Newtonian binaries

A non-relativistic type of source of considerable interest is the newtonian binary: two massive bodies (stars) in bound orbit with velocities much less than that of light. The masses nd positions of the two bodies are $(m_1, \mathbf{r}_1)$ and $(m_2, \mathbf{r}_2)$, respectively. Take the center of mass to be the origin of our co-ordinate system:

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = 0. \tag{1}$$

The relative distance vector is

$$\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1. \tag{2}$$

Problem 13.1
With $M = m_1 + m_2$, show that

$$\mathbf{r}_1 = -\frac{m_2}{M}\,\mathbf{R}, \quad \mathbf{r}_2 = \frac{m_1}{M}\,\mathbf{R} \tag{3}$$

and that Newton's law of gravity implies

$$\ddot{\mathbf{R}} = -\frac{GM}{R^2}\,\hat{\mathbf{R}}, \tag{4}$$

where $\hat{\mathbf{R}} = \mathbf{R}/R$ is th unit vector in the direction of $\mathbf{R}$.

The solutions of equation (4) are the well-known Kepler orbits. The bound states are closed planar orbits: ellipses, in the simplest case circles.

Problem 13.2
a. Check that angular momentum is conserved:

$$\mathbf{L} = m_1 \mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2 \mathbf{r}_2 \times \dot{\mathbf{r}}_2 = \mu \mathbf{R} \times \dot{\mathbf{R}} = \text{constant}, \tag{5}$$

where the reduced mass is

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}.$$

Explain why this implies that the motion is planar.
b. Take a circular orbit in the x-y-plane:

$$\mathbf{R} = R(\cos\omega t, \sin\omega t, 0). \tag{6}$$

Show that the angular velocity is given by

$$\omega^2 = \frac{GM}{R^3}. \tag{7}$$

c. Compute the quadrupole moment:

$$\begin{aligned} Q_{ij} &= m_1 \left(r_{1i} r_{1j} - \frac{1}{3}\,\delta_{ij}\,\mathbf{r}_1^2 \right) + m_2 \left(r_{2i} r_{2j} - \frac{1}{3}\,\delta_{ij}\,\mathbf{r}_2^2 \right) \\ &= \frac{\mu R^2}{2} \begin{pmatrix} \cos 2\omega t + \frac{1}{3} & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}. \end{aligned} \tag{8}$$

d. Derive the power emitted by the binary system as a function of direction:

$$\frac{d\mathcal{E}}{d^2\Omega dt} = -\frac{4G^4 m_1^2 m_2^2 M}{\pi c^5 R^5}\left(\cos^2\theta + \frac{1}{4}\sin^4\theta\sin^2 2(\varphi-\omega t)\right), \tag{9}$$

and after integrating over all angles:

$$\frac{d\mathcal{E}}{dt} = -\frac{32G^4}{5c^5}\frac{m_1^2 m_2^2 M}{R^5}. \tag{10}$$

The total energy of the binary system in non-relativistic approximation is

$$\mathcal{E} = \mathcal{E}_0 + T + V = \mathcal{E}_0 - \frac{Gm_1 m_2}{2R}, \tag{11}$$

where $\mathcal{E}_0$ is the constant rest energy. Hence

$$\frac{d\mathcal{E}}{dt} = -\frac{Gm_1 m_2}{R^2}\frac{dR}{dt}. \tag{12}$$

Identifying this with the radiative energy loss (10) derive the radial inspiral velocity

$$\frac{dR}{dt} = -\frac{64G^3}{5c^5}\frac{m_1 m_2 M}{R^3}. \tag{13}$$

Using the result (7) this can be converted in an equation for the frequency change during inspiral:

$$\frac{d\omega}{dt} = \frac{96G^{7/2}}{5c^5}\frac{m_1 m_2 M^{3/2}}{R^{11/2}} = \frac{96}{5c^5}\frac{G^{5/3} m_1 m_2}{M^{1/3}}\omega^{11/3}. \tag{14}$$