Nikhef Universiteit Leiden

# General Relativity

*Einstein's dynamical field theory of gravity*

*J.W. van Holten* *Nikhef 19-06-2017*

## Equivalence principle

Gravitational acceleration is independent of mass or composition of falling body:

$$\frac{\Delta a}{a} = (0.3 \pm 1.8) \times 10^{-13}$$

(S. Schlamminger et al., PRL(2008), 100.041101; arXiv:0712.0607)

JOHN PHILOPONUS (517 AD)
COMMENTARY ON ARISTOTLE'S PHYSICS

DE
BEGHINSELEN
DER WEEGHCONST
BESCHREVEN DVER
SIMON STEVIN
van Brugghe.

TOT LEYDEN,
Inde Druckerye van Christoffel Plantijn,
By Françoys van Raphelinghen.
clↃ. IↃ. LXXXVI.

*Simon Stevin* (1586)

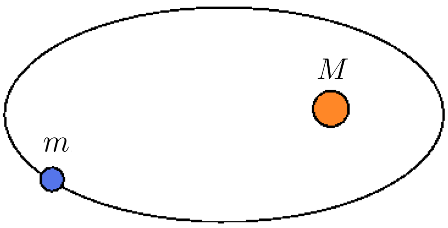
DISCORSI
E
DIMOSTRAZIONI
MATEMATICHE,
*intorno à due nuoue scienze*
Attenenti alla
MECANICA & I MOVIMENTI LOCALI,
*del Signor*
GALILEO GALILEI LINCEO,
Filosofo e Matematico primario del Serenissimo
Grand Duca di Toscana.
*Con vna Appendice del centro di grauità d'alcuni Solidi.*

IN LEIDA,
Appresso gli Elsevirii. M. D. C. XXXVIII.

*Galileo* (1638)

Newton

## static gravitational force




$$\mathbf{F} = m\mathbf{a} = G\,\frac{mM}{r^2}\,\hat{\mathbf{r}}$$

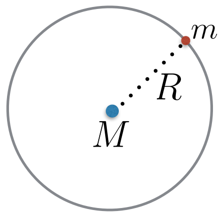
- *equivalence of gravitational and inertial mass*

- *universal attraction*

- *direct action at a distance (violates relativity)*

## Kepler's law

circular orbit of test mass $m \ll M$:




$$a = \frac{GM}{R^2} = \frac{v^2}{R}$$

*gravitational acceleration* *centripetal acceleration*

$$v = \frac{2\pi R}{T} \longrightarrow T^2 = \frac{4\pi^2}{GM} R^3$$

N.B: similar mass binaries $m \sim M$:

$$M \longrightarrow \mu = \frac{mM}{m+M}$$

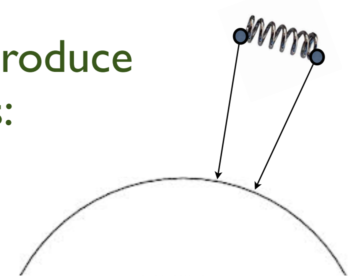
## Universality of gravity

universality of gravitational acceleration

$\longrightarrow$ with the same initial conditions (position and velocity) all massive (test) bodies follow the same trajectory

$\longrightarrow$ in *homogeneous* gravitational fields nearby bodies in free fall are not accelerated w.r.t. each other: *they move in a local inertial frame*

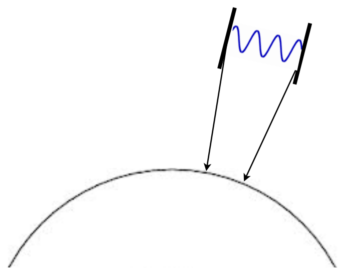
$\longrightarrow$ *non-homogeneous* gravitational fields produce relative acceleration of nearby bodies: *tidal forces*



# Light rays

- In local inertial frames light travels in straight lines and the velocity of light is a universal constant *c*
  *(~300 000 km/sec)*

- light rays are affected by tidal forces:
  light rays in free fall in one frame are red/blueshifted as seen by a distant observer in a different inertial frame if there is *relative* acceleration between those frames

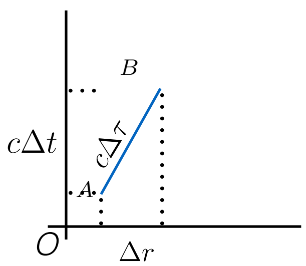


## Inertial frames:

### Minkowski space times

proper time: invariant measure of time

*measured by co-moving clock in instantaneous inertial rest frame (co-moving in free fall)*



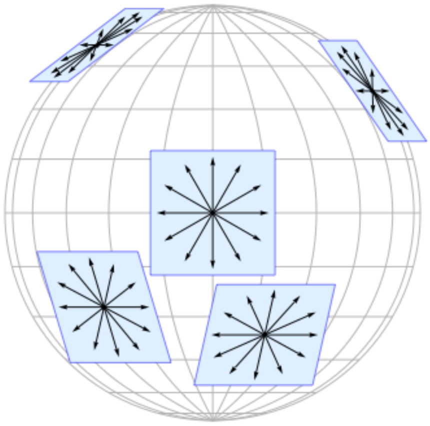

$$c^2 d\tau^2 = \underline{c^2 dt^2 - d\mathbf{r}^2} \quad \textit{Lorentz invariant}$$

$$= c^2 dt^2 \left(1 - \frac{\mathbf{v}^2}{c^2}\right) \quad \textit{with} \quad \mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$\Delta\tau = \int_A^B d\tau = \int_A^B dt \underbrace{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}_{\textit{time dilation}}$$

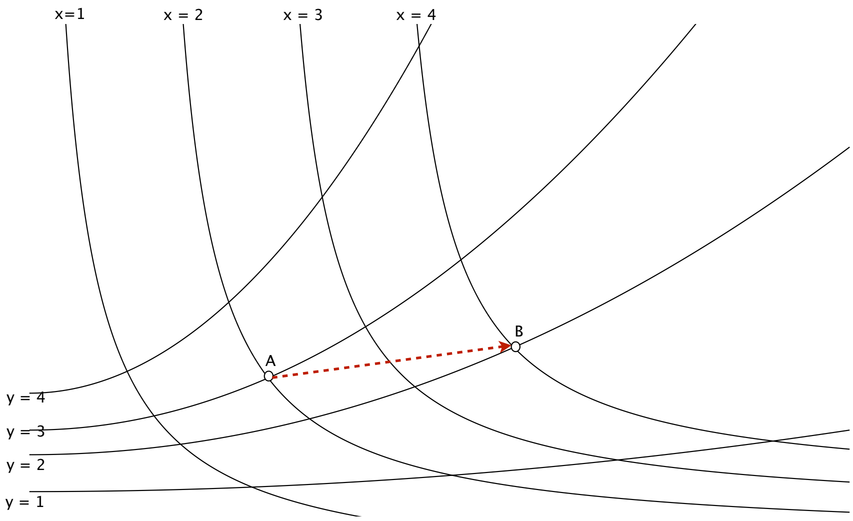
light cone: $\mathbf{v}^2 = c^2 \quad \Leftrightarrow \quad d\mathbf{r}^2 = c^2 dt^2 \quad \Leftrightarrow \quad d\tau^2 = 0$

Space-time is a mesh of local inertial frames (free-falling coordinate systems) glued together to form a smooth 3+1 dimensional manifold



the shortest path between points is constructed by glueing short sections of straight lines in each local inertial frame $\longrightarrow$ geodesics

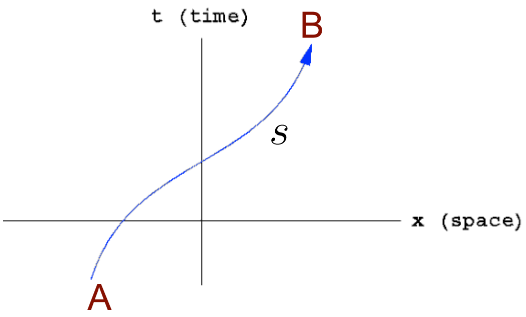
## Distances and co-ordinates




$$[d(A,B)]^2 = \alpha\Delta x^2 + 2\gamma\Delta x\Delta y + \beta\Delta y^2$$

# Geodesics

**Space-time metric:** $$c^2 d\tau^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$




world line $x^\mu(s)$

point B is in the future of A

$$c\Delta\tau = \int_A^B ds \sqrt{g_{\mu\nu}(x) \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}$$

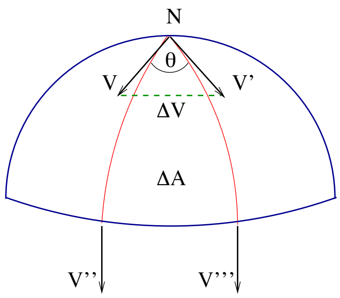
extremal: $\delta\Delta\tau = 0 \quad \Rightarrow \quad \dfrac{d^2x^\mu}{d\tau^2} + \Gamma_{\lambda\nu}{}^{\mu}(x)\dfrac{dx^\lambda}{d\tau}\dfrac{dx^\nu}{d\tau} = 0$

with $$\Gamma_{\lambda\nu}{}^{\mu} = \frac{1}{2}\, g^{\mu\kappa}\left(\frac{\partial g_{\kappa\nu}}{\partial x^\lambda} + \frac{\partial g_{\kappa\lambda}}{\partial x^\nu} - \frac{\partial g_{\lambda\nu}}{\partial x^\kappa}\right)$$

connection

inverse metric

# Intrinsic curvature




$$\Delta V = V\Delta\theta = V\,\frac{\Delta A}{R^2}$$

*generalize:*

$$\Delta V^\mu = \frac{1}{2}\,\Delta A^{\kappa\lambda}\, V^\nu R_{\kappa\lambda\nu}{}^{\mu}$$

Riemann tensor $$R_{\kappa\lambda\nu}{}^{\mu} = \frac{\partial \Gamma_{\lambda\nu}{}^{\mu}}{\partial x^\kappa} - \frac{\partial \Gamma_{\kappa\nu}{}^{\mu}}{\partial x^\lambda} - \Gamma_{\kappa\nu}{}^{\sigma}\Gamma_{\sigma\lambda}{}^{\mu} + \Gamma_{\lambda\nu}{}^{\sigma}\Gamma_{\sigma\kappa}{}^{\mu}$$

Ricci tensor: $R_{\mu\nu} \equiv R_{\mu\lambda\nu}{}^{\lambda}$

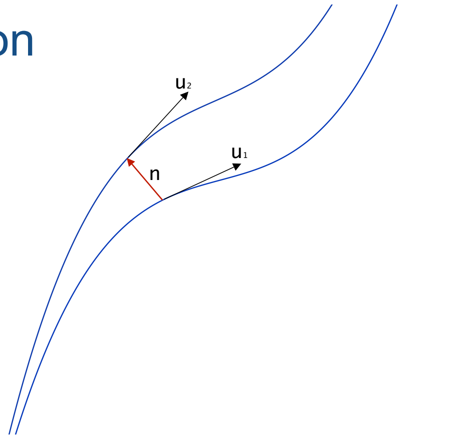
Riemann scalar $R \equiv R_\mu{}^\mu = g^{\mu\nu} R_{\mu\nu}$

## geodesic deviation

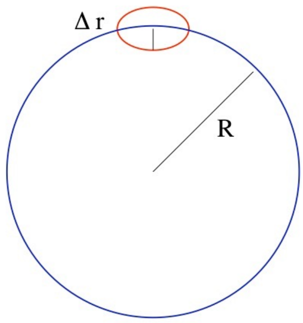
starting from known reference geodesic $x^\mu(\tau)$ construct a class of geodesics

$$x^\mu(\tau,\sigma) = x^\mu(\tau) + \sigma n^\mu(\tau) + \ \ldots$$

$$D_\tau^2 n^\mu = R_{\kappa\nu\lambda}{}^{\mu} u^\kappa u^\lambda n^\nu$$




eccentric orbits:
precession of perihelium




## Dynamical geometry

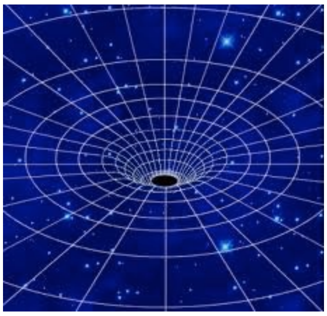
The evolution of space-time geometry is described by the Einstein equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\boxed{\frac{8\pi G}{c^4}} T_{\mu\nu}$$

$$\boxed{2.1 \times 10^{-41}\,\mathrm{kg}^{-1}\ \mathrm{m}^{-1}\ \mathrm{s}^2}$$

Here $T_{\mu\nu}$ is the energy-momentum tensor of matter and radiation

*\`Matter tells space how to curve, space tells matter how to move\`*



# Static and spherically symmetric space-time

$$c^2 d\tau^2 = \left(1 - \frac{2\mu}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - 2\mu/r} - r^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

with $\mu = \dfrac{GM}{c^2}$

At constant $r$ and $t$:

circle length: $L = \displaystyle\int_0^{2\pi} r d\varphi = 2\pi r$

spherical surface area: $A = \displaystyle\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta\, r^2 \sin\vartheta = 4\pi r^2$

## acceleration of static point mass

proper velocity $\dfrac{dx^\mu}{d\tau} = u^\mu = (u^t, 0, 0, 0)$ (static)

$$g_{\mu\nu} u^\mu u^\nu = g_{tt} u^{t\,2} = c^2 \quad \text{with} \quad u^t = \frac{c}{\sqrt{1 - 2\mu/R}}$$

(fixed)

proper acceleration $a^\mu = \dfrac{du^\mu}{d\tau} + \Gamma_{\lambda\nu}{}^{\mu} u^\mu u^\nu = 0 + \Gamma_{tt}{}^{\mu} u^{t\,2}$

(deviation from free fall)

$\longrightarrow$ $a^\mu = (0, a^r, 0, 0)$ with $a^r = \dfrac{\mu c^2}{R^2} = \dfrac{GM}{R^2}$

Newton

## Gravitational physics

Static spherically symmetric geometry around a mass $M$:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \frac{dr^2}{1 - 2GM/c^2 r} - r^2 \left(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2\right)$$

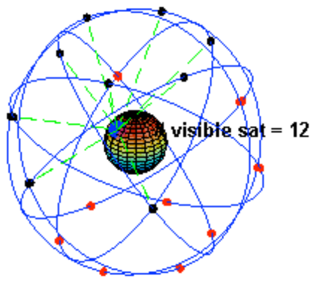
In limit $GM/c^2 r \ll 1$ and with $d\mathbf{r} = \mathbf{v} dt$

$$d\tau \approx dt \left(1 - \frac{GM}{c^2 r} - \frac{v^2}{2c^2} + \ldots\right)$$

gravitational time dilation

kinematic time dilation (special relativity)

## GPS satellites




$$\frac{GM_\oplus}{c^2} = 4.5 \times 10^{-3}\,\mathrm{m}$$

$$R_\oplus = 6.4 \times 10^6\,\mathrm{m}$$

$$R_{GPS} = 2.7 \times 10^7\,\mathrm{m}$$

Kinematic: $$\left.\frac{v^2}{R}\right|_{GPS} = \frac{GM_\oplus}{R_{GPS}^2} \quad \Rightarrow \quad \left.\frac{v^2}{c^2}\right|_{GPS} = \frac{GM_\oplus}{c^2 R_{GPS}}$$

Result: $$\boxed{\left.\frac{dt}{d\tau}\right|_\oplus - \left.\frac{dt}{d\tau}\right|_{GPS} = 0.45 \times 10^{-9}}$$