Neutrinos and their interactions

J.W. van Holten

Nikhef, Amsterdam NL

Lectures

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Topics

Conventions
1. Natural units: $c = \hbar = 1$.
2. Minkowski metric: $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.
Equivalent: invariant mass

$$\eta_{\mu\nu}\, p^\mu p^\nu = -E^2 + \mathbf{p}^2 = -m^2. \tag{1}$$

3. Pauli matrices (2×2):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_i^\dagger = \sigma_i. \tag{2}$$

Then

$$\{\sigma_i, \sigma_j\} \equiv \sigma_i \sigma_j + \sigma_j \sigma_i = 2\, \delta_{ij}\, 1. \tag{3}$$

In particular if $i = j$: $\sigma_i^2 = 1$; and if $i \neq j$: $\sigma_i \sigma_j = -\sigma_j \sigma_i$.
Note:

$$\sigma_1 \sigma_2 \sigma_3 = \frac{1}{3!}\, \varepsilon_{ijk} \sigma_i \sigma_j \sigma_k = i\, 1.$$

4. Dirac matrices (4×4):

$$\gamma_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad i = (1, 2, 3), \tag{4}$$

Then

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\, \eta_{\mu\nu}\, 1, \qquad \gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0. \tag{5}$$

In particular if $\mu = \nu$:

$$\gamma_0^2 = -1, \qquad \gamma_i^2 = +1, \quad i = (1, 2, 3);$$

and if $\mu \neq \nu$: $\gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu$.
Definition:

$$\gamma_5 \equiv -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_4 = -\frac{i}{4!}\, \varepsilon^{\mu\nu\kappa\lambda} \gamma_\mu \gamma_\nu \gamma_\kappa \gamma_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma_5^\dagger. \tag{6}$$

Then

$$\gamma_5 \gamma_\mu = -\gamma_\mu \gamma_5 \quad \Leftrightarrow \quad \gamma_5 \gamma_\mu + \gamma_\mu \gamma_5 = 0, \qquad \gamma_5^2 = 1. \tag{7}$$

5. Dirac equation for free fermions:

$$(-i\gamma \cdot p + m)\, \psi(p) = \begin{pmatrix} m & p^0 - \boldsymbol{\sigma} \cdot \mathbf{p} \\ p^0 + \boldsymbol{\sigma} \cdot \mathbf{p} & m \end{pmatrix} \psi(p) = 0. \tag{8}$$

It follows that

$$(i\gamma \cdot p + m)\, (-i\gamma \cdot p + m)\, \psi(p) = \left((\gamma \cdot p)^2 + m^2 \right) \psi(p) = 0,$$

with
$$(\gamma \cdot p)^2 = \gamma_\mu \gamma_\nu \, p^\mu p^\nu = \frac{1}{2} \left(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu\right) p^\mu p^\nu = \eta_{\mu\nu} \, p^\mu p^\nu = -E^2 + \mathbf{p}^2.$$

6. *Chiral fermions*

Define
$$\psi_L = \frac{1}{2} \left(1 + \gamma_5\right) \psi, \qquad \psi_R = \frac{1}{2} \left(1 - \gamma_5\right) \psi. \tag{9}$$

Then
$$\gamma_5 \, \psi_L = \psi_L, \qquad \gamma_5 \, \psi_R = -\psi_R. \tag{10}$$

Equivalent:
$$\psi = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \chi_1 \\ \chi_2 \end{bmatrix} \quad \Rightarrow \quad \psi_L = \begin{bmatrix} \eta_1 \\ \eta_2 \\ 0 \\ 0 \end{bmatrix}, \quad \psi_R = \begin{bmatrix} 0 \\ 0 \\ \chi_1 \\ \chi_2 \end{bmatrix}. \tag{11}$$

and
$$\psi = \psi_L + \psi_R.$$

Dirac equation:
$$m\psi_L(p) = \frac{i}{2} \left(1 + \gamma_5\right) \gamma \cdot p \, \psi(p) = \frac{i}{2} \, \gamma \cdot p \left(1 - \gamma_5\right) \psi(p) = i\gamma \cdot p \, \psi_R(p),$$

or
$$i\gamma \cdot p \, \psi_R(p) = m\psi_L(p), \qquad i\gamma \cdot p \, \psi_L(p) = m\psi_R(p). \tag{12}$$

Conclusion:

A fermion can be purely chiral (L or R) only if it is massless: $m = 0$ and $p^2 \equiv \eta_{\mu\nu} p^\mu p^\nu = 0$. As follows also from (8), a non-zero mass mixes η- and χ-components.

7. *Charge conjugation*

Note for Pauli matrices:
$$\sigma_2 = \sigma_2^{-1} = -\sigma_2^T \quad \text{and} \quad \sigma_{1,3}^T = \sigma_{1,3}.$$

Then
$$\sigma_2^{-1} \sigma_{1,3} \sigma_2 = \sigma_2 \sigma_{1,3} \sigma_2 = -\sigma_2^2 \sigma_{1,3} = -\sigma_{1,3}^T,$$

and
$$\sigma_2^{-1} \sigma_2 \sigma_2 = \sigma_2^2 \sigma_2 = -\sigma_2^T.$$

Together:
$$\sigma_2^{-1} \sigma_i \sigma_2 = -\sigma_i^T. \tag{13}$$

Equivalent:
$$\left(\sigma_i \sigma_2\right)^T = \sigma_2^T \sigma_i^T = -\sigma_2 \sigma_i^T = \sigma_i \sigma_2. \tag{14}$$

For Dirac matrices define
$$C = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} = C^{-1} = -C^T. \tag{15}$$

Then

$$(\gamma_\mu C)^T = \gamma_\mu C \quad \Leftrightarrow \quad C^{-1}\gamma_\mu C = -\gamma_\mu^T. \tag{16}$$

Also:

$$\begin{aligned} C^{-1}\gamma_5 C &= -i(C^{-1}\gamma_0 C)(C^{-1}\gamma_1 C)(C^{-1}\gamma_2 C)(C^{-1}\gamma_3 C) = -i\gamma_0^T\gamma_1^T\gamma_2^T\gamma_3^T \\ &= -i\left(\gamma_3\gamma_2\gamma_1\gamma_0\right)^T = -i(\gamma_0\gamma_1\gamma_2\gamma_3)^T = \gamma_5^T, \end{aligned}$$

or

$$(\gamma_5 C)^T = -\gamma_5 C. \tag{17}$$

Charge conjugation of fermions:

$$\bar\psi(p) = \psi^\dagger(p)\gamma_0, \qquad \psi^c(p) = C\bar\psi^T(-p) = C\gamma_0^T\psi^*(-p) = -\gamma_0 C\psi^*(-p). \tag{18}$$

Then

$$\psi^c(p) = \left(\begin{array}{cc} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{array}\right)\left[\begin{array}{c} \eta^*(-p) \\ \chi^*(-p) \end{array}\right] = \left[\begin{array}{c} i\sigma_2\chi^*(-p) \\ -i\sigma_2\eta^*(-p) \end{array}\right]$$

or

$$\eta^c(p) = i\sigma_2\chi^*(-p), \qquad \chi^c(p) = -i\sigma_2\eta^*(-p).$$

Dirac equation:

$$\begin{aligned} &(-i\gamma\cdot p + m)\,\psi^c(p) = (-i\gamma\cdot p + m)\,C\bar\psi^T(-p) = C\left(i\gamma^T\cdot p + m\right)\bar\psi^T(-p) \\ &\quad = C\left[\bar\psi(-p)\,(i\gamma\cdot p + m)\right]^T. \end{aligned}$$

Now the hermitean conjugate of the Dirac equation is

$$0 = \psi^\dagger(p)\left(i\gamma^\dagger\cdot p + m\right) = -\bar\psi(p)\gamma_0\left(i\gamma^\dagger\cdot p + m\right) = -\bar\psi(p)\,(-i\gamma\cdot p + m)\,\gamma_0$$

and therefore, for the reversed momentum p:

$$\bar\psi(-p)\,(i\gamma\cdot p + m) = 0 \quad \Rightarrow \quad (-i\gamma\cdot p + m)\,\psi^c(p) = 0.$$

Thus if $\psi(p)$ is a solution of the free Dirac equation, then $\psi^c(p)$ is also a solution with the same mass m; $\psi^c(p)$ describes the anti-particle of $\psi(p)$.

Comment: The Dirac equation in position space is

$$(\gamma\cdot\partial + m)\,\Psi(x) = 0, \qquad \Psi(x) = \int d^4p\,\psi(p)e^{-ip\cdot x}. \tag{19}$$

In the presence of an electromagnetic potential $A_\mu(x)$ this is generalized to

$$\left[\gamma\cdot(\partial - ieA) + m\right]\Psi(x) = 0. \tag{20}$$

For the charge conjugate $\Psi^c = C\bar{\Psi}^T$ the Dirac equation for the charge-conjugate field becomes

$$[\gamma \cdot (\partial + ieA) + m]\, \Psi^c(x) = 0. \tag{21}$$

This describes indeed a particle of the same mass and opposite charge.

8. *Majorana fermions*

A Majorana fermion is a fermion which is its own anti-particle. Therefore $\psi^c(p) = \psi(p)$ and $\Psi^c(x) = \Psi(x)$. This can only happen if the charge vanishes: $e = 0$; therefore Majorana fermions are necessarily *neutral* particles: $e = 0$.

In components:

$$\psi(p) = \psi^c(p) \quad \Leftrightarrow \quad \eta(p) = i\sigma_2\chi^*(-p), \quad \chi(p) = -i\sigma_2\eta^*(-p), \tag{22}$$

and

$$\psi(p) = \left[\begin{array}{c} \eta(p) \\ -i\sigma_2\eta^*(-p) \end{array} \right].$$

A Majorana fermion is therefore described by 2 complex components, whereas a Dirac fermion requires 4 complex components; this is obvious as a Dirac fermion can be either a particle or a different anti-particle, whilst for a Majorana fermion particle and anti-particle are the same. The relation $\chi(p) = -i\sigma_2\eta^*(-p)$ indicates that a Majorana fermion with negative chirality is the anti-particle of the same fermion with positive chirality.

9. *Currents*

As by construction

$$(C\gamma_\mu)_{ab} = (C\gamma_\mu)_{ba}$$

and fermion fields anti-commute:

$$\psi_a\psi_b = -\psi_b\psi_a$$

it follows that for Majorana fermions

$$\bar{\psi}\gamma_\mu\psi = \psi^T C^T \gamma_\mu \psi = -\psi_a (C\gamma_\mu)_{ab}\psi_b = 0.$$

Therefore Majorana fermions can not form vector currents, like electric currents: they cannot interact with photons. This parallels the observation that Majorana fermions cannot carry electric charge. However, Majorana fermions can form axial vector currents, and interact with axial vector fields:

$$\gamma_5 C = -(\gamma_5 C)^T \quad \Rightarrow \quad C\gamma_\mu\gamma_5 = -(C\gamma_\mu\gamma_5)^T,$$

and one can construct non-zero axial currents of Majorana fermions:

$$\bar{\psi}\gamma_\mu\gamma_5\psi.$$

Equivalently, the Dirac equation for a fermion coupling to an axial vector field B_μ

$$[\gamma \cdot (\partial - g\gamma_5 B) + m]\, \Psi = 0$$

implies the same equation for the charge-conjugate fermion

$$[\gamma \cdot (\partial - g\gamma_5 B) + m]\, \Psi^c = 0.$$

Then there is no obstruction to have $\Psi^c(x) = \Psi(x)$.

10. *Electro-weak interactions.*
According to the Standard Model (SM) of particle physics all fermions and gauge bosons were originally massless. This applies in practice to the first instants after the Big Bang. The symmetries of the SM: $SU_c(3) \times SU_L(2) \times U_Y(1)$ were all exactly realized, all quarks and leptons of any color charge, isospin, hypercharge and chirality were present in equal numbers. When as a result of its expansion the temperature of the universe dropped sufficiently for the Brout-Englert-Higgs (BEH) scalar-fields to develop a vacuum expectation value and break the $SU_L(2) \times U_Y(1)$ symmetries spontaneously, some gauge bosons and presumably all fermions (except maybe the lightest of the neutrinos) became massive.

First consider the original SM before symmetry breaking. All fermions were eigenstates of color, isospin and hypercharge; i.e., they carried precise quantum numbers corresponding to these charges. As the color symmetry $SU_c(3)$ was never broken, quarks of different color charge but similar isospin and hypercharge are still indistinguishable as concerns electro-weak interactions today. Leptons have no color charge, so for them $SU_c(3)$ is irrelevant. Therefore for our purpose it is sufficient to recall that quarks appear in triplets of different color, and leptons in singlets of no color.

Next turn to $SU_L(2)$. This symmetry acts only on left-handed chiral fermions. The rule is simple: massless left-handed quarks and leptons form doublets of isospin $T_3 = \pm 1/2$, whilst massless right-handed quarks and leptons are singlets with $T_3 = 0$.

Finally, both left- and right-handed fermions carry hypercharges Y, but in different amounts, distinguishing the various left- and right-handed quarks and leptons even apart from the color charges.

After the BEH-fields developed their vacuum expectation value, the left- and right-handed fermions combined into massive states, which can no longer have exact isospin and hypercharge as those of their left-and right-handed components differ. However, a particular combination of isospin and hypercharge survives: the electric charge Q which happens to be the same for the left- and right-handed components of the massive fermions. All these quantum numbers are listed in table 10.1. Right-handed neutrinos have been included in the table, even though they don't carry any charges of the SM gauge interactions: they are singlets of color and isospin, without hypercharge.

	$SU_c(3)$	$SU_L(2)$	T_3	Y	Q
U_L	$\underline{3}$	$\underline{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$
D_L	$\underline{3}$	$\underline{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}$
U_R	$\underline{3}$	$\underline{1}$	0	$\frac{2}{3}$	$\frac{2}{3}$
D_R	$\underline{3}$	$\underline{1}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$
N_L	$\underline{1}$	$\underline{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0
E_L	$\underline{1}$	$\underline{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
N_R	$\underline{1}$	$\underline{1}$	0	0	0
E_R	$\underline{1}$	$\underline{1}$	0	-1	-1

Table 10.1: Multiplet structure and quantum numbers of quarks and leptons

11. *Electro-weak interactions of leptons*

The Dirac equation for a pair of massless charged and neutral left-handed fermions reads

$$\begin{aligned} \gamma \cdot \left(\partial - \frac{ig_2}{2}\, W^3 + \frac{ig_1}{2}\, B \right) N_L &= \frac{ig_2}{\sqrt{2}}\, \gamma \cdot W^+ E_L, \\ \gamma \cdot \left(\partial + \frac{ig_2}{2}\, W^3 + \frac{ig_1}{2}\, B \right) E_L &= \frac{ig_2}{\sqrt{2}}\, \gamma \cdot W^- N_L. \end{aligned} \tag{23}$$

For a pair of similar right-handed fermions there is no coupling to the W-bosons, whilst the hypercharges for coupling to B-bosons are different:

$$\begin{aligned} &\gamma \cdot \partial\, N_R = 0, \\ &\gamma \cdot (\partial + ig_1 B)\, E_R = 0. \end{aligned} \tag{24}$$

Now switching on the symmetry breaking BEH-field creates a massive Z-boson and a massless photon A, corresponding to a rotation of the W^3- and B-fields by the weak mixing angle θ_w:

$$W^3 = \cos\theta_w\, Z + \sin\theta_w\, A, \qquad B = -\sin\theta_w\, Z + \cos\theta_w\, A, \tag{25}$$

After this rotation the equations for the left-handed particles take the required form

$$\gamma\cdot\left(\partial + \frac{ie}{2\sin 2\theta_w}\, Z\right) N_L = \frac{ie}{\sqrt{2}\sin\theta_w}\,\gamma\cdot W^+ E_L,$$

$$\gamma\cdot\left(\partial + ieA + ie\,\mathrm{cotan}\, 2\theta_w\, Z\right) E_L = \frac{ie}{\sqrt{2}\sin\theta_w}\,\gamma\cdot W^- N_L,$$

provided

$$e = g_2\sin\theta_w = g_1\cos\theta_w \quad\Rightarrow\quad \tan\theta_w = \frac{g_1}{g_2}. \tag{26}$$

Then also

$$\gamma\cdot\partial\, N_R = 0,$$

$$\gamma\cdot\left(\partial + ieA - ie\tan\theta_w\, Z\right) E_R = 0.$$

If we now combine the left- and righthanded component and add mass terms m_N and m_E, the corresponding massive Dirac equations for $N = N_L + N_R$ and $E = E_L + E_R$ read

$$\gamma\cdot\left[\partial + \frac{ie}{2\sin 2\theta_w}\,(1+\gamma_5)\, Z + m_N\right] N = \frac{ie}{2\sqrt{2}\sin\theta_w}\,\gamma\cdot W^+\,(1+\gamma_5)\, E,$$

$$\gamma\cdot\left[\partial + ieA + \frac{ie}{2\sin 2\theta_w}\,(2\cos 2\theta_w - 1 + \gamma_5)\, Z + m_E\right] E = \frac{ie}{2\sqrt{2}\sin\theta_w}\,\gamma\cdot W^-\,(1+\gamma_5)\, N. \tag{27}$$

With these relations between (e, θ_w) and (g_2, g_1) by construction the neutrinos couple only to the Z- and W-bosons, with chiral (γ_5-dependent) couplings, whilst the charged leptons have a non-chiral vector coupling to the photon with charge $-e$, and chiral coupling to the Z- and W-bosons which differ for the Z-bosons.

12. *The mass matrix*

The standard model includes 3 families of quarks and leptons. We label the families with an index $i = 1, 2, 3$ for (e, μ, τ). All charged leptons have the same gauge interactions, and so do the neutrinos. Therefore the only difference is in the masses, which arise from symmetry breaking. The equations (27) for 3 families of leptons then takes the general form

$$\gamma\cdot\left[\partial + \frac{ie}{2\sin 2\theta_w}\,(1+\gamma_5)\, Z\right] N_i + m_{Nij} N_j = \frac{ie}{2\sqrt{2}\sin\theta_w}\,\gamma\cdot W^+\,(1+\gamma_5)\, E_i,$$

$$\begin{aligned}\gamma\cdot\left[\partial + ieA + \frac{ie}{2\sin 2\theta_w}\,(2\cos 2\theta_w - 1 + \gamma_5)\, Z\right] E_i + m_{Eij} E_j \\ = \frac{ie}{2\sqrt{2}\sin\theta_w}\,\gamma\cdot W^-\,(1+\gamma_5)\, N_i,\end{aligned} \tag{28}$$

with m_{Nij} and m_{Eij} hermitean matrices with real eigenvalues. Note that we employ the summation convention for repeated indices i, j. Also note that by convention we have defined the neutrinos N_i to be the ones that correspond to the E_i with the same label i in transitions caused by emission or absorption of a $W^{\pm}$-boson. However, with this choice it is not guaranteed that the mass matrices m_{Nij} and m_{Eij} are diagonal. Of course, we can always define a lepton family by identifying one component of each family, for example E_i, with an eigenstate of the mass matrix, m_{Eij}. Then by choice this matrix is diagonal: $m_{Eij} = m_{Ei}\,\delta_{ij} = \mathrm{diag}(m_e, m_\mu, m_\tau)$. But nothing guarantees that in this basis the mass matrix m_{Nij} will be diagonal as well; in fact, in the standard model it is *not*. The eigenstates of the neutrino mass matrix, which are denoted by ν_i, will then be mixtures of interaction eigenstates N_i:

$$\nu_i = U_{ij} N_j \quad \Leftrightarrow \quad N_i = U^{-1}_{ij} \nu_j = U^{\dagger}_{ij}\, \nu_j, \tag{29}$$

with U a unitary matrix so a to guarantee that the vectors ν_i and N_i have the same normalization.

Consider this situation with the charged-lepton states defined by mass eigenstates; then in terms of the neutrino mass eigenstates

$$\gamma \cdot \left[\partial + \frac{ie}{2\sin 2\theta_w} \left(1+\gamma_5\right) Z \right] U^{-1}_{ij} \nu_j + m_{Nij} U^{-1}_{jk} \nu_k = \frac{ie}{2\sqrt{2}\sin\theta_w}\, \gamma \cdot W^{+} \left(1+\gamma_5\right) E_i,$$

$$\gamma \cdot \left[\partial + ieA + \frac{ie}{2\sin 2\theta_w} \left(2\cos 2\theta_w - 1 + \gamma_5\right) Z \right] E_i + m_{Ei} E_i$$

$$= \frac{ie}{2\sqrt{2}\sin\theta_w}\, \gamma \cdot W^{-} \left(1+\gamma_5\right) U^{-1}_{ij} \nu_j.$$

Multiplying the first equation by U it becomes

$$\gamma \cdot \left[\partial + \frac{ie}{2\sin 2\theta_w} \left(1+\gamma_5\right) Z \right] \nu_i + \left(U \cdot m_N \cdot U^{-1} \right)_{ij} \nu_j = \frac{ie}{2\sqrt{2}\sin\theta_w}\, \gamma \cdot W^{+} \left(1+\gamma_5\right) U_{ij} E_j.$$

Now chosing U such that ν_i are mass eigenstates, the mass matrix becomes diagonal:

$$\left(U \cdot m_N \cdot U^{-1} \right)_{ij} = m_{Ni} \delta_{ij}, \tag{30}$$

and we finally end up with the equations for the mass eigenstates of the leptons

$$\begin{aligned}
&\gamma \cdot \left[\partial + \frac{ie}{2\sin 2\theta_w} \left(1+\gamma_5\right) Z + m_{Ni} \right] \nu_i = \frac{ie}{2\sqrt{2}\sin\theta_w}\, \gamma \cdot W^{+} \left(1+\gamma_5\right) U_{ij} E_j, \\
&\gamma \cdot \left[\partial + ieA + \frac{ie}{2\sin 2\theta_w} \left(2\cos 2\theta_w - 1 + \gamma_5\right) Z + m_{Ei} \right] E_i \\
&\qquad = \frac{ie}{2\sqrt{2}\sin\theta_w}\, \gamma \cdot W^{-} \left(1+\gamma_5\right) U^{-1}_{ij} \nu_j.
\end{aligned} \tag{31}$$

The unitary matrix U which appears in the coupling of leptons to the charged $W^{\pm}$-bosons is known as the PMNS-matrix.

13. *Parametrizing the PMNS matrix*
If all leptons (ν_i, E_i) are Dirac fermions, both sets of equations (31) can be multiplied by independent phase factors $e^{i\alpha_i}$ and $e^{i\beta_i}$. As Dirac fields are complex, their states can be redefined by such global phases

$$\nu_i \to \nu_i' = e^{i\alpha_i}\nu_i, \qquad E_i \to E_i' = e^{i\beta_i}E_i,$$

with the result

$$\gamma \cdot \left[\partial + \frac{ie}{2\sin 2\theta_w}\left(1+\gamma_5\right)Z + m_{Ni}\right]\nu_i' = \frac{ie}{2\sqrt{2}\sin\theta_w}\gamma \cdot W^+\left(1+\gamma_5\right)U_{ij}'E_j',$$

$$\gamma \cdot \left[\partial + ieA + \frac{ie}{2\sin 2\theta_w}\left(2\cos 2\theta_w - 1 + \gamma_5\right)Z + m_{Ei}\right]E_i'$$

$$= \frac{ie}{2\sqrt{2}\sin\theta_w}\gamma \cdot W^-\left(1+\gamma_5\right)U_{ij}'^{-1}\nu_j'.$$

where the PMNS matrix is redefined by

$$U_{ij}' = e^{i\alpha_i}U_{ij}e^{-i\beta_j}.$$

Therefore U is unique only up to 5 independent phase factors $e^{i(\alpha_i-\beta_i)}$. Now a unitary 3×3 matrix with 9 complex components, satisfying 9 real equations

$$\sum_k U_{ik}U_{kj}^* = \delta_{ij},$$

depends on 9 independent real parameters. The real phase angles $\alpha_i - \beta_i$ can be used to eliminate 5 real parameters, leaving the PMNS matrix to depend on 4 real parameters which are physically relevant. These can be chosen to correspond to 3 real rotations in the 3-dimensional family space and 1 additional phase factor. The physical PMNS matrix U' can therefore be parametrized (dropping the prime notation) by 3 angles $(\theta_{12}, \theta_{13}, \theta_{23})$ and a real phase δ in the standard form

$$U[\theta_{ij}, \delta] = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_1 \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{32}$$

with the abbreviation $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$.

In an alternative to the above scenario the neutrino mass eigenstates ν_i can be taken to be Majorana fermions, satisfying

$$\nu_i = \nu_i^c = C\bar{\nu}_i^T.$$

Such Majorana fermion states can not be redefined by a phase factor and therefore $\alpha_i = 0$. Only the 3 phases β_i of the charged leptons can be used to redefine the PMNS matrix. This allows for 2 additional phases to be physically relevant, and the PMNS matrix is parametrized by additional phase factors. The standard form is

$$U \to U_M[\theta_{ij}, \gamma_{ij}] = U[\theta_{ij}, \delta = \gamma_{23}] \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\gamma_{12}} & 0 \\ 0 & 0 & e^{i\gamma_{13}} \end{pmatrix}.$$

14. *Neutrino oscillations*

The mixing of weak-interaction states in the mass eigenstates of neutrinos leads to the phenomenon of neutrino oscillations. Consider a neutrino N_i of created at time $t = 0$ with energy $\mathcal{E}$ in a weak interaction as partner of a charged lepton E_i; it is a linear combination of mass eigenstates described by the quantum state

$$|\Psi(0)\rangle = |N_i\rangle = U^\dagger_{ik}\,|\nu_k\rangle.$$

As the ν_i are eigenstates of well-defined mass, with the given energy related to their momentum $\mathbf{p}_i$ by $\mathcal{E} = \sqrt{m_i^2 + \mathbf{p}_i^2}$ during propagation they evolve as

$$|\nu_k(t)\rangle = e^{i(\mathbf{p}_k\cdot\mathbf{r} - \mathcal{E}t)}|\nu_k\rangle.$$

Therefore the interaction eigenstate evolves as

$$|\Psi(t)\rangle = e^{i(\mathbf{p}_k\cdot\mathbf{r} - \mathcal{E}t)}U^\dagger_{ik}|\nu_k\rangle. \tag{33}$$

The probability amplitude for this state to be identified with an interaction eigenstate N_j is (writing out summations explicitly)

$$\langle N_j|\Psi(t)\rangle = \sum_k e^{i(\mathbf{p}_k\cdot\mathbf{r} - \mathcal{E}t)}U^\dagger_{ik}\langle N_j|\nu_k\rangle = \sum_k e^{i(\mathbf{p}_k\cdot\mathbf{r} - \mathcal{E}t)}U^\dagger_{ik}U^\dagger_{jk} = \sum_k e^{i(\mathbf{p}_k\cdot\mathbf{r} - \mathcal{E}t)}U^\dagger_{ik}U_{kj}. \tag{34}$$

The probability to observe a neutrino created with flavor i at later time t at position $\mathbf{r}$ in an interaction as a neutrino of flavor j is therefore

$$P_{i\to j}(t, \mathbf{r}) = \left|\sum_k e^{i(\mathbf{p}_k\cdot\mathbf{r} - \mathcal{E}t)}U^\dagger_{ik}U_{kj}\right|^2 = \left|\sum_k e^{i\mathbf{p}_k\cdot\mathbf{r}}U^\dagger_{ik}U_{kj}\right|^2 \tag{35}$$

The point is, that different mass eigenstates propagate with different momentum$\mathbf{p}_k$ and therefore the relative phases of the different mass eigenstates vary with distance. This is seen clearly in the simplified example of having two neutrino flavors, say N_e and N_μ. In that case the mixing matrix can be taken real with a single mixing angle θ:

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and the amplitude can be factorized as

$$\langle N_\mu|\Psi(t)\rangle = e^{i(\mathbf{p}_e\cdot\mathbf{r}-\mathcal{E}t)}\left(U^\dagger_{ee}U_{e\mu} + e^{i\Delta\mathbf{p}\cdot\mathbf{r}}U^\dagger_{e\mu}U_{\mu\mu}\right),$$

with $\Delta\mathbf{p} = \mathbf{p}_\mu - \mathbf{p}_e$. As the $\mathbf{p}_\mu$ and $\mathbf{p}_e$ both point in the direction of propagation, after a distance L

$$\Delta\mathbf{p}\cdot\mathbf{r} = \Delta p\, L = \left(\sqrt{\mathcal{E}^2 - m_\mu^2} - \sqrt{\mathcal{E}^2 - m_e^2}\right) L = -\frac{\Delta m^2 L}{2\mathcal{E}},$$

where $\Delta m^2 = m_\mu^2 - m_e^2$. The probability for a muon neutrino to change into an electron neutrino after a distance L in the direction of propagation then is

$$\begin{aligned} P_{\mu\to e}(t,L) &= \left|\left(1-e^{i\Delta pL}\right)\sin\theta\cos\theta\right|^2 = 2\left(1-\cos\Delta pL\right)\sin^2\theta\cos^2\theta \\ &= \sin^2 2\theta\,\sin^2\frac{\Delta m^2 L}{4\mathcal{E}}. \end{aligned} \tag{36}$$

If L is expressed in km, Δm^2 in eV and E in GeV this becomes numerically

$$P_{\mu\to e}(t,L) = \sin^2 2\theta\,\sin^2 1.27\,\frac{\Delta m^2 L}{\mathcal{E}}.$$

15. *Yukawa couplings*

The masses of the W- and Z-bosons, which break the isospin and hypercharge symmetry, are generated dynamically by a vacuum expectation value of the BEH scalar fields. In the standard model there is a doublet of complex BEH scalars (φ_1, φ_2) with quantum numbers

	$SU_c(3)$	$SU_L(2)$	T_3	Y	Q
φ_1	$\underline{0}$	$\underline{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
φ_2	$\underline{0}$	$\underline{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0

Table 15.1: Multiplet structure of BEH scalar fields

Before the electrically neutral field φ_2 develops a vacuum expectation value and the $SU_L(2)\times U_Y(1)$ symmetries are still exact, the Dirac equations for the leptons can be extended to include Yukawa couplings to the scalar fields by[1]

$$\begin{aligned} \gamma\cdot\left(\partial - \frac{ig_2}{2}W^3 + \frac{ig_1}{2}B\right)N_L - \frac{ig_2}{\sqrt{2}}\gamma\cdot W^+E_L &= -f_N^*\,\varphi_2\,N_R - f_E^*\,\varphi_1\,E_R, \\ \gamma\cdot\left(\partial + \frac{ig_2}{2}W^3 + \frac{ig_1}{2}B\right)E_L - \frac{ig_2}{\sqrt{2}}\gamma\cdot W^-N_L &= f_N^*\,\varphi_1\,N_R - f_E^*\,\varphi_2\,E_R. \end{aligned} \tag{37}$$

[1]The minus signs on the r.h.s. appear so as to get equations covariant under non-diagonal isospin transformations.

and

$$\gamma\cdot\partial\, N_R = -f_N\left(\varphi_2 N_L - \varphi_1 E_L\right),$$
$$\gamma\cdot\left(\partial + ig_1 B\right) E_R = -f_E\left(\varphi_1^* N_L + \varphi_2^*\, E_L\right). \tag{38}$$

Now for independent fermion states (N_L, E_L) and (N_R, E_R) their relative phases and the phases of the the scalar fields (φ_1, φ_2) can be chosen such that $f_{N,E}$ and the vacuum expectation value $\langle(\varphi_1, \varphi_2)\rangle = (0, v)$ are real and generate mass terms

$$\gamma\cdot\left(\partial - \frac{ig_2}{2}\, W^3 + \frac{ig_1}{2}\, B\right) N_L - \frac{ig_2}{\sqrt{2}}\,\gamma\cdot W^+ E_L = -m_N\, N_R,$$

$$\gamma\cdot\left(\partial + \frac{ig_2}{2}\, W^3 + \frac{ig_1}{2}\, B\right) E_L - \frac{ig_2}{\sqrt{2}}\,\gamma\cdot W^- N_L = -m_E E_R,$$

and

$$\gamma\cdot\partial\, N_R = -m_N\, N_L,$$

$$\gamma\cdot\left(\partial + ig_1 B\right) E_R = -m_E\, E_L,$$

with $m_N = -f_N\, v$ and $m_E = -f_E\, v$. With these conventions[2] and rotating the fields (W^3, B) as in (25) the above equations can be combined to take the form of the massive Dirac equations (27). The 3-family generalization follows the same pattern, but the Yukawa-couplings can mix the the chiral families to give

$$\gamma\cdot\left(\partial - \frac{ig_2}{2}\, W^3 + \frac{ig_1}{2}\, B\right) N_{Li} - \frac{ig_2}{\sqrt{2}}\,\gamma\cdot W^+ E_{Li} = f^\dagger_{Nij}\varphi_2^*\, N_{Rj} + f^\dagger_{Eij}\varphi_1\, E_{Rj},$$
$$\gamma\cdot\left(\partial + \frac{ig_2}{2}\, W^3 + \frac{ig_1}{2}\, B\right) E_{Li} - \frac{ig_2}{\sqrt{2}}\,\gamma\cdot W^- N_{Li} = -f^\dagger_{Nij}\varphi_1^*\, N_{Rj} + f^\dagger_{Eij}\varphi_2\, E_{Rj}, \tag{39}$$

and

$$\gamma\cdot\partial\, N_{Ri} = f_{Nij}\left(\varphi_2\, N_{Lj} - \varphi_1\, E_{Lj}\right),$$
$$\gamma\cdot\left(\partial + ig_1 B\right) E_{Ri} = f_{Eij}\left(\varphi_1^* N_{Lj} + \varphi_2^*\, E_{Lj}\right). \tag{40}$$

The phases can now be chosen such that there is a mixing of families in the massive fermion states by hermitean mass matrices

$$m_{Nij} = -\langle f_{Nij}\varphi_2\rangle = -\langle f^\dagger_{Nij}\varphi_2^*\rangle, \qquad m_{Eij} = -\langle f_{Eij}\varphi_2\rangle = -\langle f^\dagger_{Eij}\varphi_2^*\rangle. \tag{41}$$

16. *Majorana masses*

The exact $SU_L(2)\times U_Y(1)$ isospin symmetry of the standard model acts differently on left-handed fermions and right-handed anti-fermions. But as we have seen in par. 6 the

[2] To be able to perform these steps it is necessary that the phases of the L- and R-handed states are independent.

distinction between left- and right-handed fermions can be made at a fundamental level only for massless fermions. Therefore it is *necessary* that masses for fermions carrying isospin- and hypercharges in the standard model are generated dynamically through coupling to the BEH-scalars. There is only one potential exception: the right-handed neutrinos N_R carry neither isospin nor hypercharge. Therefore they could in principle have a mass of their own, apart from their coupling to the left-handed neutrino N_L via the BEH-scalars. Of course, as implied by eq. (12) such a mass term would connect N_L to a right-handed counterpart; but such a counterpart could be formed from N_L by charge conjugation without introducing a new fermion field. Indeed, by definition a left-handed fermion field Ψ satisfies

$$\Psi = \gamma_5 \Psi \quad \Rightarrow \quad \bar{\Psi} = (\gamma_5 \Psi)^\dagger \gamma_0 = \Psi^\dagger \gamma_5^\dagger \gamma_0 = -\bar{\Psi} \gamma_5.$$

Therefore

$$\gamma_5 \Psi^c = \gamma_5 C \bar{\Psi}^T = -(\gamma_5 C)^T \bar{\Psi}^T = -C^T \gamma_5^T \bar{\Psi}^T = C \left(\bar{\Psi} \gamma_5 \right)^T = -C \bar{\Psi}^T = -\Psi^c.$$

Conclusion: if Ψ is a left-handed spinor, then Ψ^c is a right-handed spinor. Now define the Majorana spinor Φ by

$$\Phi = \frac{1}{2} \left(\Psi + \Psi^c \right) = \Phi^c.$$

Then by construction

$$\Phi_L = \frac{1}{2} \left(1 + \gamma_5 \right) \Phi = \Psi, \qquad \Phi_R = \frac{1}{2} \left(1 - \gamma_5 \right) \Phi = \Psi^c = (\Phi_L)^c.$$

If the Majorana fermion Φ satisfies the Dirac equation

$$\left(\gamma \cdot \partial + M \right) \Phi = 0,$$

this is equivalent with

$$\gamma \cdot \partial \, \Phi_L = -M \Phi_R = -M (\Phi_L)^c \quad \text{and} \quad \gamma \cdot \partial \, \Phi_R = -M \Phi_L = -M \left(\Phi_R \right)^c.$$

We can use this for the right-handed neutrino N_R by introducing a Majorana spinor Φ such that

$$N_R = \frac{1}{2} \left(1 - \gamma_5 \right) \Phi, \qquad (N_R)^c = \frac{1}{2} \left(1 + \gamma_5 \right) \Phi. \tag{42}$$

We then see that it is possible to extend the Dirac equation (38) for N_R by an additional mass term

$$\gamma \cdot \partial N_R = f_N \left(\varphi_2 N_L - \varphi_1 E_L \right) - M \left(N_R \right)^c, \tag{43}$$

without breaking isospin and hypercharge assignments. In the BEH-vacuum this leads to effective massive fermion states satisfying

$$\gamma \cdot \partial N_{Ri} = -m_{Nij} N_{Lj} - M_{ij} \left(N_{Rj} \right)^c. \tag{44}$$

Of course, as the N_{Ri} are independent fermion fields we can choose them to be eigenstates of the Majorana mass matrix M_{ij}, in which case this mass matrix becomes diagonal: $M_{ij} =$

diag(M_1, M_2, M_3). Note however, that such a mass violates the conservation of lepton number, as the Dirac equation mixes the right-handed neutrino with its anti-particle. Thus N_R would have to be assigned a lepton number $L_{N_R} = 0$ which violates lepton number as a conserved quantity in weak interactions as follows from eqs. (39), (40).

17. *Neutrino mass eigenstates and the seesaw mechanism*
To see how all this affects the massive neutrino states consider the situation with single left- and righthanded neutrinofields N_L and N_R. In the absence of external fields they satisfy the Dirac equations

$$\gamma \cdot \partial N_L = -m_N N_R, \quad \gamma \cdot \partial (N_L)^c = -m_N (N_R)^c,$$

$$\gamma \cdot \partial N_R = -m_N N_L - M\,(N_R)^c, \quad \gamma \cdot \partial (N_R)^c = -m_N (N_L)^c - M N_R.$$

Now define the Majorana neutrinos

$$\Phi_1 = \frac{1}{2}\left[N_L + (N_L)^c\right], \quad \Phi_2 = \frac{1}{2}\left[N_R + (N_R)^c\right].$$

Then the above equations can be summarized in matrix notation as

$$-\gamma \cdot \partial \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 0 & m_N \\ m_N & M \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \tag{45}$$

The eigenvalues of this mass matrix are

$$m_\pm = \frac{1}{2}\, M \left(1 \pm \sqrt{1 + \frac{4 m_N^2}{M^2}} \right). \tag{46}$$

The corresponding mass eigenstates of the Majorana neutrinos are

$$\Phi_\pm = \frac{1}{\sqrt{1 + \frac{m_\pm^2}{m_N^2}}} \left(\Phi_1 + \frac{m_\pm}{m_N}\, \Phi_2 \right). \tag{47}$$

To generalize this to 3 families, construct Majorana neutrino states

$$\Phi_{1i} = \frac{1}{2}\left[N_{Li} + (N_{Li})^c\right], \quad \Phi_{2i} = \frac{1}{2}\left[N_{Ri} + (N_{Ri})^c\right], \tag{48}$$

which satisfy the 3-dimensional generalization of eq. (45) in which m_N and M now represent 3×3 matrices. Consider linear combinations

$$\Phi = C \cdot (\Phi_1 + A \cdot \Phi_2)$$

These states are mass eigenstates of the generalized Dirac equation (45) provided

$$-\gamma \cdot \partial \left(\Phi_1 + A \cdot \Phi_2\right)_i = m_i \left(\Phi_1 + A \cdot \Phi_2\right)_i$$

which happens if $A_\pm$ are the solutions of the matrix equation

$$A \cdot m_N \cdot A = m_N + A \cdot M,$$

and requiring C to diagonalize

$$C \cdot A \cdot m_N \cdot C^{-1},$$

with diagonal elements m_i as the mass eigenvalues.

Whether or not the right-handed neutrinos possess Majorana masses M only experiment can decide. Presently there is no decisive theoretical argument that forbids such mass terms. In fact, there are reasons to argue in favor of Majorana masses, as they could explain the large hierarchy in masses between neutrinos and charged leptons in the standard model.

Consider the limit $M \gg m_N$; in this limit the Majorana neutrinos the mass eigenvalues tend to:

$$m_+ \simeq M, \qquad m_- \simeq \frac{m_N^2}{M}.$$

The first mass m_+ gets very large, the second mass m_- gets much smaller than m_N. Such a scenario is called the seesaw scenario, and can explain a triplet of very small physical neutrino masses even if the dynamical mass generated by the BEH-scalars via the Yukawa couplings are comparable. Suppose that m_N and m_E in eq. (41) are of the same order of magnitude; the actual neutrino masses are at least a factor 10^{-6} smaller. This can be explained by the seesaw scenario if

$$\frac{m_N}{M} \lesssim 10^{-6}.$$

The electron being the lightest charged lepton, we may suppose

$$m_N \gtrsim 1\,\mathrm{MeV}.$$

In this case it follows that the mass of the lightest heavy Majorana neutrino must satisfy

$$M \gtrsim 10^6\, m_N \gtrsim 1\,\mathrm{TeV}.$$

The masses of the heavier ones might still go higher by a factor of the order of 10^3-10^4. In the seesaw scenario it follows easily from eq. (47) that the light mass eigenstates Φ_- are predominantly of the form Φ_1, which have weak interactions similar to N_L, whereas the heavy mass eigenstates are predominantly are of the form Φ_2, and like the N_R do not participate in the weak interactions. Such non-interacting massive neutrinos are referred to as *sterile* neutrinos. Sterile neutrinos could be a component of dark matter, which is postulated by astrophysicists and cosmologists to explain observations of the dynamics of galaxies and the cosmic microwave background (CMB).

18. *A test for Majorana neutrinos*

Some present experiments to test for a Majorana nature of neutrinos look for neutrinoless double-β decay ($0\nu\beta\beta$) of certain unstable nuclei such as ^{76}Ge. In this process two neutrons simultaneously decay into protons, producing a pairs of W-bosons. The W-bosons decay leptonically into electrons, but the accompanying neutrino can be exchanged between the W's if the neutrino is its own antiparticle, i.e. if it is a Majorana neutrino; see fig. 18.1.

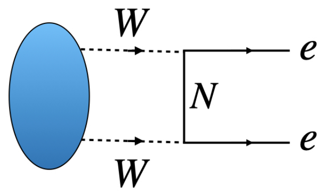

Fig. 18.1: Neutrinoless double-β decay.

The analysis of this process, apart from being very rare, is complicated by a range of nuclear physics effects to be taken into account. A cleaner test exists in principle by turning the process around: start with two leptons (electrons, muons) of the same charge. They can scatter by exchanging a Majorana neutrino (if it exists), producing a pair of same-sign W-bosons; the Feynman diagrams for this process are shown in fig. 18.2.

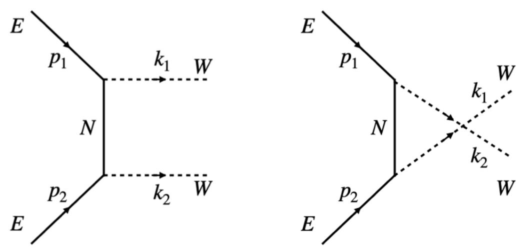

Fig. 18.2: Lepton scattering by neutrino exchange.

Obviously this process violates conservation of lepton number by -2, which is a hallmark of Majorana neutrinos. To produce the W-bosons the scattering energy $\sqrt{s}$ of the initial leptons has to be very large. As a result the mass of the exchanged neutrino becomes negligeable and the amplitude for the process takes the form[3]

$$\mathcal{M}_{ij} = -\frac{g_2^2 m_{Nij}}{2(p_1 - k_1)^2} \left(e(k_1) \cdot e(k_2) \right) \bar{u}_i(p_1) \left(1 - \gamma_5 \right) u_j^c(p_2).$$

In this expression g_2 is the W-boson coupling constant with m_{Nij} the neutrino mass-matrix element for neutrinos of type i and j; furthermore $e(k_{1,2})$ are the W-boson polarization vectors and $u_i(p)$ is the spinor for an incoming charged lepton of type i. The total cross section for the process computed from this amplitude reads

$$\sigma(ee \rightarrow WW) = \frac{G_F^2 |m_{Nij}|^2}{2\pi} \sqrt{1 - \frac{4M_W^2}{s}} \left(3 + \frac{s(s - 4M_W^2)}{4M_W^4} \right). \tag{49}$$

Here $G_F = g_2^2 \sqrt{2}/8M_W^2$ is the Fermi constant of weak interactions. In terms of the neutrino mass eigenvalues m_i and the PMNS matrix U the mass matrix elements are

$$m_{Nij} = \sum_k m_k U_{ik} U_{jk}.$$

[3] J.W. van Holten, preprint NIKHEF-2025-13; arXiv:2509.02035.

Note that for very high CM energies the total cross section increases as s^2/M_W^4.

19. *Beyond the standard model*

The existence of neutrino masses necessitates the inclusion of right-handed neutrinos in the standard model. The possibility of their Majorana mass implies the appearance of another mass scale into the model, independent of the vacuum expectation value of the BEH scalars (φ_1, φ_2). That is an indication of physics beyond the standard model. Therefore in the next sections we investigate a minimal extension of the standard model that can accomodate right-handed neutrinos and their Majorana masses. At the same time it also indicates how more complex extensions might arise, explaining other aspects of the standard model than only neutrino masses.

As a preparation we first recall an important property of the standard model connecting leptons and quarks. This property concerns the appearance of anomalies at higher-orders in quantum field theory; more precisely, anomalous breaking of symmetries by contributions of loop diagrams in quantum field theory. Consider the process of fig. 19.1, involving three gauge bosons interacting through a loop of fermions.

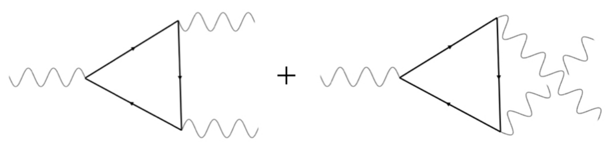

Fig. 19.1: 3-boson interaction through a fermion loop

The Bose-symmetry if the vector fields has been taken into account explicitly by including the second diagram (note: this diagram is equivalent to a diagram like the first one, with the fermions running in the opposite direction).

At each vertex a boson field A_μ^a is connected to the fermions by an interaction of the form

$$(a\gamma_\mu + b\gamma_5\gamma_\mu)\, T_a,$$

where T_a is the generator (the matrix of gauge couplings) of the symmetry associated with the gauge boson A_μ^a: Gell-Mann matrices λ_a for the $SU_c(3)$ gauge couplings of quarks, Pauli matrices τ_i for the gauge couplings of the $SU_L(2)$ gauge fields of weak interactions to left-handed quarks and leptons, and the real eigenvalues Y for the hypercharges of all left- and righthanded fermions.

In the standard model, before the BEH scalar fields develop an expectation value all fermions are chiral: left- or righthanded, such that for the L-fermions we have $a = b = 1$, and for the R-fermions $a = -b = 1$. If the vacuum expectation value of the scalar fields is switched on, the $SU_L(2) \times U_Y(1)$ electroweak symmetries are broken spontaneously, meaning that the full symmetry still governs the field equations, but is not respected by the solution $\langle \varphi_2 \rangle = v$ of the scalar field, which gives rise to massive vector bosons, a massive Higgs scalar and massive fermions. Of the $SU_L(2) \times U_Y(1)$ currents and charges only the electric current is still conserved, the other isospin- and hypercharges can be absorbed by or created from the BEH-condensate in the ground state of the theory. Nevertheless the symmetries are not fundamentally violated; heating up the BEH-condensate would release the isospin- and hypercharges it contains and restore the fully symmetric massless version of the standard model.

However, the quantum corrections to the gauge-boson interactions arising from the diagrams in fig. 19.1 fundamentally do not respect the gauge symmetries associated with the gauge bosons on the external lines whenever the fermion loop contains an odd number of γ_5 matrices (i.e., one or three). They represent non-renormalizable gauge symmetry-breaking interactions. Therefore we cannot allow such quantum contributions to the theory to be present and must impose as a consistency condition on the standard model that the total contribution of all fermions to the loops in these triangle diagrams vanish.

As it turns out, these conditions are indeed satisfied by the standard model, but not by the quarks and leptons separately: only by including both quarks and leptons in the loops do their contributions cancel. A standard model of only quarks or leptons would be non-renormalizable and inconsistent. Thus there exists a curious but important relation between the number of quark colors and the isospins and the hypercharges of both quarks and leptons.

For our purpose it is not necessary to calculate the precise form of the non-renormalizable 3-boson interactions represented by fig. 19.1. It suffices to know that with an odd number of γ_5 matrices from the vertices in the loop the total contribution of the two diagrams is proportional to

$$\pm \mathrm{Tr}\left[T_a(T_bT_c + T_cT_b)\right],$$

the sign being opposite for loops of left- and right-handed fermions. Therefore the consistency condition is met if the contributions of left- and right-handed fermions, with gauge charges $T_a^{(l)}$ and $T_a^{(r)}$ respectively, are equal:

$$\sum_l \mathrm{Tr}\left[T_a^{(l)}(T_b^{(l)}T_c^{(l)} + T_c^{(l)}T_b^{(l)})\right] = \sum_r \mathrm{Tr}\left[T_a^{(r)}(T_b^{(r)}T_c^{(r)} + T_c^{(r)}T_b^{(r)})\right]. \qquad (50)$$

Now an odd number of γ_5 matrices appear only in diagrams with at least one $SU_L(2)\times U_Y(1)$ gauge bosons. The $SU_c(3)$-charges are equal for left- and right-handed quarks, therefore the condition (50) is automatically satisfied for the case of 3 gluons at the triangle vertices.

Consider therefore first 3 hypercharge-bosons B coupling to a loop of left- and right-handed fermions of a single family as listed in table 10.1; then the trace terms in the fermion loops are

$$\sum_l Y^{(l)\,3} = 2 \times 3 \times \frac{1}{6^3} - 2 \times \frac{1}{2^3} = -\frac{2}{9},$$

$$\sum_r Y^{(r)\,3} = 3 \times \left(\frac{2}{3}\right)^3 - 3 \times \left(\frac{1}{3}\right)^3 - 1 = -\frac{2}{9},$$

satisfying (50). Next, for two W-bosons and a hypercharge boson B only left-handed fermions contribute; the trace of a symmetric combination of Pauli matrices then leads to

$$\sum_l Y^{(l)} (\tau_i\tau_j + \tau_j\tau_i) = 2\,\delta_{ij} \sum_l Y^{(l)} = 2 \times 3 \times \frac{1}{6} - 2 \times \frac{1}{2} = 0,$$

not requiring compensation from (non-existing) isospin charges of right-handed fermions. Finally, for two gluons and a single hypercharge boson the diagrams give contributions

$$\sum_l Y^{(l)} (\lambda_a\lambda_b + \lambda_b\lambda_a) = 2\,\delta_{ab} \left(2 \times 3 \times \frac{1}{6} \right) = 2\,\delta_{ab},$$

$$\sum_r Y^{(r)} (\lambda_a\lambda_b + \lambda_b\lambda_a) = 2\,\delta_{ab} \left(3 \times \frac{2}{3} - 3 \times \frac{1}{3} \right) = 2\,\delta_{ab},$$

balancing as required. No other non-trivial diagrams arise with single gluons or W-bosons, as the traces of Gell-Mann and Pauli matrices vanish: $\mathrm{Tr}\lambda_a = \mathrm{Tr}\,\tau_i = 0$.

A consistency condition on any extension of the standard model is now, that it respects the absence of anomalies from triangle diagrams as in fig. 19.1.

20. *Gauge interactions of right-handed neutrinos*

Obviously the right-handed neutrinos do not contribute to the above loop diagrams, as they do not couple to any of the $SU_c(3) \times SU_L(2) \times U_Y(1)$ gauge bosons. However, additional gauge symmetries and interactions with associated gauge bosons may be incorporated if any anomalous contribution to the triangle diagram from the other fermions coupling to the new gauge bosons can be canceled by a compensating contribution from the right-handed neutrinos. Thus in such extensions of the standard model the right-handed neutrinos would also have gauge interactions.

Extensions of the standard model with an enlarged group of gauge symmetries are indeed possible, the simplest one being an extension with an additional $U(1)$ gauge symmetry and a single associated new gauge boson. A direct proof is obtained from the explicit example of a $U_R(1)$ gauge group acting only on right-handed fermions by the R-charge with eigenvalues as in table 20.1:

particle	U_R	D_R	N_R	E_R
R	1	-1	1	-1

Table 20.1: R-charges of right-handed fermions.

Indeed with these assignments

$$\sum_r R^{(r)\,3} = 3 \times 1 - 3 \times 1 + 1 - 1 = 0,$$

and also

$$\sum_r Y^{(r)} R^{(r)\,2} = \sum_r Y^{(r)} = 3 \times \frac{2}{3} - 3 \times \frac{1}{3} + 0 - 1 = 0,$$

$$\sum_r Y^{(r)\,2} R^{(r)} = 3 \times \left(\frac{2}{3}\right)^2 - 3 \times \left(\frac{1}{3}\right)^2 + 0 - 1 = 0.$$

Given this result, it follows directly that any combination

$$X = \alpha Y + \beta R, \tag{51}$$

is an anomaly-free charge characterizing the coupling of left- and right-handed fermions to the associated linear combination of gauge bosons $X_\mu = \alpha B_\mu + \beta R_\mu$. Indeed

$$\sum_r \left(\alpha Y^{(r)} + \beta R^{(r)}\right)^3 = \alpha^3 \sum_r Y^{(r)\,3} + 3\alpha^2\beta \sum_r Y^{(r)\,2} R^{(r)} + 3\alpha\beta^2 \sum_r Y^{(r)} R^{(r)\,2}$$
$$+\, \beta^3 \sum_r R^{(r)\,3} = \alpha^3 \sum_r Y^{(r)\,3} = \sum_l \left(\alpha Y^{(l)} + \beta R^{(l)}\right)^3,$$

and so on. Note that the particular combination $\alpha = -2\beta = 2$ actually equals a special linear combination of baryon and lepton number:

$$2Y - R = B - L, \tag{52}$$

with $B - L = 1/3$ for quarks and $B - L = -1$ for leptons. Hence this combination of B and L is anomaly-free, in contrast with $B + L$ which is not an anomaly-free combination and cannot be associated with a conserved $U_X(1)$ gauge charge.

If any such gauge interaction mediated by bosons C_μ is to exist, these bosons must be massive in order for the interactions to be too weak to be detected as yet. This mass can not be created by the standard BEH scalars, as they are necessary to create the mass of the W- and Z-bosons. The minimal solution is for an additional single scalar field ϕ coupling to X_μ generating its mass m_X by its vacuum expectation value. This scalar field could at the same time generate a Majorana mass for the right-handed neutrinos. This is the scenario we will investigate next.

21. *An extended electro-weak gauge theory*

Consider the gauge theory of $SU_c(3) \times SU_L(2) \times U_Y(1) \times U_X(1)$ broken to $SU_c(3) \times U_{em}(1)$ as the minimal extension of the standard model. The scalar sector of this theory consists of the standard BEH-doublet of complex scalars $\Phi = (\varphi_1, \varphi_2)$ and an additional complex singlet scalar ϕ. This singlet cannot couple to $SU_c(3)$ or $SU_L(2)$, but it can couple to both of he $U_Y(1) \times U_X(1)$ gauge fields. Its hypercharge is not yet fixed, we denote it by η; its X-charge is taken to be $X = 1$ by definition. Then the covariant derivative of ϕ with coupling constants g_x and g_1 for the gauge fields X and B is

$$D\phi = \left(\partial - i(g_x C + g_1 \eta B)\right)\phi. \tag{53}$$

The standard BEH fields form a doublet under $SU_L(2)$ with hypercharge $Y = 1/2$ as given in table 15.1. Its X-charge is still to be fixed and parametrized by $X = \xi/2$. Then

$$D\Phi = \left(\partial - \frac{i}{2}\left(g_x \xi C + g_1 B + g_2 \mathbf{W} \cdot \boldsymbol{\tau}\right)\right)\Phi, \tag{54}$$

with

$$\mathbf{W}\cdot\boldsymbol{\tau} = \left(\begin{array}{cc} W_3 & \sqrt{2}\,W^+ \\ \sqrt{2}\,W^- & -W_3 \end{array} \right), \qquad W^{\pm} = \frac{W_1 \mp iW_2}{\sqrt{2}}.$$

Finally the dynamics of the scalar fields is governed by the invariant potential

$$V[\Phi,\phi] = \frac{\lambda_1}{4}\left(|\Phi|^2 - v_1^2\right)^2 + \frac{\lambda_2}{4}\left(|\phi|^2 - v_2^2\right)^2 + \frac{\lambda_m}{4}\left(|\phi|^2 - v_2^2\right)\left(|\Phi|^2 - v_1^2\right), \tag{55}$$

which has its absolute minimum $V = 0$ for scalar vacuum expectations

$$\langle |\Phi|^2 \rangle = v_1^2, \qquad \langle |\phi|^2 \rangle = v_2^2. \tag{56}$$

We will therefore parametrize the physical scalar fields by expanding around the vacuum expectation values in terms of real Higgs scalars h and a as

$$\Phi = \left(\begin{array}{c} 0 \\ v_1 + h/\sqrt{2} \end{array} \right), \qquad \phi = v_2 + a/\sqrt{2}. \tag{57}$$

The scalar-field action is

$$S_{sc}[\Phi,\phi] = \int d^4x \left(-|D\Phi|^2 - |D\phi|^2 - V[\Phi,\phi] \right), \tag{58}$$

and expanding around the vacuum expectation values this becomes

$$\begin{aligned} S_{sc} \;\; = \;\; & \int d^4x \left(-\frac{g_2^2 v_1^2}{2}\, W^+W^- - \frac{v_1^2}{4}\left(g_x\xi C + g_1 B - g_2 W_3\right)^2 - v_2^2\left(g_x C + g_1\eta B\right)^2 \right. \\ & \left. + \text{terms involving Higgs fields } (h,a) \right). \end{aligned}$$

This implies the existence of 3 sets of massive vector bosons:
a. charged $W^{\pm}$-bosons with mass

$$m_W^2 = \frac{g_2^2 v_1^2}{2}; \tag{59}$$

b. two neutral vector bosons

$$Z = \frac{g_2 W_3 - g_1 B - g_x \xi C}{\sqrt{g_2^2 + g_1^2 + g_x^2 \xi^2}}, \qquad Z' = \frac{g_x C + g_1 \eta B}{\sqrt{g_x^2 + g_1^2 \eta^2}}, \tag{60}$$

with masses

$$m_Z^2 = \frac{v_1^2}{2}\left(g_2^2 + g_1^2 + g_x^2 \xi^2\right), \qquad m_{Z'}^2 = 2 v_2^2 \left(g_x^2 + g_1^2 \eta^2\right). \tag{61}$$

A third independent linear combination of (W_3, B, C) remains massless and represents the photon γ. It can be obtained by constructing the three normalized vector fields obtained from a rotation of (W_3, B, C) and reproducing the combinations (60):

$$\begin{aligned} \gamma &= W_3 \sin\theta_w + (B\cos\delta - C\sin\delta)\cos\theta_w, \\ Z &= W_3 \cos\theta_w - (B\cos\delta - C\sin\delta)\sin\theta_w, \\ Z' &= B\sin\delta + C\cos\delta; \end{aligned} \tag{62}$$

Equivalently:

$$\begin{aligned} W_3 &= \gamma \sin\theta_w + Z \cos\theta_w, \\ B &= Z' \sin\delta + (\gamma \cos\theta_w - Z \sin\theta_w) \cos\delta, \\ C &= Z' \cos\delta - (\gamma \cos\theta_w - Z \sin\theta_w) \sin\delta, \end{aligned} \tag{63}$$

with

$$\cos\delta = \frac{g_x}{\sqrt{g_x^2 + g_1^2 \eta^2}}, \qquad \sin\delta = \frac{g_1 \eta}{\sqrt{g_x^2 + g_1^2 \eta^2}},$$

$$\cos\theta_w = \frac{g_2}{\sqrt{g_2^2 + g_1^2 + g_x^2 \xi^2}}, \quad \sin\theta_w = \frac{g_1}{g_x} \sqrt{\frac{g_x^2 + g_1^2 \eta^2}{g_2^2 + g_1^2 + g_x^2 \xi^2}}.$$

However, from the definition of Z it also follows that

$$\sin\delta = -\frac{g_x^2 \xi}{g_1 \sqrt{g_x^2 + g_1^2 \eta^2}}.$$

The two expressions for $\sin\delta$ agree only if

$$\frac{\xi}{\eta} = -\frac{g_1^2}{g_x^2}. \tag{64}$$

Thus the hypercharge and X-charge of the scalar bosons Φ and ϕ are related. As a result, the masses of the neutral vector bosons are

$$m_\gamma^2 = 0, \qquad m_Z^2 = \frac{g_2^2 v_1^2}{2 \cos^2\theta_w} = \frac{m_W^2}{\cos^2\theta_w}, \qquad m_{Z'}^2 = \frac{2 g_x^2 v_2^2}{\cos^2\delta}. \tag{65}$$

The relation between the W- and Z-mass in terms of the weak mixing angle θ_w is therefore the same as in the original non-extended standard model.

22. *Scalar masses*

By expanding around the minimum of the scalar potential $V[v_1, v_2] = 0$ the lowest-order terms are the quadratic terms from which the mass eigenvalues can be found:

$$V[h, a] = \frac{\lambda_1 v_1^2}{2}\, h^2 + \frac{\lambda_2 v_2^2}{2}\, a^2 + \frac{\lambda_m v_1 v_2}{2}\, ha + \mathcal{O}[(h, a)^3]$$

To find the mass eigenvalues we diagonalize these terms by a rotation

$$h = h_- \cos\theta_s + h_+ \sin\theta_s, \qquad a = -h_- \sin\theta_s + h_+ \cos\theta_s, \tag{66}$$

after which the potential becomes

$$V[h_-, h_+] = \frac{1}{2}\, m_-^2 h_-^2 + \frac{1}{2}\, m_+^2 h_+^2 + \mathcal{O}[(h_-, h_+)^3],$$

provided

$$\tan 2\theta_s = \frac{\lambda_m v_1 v_2}{\lambda_1 v_1^2 - \lambda_2 v_2^2}, \tag{67}$$

and with the mass eigenvalues

$$m_\pm^2 = \frac{1}{2}\left(\lambda_1 v_1^2 + \lambda_2 v_2^2 \pm \sqrt{(\lambda_1 v_1^2 - \lambda_2 v_2^2)^2 + \lambda_m^2 v_1^2 v_2^2}\right). \tag{68}$$

To compare with directly measurable quantities, recall that

$$v_1^2 = \frac{2m_Z^2 \cos^2\theta_w}{g_2^2} = \frac{2m_W^2}{g_2^2}, \qquad v_2^2 = \frac{m_{Z'}^2 \cos^2\delta}{2g_x^2}. \tag{69}$$

23. *Fermions and Yukawa couplings*
In the standard model extended with $U_X(1)$ the covariant Dirac operators (using covariant derivatives) of the left- and righthanded fermions are of the form $\gamma \cdot D$ with

$$D\Psi_R = (\partial - ig_x XC - ig_1 YB)\,\Psi_R, \quad D\Psi_L = \left(\partial - ig_x XC - ig_1 YB - \frac{ig_2}{2}\,\mathbf{W}\cdot\boldsymbol{\tau}\right)\Psi_L,$$

where X and Y are the X-charge (51) and the hypercharge, respectively. Note that according to table 20.1 the X-charges of the right-handed fermions (U_R, D_R, N_R, E_R) are of the form $X_R = \alpha Y \pm \beta$, whereas for left-handed quark and lepton doublets $Q_L = (U_L, D_L)$ and $L_L = (N_L, E_L)$ the R-charge vanishes and therefore the X-charge is directly proportional to the hypercharge: $X_L = \alpha Y$. Thus the X-charges are parametrized as in table 23.1:

particle	Q_L	U_R	D_R	L_L	N_R	E_R
X	$\alpha/6$	$2\alpha/3+\beta$	$-\alpha/3-\beta$	$-\alpha/2$	β	$-\alpha-\beta$

Table 23.1: X-charges of the standard-model fermions

By comparison with the Dirac equations (37) and (38) it follows that the Yukawa couplings in these equations are consistent with every one of the X-charge assignments if

$$\xi = -\left(\alpha + 2\beta\right). \tag{70}$$

This provides one more consistency check on the $U_X(1)$-extension of the standard model. Equivalently, the Lagrange density (58) can be extended with the standard Yukawa coupling terms

$$\begin{aligned} S_{Yuk}[\Phi; U, D, N, L] = \frac{i}{2}\int d^4x\, \Big[& f_N \bar{N}_R\left(\varphi_2 N_L - \varphi_1 E_L\right) + f_N^*\left(\varphi_2^* \bar{N}_L - \varphi_1^* \bar{E}_L\right) N_R \\ & + f_E \bar{E}_R\left(\varphi_1^* N_L + \varphi_2^* E_L\right) + f_E^*\left(\varphi_1 \bar{N}_L + \varphi_2 \bar{E}_L\right) E_R \\ & + f_U \bar{U}_R\left(\varphi_2 U_L - \varphi_1 D_L\right) + f_U^*\left(\varphi_2^* \bar{U}_L - \varphi_1^* \bar{D}_L\right) U_R \\ & + f_D \bar{D}_R\left(\varphi_1^* U_L + \varphi_2^* D_L\right) + f_D^*\left(\varphi_1 \bar{U}_L + \varphi_2 \bar{D}_L\right) D_R \Big]. \end{aligned} \tag{71}$$

It is straightforward to check that the combinations

$$\bar{N}_R L_L,\ \bar{L}_L E_R,\ \bar{U}_R Q_L,\ \bar{Q}_L D_R,$$

all have the total X-charge $\alpha/2 + \beta = -\xi/2$, whereas

$$\bar{L}_L N_R,\ \bar{E}_R L_L,\ \bar{Q}_L U_R,\ \bar{D}_R Q_L,$$

all have X-charge $-(\alpha/2 + \beta) = \xi/2$. Therefore the above Yukawa terms turn out to be invariant under $U_X(1)$.

With the addition of the singlet scalar ϕ which has $X = 1$ and $Y = \eta$ it is possible in principle to construct new Yukawa terms with bilinear combinations for which $X = -1$ and $Y = -\eta$, with their complex conjugates. However, no non-trivial $SU_L(2)$ singlets can be constructed directly from the fermions, except the Majorana-type coupling $N_R^T C N_R$ in the special case $\eta = \xi = 0$ implying also $\delta = 0$. From (70) it follows directly that $\alpha = -2\beta$. Then the invariance of the Majorana-Yukawa term

$$S_{MYuk}[\phi, N_R] = \frac{i}{2} \int d^4x \left(f_M \phi^* N_R^T C N_R + f_M^* \phi\, \bar{N}_R C \bar{N}_R^T \right), \tag{72}$$

requires

$$2\beta = -1 \quad \Rightarrow \quad \alpha = 1, \tag{73}$$

and the X-charges of the fermions become

particle	Q_L	U_R	D_R	L_L	N_R	E_R
X	1/6	1/6	1/6	−1/2	−1/2	−1/2

Table 23.2: X-charges of the standard-model fermions allowing for a Majorana neutrino mass

or simply

$$X = \frac{1}{2}(B - L). \tag{74}$$

Note that the vacuum expectation value of ϕ generating a Majorana mass for the right-handed neutrinos automatically breaks the $B - L$ symmetry spontaneously. Thus we find that the possibility of adding Majorana mass terms for the right-handed neutrinos is directly linked to non-conservation of $B - L$ in the neutrino sector. The corresponding gauge boson, not mixing with the other electro-weak vector bosons, is $Z' = C$ with mass

$$m_{Z'}^2 = 2g_x^2 v_2^2. \tag{75}$$

However, for $\lambda_m \neq 0$ the mixing of the Higgs bosons still survives and is reflected in the mass terms (68). Note that in the limit of very large $m_{Z'}^2$, with $v_1^2/v_2^2 \ll 1$, the standard Higgs mass is shifted from the bare standard-model result by

$$m_-^2 = \lambda_1 v_1^2 \left(1 - \frac{\lambda_m^2}{4\lambda_1\lambda_2} + \mathcal{O}[v_1^2/v_2^2] \right) = \frac{2\lambda_1 m_Z^2 \cos^2\theta_w}{g_2^2} \left(1 - \frac{\lambda_m^2}{4\lambda_1\lambda_2} + \mathcal{O}[v_1^2/v_2^2] \right). \tag{76}$$

24. *Grand Unification*

From the analysis of anomalies it follows that a $U_X(1)$ is the only possible additional gauge symmetry of the standard model.[4] However, it is very well possible to extend the symmetries of the standard model by having $G_{sm} = SU_c(3) \times SU_L(2) \times U_Y(1)$ as the subgroup of a larger symmetry group in which all anomalous contributions also cancel among themselves. One of the simplest extensions which presents itself in this context is an $SU(5)$ gauge symmetry. The simplest realization of this symmetry is the set of unitary 5×5-matrices with unit determinant; they can be written in the form

$$U = e^{ia^\alpha T_\alpha},$$

with a^i arbitrary real parameters and T_α a set of linearly independent traceless hermitean 5×5 matrices: $\mathrm{Tr}\, T_\alpha = 0$ and $T_\alpha^\dagger = T_\alpha$.[5] Now there are 25 independent hermitean 5×5 matrices, but only 24 can be made traceless by taking linear combinations. Therefore $\alpha = 1, ..., 24$. A subset of these are the block-diagonal 5×5-matrices

$$\tilde{T}_\alpha = \left\{ \left(\begin{array}{cc} \lambda_a & 0 \\ 0 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & 0 \\ 0 & \tau_i \end{array} \right) \right\}, \quad a = 1, ...8; \quad i = 1, 2, 3,$$

with λ_a the Gell-Mann matrices and τ_i the Pauli matrices. Clearly the upper block is a complete set of traces hermitean 3×3 matrices, which generate the group $SU(3)$, and the lower 2×2 block similarly generates $SU(2)$. $U(1)$ can be generated by a properly normalized matrix

$$\tilde{T}_1 \propto \left(\begin{array}{cc} \frac{1}{3}\, 1_3 & 0 \\ 0 & -\frac{1}{2}\, 1_2 \end{array} \right),$$

with 1_n the unit matrix in n dimensions. In this way G_{sm} can be embedded in $SU(5)$. In this larger gauge theory the fermions of the standard model transform not only under $SU_c(3) \times SU_L(2) \times U_Y(1)$, but also under the remaining off-diagonal elements of $SU(5)$ which mix quarks and leptons. Therefore in such a theory quarks can be transformed into leptons and vice-versa, such that baryon number B and lepton number L are not conserved separately. Of course, as such transitions have never been observed, they must be heavily suppressed, which implies that the vector bosons corresponding to the off-diagonal elements of $SU(5)$ mediating these transitions should have very large masses.

This idea is actually supported by the experimental results for the coupling constants of the standard model gauge interactions. In the lagrangean of the standard model the coupling constants (g_1, g_2, g_3) are just numerical parameters. However, the coupling of fermions to gauge bosons, or of gauge bosons to other gauge bosons and scalars, is determined experimentally by measuring the cross section for various processes involving the

[4] Apart from local co-ordinate invariance in the context of general relativity, which introduces gravity into the theory.

[5] Note that hermitean matrices have real eigenvalues λ; they are traceless if $\sum \lambda = 0$. Therefore diagonalizing $a^i T_i$ makes U diagonal with all eigenvalues $e^{i\lambda}$ of unit modulus, such that $U^{-1} = U^\dagger$ and $\det U = e^{i \sum \lambda} = 1$.

interaction of these particles. But these cross sections vary with the energies involved, and therefore the strengths of the interactions depend on the energy scale at which they are measured.

In the quantum field theories for these interactions such an energy dependence arises because the quantum corrections to processes, involving virtual particles, also depend on the energy scale. Thus the actual value of the coupling constant to be used depends on the scale at which the theory is compared with the experiment; this effect is known as the *running* of the coupling constant. The strengths of the coupling at an energy scale μ is denoted by the dimensionless coupling strength

$$\alpha(\mu) = \frac{g^2(\mu)}{4\pi}.$$

The running has been established experimentally. For the standard model interactions their behaviour can be extrapolated to high energies, as shown in fig. 24.1. From the figure

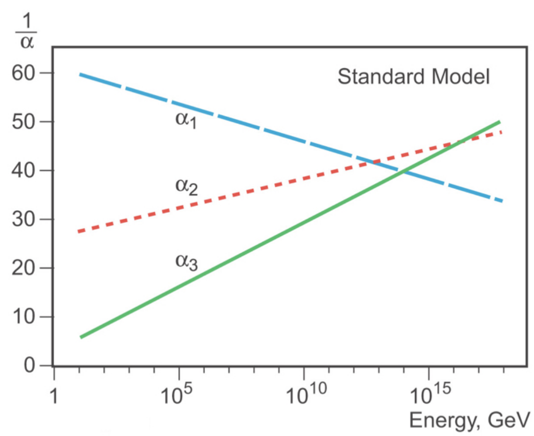

Fig. 24.1: Running couplings of $SU_c(3) \times SU_L(2) \times U_Y(1)$ in the standard model[6]

one sees they tend to end up in the same range of values at energy scales around 10^{15} GeV. This would then be a natural scale at which the standard model interactions may be unified in a larger gauge theory like $SU(5)$.

As concerns the extension of the standard model with $U_X(1)$, the gauge group $G_{sm} \times U_X(1)$ is not a subgroup of $SU(5)$, and gauge interactions of right-handed neutrinos with a Z' accompanied by Majorana masses still require a similar extension to $SU(5) \times U_X(1)$. In this respect, as well as with a view to anomalies, a grand unification of $SU_c(3) \times SU_L(2) \times U_Y(1) \times U_X(1)$ as a subgroup of $SO(10)$ is much better. In fact, $SO(10)$ includes $SU(5) \times U(1)$ as a subgroup. Its chiral fermion multiplets are all intrinsically anomaly-free and incorporate right-handed neutrinos and their gauge interactions from the start; in particular one of the gauge charges they carry is proportional to $B - L$. Therefore again Majorana masses for the right-handed neutrinos can be introduced at the price of

[6]Credit: www.nobelprize.org, 2004

spontaneously broken $B - L$ invariance. This is in fact part of the breaking of the $SO(10)$ unified gauge invariance.

25. *Neutrinos in cosmology*
In the very early universe, at temperatures above the Fermi scale, all standard-model particles were massless. Thermal equilibrium then dictates that their number densities were equal and large. When the universe expanded and cooled, masses were generated and the more massive particles started to decay into lighter ones. Either because of the rapid expansion of the universe itself, or during phase transitions, CP-violating processes created a small excess of quarks over anti-quarks and eventually baryons over anti-baryons. At temperatures well above 1 MeV the weak interactions were still strong enough to maintain equilibrium between photons, charged leptons and neutrinos, as well as between protons and neutrons. The balance was maintained by reactions like

$$e^+ + e^- \leftrightarrow \gamma + \gamma,$$

$$p^+ + e^- \leftrightarrow n + \nu_e,$$

$$n + e^+ \leftrightarrow p^+ + \bar{\nu}_e.$$

In this plasma the number densities of photons and relativistic electrons, positrons and neutrinos were all equal. In contrast, quarks and anti-quarks had already mostly annihilated each other, except for the small excess of quarks from which all present baryonic matter was formed. This resulted in the baryon density of the universe being extremely small compared to the number density of photons, roughly 1 nucleon per 10^9 photons. However, especially as the universe was still young compared to the life time of the neutron, the reactions above still maintained the balance between protons and neutrons.

When the temperature dropped further to values where the electrons and positrons were no longer relativistic, the weak interactions became very weak indeed and neutrinos decoupled. The positrons and electrons annilated, thereby producing large numbers of additional photons: the reaction $e^+ + e^- \rightarrow \gamma + \gamma$ now proceeded only in one direction. About 1 in 10^9 electrons was left over, balancing the charge of the protons to make the universe electrically neutral. In principle the production of neutrinos would have been possible: $e^+ + e^- \rightarrow \bar{\nu}_e + \nu_e$, but the large Z-mass prevented this from happening at any appreciable rate. As a result during that period of cosmic expansion the number density and temperature of the photons decreased slower than the of the decoupled neutrino gas.

At still much lower temperatures, first deuterium and α-particles were created in a matter of minutes; and after about 380 000 years, at temperatures of about 3000 K, finally neutral hydrogen and helium atoms could form and the photons also decoupled from baryonic matter. These photons are now observed as the cosmic microwave background (CMB). Its present temperature is 2.73 K, implying a number density of 411 photons/cm^3. In contrast, the present number density of baryons is 2.5×10^{-7}/cm^3. The scenario just sketched can be worked out quantitatively and the theory is in very good agreement with the observed relative amounts of hydrogen, helium, deuterium and other light nuclei in the present universe. As a result, it is expected that next to the CMB there is also a cosmic background of neutrinos (CνB), at a still somewhat lower temperature. For massless neutrinos it would be about 1.9 K. Thus we expect large number densities of very light low-energy neutrinos to be present throughout the universe, of the order of 200 neutrinos/cm^3. Unfortunately, due to the extreme weakness of their interactions and in spite of their relatively large numbers it has as yet not been possible to test this prediction.

In some cosmological scenarios also heavy neutrinos play a role. If they are quasi-stable they are a potential candidate for dark matter. If they have decayed they may have played a crucial role in the creation of the small excess of quarks over anti-quarks which is responsible for the present baryonic matter density in the universe. In particular, if right-handed neutrinos have a large Majorana mass well above the Fermi scale, they can decay by the Yukawa interactions

$$N_R \to \varphi^* L_L \quad \text{or} \quad N_R \to \bar{L}_L \varphi.$$

Now the Yukawa couplings contain the CP-violating phases that we have seen to show up in the PMNS matrix. Therefore these decays may proceed at unequal rates. If the decay of the massive Majorana neutrinos N_R is slow compared to the expansion of the universe, the drop in temperature will not allow the inverse processes, implying that the decay takes place out of thermal equilibrium, and different numbers of leptons and anti-leptons can be created. This results in a lepton number density ΔL. This scenario is called leptogenesis. Finally this lepton number can be transferred to baryons and explain the observed density of baryons in the universe, in absence of anti-baryons.

The transfer process involves a non-perturbative weak interaction caused by a configuration of weak-interaction fields called sphalerons. These configurations allow transitions between states with different particle content, whereby leptons can be converted into quarks and anti-quarks or quarks into leptons and anti-leptons. The basic argument is that baryon-number changing processes are split into those in which $B - L$ is conserved and those which change $B + L$, such that

$$\Delta B(t) = \frac{1}{2}\,\Delta(B - L)(t) + \frac{1}{2}\,\Delta(B + L)(t).$$

It is then argued that the first term is time-independent and is equal to its initial value: $\Delta(B - L)(t) = -\Delta L(0)$, whilst the second term becomes almost negligible at the time of the electro-weak phase transition, when the sphalerons are active. If $\Delta(B+L)$ is neglected, the final result at late times $t \to \infty$ is a change in the baryon number given by

$$\Delta B(\infty) = -\frac{1}{2}\,\Delta L(0),$$

i.e. the net change in baryon number equals half the net change in anti-leptons. In more careful modelling the factor 1/2 is modified and depending on assumptions about the circumstances becomes a number closer to 1/3. In any case the leptogenesis thus gives rise to baryogenesis. Whether this scenario also is in quantitative agreement with the observed baryon density depends on many factors and assumptions in the modelling. At present no universally agreed model of baryogenesis has come forward.

Neutrinos, they are very small.
They have no charge and have no mass
And do not interact at all.
The earth is just a silly ball
To them, through which they simply pass,
Like dustmaids through a drafty hall
Or photons through a sheet of glass.
They snub the most exquisite gas,
Ignore the most substantial wall,
Cold-shoulder steel and sounding brass,
Insult the stallion in his stall,
And scorning barriers of class,
Infiltrate you and me! Like tall
And painless guillotines, they fall
Down through our heads into the grass.
At night, they enter at Nepal
And pierce the lover and his lass
From underneath the bed – you call
It wonderful; I call it crass.

John Updike