Gross, Wilczek; Politzer]
- β_1:
  - 1974 [Caswell; Jones]
  - 1979 [Egorian, Tarasov]

History of the QCD beta function

- β_0:
  - 1965 [Vanyashin, Terentev]; 1970 [Khriplovich];1972 [T'Hooft];
  - 1973 [Gross, Wilczek; Politzer]
- β_1:
  - 1974 [Caswell; Jones]
  - 1979 [Egorian, Tarasov]
- β_2:
  - 1980 [Tarasov, Vladimirov, Zharkov]
  - 1993 [Larin, Vermaseren]

History of the QCD beta function

- β_0:
  - 1965 [Vanyashin, Terentev]; 1970 [Khriplovich];1972 [T'Hooft];
  - 1973 [Gross, Wilczek; Politzer]
- β_1:
  - 1974 [Caswell; Jones]
  - 1979 [Egorian, Tarasov]
- β_2:
  - 1980 [Tarasov, Vladimirov, Zharkov]
  - 1993 [Larin, Vermaseren]
- β_3:
  - 1997 [van Ritbergen, Vermaseren, Larin]
  - 2005 [Czakon]

History of the QCD beta function

- β_0:
  - 1965 [Vanyashin, Terentev]; 1970 [Khriplovich];1972 [T'Hooft];
  - 1973 [Gross, Wilczek; Politzer]
- β_1:
  - 1974 [Caswell; Jones]
  - 1979 [Egorian, Tarasov]
- β_2:
  - 1980 [Tarasov, Vladimirov, Zharkov]
  - 1993 [Larin, Vermaseren]
- β_3:
  - 1997 [van Ritbergen, Vermaseren, Larin]
  - 2005 [Czakon]
- β_4:
  - 2016 [Baikov, Chetyrkin, Kühn]
  - 2017 [FH, Ruijl, Ueda, Vermaseren, Vogt]

History of the QCD beta function

- β_0:
  - 1965 [Vanyashin, Terentev]; 1970 [Khriplovich];1972 [T'Hooft];
  - 1973 [Gross, Wilczek; Politzer]
- β_1:
  - 1974 [Caswell; Jones]
  - 1979 [Egorian, Tarasov]
- β_2:
  - 1980 [Tarasov, Vladimirov, Zharkov]
  - 1993 [Larin, Vermaseren]
- β_3:
  - 1997 [van Ritbergen, Vermaseren, Larin]
  - 2005 [Czakon]
- β_4:
  - 2016 [Baikov, Chetyrkin, Kühn]
  - 2017 [FH, Ruijl, Ueda, Vermaseren, Vogt]

1 year

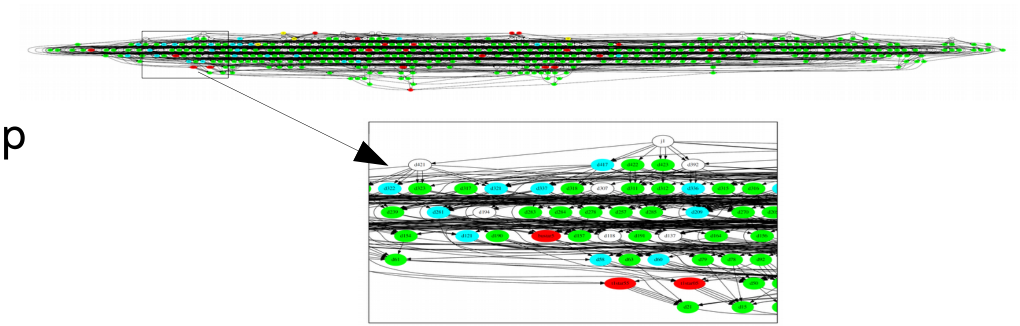
4 days

Computing time

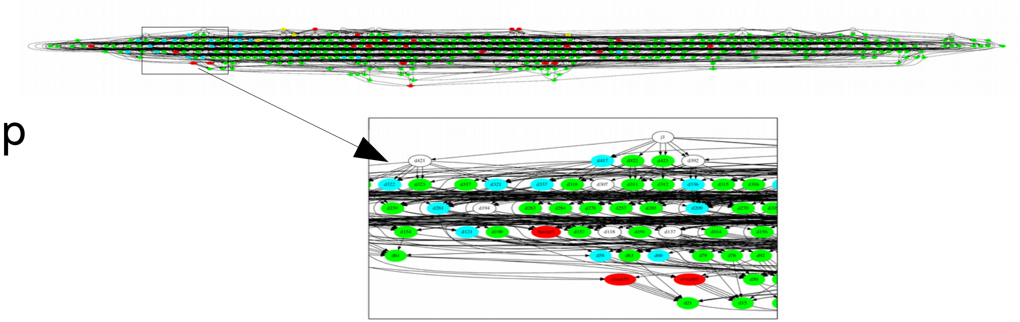
Calculation

Calculation

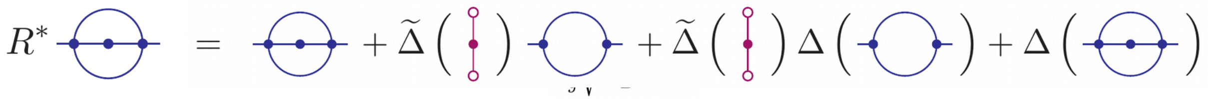
- Forcer arXiv:1704.06650
  - Parameteric solution of IBPs for up to 4-loop massless self energy graphs

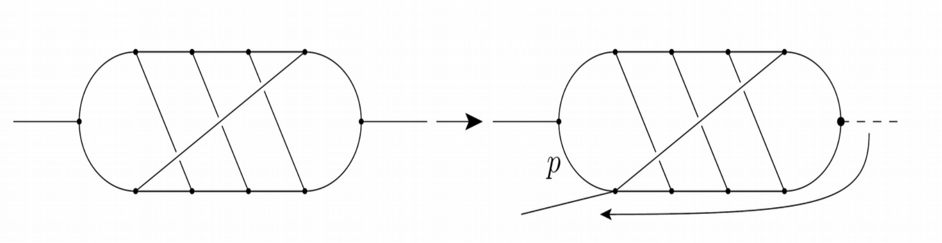


Calculation



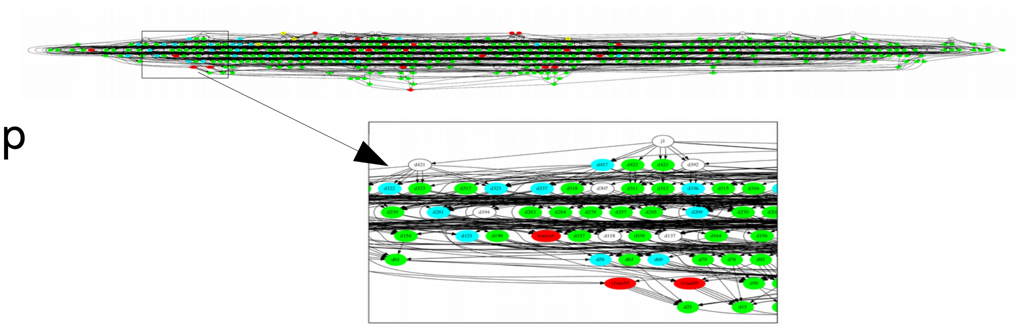
- Forcer arXiv:1704.06650
  - Parameteric solution of IBPs for up to 4-loop massless self energy graphs
- Local R*-operation arXiv:1703.03776
  - fully automated in Form (& Maple for tests)
  - Allows extraction of poles of L loop euclidean integrals from L-1 loops via infrared rearrangement:



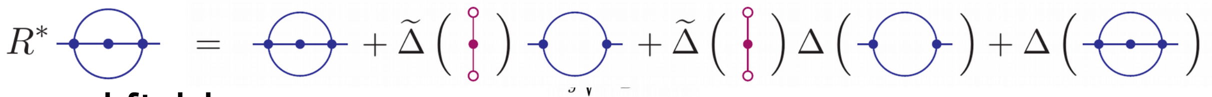


Calculation

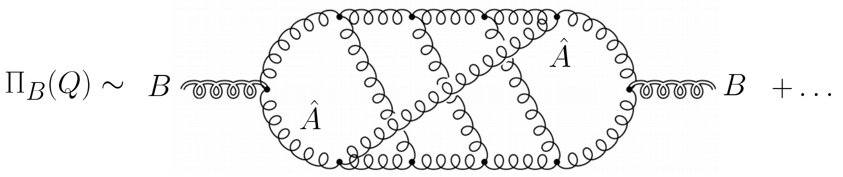
- Forcer arXiv:1704.06650
  - Parameteric solution of IBPs for up to 4-loop massless self energy graphs



- Local R*-operation arXiv:1703.03776
  - fully automated in Form (& Maple for tests)
  - Allows extraction of poles of L loop euclidean integrals from L-1 loops via infrared rearrangement:



- Background field gauge:
  - Extract beta from background field self energy



$$Z_g \sqrt{Z_B} = 1$$

General gauge group

Besides C_A, C_F we only need the symmetric group invariant tensors

T^a are the generators of the fermionic representation.

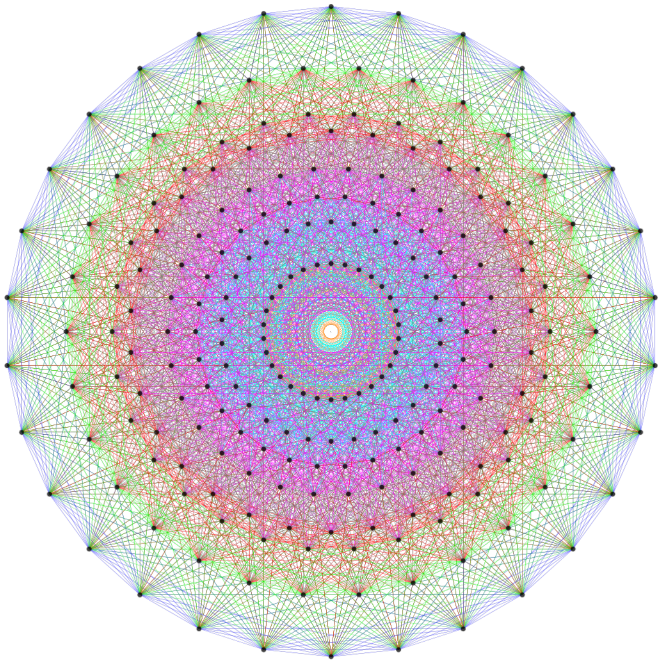
C^a are the generators of the adoint representation.

$$d_F^{abcd} = \frac{1}{6} \mathrm{Tr}(T^a T^b T^c T^d + \text{ five } bcd \text{ permutations })$$
$$d_A^{abcd} = \frac{1}{6} \mathrm{Tr}(C^a C^b C^c C^d + \text{ five } bcd \text{ permutations })$$

UM-TH-98-01
NIKHEF-98-004

Group theory factors for Feynman diagrams

T. van Ritbergen[a], A.N. Schellekens[b], J.A.M. Vermaseren[b]



A projection of the 8 dimensional 4_{21} polytope invariant under the the E_8 Lie group

Old Results

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f ,$$

Old Results

$$\beta_0 = \frac{11}{3}\,C_A - \frac{4}{3}\,T_F\,n_f \ ,$$

$$\beta_1 = \frac{34}{3}\,C_A^2 - \frac{20}{3}\,C_A\,T_F\,n_f - 4\,C_F\,T_F\,n_f \ ,$$

Old Results

$$\beta_0 = \frac{11}{3}\,C_A - \frac{4}{3}\,T_F\,n_f \ ,$$

$$\beta_1 = \frac{34}{3}\,C_A^2 - \frac{20}{3}\,C_A\,T_F\,n_f - 4\,C_F\,T_F\,n_f \ ,$$

$$\beta_2 = \frac{2857}{54}\,C_A^3 - \frac{1415}{27}\,C_A^2\,T_F\,n_f - \frac{205}{9}\,C_F\,C_A\,T_F\,n_f + 2\,C_F^2\,T_F\,n_f + \frac{44}{9}\,C_F\,T_F^2\,n_f^2 + \frac{158}{27}\,C_A\,T_F^2\,n_f^2 \ ,$$

Old Results

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f ,$$

$$\beta_1 = \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_F n_f - 4 C_F T_F n_f ,$$

$$\beta_2 = \frac{2857}{54} C_A^3 - \frac{1415}{27} C_A^2 T_F n_f - \frac{205}{9} C_F C_A T_F n_f + 2 C_F^2 T_F n_f + \frac{44}{9} C_F T_F^2 n_f^2 + \frac{158}{27} C_A T_F^2 n_f^2 ,$$

$$\begin{aligned}\beta_3 = {} & C_A^4 \left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right) + \frac{d_A^{abcd} d_A^{abcd}}{N_A}\left(-\frac{80}{9} + \frac{704}{3}\zeta_3\right) \\ & + C_A^3 T_F n_f \left(-\frac{39143}{81} + \frac{136}{3}\zeta_3\right) + C_A^2 C_F T_F n_f \left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right) \\ & + C_A C_F^2 T_F n_f \left(-\frac{4204}{27} + \frac{352}{9}\zeta_3\right) + \frac{d_F^{abcd} d_A^{abcd}}{N_A} n_f \left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right) \\ & + 46 C_F^3 T_F n_f + C_A^2 T_F^2 n_f^2 \left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right) + C_F^2 T_F^2 n_f^2 \left(\frac{1352}{27} - \frac{704}{9}\zeta_3\right) \\ & + C_A C_F T_F^2 n_f^2 \left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right) + \frac{d_F^{abcd} d_F^{abcd}}{N_A} n_f^2 \left(-\frac{704}{9} + \frac{512}{3}\zeta_3\right) \\ & + \frac{424}{243} C_A T_F^3 n_f^3 + \frac{1232}{243} C_F T_F^3 n_f^3 ,\end{aligned}$$

New Result

$$
\begin{aligned}
\beta_4 \;=\;& C_A^5\left(\frac{8296235}{3888}-\frac{1630}{81}\zeta_3+\frac{121}{6}\zeta_4-\frac{1045}{9}\zeta_5\right)\\
&+\frac{d_A^{abcd}d_A^{abcd}}{N_A}C_A\left(-\frac{514}{3}+\frac{18716}{3}\zeta_3-968\zeta_4-\frac{15400}{3}\zeta_5\right)\\
&+C_A^4T_Fn_f\left(-\frac{5048959}{972}+\frac{10505}{81}\zeta_3-\frac{583}{3}\zeta_4+1230\zeta_5\right)\\
&+C_A^3C_FT_Fn_f\left(\frac{8141995}{1944}+146\zeta_3+\frac{902}{3}\zeta_4-\frac{8720}{3}\zeta_5\right)\\
&+C_A^2C_F^2T_Fn_f\left(-\frac{548732}{81}-\frac{50581}{27}\zeta_3-\frac{484}{3}\zeta_4+\frac{12820}{3}\zeta_5\right)\\
&+C_AC_F^3T_Fn_f\left(3717+\frac{5696}{3}\zeta_3-\frac{7480}{3}\zeta_5\right)-C_F^4T_Fn_f\left(\frac{4157}{6}+128\zeta_3\right)\\
&+\frac{d_A^{abcd}d_A^{abcd}}{N_A}T_Fn_f\left(\frac{904}{9}-\frac{20752}{9}\zeta_3+352\zeta_4+\frac{4000}{9}\zeta_5\right)\\
&+\frac{d_F^{abcd}d_A^{abcd}}{N_A}C_An_f\left(\frac{11312}{9}-\frac{127736}{9}\zeta_3+2288\zeta_4+\frac{67520}{9}\zeta_5\right)\\
&+\frac{d_F^{abcd}d_A^{abcd}}{N_A}C_Fn_f\left(-320+\frac{1280}{3}\zeta_3+\frac{6400}{3}\zeta_5\right)\\
&+C_A^3T_F^2n_f^2\left(\frac{843067}{486}+\frac{18446}{27}\zeta_3-\frac{104}{3}\zeta_4-\frac{2200}{3}\zeta_5\right)\\
&+C_A^2C_FT_F^2n_f^2\left(\frac{5701}{162}+\frac{26452}{27}\zeta_3-\frac{944}{3}\zeta_4+\frac{1600}{3}\zeta_5\right)\\
&+C_F^2C_AT_F^2n_f^2\left(\frac{31583}{18}-\frac{28628}{27}\zeta_3+\frac{1144}{3}\zeta_4-\frac{4400}{3}\zeta_5\right)\\
&+C_F^3T_F^2n_f^2\left(-\frac{5018}{9}-\frac{2144}{3}\zeta_3+\frac{4640}{3}\zeta_5\right)\\
&+\frac{d_F^{abcd}d_A^{abcd}}{N_A}T_Fn_f^2\left(-\frac{3680}{9}+\frac{40160}{9}\zeta_3-832\zeta_4-\frac{1280}{9}\zeta_5\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{d_F^{abcd}d_F^{abcd}}{N_A}C_An_f^2\left(-\frac{7184}{3}+\frac{40336}{9}\zeta_3-704\zeta_4+\frac{2240}{9}\zeta_5\right)\\
&+\frac{d_F^{abcd}d_F^{abcd}}{N_A}C_Fn_f^2\left(\frac{4160}{3}+\frac{5120}{3}\zeta_3-\frac{12800}{3}\zeta_5\right)\\
&+C_A^2T_F^3n_f^3\left(-\frac{2077}{27}-\frac{9736}{81}\zeta_3+\frac{112}{3}\zeta_4+\frac{320}{9}\zeta_5\right)\\
&+C_AC_FT_F^3n_f^3\left(-\frac{736}{81}-\frac{5680}{27}\zeta_3+\frac{224}{3}\zeta_4\right)\\
&+C_F^2T_F^3n_f^3\left(-\frac{9922}{81}+\frac{7616}{27}\zeta_3-\frac{352}{3}\zeta_4\right)\\
&+\frac{d_F^{abcd}d_F^{abcd}}{N_A}T_Fn_f^3\left(\frac{3520}{9}-\frac{2624}{3}\zeta_3+256\zeta_4+\frac{1280}{3}\zeta_5\right)\\
&+C_AT_F^4n_f^4\left(\frac{916}{243}-\frac{640}{81}\zeta_3\right)-C_FT_F^4n_f^4\left(\frac{856}{243}+\frac{128}{27}\zeta_3\right)
\end{aligned}
$$

QCD

[agrees with result of Baikov, Chetyrkin and Kühn]

$$\beta_0 = 11 - \frac{2}{3} n_f \ , \qquad \beta_1 = 102 - \frac{38}{3} n_f \ ,$$

$$\beta_2 = \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \ ,$$

$$\beta_3 = \frac{149753}{6} + 3564\,\zeta_3 + n_f \left(-\frac{1078361}{162} - \frac{6508}{27}\zeta_3\right) + n_f^2 \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) + \frac{1093}{729} n_f^3$$

$$\begin{aligned}\beta_4 = {} & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\,\zeta_5 \\ & + n_f \left(-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right) \\ & + n_f^2 \left(\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\right) \\ & + n_f^3 \left(-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\right) + n_f^4 \left(\frac{1205}{2916} - \frac{152}{81}\zeta_3\right)\end{aligned}$$

QED

[agrees with result of Baikov, Chetyrkin, Kühn and Rittinger]

$$\beta_0 = \frac{4}{3}\, n_f \;, \quad \beta_1 = 4\, n_f \;, \quad \beta_2 = -\,2\, n_f - \frac{44}{9}\, n_f^2 \;,$$

$$\beta_3 = -\,46\, n_f + n_f^2 \left(\frac{760}{27} - \frac{832}{9}\, \zeta_3 \right) - \frac{1232}{243}\, n_f^3$$

$$\begin{aligned}\beta_4 = {} & n_f \left(\frac{4157}{6} + 128\, \zeta_3 \right) + n_f^2 \left(-\frac{7462}{9} - 992\, \zeta_3 + 2720\, \zeta_5 \right) \\ & + n_f^3 \left(-\frac{21758}{81} + \frac{16000}{27}\, \zeta_3 - \frac{416}{3}\, \zeta_4 - \frac{1280}{3}\, \zeta_5 \right) + n_f^4 \left(\frac{856}{243} + \frac{128}{27}\, \zeta_3 \right)\end{aligned}$$

Same method applies to decay rates

1 [hep-ph] 4 Jul 2017

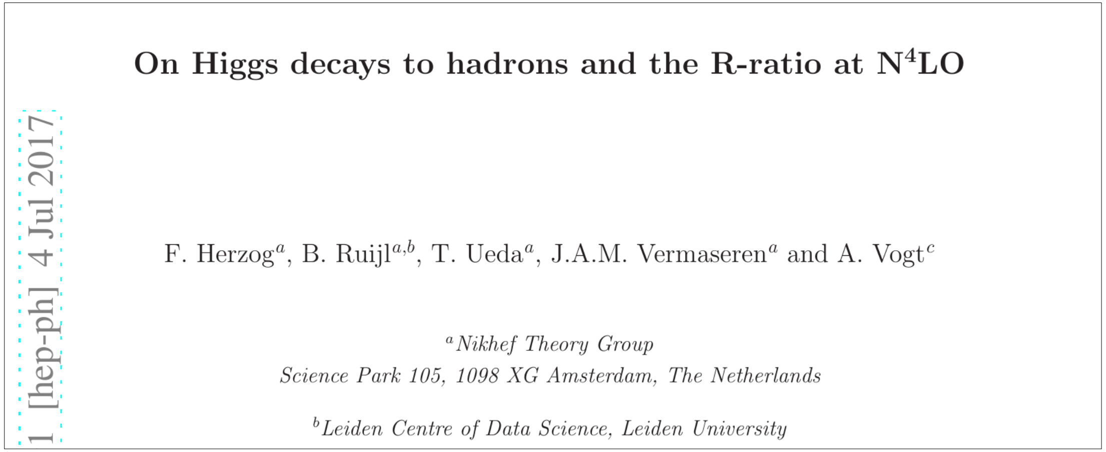
On Higgs decays to hadrons and the R-ratio at N^4LO

F. Herzog[a], B. Ruijl[a,b], T. Ueda[a], J.A.M. Vermaseren[a] and A. Vogt[c]

[a] *Nikhef Theory Group*
Science Park 105, 1098 XG Amsterdam, The Netherlands

[b] *Leiden Centre of Data Science, Leiden University*

Same method applies to decay rates



1 [hep-ph] 4 Jul 2017

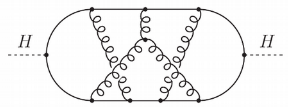
On Higgs decays to hadrons and the R-ratio at N^{4}LO

F. Herzog[a], B. Ruijl[a,b], T. Ueda[a], J.A.M. Vermaseren[a] and A. Vogt[c]

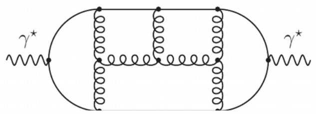
[a]*Nikhef Theory Group*
Science Park 105, 1098 XG Amsterdam, The Netherlands

[b]*Leiden Centre of Data Science, Leiden University*

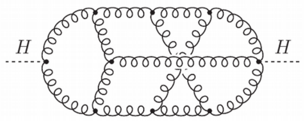
- $H \to bb$ at N4LO



- Hadronic R-ratio at N4LO



- $H \to gg$ at N4LO



Same method applies to decay rates

On Higgs decays to hadrons and the R-ratio at N^4LO

arXiv: ...1 [hep-ph] 4 Jul 2017

F. Herzog[a], B. Ruijl[a,b], T. Ueda[a], J.A.M. Vermaseren[a] and A. Vogt[c]

[a] *Nikhef Theory Group*
Science Park 105, 1098 XG Amsterdam, The Netherlands

[b] *Leiden Centre of Data Science, Leiden University*

- $H \to bb$ at N4LO
- Hadronic R-ratio at N4LO

First independent confirmation of results by Baikov, Chetyrkin and Kuehn

- $H \to gg$ at N4LO

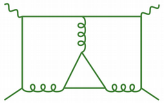
NEW!

Outlook

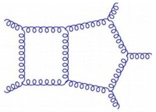
- R*:
  - Develop general Framework for computing Renormalisation constants in **any** local QFT
    - With Forcer up to 5 loops!
    - Moments of splitting functions at 4 and 5 loops.
    - SM EFT at 2 loops, even beyond ?
    - Supersymmetry, Gravity, ..
- VIDI starting April 2018:
  - Generalise R* to massless external states:
    - Develop methods for high order LHC cross sections

$$R^*(\Gamma) = \sum_{\substack{S\subseteq\Gamma,\tilde{S}\subseteq\Gamma \\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma/S\setminus\tilde{S}$$

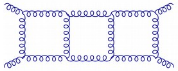
4-loop splitting functions



3 jets at 2 loops



2 jets at 3 loops



- Discover new particles?!
- Constrain BSM parameters!
- Improve SM Measurements!

n_f dependence

$$\widetilde{\beta} \equiv -\beta(a_s)/(a_s^2 \beta_0)$$

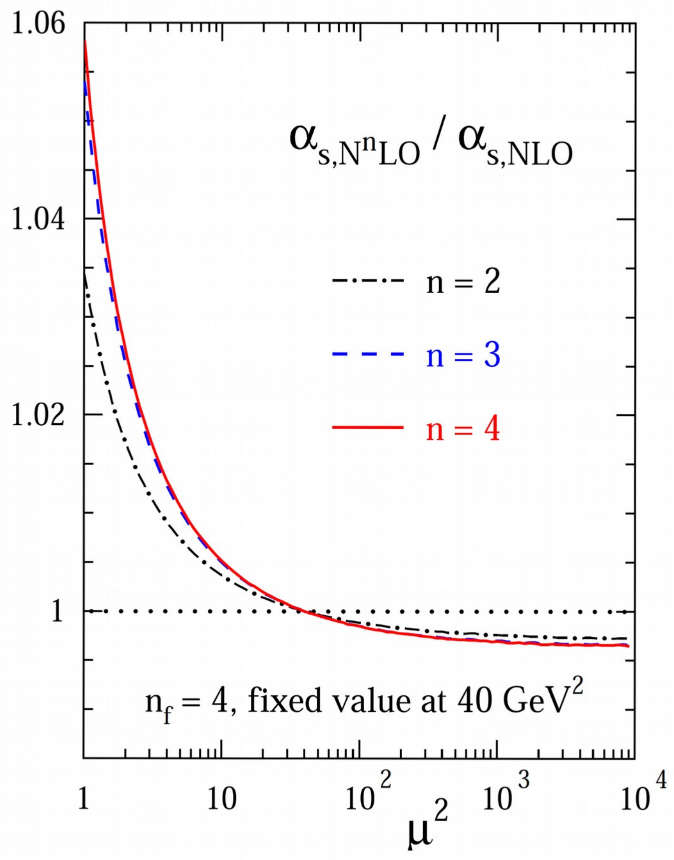
$$\begin{aligned}
\widetilde{\beta}(\alpha_s, n_f\!=\!3) &= 1 + 0.565884\,\alpha_s + 0.453014\,\alpha_s^2 + 0.676967\,\alpha_s^3 + 0.580928\,\alpha_s^4 \\
\widetilde{\beta}(\alpha_s, n_f\!=\!4) &= 1 + 0.490197\,\alpha_s + 0.308790\,\alpha_s^2 + 0.485901\,\alpha_s^3 + 0.280601\,\alpha_s^4 \\
\widetilde{\beta}(\alpha_s, n_f\!=\!5) &= 1 + 0.401347\,\alpha_s + 0.149427\,\alpha_s^2 + 0.317223\,\alpha_s^3 + 0.080921\,\alpha_s^4 \\
\widetilde{\beta}(\alpha_s, n_f\!=\!6) &= 1 + 0.295573\,\alpha_s - 0.029401\,\alpha_s^2 + 0.177980\,\alpha_s^3 + 0.001555\,\alpha_s^4
\end{aligned}$$

Convergence enhanced for larger n_f

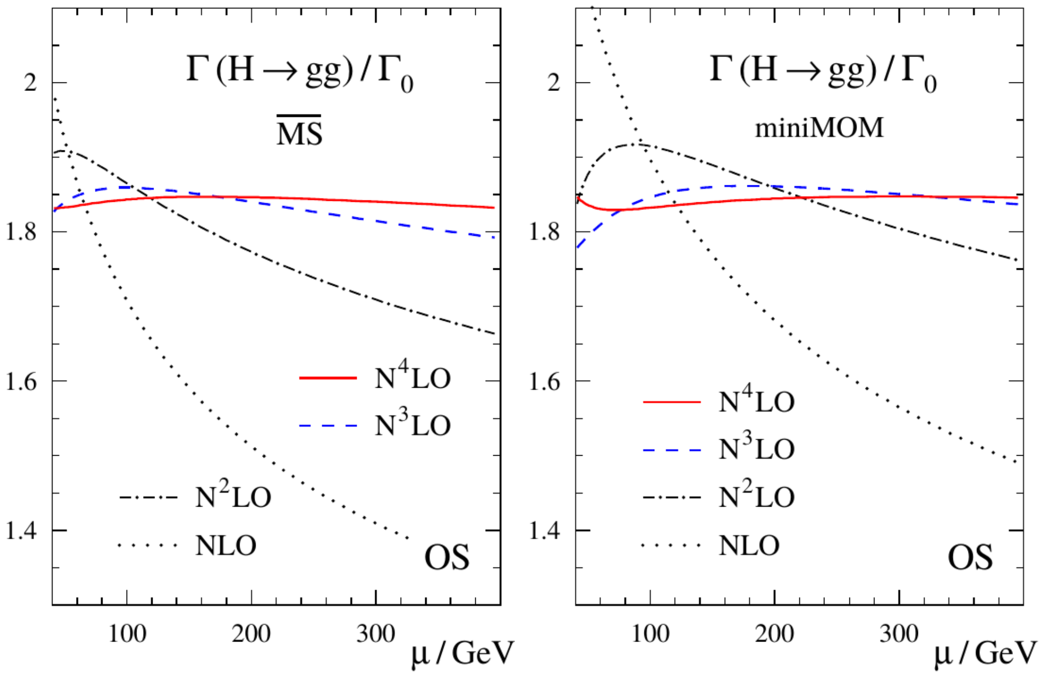
Scale Evolution at low scales

Let us hypothetically fix

$$\alpha_s(6.3GeV) = 0.2$$



Scale and scheme dependence of $\Gamma(H \to gg)$



$$\Gamma_{\mathrm{N^4LO}}(H \to gg) \;=\; \Gamma_0 \left(1.844 \,\pm\, 0.011_{\,\mathrm{series}} \,\pm\, 0.045_{\,\alpha_s(M_Z),1\%}\right)$$