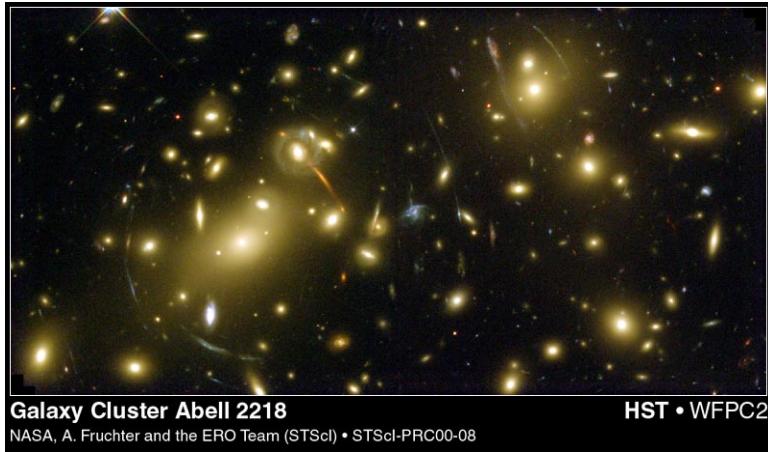
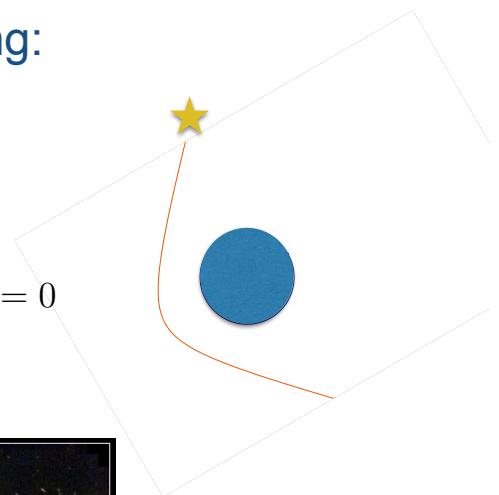


## Gravitational lensing: deflection of light

in equatorial plane:

$$-\left(1 - \frac{2\mu}{r}\right) c^2 dt^2 + \frac{dr^2}{1 - 2\mu/r} + r^2 d\varphi^2 = 0$$



## Radial distances

at constant  $\varphi$  and  $t$ :

$$\begin{aligned} L &= \int_{2\mu}^r \frac{dr'}{\sqrt{1 - 2\mu/r'}} \\ &= \sqrt{r(r - 2\mu)} + \mu \ln \left( \frac{r}{\mu} - 1 + \frac{1}{\mu} \sqrt{r(r - 2\mu)} \right) \end{aligned}$$

measures distance from lower bound  $r = 2\mu$

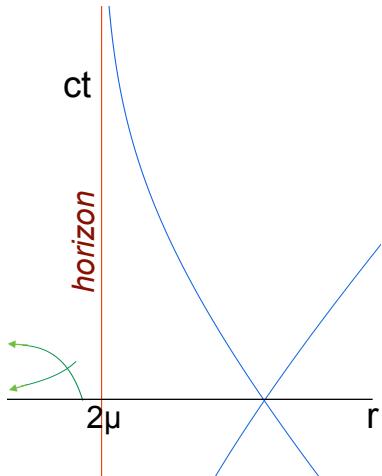
## Time dilation

at fixed position  $(r, \theta, \varphi)$  time dilation is

$$\Delta\tau = \int_0^{\Delta t} \left(1 - \frac{2\mu}{r}\right) dt = \left(1 - \frac{2\mu}{r}\right) \Delta t \rightarrow 0$$

horizon

## Light cones



radial light rays:

$$\left(1 - \frac{2\mu}{r}\right) c^2 dt^2 = \frac{dr^2}{1 - 2\mu/r}$$

$$\Rightarrow \frac{dr}{cdt} = \pm \left(1 - \frac{2\mu}{r}\right)$$

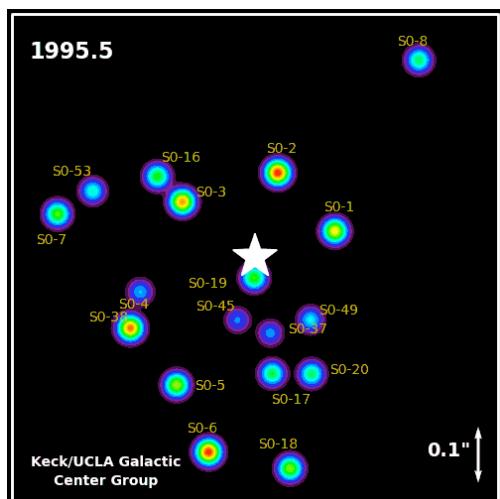
$$r + 2\mu \ln \left( \frac{r}{2\mu} - 1 \right) = \pm ct \quad (r > 2\mu)$$

If a mass  $M$  is concentrated in a sphere of surface area

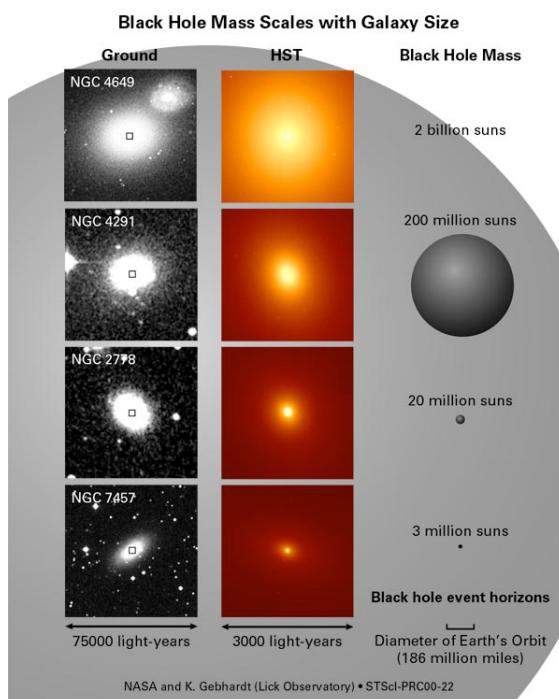
$$A < \frac{16\pi G^2 M^2}{c^4}$$

the object becomes a *black hole*

## Galactic black holes



galactic center (Sag A)



# Cosmology: dynamical geometry

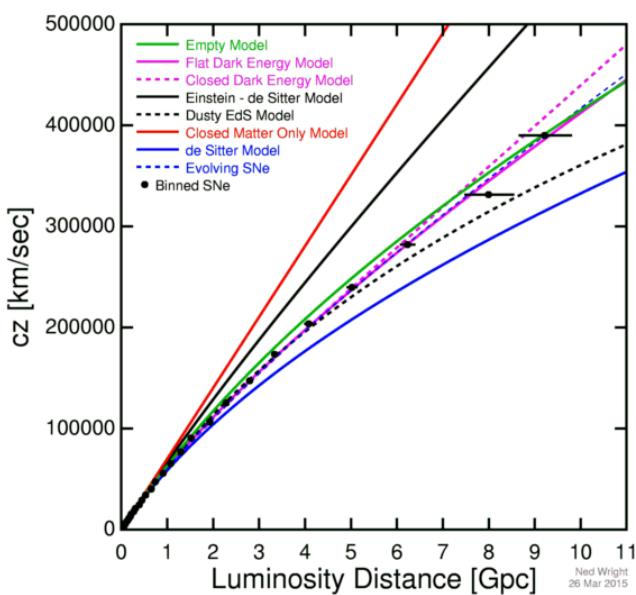
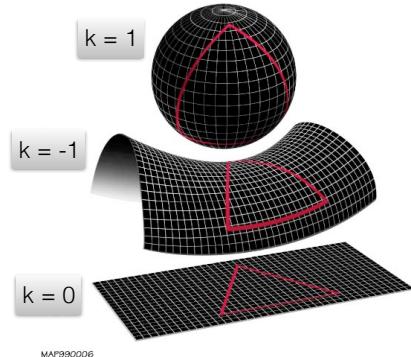
On large scales our universe is *homogeneous and isotropic*  
 → line element of cosmic geometry:

$$ds^2 = -c^2 dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right)$$

scale factor      curvature constant

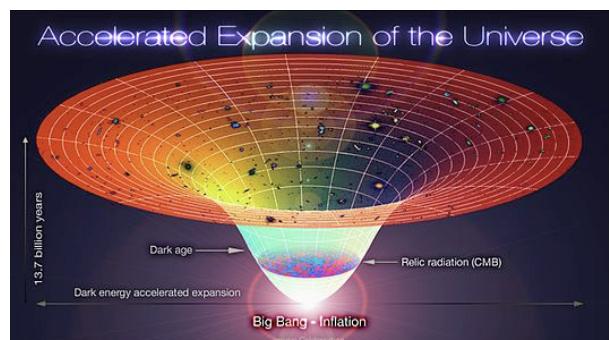
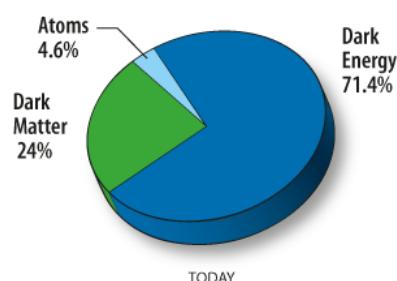
Expansion rate  
 → Hubble parameter

$$H = \frac{1}{a} \frac{da}{dt}$$



supernova redshift

$$H_0 = 71 \text{ km/s/Mpc}$$



# Gravitational waves

- fluctuations of metric in the form of traveling waves
- asymptotic form: waves on Minkowski background

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2h_{\mu\nu}$$

↓

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}$$

linear approximation of Einstein equations:

$$\square h_{\mu\nu} - \partial_\mu \partial_\lambda h_\nu^\lambda - \partial_\nu \partial_\lambda h_\mu^\lambda + \partial_\mu \partial_\nu h_\lambda^\lambda - \eta_{\mu\nu} (\square h_\lambda^\lambda - \partial_\kappa \partial_\lambda h^{\kappa\lambda}) = -8\pi G T_{\mu\nu}$$

this implies energy-momentum conservation:

$$\partial^\mu T_{\mu\nu} = 0$$

and is invariant under gauge transformations

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

## polarization

*De Donder gauge:*  $\partial^\nu h_{\mu\nu} = \frac{1}{2} \partial_\mu h_\nu^\nu \quad \longrightarrow$

$$\square \bar{h}_{\mu\nu} \equiv \square \left( h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h_\lambda^\lambda \right) = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \text{with} \quad \partial^\mu \bar{h}_{\mu\nu} = 0$$

inhomogeneous massless (dispersion-free) wave equation

in empty space:  $\bar{h}_{\mu\nu} = h_{\mu\nu} = \int_k \varepsilon_{\mu\nu}(k) e^{ik \cdot x}$

dispersion law  $k^2 = 0 \Leftrightarrow \omega^2 = \mathbf{k}^2 c^2$

amplitude polarization  $k^\mu \varepsilon_{\mu\nu}(k) = 0, \quad \varepsilon_\lambda^\lambda(k) = 0$

→ 2 transverse polarization states



## energy-momentum of gravitational waves

integral of energy density = hamiltonian

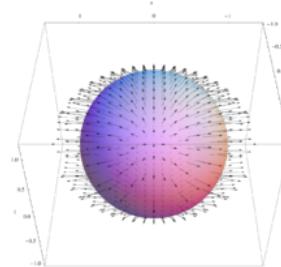
$$E_V = \int_V d^3x \mathcal{E} = \int_V d^3x \left( \frac{1}{c} \frac{\partial h_{\mu\nu}}{\partial t} \right)^2 + (\nabla h_{\mu\nu})^2$$

equation of continuity

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} = \nabla \cdot \Phi}$$

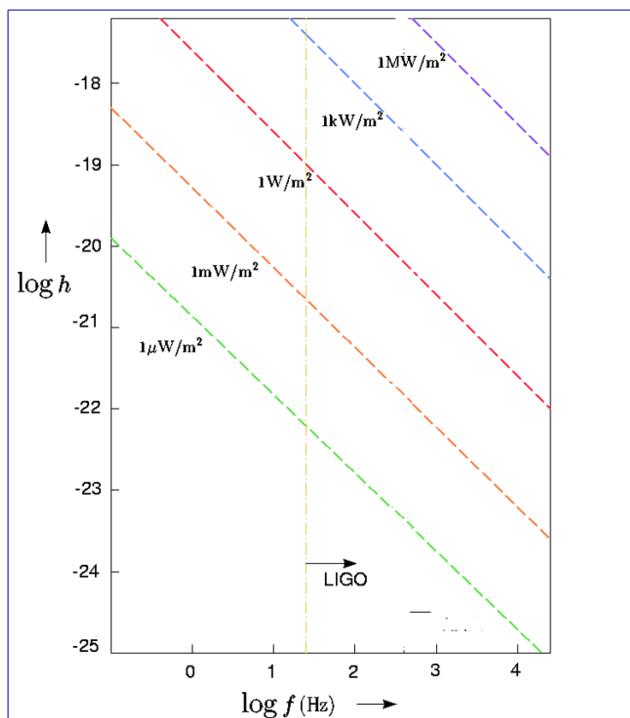
with  $\Phi = \frac{c^4}{8\pi G} \frac{\partial h^{\mu\nu}}{\partial t} \nabla h_{\mu\nu}$

$\longrightarrow \frac{dE_V}{dt} = \oint_{\partial V} d^2\sigma \Phi_n$



*energy conservation:  
change in field-energy in V =  
flux of gravitational-wave energy across the boundary*

## Energy flux of plane gravitational waves



$$h_{ij} = e_{ij} e^{i(\vec{k} \cdot \vec{r} - 2\pi f t)}$$

$$\Phi = \frac{d^2 E}{d A dt} = \frac{\pi c^3 f^2}{8G} |e_{ij}|^2$$

## Non-relativistic sources

No dipole radiation:  $\frac{d}{dt} \left( \sum_i m_i \mathbf{r}_i \right) = \frac{d\mathbf{P}}{dt} = 0$   
*(momentum conservation)*

Quadrupole radiation:

$$\frac{dE}{d\Omega dt} = -\frac{G}{8\pi c^5} \left[ \ddot{Q}_{ij}^2 - 2\hat{r} \cdot \ddot{Q}^2 \cdot \hat{r} + \frac{1}{2} (\hat{r} \cdot \ddot{Q} \cdot \hat{r})^2 \right]$$

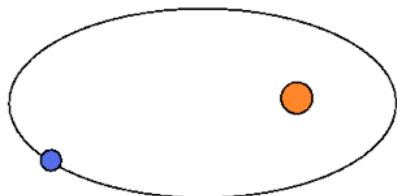
where  $Q_{ij}$  is the retarded mass quadrupole:

$$Q_{ij} = \int d^3x' \left( x'_i x'_j - \frac{1}{3} \delta_{ij} \mathbf{x}'^2 \right) \rho(\mathbf{x}', t - r/c)$$

Total energy flux

$$\boxed{\frac{dE}{dt} = -\frac{G}{5c^5} \ddot{Q}_{ij}^2}$$

Newtonian binaries:

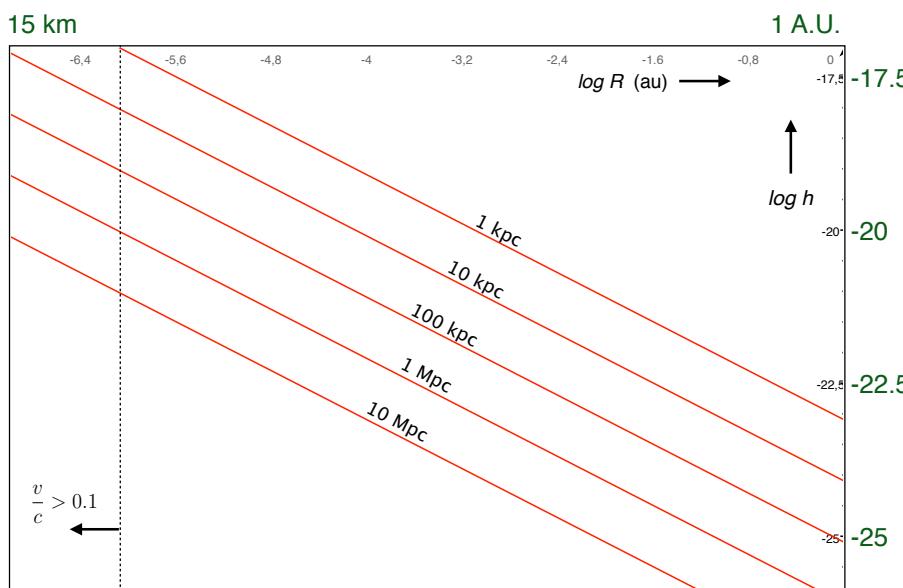


binary NS /  $1.5 M_{\text{sun}}$

$$Q_{ij} = \frac{\mu R^2}{2} \begin{pmatrix} \cos 2\omega t + \frac{1}{3} & -\sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

$$\omega^2 = \left( \frac{2\pi}{T} \right)^2 = \frac{GM}{R^3} \quad (\text{Kepler})$$

15 km

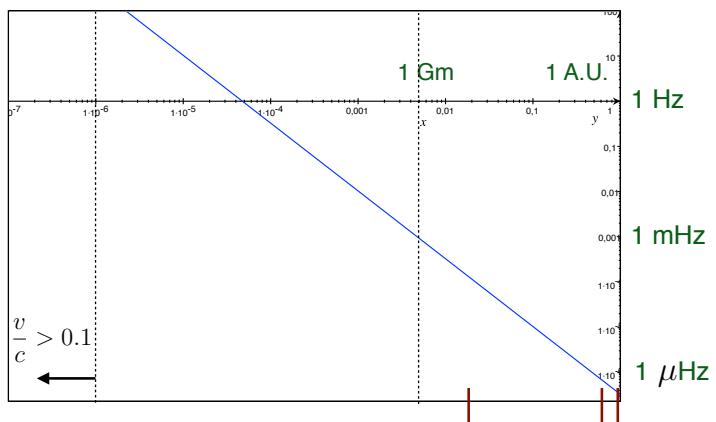


flux:

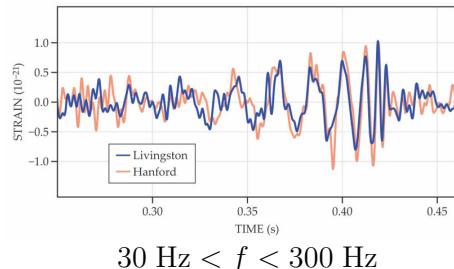
$$\begin{aligned} & \frac{1}{4\pi D^2} \left| \frac{dE}{dt} \right| \\ &= \frac{8G\mu^2 R^4 \omega^6}{5\pi c^5 D^2} \\ &= \frac{8G^4}{5\pi c^5} \frac{m_1^2 m_2^2 M}{R^5 D^2} \end{aligned}$$

frequency  
of newtonian binaries:

$$f = \frac{\omega}{\pi} = \frac{1}{\pi} \sqrt{\frac{GM}{R^3}}$$



GW150914



$30 \text{ Hz} < f < 300 \text{ Hz}$

PSR1913+16  
Mercury  
Earth

PSR1913+16

Binary pulsar data:

masses  $1.441 M_{\odot}, 1.387 M_{\odot}$

period 7.75 hr

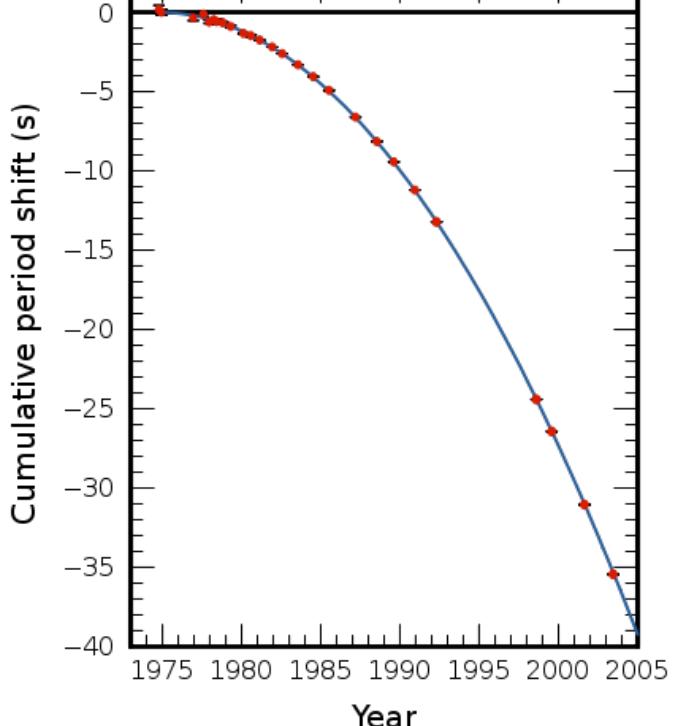
radius  $\langle R \rangle = 1950\,000 \text{ km}$

distance 6.4 kpc

expected  
gravitational waves:

$$f = 0.7 \times 10^{-4} \text{ Hz}$$

$$h = 10^{-22}$$



## Non-binary sources

- continuous sources                    *msec pulsars*
- transient sources                    *supernovae*
- cosmological sources                    *tensor modes in CMB:*
  - *B-mode polarization*
  - *quantum gravity*

