

Gravitational wave data analysis: Detection

Gravitational waves

- Linearized general relativity: Einstein equations become a *wave equation*

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- Gravitational radiation is primarily generated by *time-dependent mass quadrupole moment*:

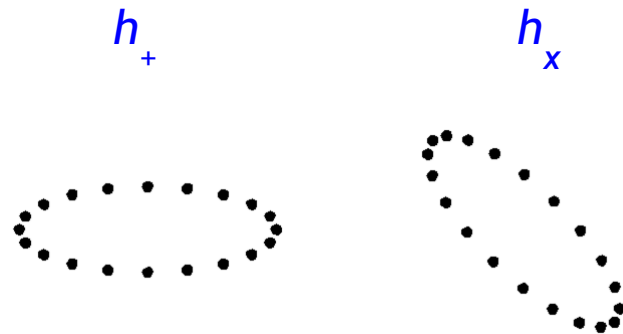
$$[h_{ij}^{\text{TT}}(t, \mathbf{x})]_{\text{quad}} = \frac{1}{r} \frac{2G}{c^2} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \ddot{M}^{kl}(t - r/c)$$

- Nearby point particles in free fall, small separation ζ^μ

→ *Tidal effect*:

$$\ddot{\zeta}^i = \frac{1}{2} \ddot{h}_{ij}^{\text{TT}} \zeta^j$$

"plus" and "cross" polarizations:

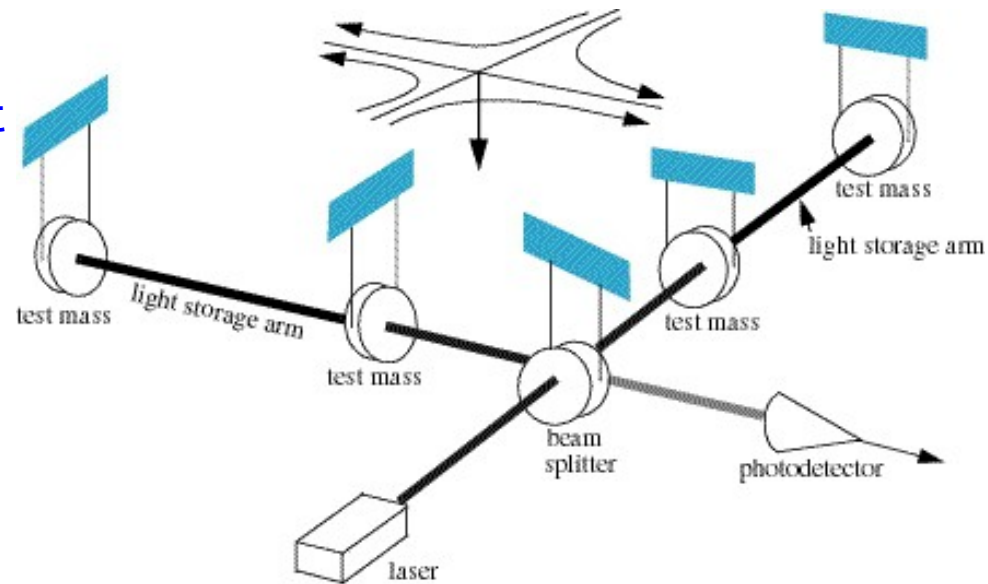


Interferometric detectors

$$\ddot{\zeta}^i = \frac{1}{2} \ddot{h}_{ij}^{\text{TT}} \zeta^j$$

● Interferometric detectors

- Laser beam through beam splitter
- Power built up in long (several km) cavities
- At output: destructive interference
... unless gravitational wave present



Interferometric detectors

$$\ddot{\zeta}^i = \frac{1}{2} \ddot{h}_{ij}^{\text{TT}} \zeta^j$$

- Motion of the end mirrors

$$\delta \ddot{\zeta}_1^x = \frac{1}{2} \ddot{h}_{xx} (L + \cancel{\delta \zeta_1^x})$$

$$\delta \ddot{\zeta}_2^y = \frac{1}{2} \ddot{h}_{yy} (L + \cancel{\delta \zeta_2^y})$$

... hence

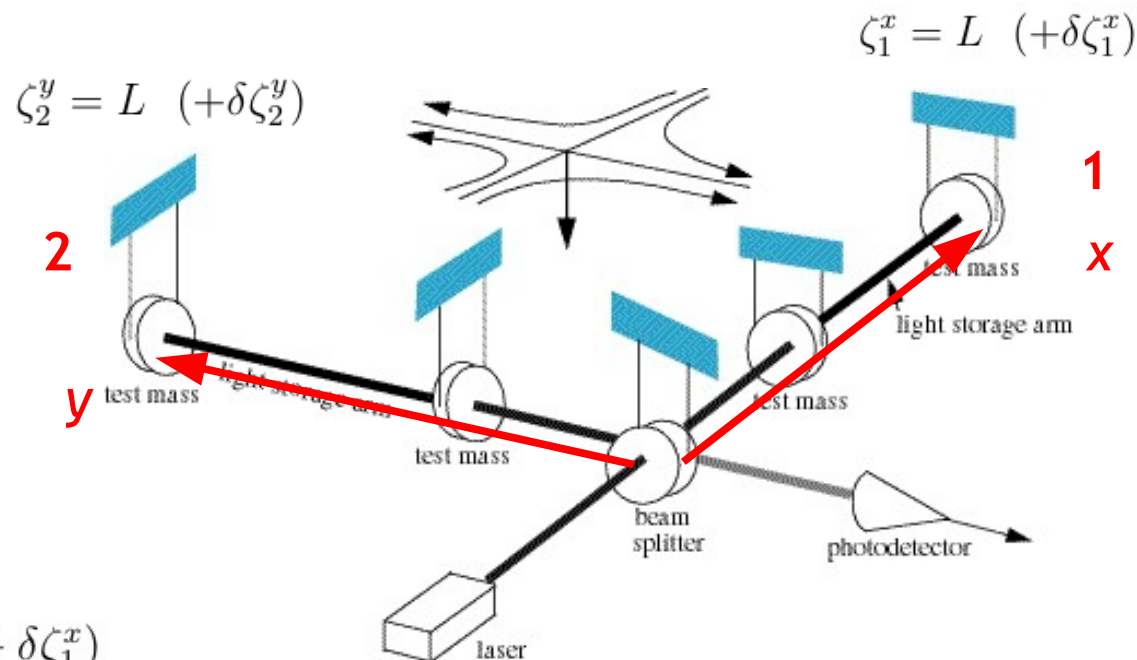
$$\delta \zeta_1^x = \frac{1}{2} h_{xx} L$$

$$\delta \zeta_2^y = \frac{1}{2} h_{yy} L$$

- Measured strain:

$$h(t) \equiv \frac{\Delta L}{L} = \frac{(L + \delta \zeta_2^y) - (L + \delta \zeta_1^x)}{L}$$

$$h(t) = \frac{1}{2} (h_{xx} - h_{yy})$$



Detection of gravitational waves

- For L-shaped detectors:

$$h(t) = \frac{1}{2}(h_{xx} - h_{yy})$$

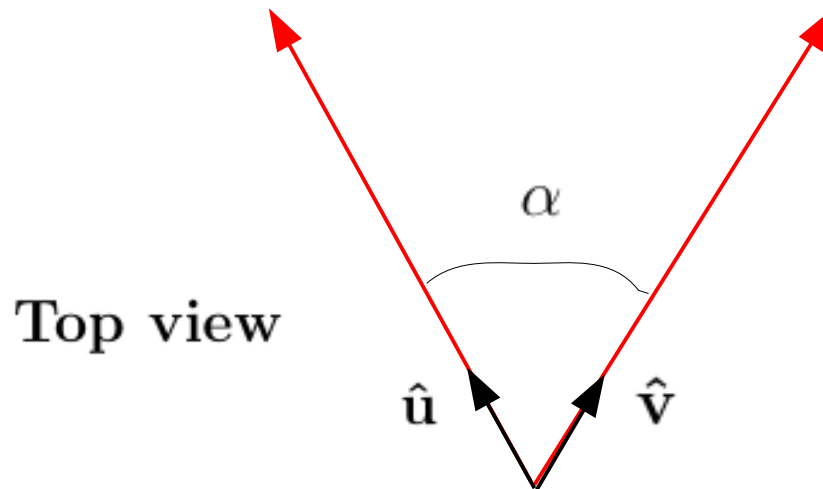
- More generally:

$$h(t) = \frac{1}{2}(h_{uu} - h_{vv})$$

- Detector tensor:

$$D^{ij} = \frac{1}{2}(u^i u^j - v^i v^j)$$

$$h(t) = D^{ij} h_{ij}$$



Detection of gravitational waves

$$h(t) = \frac{1}{2}(h_{xx} - h_{yy})$$

- Signal from arbitrary direction

In transverse-traceless frame:

$$h_{ij}^{\hat{n}=\hat{z}'} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

- If $x' = x$, $y' = y$, $z' = z$ then

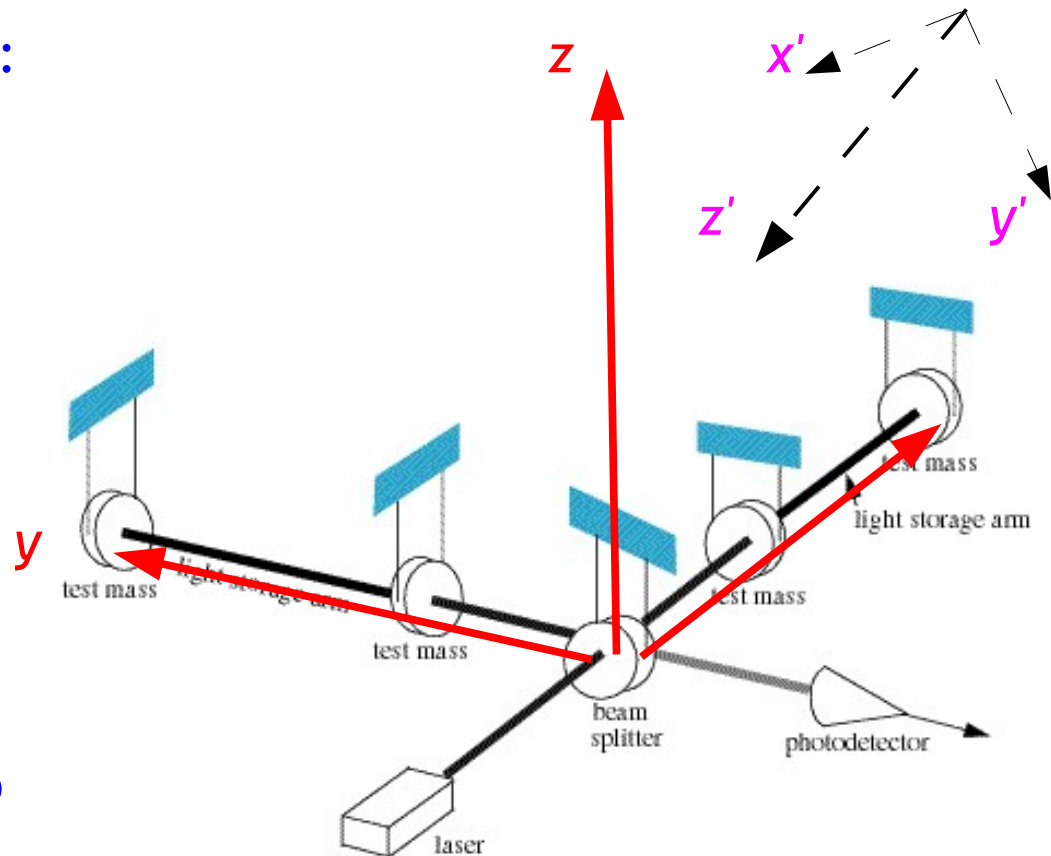
$$h(t) = h_+(t)$$

- In general, need to apply linear transformation

$$h_{ij}^{\text{det}} = \mathcal{R}_{ik} \mathcal{R}_{jl} h_{kl}^{\hat{n}=\hat{z}'}$$

- Projection onto detector also linear, hence

$$h(t) = D^{ij} h_{ij}^{\text{det}} = D^{ij} \mathcal{R}_{ik} \mathcal{R}_{jl} h_{kl}^{\hat{n}=\hat{z}'} = F_+ h_+(t) + F_\times h_\times(t)$$



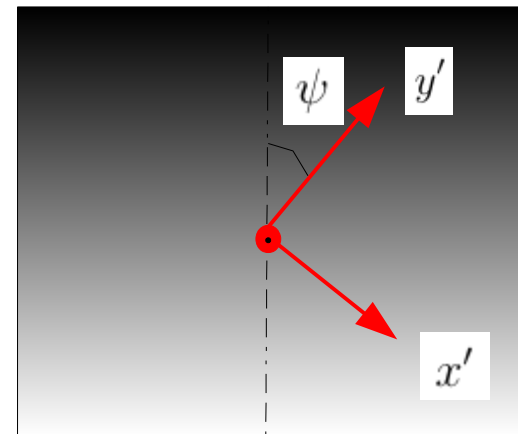
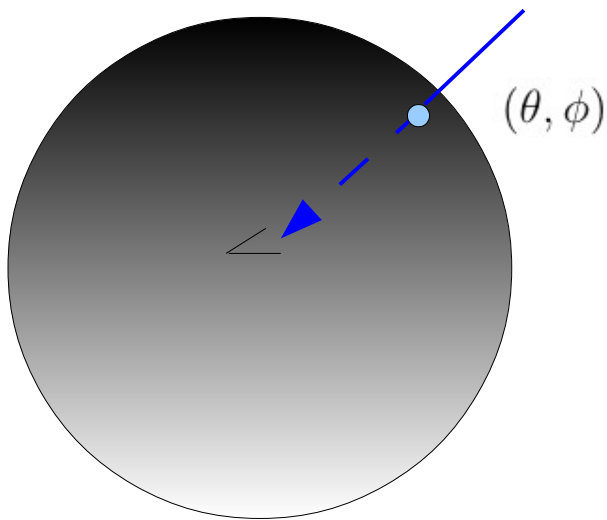
Detection of gravitational waves

$$h(t) = F_+ h_+(t) + F_\times h_\times(t)$$

- For an L-shaped interferometer:

$$F_+ = \frac{1}{2}(1 + \cos^2(\theta)) \cos(2\phi) \cos(2\psi) - \cos(\theta) \sin(2\phi) \sin(2\psi)$$

$$F_\times = \frac{1}{2}(1 + \cos^2(\theta)) \cos(2\phi) \sin(2\psi) + \cos(\theta) \sin(2\phi) \cos(2\psi)$$



- (θ, ϕ) sky position; ψ polarization angle

Digging a signal out of noise

- If there is a signal: measured strain is *sum of noise and signal*:

$$s(t) = n(t) + h(t)$$

- If shape of signal approximately known: integrate against the output

$$\frac{1}{T} \int_0^T s(t) h(t) dt = \frac{1}{T} \int_0^T n(t) h(t) dt + \frac{1}{T} \int_0^T h(t)^2 dt$$

oscillatory positive definite

$$\sim \left(\frac{\tau_0}{T}\right)^{1/2} n_0 h_0 \qquad \sim h_0^2$$

τ_0 period of the GW signal,

h_0 characteristic signal amplitude

n_0 characteristic amplitude of the noise

- *To detect the signal, don't need $h_0 > n_0$ but only $h_0 > (\tau_0/T)^{1/2} n_0$*

- Binary objects: $\tau_0 \sim 10^{-2} \text{ s}$, $T \sim 100 \text{ s}$ $\rightarrow (\tau_0/T)^{1/2} \sim 10^{-2}$

- Millisecond pulsars: $\tau_0 = 1 \text{ ms}$, $T \sim 1 \text{ yr}$ $\rightarrow (\tau_0/T)^{1/2} \sim 10^{-5}$

Characterizing the noise

- Detector data comes in as time series

- If only noise:

$$(n(t_0), n(t_1), \dots, n(t_N)) \text{ where } t_{i+1} = t_i + \Delta t$$

- Often convenient to take a (discrete) Fourier transform:

$$(\tilde{n}(f_0), \tilde{n}(f_1), \dots, \tilde{n}(f_N)) \text{ where } f_{i+1} = f_i + \Delta f$$

Notation: $\tilde{n}(f_i) = \tilde{n}_i$

- Some noise realizations are more probable than others

Probability distribution in each frequency bin: $p(\tilde{n}_i)$

- We will assume that the noise is *stationary* and *Gaussian*:

$$p(\tilde{n}_i) \propto e^{-\frac{|\tilde{n}_i|^2}{2\sigma_i^2}}$$

Stationarity and Gaussianity:

$$\langle \tilde{n}_i \rangle = \int \tilde{n}_i p(\tilde{n}_i) d\tilde{n}_i = 0 \qquad \langle |\tilde{n}_i|^2 \rangle = \int |\tilde{n}_i|^2 p(\tilde{n}_i) d\tilde{n}_i = \sigma_i^2$$

- Probability density for noise realization *as a whole*:

$$p[n] = p(\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_N) = \prod_{i=1}^N p(\tilde{n}_i)$$

Characterizing the noise

- Probability density for noise realization *as a whole*:

$$p[n] = p(\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_N) = \prod_{i=1}^N p(\tilde{n}_i)$$

- For stationary, Gaussian noise:

$$p[n] = \prod_{i=1}^N p(\tilde{n}_i) = \mathcal{N} e^{-\frac{1}{2} \sum_{i=1}^N \frac{|\tilde{n}_i|^2}{\sigma_i^2}}$$

- For purpose of this lecture, convenient to take continuum limit:

$$\begin{aligned} p[n] &= \mathcal{N} e^{-\frac{1}{2} \sum_{i=1}^N \frac{|\tilde{n}_i|^2}{\sigma_i^2}} \\ &= \mathcal{N} e^{-\frac{1}{2} \sum_{i=1}^N \frac{|\tilde{n}_i|^2}{\sigma_i^2 \Delta f} \Delta f} \\ &\rightarrow \mathcal{N} e^{-\frac{1}{2} \int_{-\infty}^{\infty} \frac{|\tilde{n}(f)|^2}{\frac{1}{2} S_n(f)} df} \end{aligned}$$

Variance:

$$\sigma_i^2 \Delta f \rightarrow \frac{1}{2} S_n(f) \quad \text{or} \quad \sigma_i^2 \rightarrow \underbrace{\delta(f - f')}_{\sim \frac{1}{\Delta f}} \frac{1}{2} S_n(f)$$

$$\langle \tilde{n}^*(f) \tilde{n}(f') \rangle = \delta(f - f') \frac{1}{2} S_n(f)$$

Characterizing the noise

- Probability density and variance for noise realizations:

$$p[n] = \mathcal{N} e^{-\frac{1}{2} \int_{-\infty}^{\infty} \frac{|\tilde{n}(f)|^2}{\frac{1}{2} S_n(f)} df}$$

$$\langle \tilde{n}^*(f) \tilde{n}(f') \rangle = \delta(f - f') \frac{1}{2} S_n(f)$$

- Could also have worked in time domain

- Stationarity: $\langle n(t) \rangle = 0$

- Gaussianity: completely determined by $R(\tau) \equiv \langle n(t + \tau) n(t) \rangle$

where again $\langle \dots \rangle$ denotes average over noise realizations

Defining *noise power spectral density* as

$$\frac{1}{2} S_n(f) \equiv \int_{-\infty}^{\infty} d\tau R(\tau) e^{2\pi i f \tau}$$

one finds

$$\langle \tilde{n}^*(f) \tilde{n}(f') \rangle = \frac{1}{2} \delta(f - f') S_n(f)$$

Matched filtering

- Instead of integrating the data against waveforms, use more generic filter:

$$\hat{s} = \int_{-\infty}^{\infty} dt s(t) \underbrace{K(t)}_{\text{filter}}$$

- Define S to be the expected value if a signal $h(t)$ is present, and N the root-mean-square value if no signal present:

$$S = \langle \hat{s} \rangle_h \qquad N = [\langle \hat{s}^2 \rangle_{h=0} - \langle \hat{s} \rangle_{h=0}^2]^{1/2}$$

- Define *signal-to-noise ratio*:

$$\frac{S}{N} = \frac{\langle \hat{s} \rangle_h}{[\langle \hat{s}^2 \rangle_{h=0} - \langle \hat{s} \rangle_{h=0}^2]^{1/2}}$$

Now find out which filter $K(t)$ maximizes S/N

Matched filtering

$$\frac{S}{N} = \frac{\langle \hat{s} \rangle_h}{[\langle \hat{s}^2 \rangle_{h=0} - \langle \hat{s} \rangle_{h=0}^2]^{1/2}}$$

$$\hat{s} = \int_{-\infty}^{\infty} dt s(t) \underbrace{K(t)}_{\text{filter}}$$

Write S in the frequency domain:

$$\begin{aligned} S &= \langle \hat{s} \rangle_h \\ &= \int_{-\infty}^{\infty} dt \langle s(t) \rangle K(t) \\ &= \int_{-\infty}^{\infty} dt h(t) K(t) \\ &= \int_{-\infty}^{\infty} df \tilde{h}(f) \tilde{K}^*(f) \end{aligned}$$

... and N :

$$\begin{aligned} N &= [\langle \hat{s}^2 \rangle - \langle \hat{s} \rangle^2]_{h=0}^{1/2} \\ &= [\langle \hat{s}^2 \rangle]_{h=0}^{1/2} \\ &= \left[\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \langle n(t) n(t') \rangle K(t) K(t') \right]^{1/2} \\ &= \left[\int_{-\infty}^{\infty} df \frac{1}{2} S_n(f) |\tilde{K}(f)| \right]^{1/2} \end{aligned}$$

Matched filtering

... to arrive at:

$$\frac{S}{N} = \frac{\int_{-\infty}^{\infty} df \tilde{h}(f) \tilde{K}^*(f)}{\left[\int_{-\infty}^{\infty} df \frac{1}{2} S_n(f) |\tilde{K}(f)|^2 \right]^{1/2}}$$

● Now define the *noise-weighted inner product*

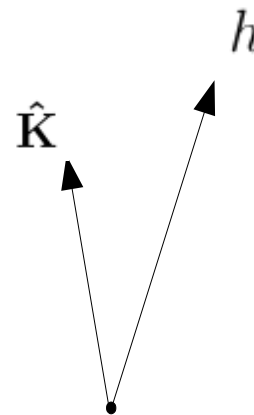
$$\begin{aligned} (A|B) &\equiv \operatorname{Re} \int_{-\infty}^{\infty} df \frac{\tilde{A}^*(f) \tilde{B}(f)}{\frac{1}{2} S_n(f)} \\ &= 4 \operatorname{Re} \int_0^{\infty} \frac{\tilde{A}^*(f) \tilde{B}(f)}{S_n(f)} \end{aligned}$$

... and rewrite S/N as

$$\frac{S}{N} = \frac{(\mathbf{K}|h)}{(\mathbf{K}|\mathbf{K})^{1/2}} \quad \mathbf{K} = \frac{1}{2} S_n(f) \tilde{K}(f)$$

... or

$$\frac{S}{N} = (\hat{\mathbf{K}}|h) \quad \hat{\mathbf{K}} = \frac{\mathbf{K}}{(\mathbf{K}|\mathbf{K})^{1/2}}$$



Maximizing S/N is equivalent to making $\hat{\mathbf{K}}$ point in the same direction as h !

Matched filtering

$$\frac{S}{N} = (\hat{\mathbf{K}}|h)$$

$$\hat{\mathbf{K}} = \frac{\mathbf{K}}{(\mathbf{K}|\mathbf{K})^{1/2}}$$

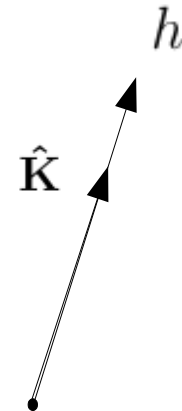
$$\mathbf{K} = \frac{1}{2} S_n(f) \tilde{K}(f)$$

- Need to make $\hat{\mathbf{K}}$ point in the same direction as h

$$\hat{\mathbf{K}} \propto h \quad \longrightarrow \quad \mathbf{K} \propto h$$



$$\tilde{K}(f) \propto \frac{\tilde{h}(f)}{S_n(f)}$$



"Wiener filter"

- Substituting back into the expression for S/N gives the highest signal-to-noise ratio one can attain:

$$\begin{aligned} \frac{S}{N} &= \frac{(h|\langle s \rangle)}{\sqrt{(h|h)}} \\ &= \left[4 \int_0^\infty df \frac{|\tilde{h}(f)|^2}{S_n(f)} \right]^{1/2} \end{aligned}$$

Matched filtering

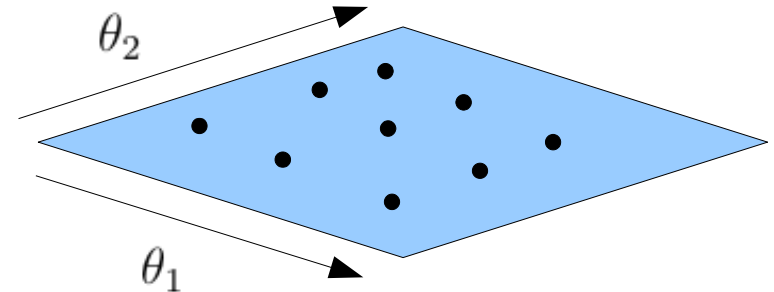
$$\begin{aligned}\frac{S}{N} &= \frac{(h|\langle s \rangle)}{\sqrt{(h|h)}} \\ &= \left[4 \int_0^\infty df \frac{|\tilde{h}(f)|^2}{S_n(f)} \right]^{1/2}\end{aligned}$$

● In practice:

- No access to *expected* $s(t)$, only the actual detector output $s(t)$
- Waveforms are characterized by source parameters $\bar{\theta} = \{\theta_1, \theta_2, \dots, \theta_N\}$
- Use optimal filter for many choices of parameters and find largest S/N:

$$\frac{S}{N} = \max_{\bar{\theta}} \frac{(h(\bar{\theta})|s)}{\sqrt{(h(\bar{\theta})|h(\bar{\theta}))}}$$

- Parameter space can only be sampled in a discrete manner:
"template banks"



- Noise not perfectly Gaussian or stationary

Matched filtering in practice: compact binary coalescence

- Simplest compact binary coalescence waveforms:

- Inspiral only
- Amplitude to leading PN order only

$$h_+ = h_0 \cos(2\varphi(t))$$

$$h_\times = h_{\frac{\pi}{2}} \sin(2\varphi(t))$$

- Detector response:

$$h(t) = F_+ h_+ + F_\times h_\times$$

- Can be written as:

$$h(t) = A(t) \cos(2\varphi(t)) \cos(\Phi_0) + A(t) \sin(2\varphi(t)) \sin(\Phi_0)$$

where

$$A(t) = \left[F_+^2 h_0^2 + F_\times^2 h_{\frac{\pi}{2}}^2 \right]^{1/2}$$

$$\cos \Phi_0 = \frac{F_+ h_0}{A},$$

$$\sin \Phi_0 = \frac{F_\times h_{\frac{\pi}{2}}}{A}.$$

Define

$$h_c = A(t) \cos 2\varphi(t)$$

$$h_s = A(t) \sin 2\varphi(t)$$

Matched filtering in practice: compact binary coalescence

$$h(t) = A(t) \cos(2\varphi(t)) \cos(\Phi_0) + A(t) \sin(2\varphi(t)) \sin(\Phi_0)$$

$$h_c = A(t) \cos 2\varphi(t)$$

$$h_s = A(t) \sin 2\varphi(t)$$

- Normalize h_c , h_s :

$$\hat{h}_c = \frac{h_c}{(h_c|h_c)} \quad \hat{h}_s = \frac{h_s}{(h_s|h_s)}$$

- Matched filtering against detector output s :

$$\begin{aligned} (s|\hat{h}) &= (s|\hat{h}_c) \cos(\Phi_0) + (s|\hat{h}_s) \sin(\Phi_0) \\ &= \left[(s|\hat{h}_c)^2 + (s|\hat{h}_s)^2 \right]^{1/2} \cos(\Phi_0 - \alpha) \end{aligned}$$

where $\alpha = \tan^{-1} \frac{(s|\hat{h}_s)}{(s|\hat{h}_c)}$

and the S/N ratio is maximized when $\Phi_0 = \alpha$

- Matched filtering reduces to:

$$(s|\hat{h}) = \left[(s|\hat{h}_c)^2 + (s|\hat{h}_s)^2 \right]^{1/2}$$

No dependence of templates on extrinsic parameters!
Only dependence on masses and spins

Placement of template banks

- Match between nearby templates:

$$M = (\hat{h}(\bar{\theta}) | \hat{h}(\bar{\theta} + \Delta\bar{\theta}))$$

- Expand in small quantities:

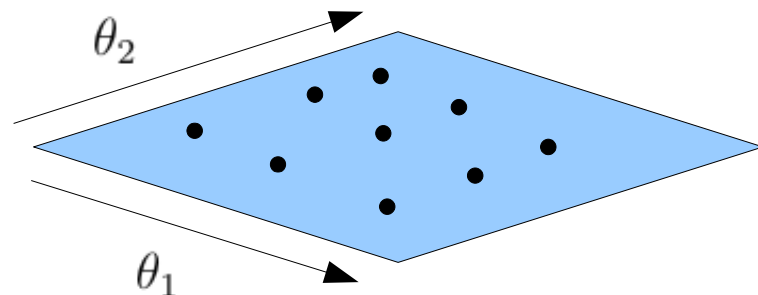
$$\begin{aligned} M &\simeq (\hat{h}(\bar{\theta}) | \hat{h}(\bar{\theta})) + \frac{\partial M}{\partial \theta^i} \Delta\theta^i + \frac{1}{2} \frac{\partial^2 M}{\partial \theta^i \partial \theta^j} \Delta\theta^i \Delta\theta^j \\ &= 1 + \frac{1}{2} \frac{\partial^2 M}{\partial \theta^i \partial \theta^j} \Delta\theta^i \Delta\theta^j \end{aligned}$$

- Define a *metric tensor* on parameter space:

$$g_{ij} \equiv -\frac{1}{2} \frac{\partial^2 M}{\partial \theta^i \partial \theta^j}$$

- *Mismatch* between nearby templates:

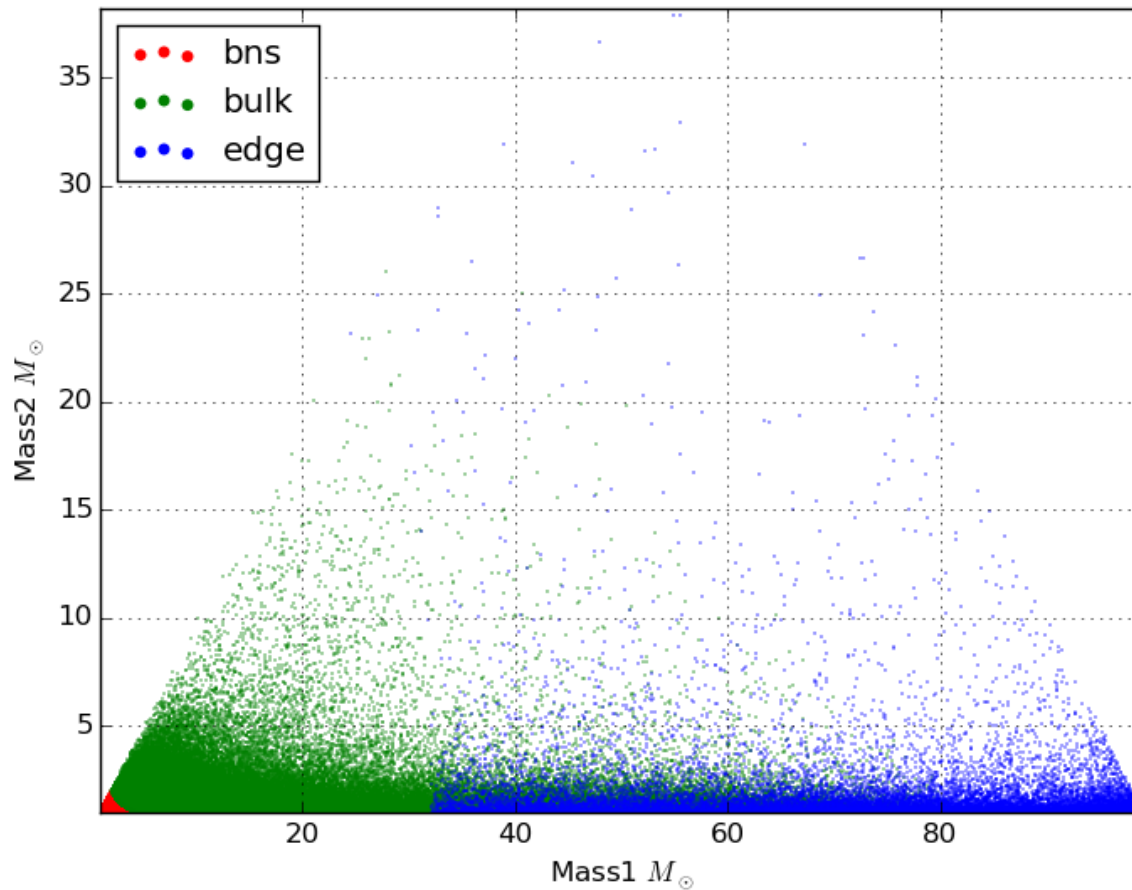
$$1 - M = g_{ij} \Delta\theta^i \Delta\theta^j$$



Place templates on parameter space such that metric distance never larger than a pre-specified mismatch, e.g. 0.03

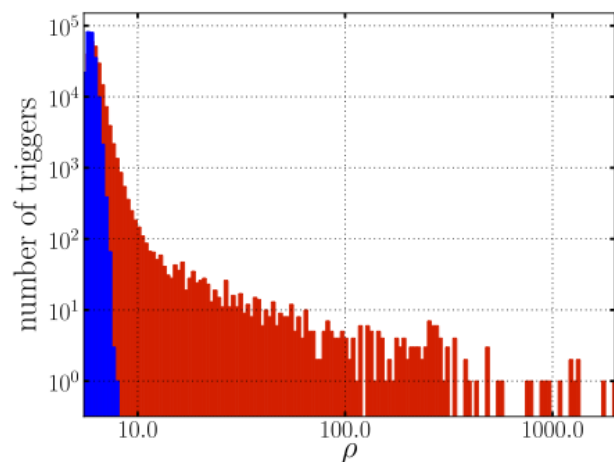
Placement of template banks

- Example of placement in (m_1, m_2)
 - Spin dimensions not shown

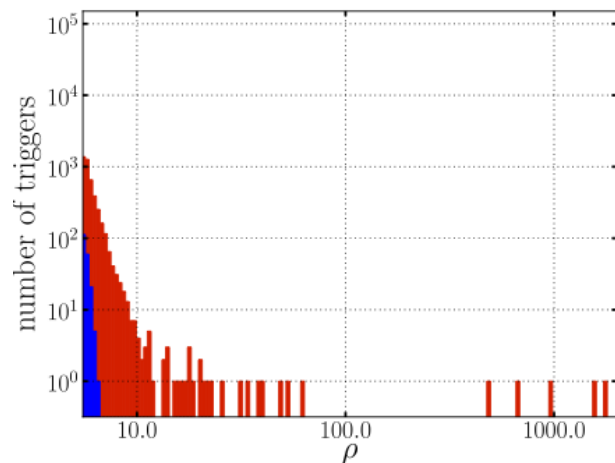


Handling of potential signals

- Glitches in the instruments can pose as gravitational wave signals!
- Distribution of SNRs for noise triggers in a single detector, in Gaussian noise (**blue**) and in real data (**red**)



- Demand that event seen in at least two detectors, with *same parameters*



Signal consistency tests

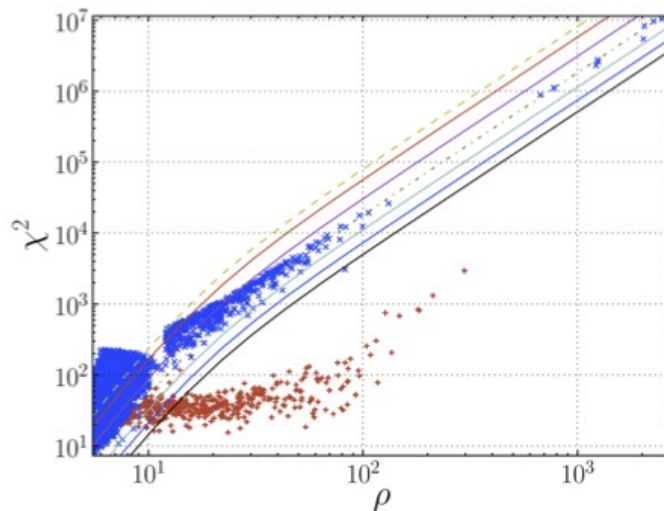
- Demand that build-up of SNR over frequency is as expected if a signal is really present
 - Divide frequency range in n bins that should contribute equally to SNR

$$(h_k|h_k) = \frac{(h|h)}{n}$$

- Construct χ^2 statistic

$$\chi^2 = \sum_{k=1}^n \left[\left((s|h_{c,k}) - \frac{(h_c|h_c)}{n} \right)^2 + \left((s|h_{s,k}) - \frac{(h_c|h_c)}{n} \right)^2 \right]$$

Noise transients will typically have high χ^2



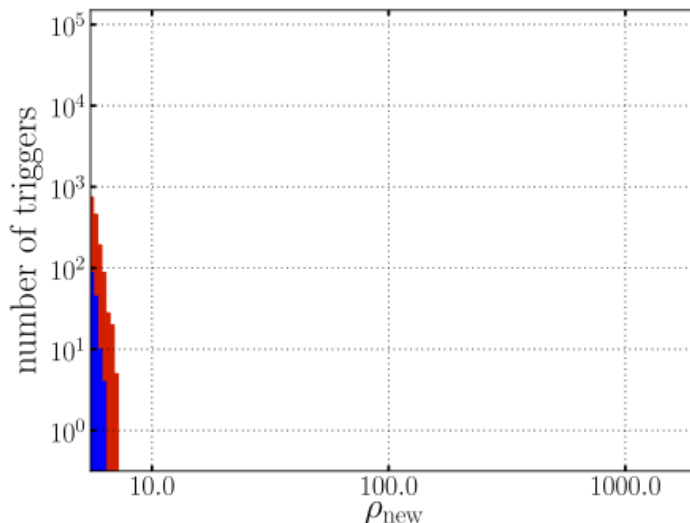
Compare distribution of noise triggers (blue) and simulated GW signals (red)

Signal consistency tests

- Construct a detection statistic that is large when SNR large and χ^2 small, e.g. “new SNR”

$$\rho_{\text{new}} = \begin{cases} \rho & \text{for } \chi^2 \leq n_{\text{dof}} \\ \rho \left[\frac{1}{2} \left(1 + \left(\frac{\chi^2}{n_{\text{dof}}} \right)^3 \right) \right]^{-1/6} & \text{for } \chi^2 > n_{\text{dof}} \end{cases}$$

- Distribution of new SNR for noise triggers in Gaussian noise (blue) and real data (red)



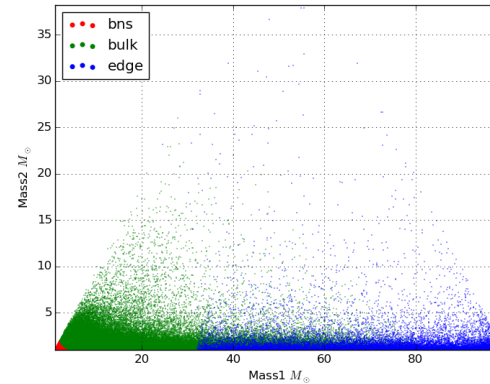
- Given data from multiple detectors, slide data stream in time w.r.t. each other
- Construct distribution of new SNR for multi-detector triggers (necessarily fake events!)
- Given a candidate signal, assess its significance by comparing with this distribution

Summary of the search pipeline

- Construction of template bank

- Matched filtering: SNR

- Signal consistency test: $\chi^<$

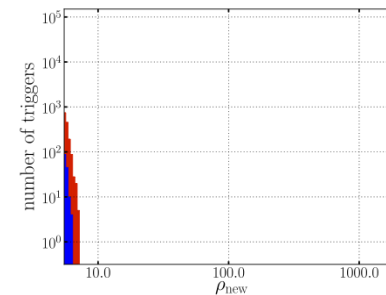


- Introduce detection statistic (“new SNR”) that is large when SNR is high and $\chi^<$ is low

$$\rho_{\text{new}} = \begin{cases} \rho & \text{for } \chi^2 \leq n_{\text{dof}} \\ \rho \left[\frac{1}{2} \left(1 + \left(\frac{\chi^2}{n_{\text{dof}}} \right)^3 \right) \right]^{-1/6} & \text{for } \chi^2 > n_{\text{dof}} \end{cases}$$

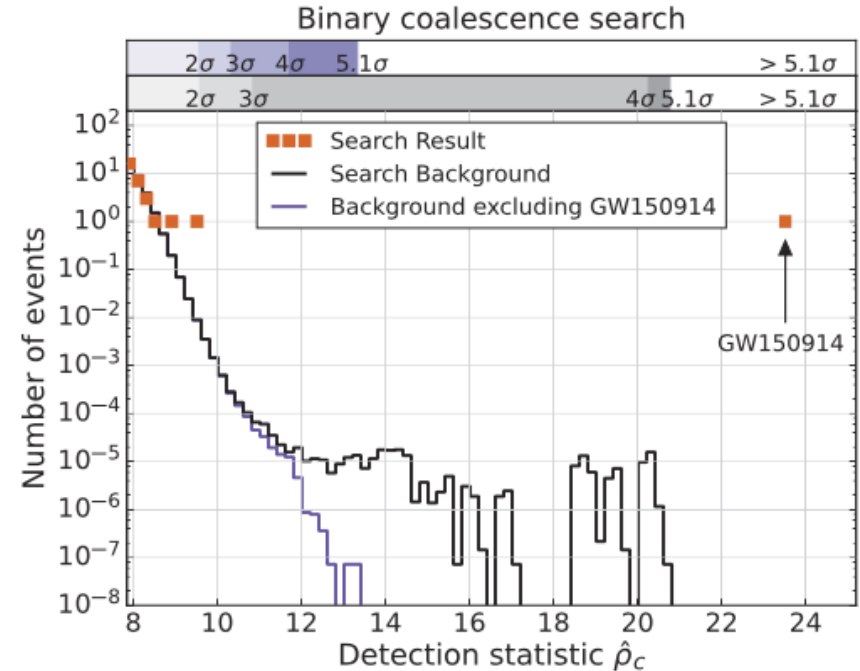
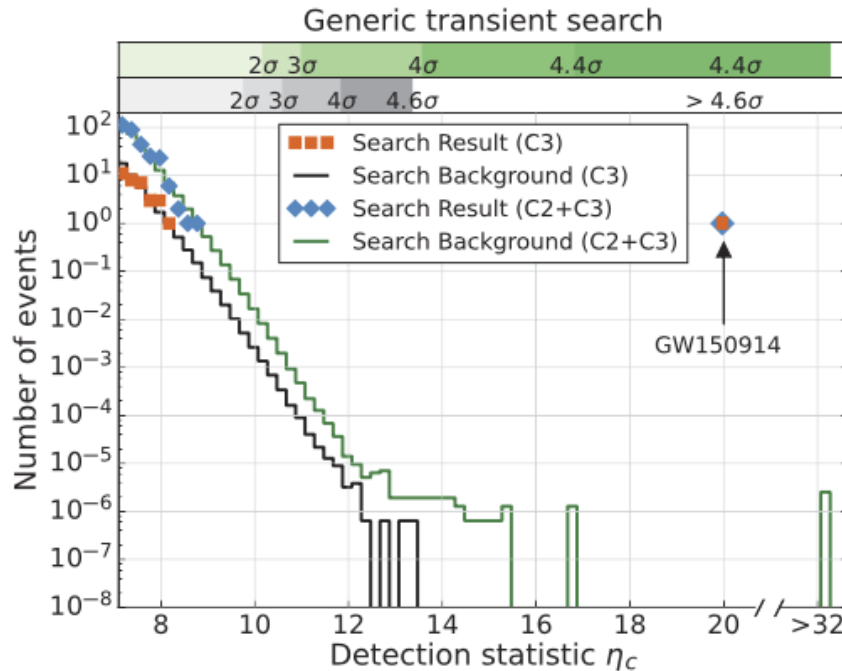
- Only accept candidate events that give the highest new SNR for the same template (i.e. same masses and spins) in at least two detectors

- Characterize distribution of new SNR for false positives by time-sliding



- Significance of candidates is assessed by comparing with this distribution

The detection: GW150914



□ Found in “unmodeled” and “modeled” searches

□ Modeled (matched-filter) search used 16 days of coincident data

- False alarm rate < 1 in 203000 years
- **Significance $> 5.1\sigma$**

The detection: GW150914

