Data analysis, continuous waves
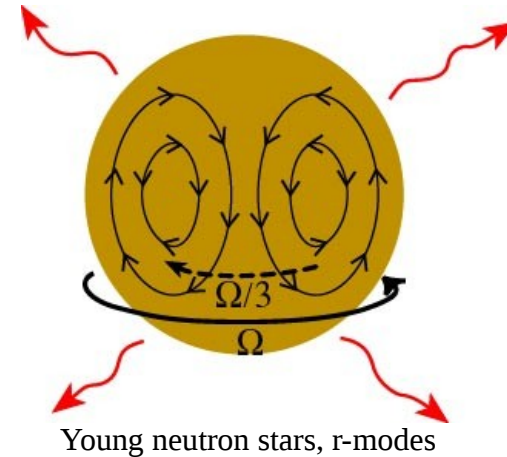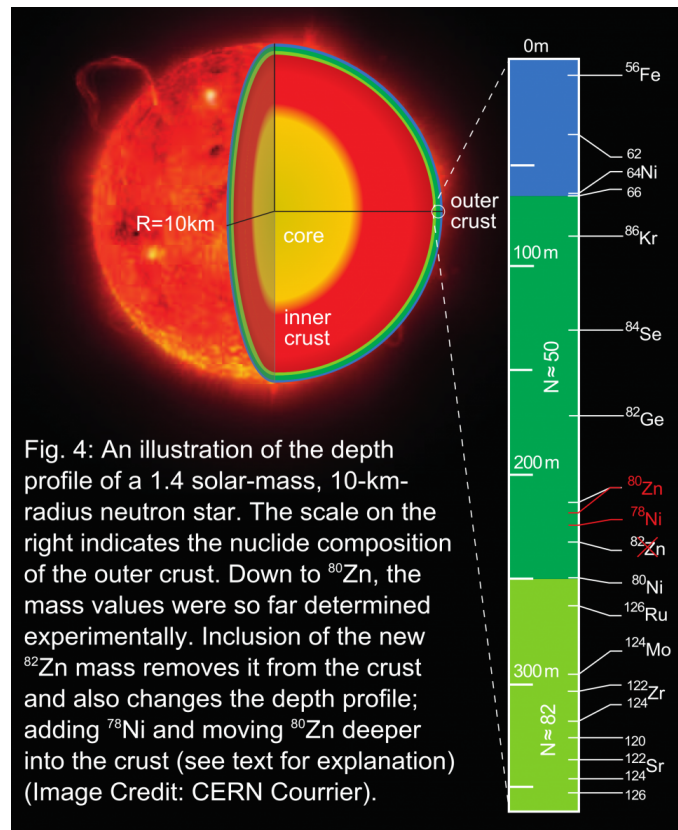
Sources
Fourier transform basics
Analysis strategies

Continuous gravitational waves

- ## Ground-based: frequencies 10 Hz – 10kHz

  - ### Galactic neutron stars

  - ### Stochastic background

Fig. 4: An illustration of the depth profile of a 1.4 solar-mass, 10-km-radius neutron star. The scale on the right indicates the nuclide composition of the outer crust. Down to ^{80}Zn, the mass values were so far determined experimentally. Inclusion of the new ^{82}Zn mass removes it from the crust and also changes the depth profile; adding ^{78}Ni and moving ^{80}Zn deeper into the crust (see text for explanation) (Image Credit: CERN Courrier).

Young neutron stars, r-modes

Binaries: accreting neutron stars
Spin up

Isolated neutron stars

Mostly composed of neutrons in superfluid state

Crust composed of dense but ordinary matter

Magnetic field
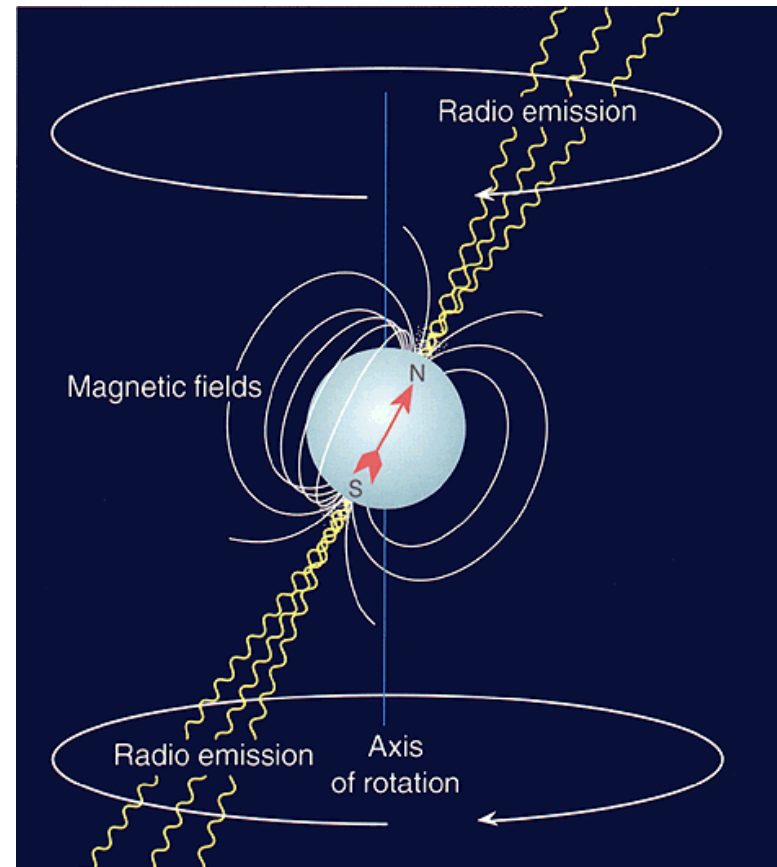
→ Precessing magnetic dipole

- Emission of EM radiation
- Carries away rotational energy
 - Star becomes less oblate

Cracking of the crust – glitches

    (neutron star spins up!)

"Mountains" (~1 mm) possible

    → Asymmetry

    → Varying mass quadrupole moment

Gravitational waves from neutron stars

Model neutron stars as rigid bodies

Characterize by *inertia tensor*

$$I_{ij} = \int d^3x\, \rho(x)\left(r^2 \delta_{ij} - x_i x_j\right)$$

Angular momentum and rotational kinetic energy:

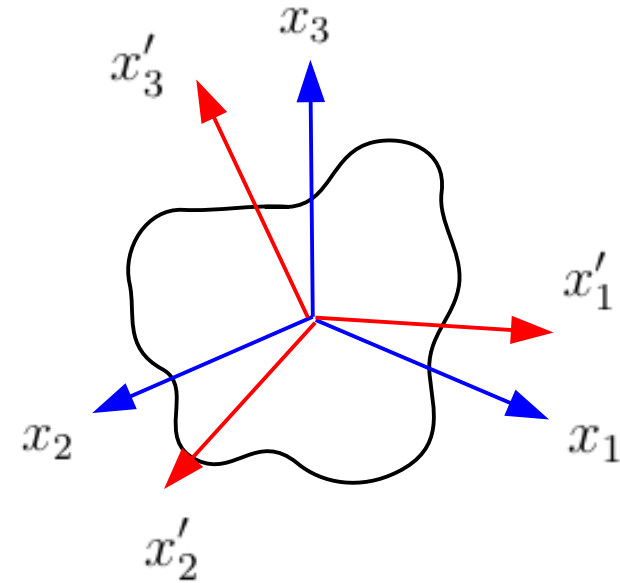$$J_i = I_{ij}\omega_j \qquad\qquad E_{rot} = \frac{1}{2}I_{ij}\omega_i\omega_j$$

"Body frame": diagonalizes inertia tensor

Principal moments of inertia:

$$I_1 = \int d^3x'\, \rho(\mathbf{x}')\left(x_2'^2 + x_3'^2\right)$$

$$I_2 = \int d^3x'\, \rho(\mathbf{x}')\left(x_3'^2 + x_1'^2\right)$$

$$I_3 = \int d^3x'\, \rho(\mathbf{x}')\left(x_1'^2 + x_2'^2\right)$$

Gravitational waves from neutron stars

$$x_3' = x_3$$

$$\omega_{\rm rot} t$$

$$x_1$$

$$x_2'$$

$$x_1'$$

$$x_2$$

Neutron stars are almost oblate:

$$I_3 > I_1 \simeq I_2$$

Assume rotation around x_3

Body frame rotates along with star

Inertia tensor in body frame:

$$I_{ij}' = \mathrm{diag}(I_1, I_2, I_3)$$

Inertia tensor in stationary frame:

$$I_{ij} = (\mathcal{R}^T I' \mathcal{R})_{ij}$$

$$\mathcal{R}_{ij}(t) = \begin{pmatrix} \cos(\omega_{\rm rot} t) & \sin(\omega_{\rm rot} t) & 0 \\ -\sin(\omega_{\rm rot} t) & \cos(\omega_{\rm rot} t) & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ij}$$
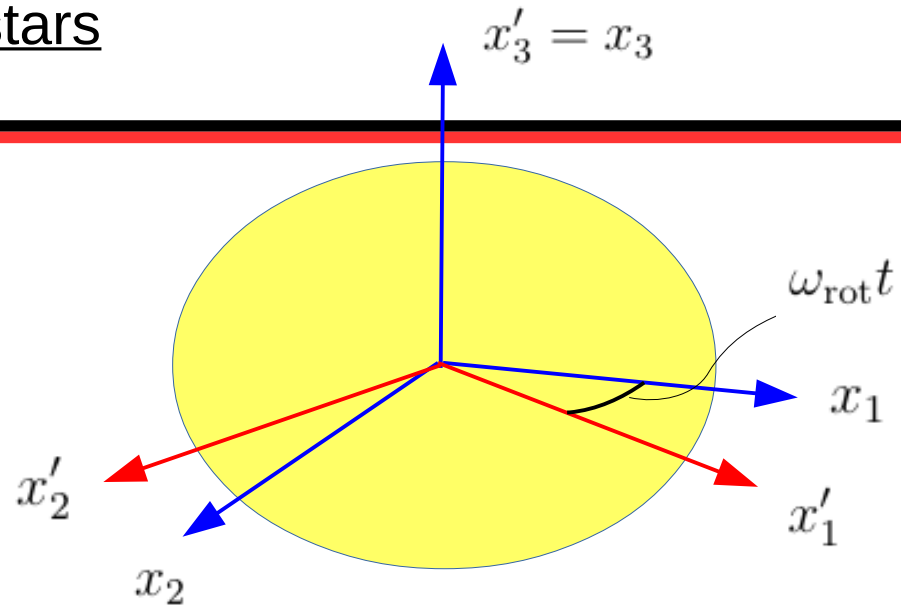
Components will be time-dependent linear combinations of I_1, I_2, I_3

Compare inertia tensor with quadrupole tensor:

$$I_{ij} = \int d^3x \, \rho(\mathbf{x}) \, (r^2 \delta_{ij} - x_i x_j) \qquad M_{ij} = \frac{1}{c^2} \int d^3x \, T^{00} x^i x^j = \int d^3x \, \rho(\mathbf{x}) \, x^i x^j$$

... hence $M_{ij} = -I_{ij} + \mathrm{Tr}(I)\, \delta_{ij}$

Gravitational waves from neutron stars

Substitute into quadrupole expressions:

$$h_+ = \frac{1}{r}\frac{G}{c^4}(\ddot{\tilde{M}}_{11} - \ddot{\tilde{M}}_{22})$$

$$h_\times = \frac{2}{r}\frac{G}{c^4}\ddot{\tilde{M}}_{12}$$

… to arrive at:

$$h_+ = \frac{1}{r}\frac{4G}{c^4}\omega_{\mathrm{rot}}^2 (I_1 - I_2)\frac{1 + \cos^2(\iota)}{2} \cos(2\omega_{\mathrm{rot}}t)$$
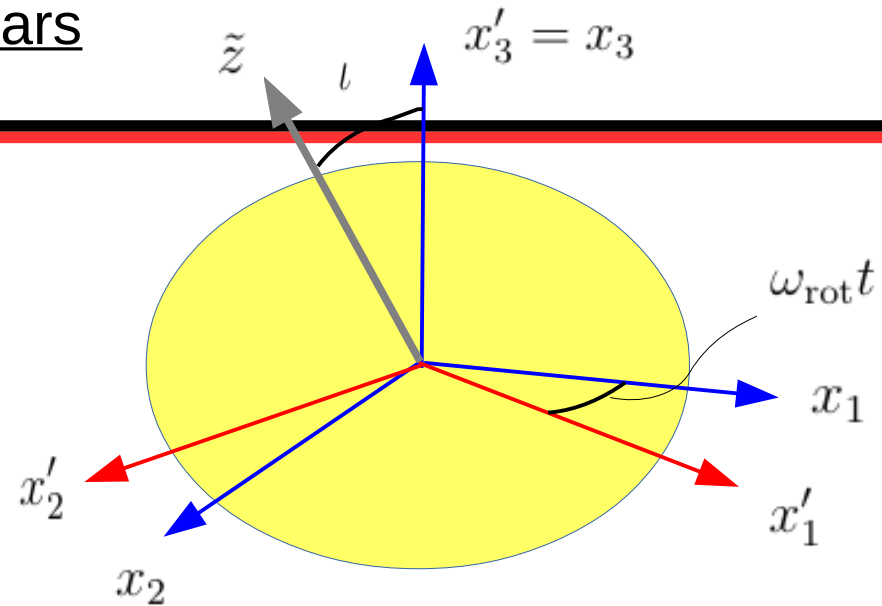
$$h_\times = \frac{1}{r}\frac{4G}{c^4}\omega_{\mathrm{rot}}^2 (I_1 - I_2) \cos(\iota)\ \sin(2\omega t)$$

Define *ellipticity* by

$$\epsilon = \frac{I_1 - I_2}{I_3}$$

Define "gravitational wave frequency"

$$f_{\mathrm{gw}} = 2 f_{\mathrm{rot}} = 2\omega_{\mathrm{rot}}/(2\pi)$$

Gravitational waves from neutron stars

Write polarizations as

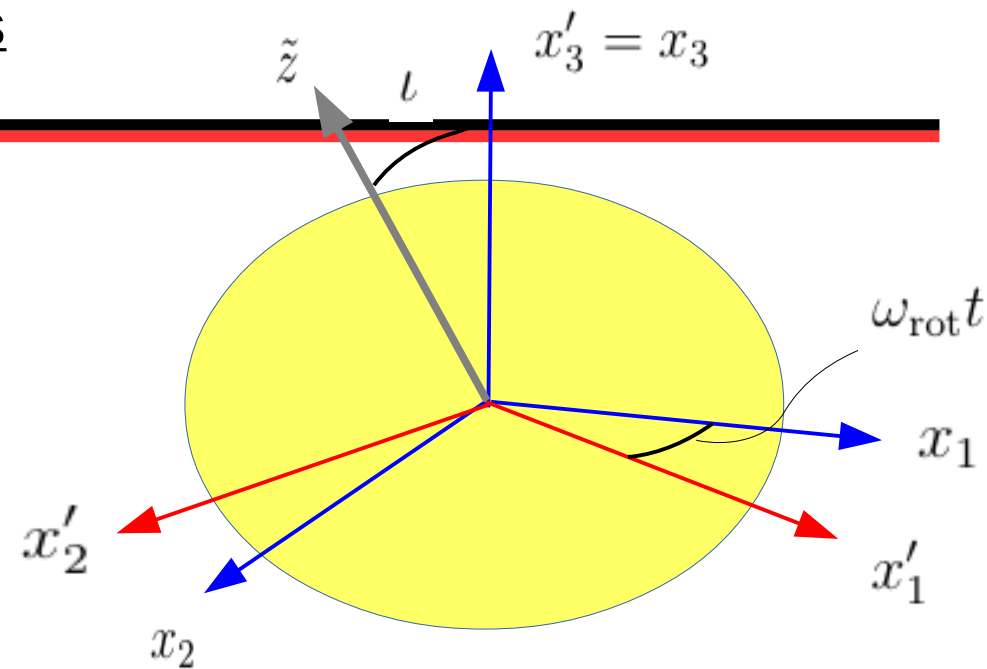$$h_+ = \mathcal{A}\frac{1+\cos^2(\iota)}{2}\cos(2\pi f_{\text{gw}}t)$$

$$h_\times = \mathcal{A}\cos(\iota)\,\sin(2\pi f_{\text{gw}}t)$$

… where

$$\mathcal{A} = \frac{1}{r}\frac{4\pi^2 G}{c^4}I_3 f_{\text{gw}}^2 \epsilon$$

Typical neutron star:

- Mass $\quad m \simeq 1.4\,M_\odot$

- Radius $\quad R \simeq 10\,\text{km}$

$\rightarrow \quad I_3 = (2/5)\,MR^2 \simeq 10^{38}\,\text{kg}\,\text{m}^2$

- Frequency ~ 1 kHz (higher end)

- Located near center of galaxy, r ~ 10 kpc

- Ellipticity unknown; $\quad \epsilon \sim 10^{-6}\quad$ possible

$$\boxed{\mathcal{A} = 1.04 \times 10^{-25}\left(\frac{10\,\text{kpc}}{r}\right)\left(\frac{I_3}{10^{38}\,\text{kg}\,\text{m}^2}\right)\left(\frac{f}{1\,\text{kHz}}\right)^2\left(\frac{\epsilon}{10^{-6}}\right)}$$

Gravitational waves from neutron stars

Energy emitted in gravitational waves:

$$
\begin{aligned}
\frac{dE_{\mathrm{GW}}}{dt} &= \frac{c^3 r^2}{16\pi G} \int_{\mathcal{S}} d\Omega \, \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \\
&= \frac{2G}{c^5} \langle \dddot{\tilde{M}}_{11}^2 + \dddot{\tilde{M}}_{12}^2 \rangle \\
&= \frac{32G}{5c^5} I_3^2 \epsilon^2 \omega_{\mathrm{rot}}^6
\end{aligned}
$$

Effect on $\quad E_{\mathrm{rot}} = \dfrac{1}{2} I_3 \, \omega_{\mathrm{rot}}^2 \quad$ *if* no other braking mechanisms:

$$
\frac{dE_{\mathrm{rot}}}{dt} = -\frac{32G}{5c^5} I_3^2 \epsilon^2 \omega_{\mathrm{rot}}^6
$$

... hence

$$
\boxed{\dot{\omega}_{\mathrm{rot}} = -\frac{32G}{5c^2} \, \epsilon^2 I_3 \, \omega_{\mathrm{rot}}^5}
$$

Observations: $\quad \dot{\omega}_{\mathrm{rot}} \propto -\omega_{\mathrm{rot}}^n \quad$ with *n* = 2 - 3 , *hence GW not dominant*

LIGO result for the Crab pulsar:

Not more than 0.2% of energy emitted in GW

Gravitational waves from neutron stars

A NEUTRON STAR: SURFACE and INTERIOR

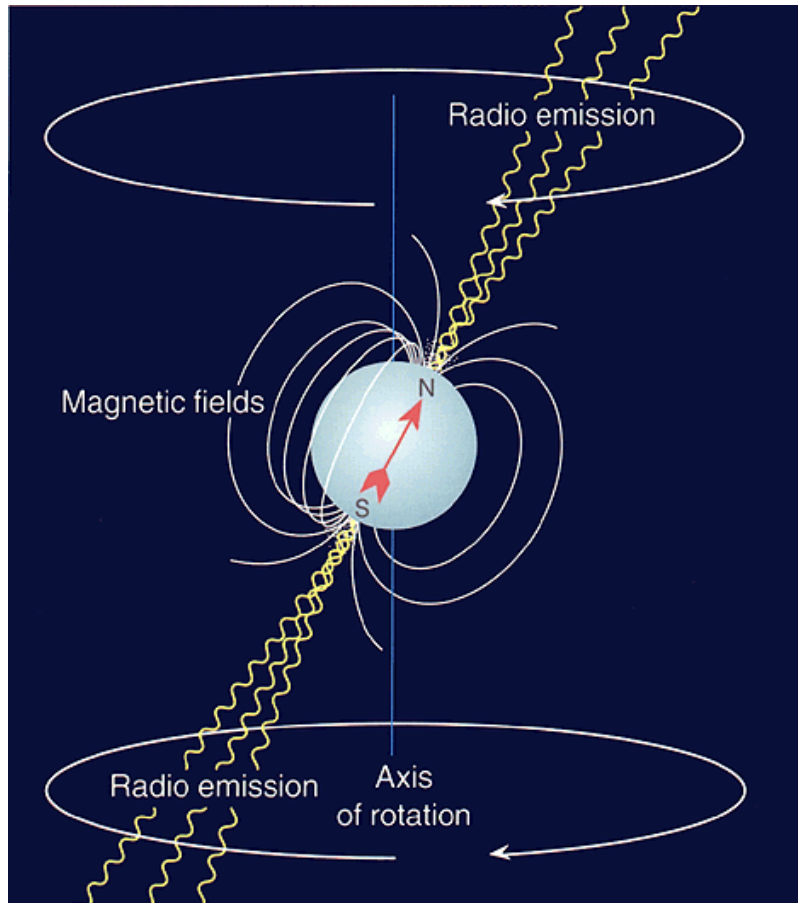https://www.astro.umd.edu/~miller/Images/NStarInt.jpeg
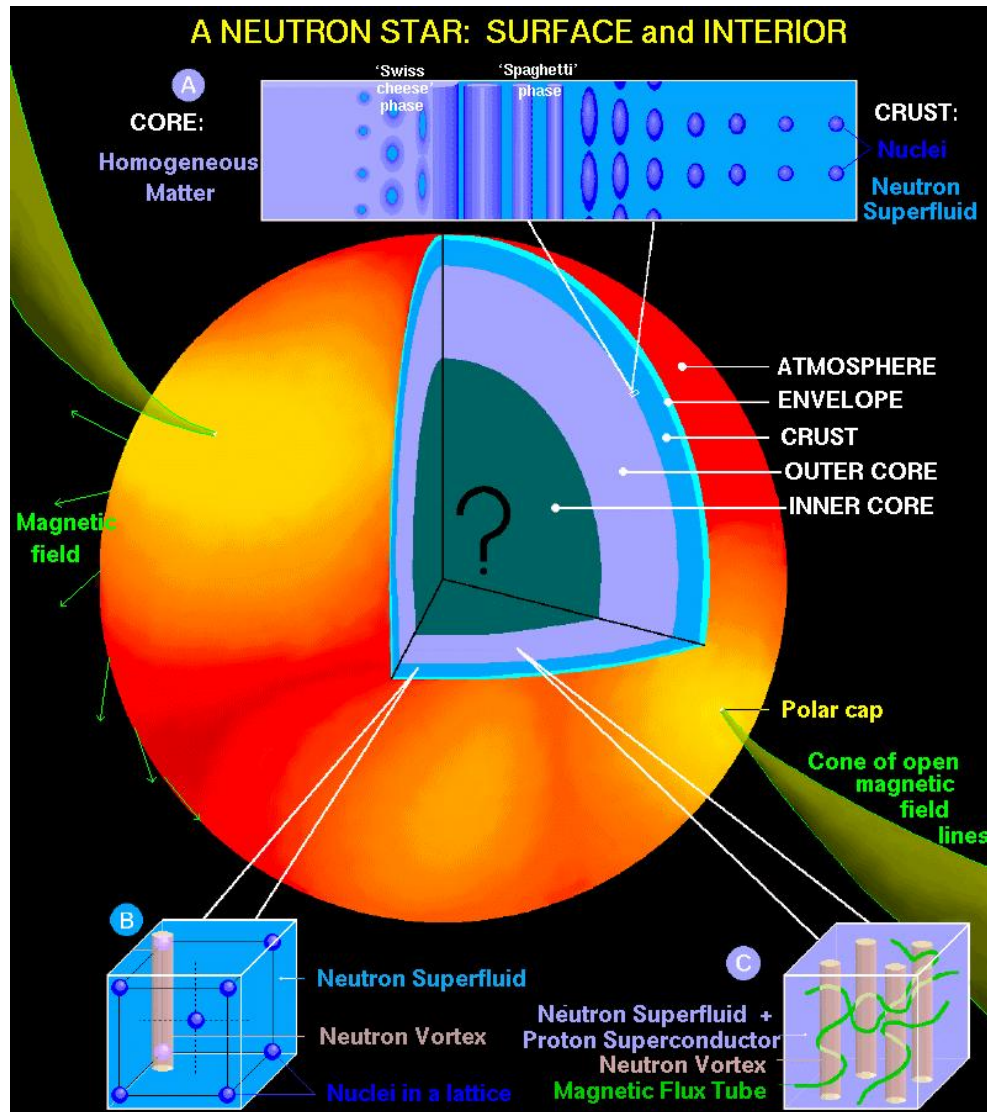
What can be learned?

Neutron star structure poorly understood:

- Structure of the crust?
- Equation of state?
- Superfluid interior
- "Pinning of fluid vortices to crust
- Origin of magnetic field?
- More exotic objects?

  (Strange quark stars?)

Gravitational waves from neutron stars

Crustal structure:

- Spin-down causes cracking/ resettling of the crust; "mountains" possible

  - determine ellipticity.

Pulsar "glitches":

- Fluid modes get excited; causes *transient* GW signal

   - Learn about equation of state of dense nuclear matter

# Continuous waves, galactic neutron stars

- Crusts can support eccentricities up to 10^{-5}

$$h_0 = \frac{16\pi^2 G \epsilon I f^2}{c^4 r} = 4.23\,10^{-26} \frac{1\,kpc}{r}\, \frac{\epsilon}{10^{-6}}\, \frac{I}{10^{38}\,kgm^2}$$

- strains in the order of $h <\sim 10^{-26}$

- Long integration times needed ($S/N \propto \sqrt{T_{obs}}$)

  - Phase and amplitude (antenna pattern) g.w. depend on several parameters:

    - Right ascension, declination, orientation spin axis, angle of principle axis for deformation, polarization of the wave, latitude, longitude, and direction of arms detector, Doppler shifts due to detector motion

- Coherent templates over long time periods: different type of analysis than in binary coalescence (non-stationary noise, small noise sources at sharp frequencies)

Doppler shifts

- Observation of Crab pulsar (radio). Line broadening due to motion of the earth-based detector.
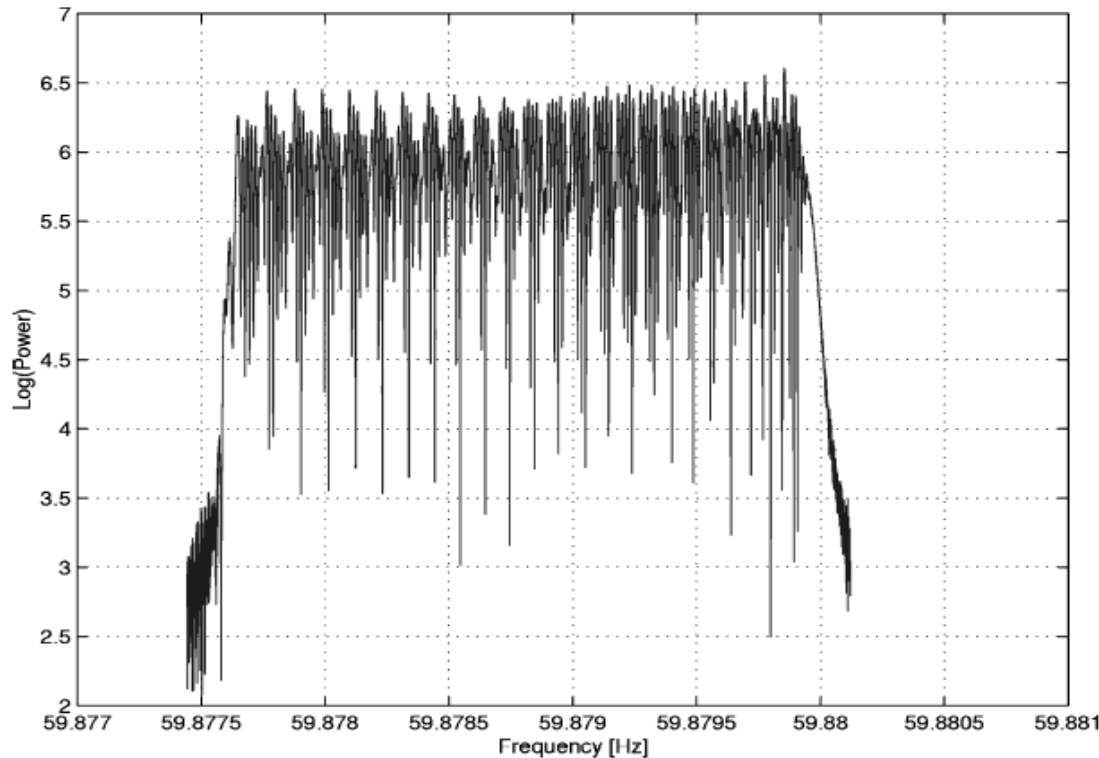
24 days of observation, 24 peaks. Each dayly peak has 5 lines due to amplitude modulation.

Targeted search, known pulsars

- For a solitary neutron star with **known** position and frequency, fit the 8 polarization-dependent amplitudes based on 5 parameters. Correct for detector-motion induced Doppler shifts and antenna pattern effects.

- Coherent integration times for as long as possible. Different analysis techniques.

- Spin-down limit: observed strain when all observed loss is due to gravitational waves

$$h_0^{sd} = \sqrt{\frac{5 G I_{zz} \dot{f}_r}{2 c^3 d^2 f_r}}$$

- Abbott et al (2017) arXiv:1701.07709
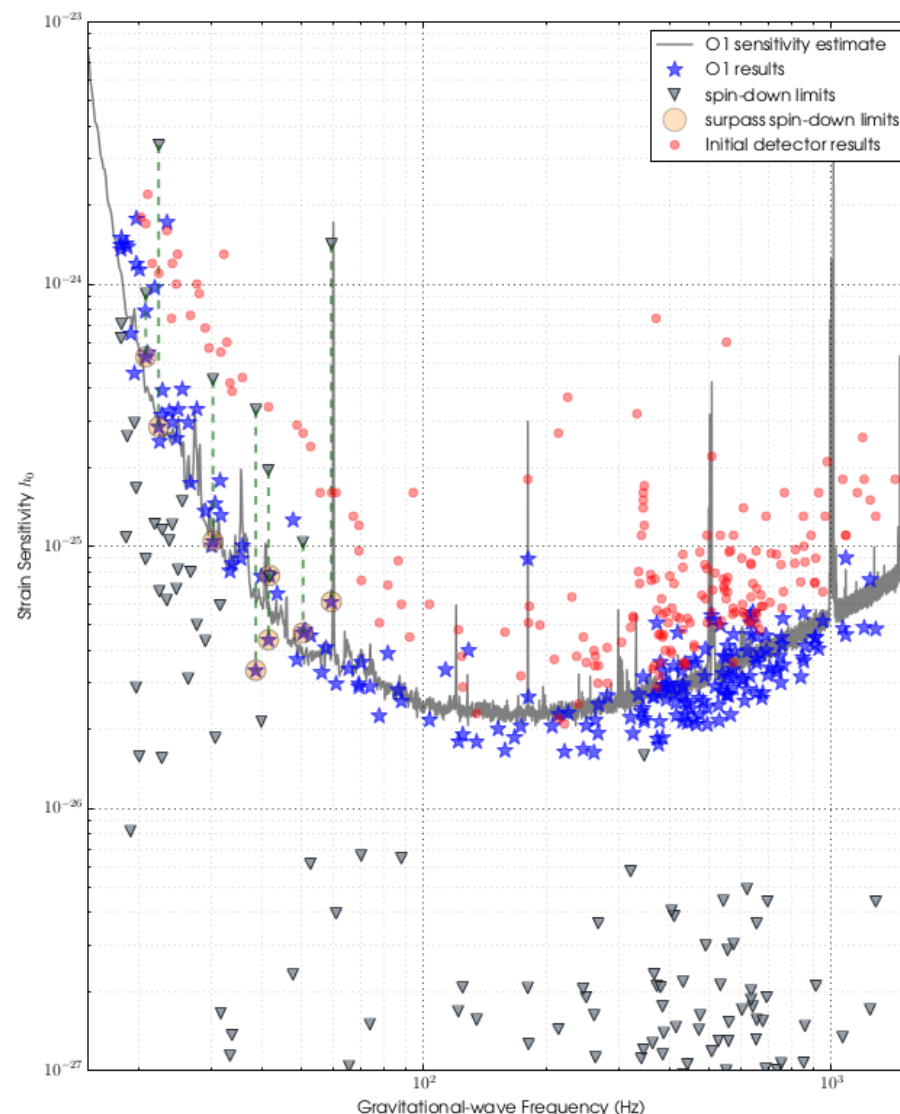
Figure 1. Stars show 95% credible upper limits on gravitational-wave amplitude, $h_0^{95\%}$, for 200 pulsars using data from the O1 run. ▼ give the spin-down limits for all pulsars (based on distance values taken from the ATNF pulsar catalog (Manchester et al. 2005), unless otherwise stated in Tables 1 and 4) and assuming the canonical moment of inertia. The upper limits shown within the shaded circles are those for which the spin-down limits (linked via the dashed vertical lines) are surpassed with our observations. The gray curve gives an estimate of the expected strain sensitivity for O1, combining representative amplitude spectral density measurements for both H1 and L1. This estimate is an angle-averaged value and for particular sources is representative only, whilst the broader range over all angles for such an estimate is shown, for example, in Figure 4 of Abbott et al. (2010a). Previous initial detector run results (Aasi et al. 2014) for 195 pulsars are shown as red circles, with 122 of these sources corresponding to sources searched for in O1.
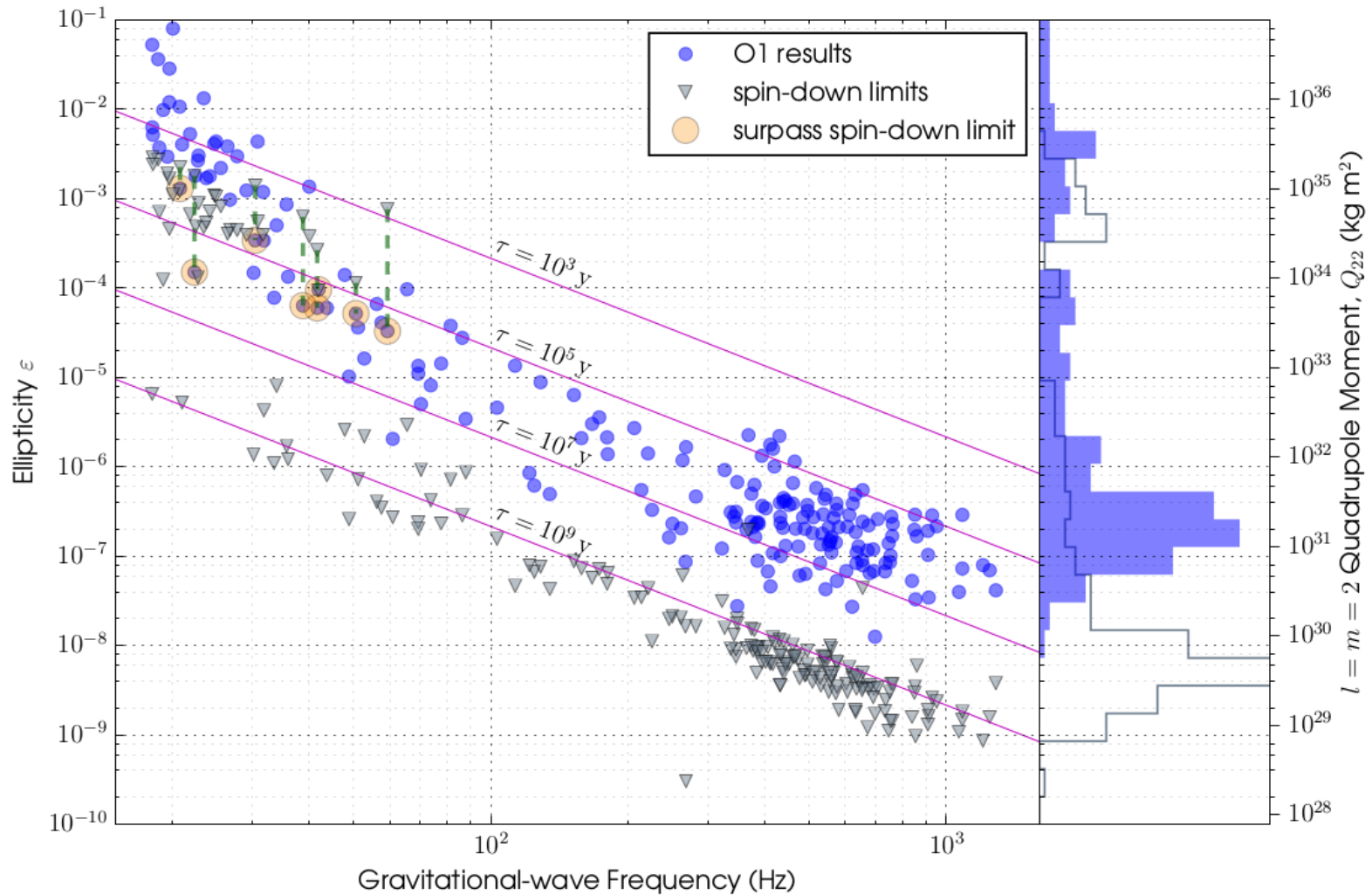
Figure 3. Limits on fiducial ellipticities (ε) and mass quadrupole moments (Q_{22}). ▼ show the values based on the spin-down limits for these pulsars. The pulsars for which the spin-down limit is surpassed are highlighted within larger shaded circles and linked to their spin-down limit values with dashed vertical lines. Also shown are diagonal lines of constant characteristic age, τ, for gravitars (with braking indices of $n = 5$) calculated via $\varepsilon^{sd} = 1.91 \times 10^5 f_{rot}^{-2}/\sqrt{(n-1)\tau I_{38}}$, where I_{38} is the principal moment of inertia in units of 10^{38} kg m² (where we set $I_{38} = 1$).

All-sky blind searches

- Only a small fraction of the neutron stars appear as pulsar

- Blind search: F-statistic [Jaranowski et al., PRD 58, 063001 (1998)]

- Einstein@home Hierarchical search, Abbott et al, PRD 94, 102002 (2016)

  - Amplitude modulation due to antenna pattern is corrected

  - Phase evolution needed

    - Computationally very challenging for coherent searches with a duration of a few days.

    - Hierachical approaches

      - Slices of 1800 seconds. Total coherence time T = 60 h.
        - Slices are analyzed with matched filters, and the results are stacked.
      - Signal to noise improves as $T^{1/4}$ (semi-coherent search)
      - Templates in a grid of ascension, declination, frequency, and df/dt
      - Combine likelihoods of hits in separate segments
      - Number of templates needed depends on total time of observation and length of data slice.

      - Einstein@home Used >10 million work units to obtain 10^{12} candidates. A small fraction has been followed-up with a fully coherent search in a narrow limit of parameters.

Blind searches, Einstein@home

- Template spacing (a frequency, df/dt grid and a sky grid).

- For a 60-hour search 50-510 Hz, 6.3×10^{16} were needed. 12.7 million Einstein@home work units (1 WU equals about 6 hours CPU time)

TABLE I. Search parameters rounded to the first decimal figure. T_{ref} is the reference time that defines the frequency and frequency derivative values.

T_{coh}	60 hours
T_{ref}	960499913.5 GPS sec
N_{seg}	90
δf	1.6×10^{-6} Hz
$\delta \dot{f}_c$	5.8×10^{-11} Hz/s
γ	230
m_{sky}	$0.3 +$ equatorial patch

FIG. 2. Number of searched templates in 50 mHz bands. The variation with frequency is due to the increasing sky resolution. $N_f \times N_{\dot{f}} \sim 3.7 \times 10^8$, where N_f and $N_{\dot{f}}$ are the number of f and $\dot{f}$ templates searched in 50 mHz bands. The total number of templates searched between 50 and 510 Hz is 6.3×10^{16}.
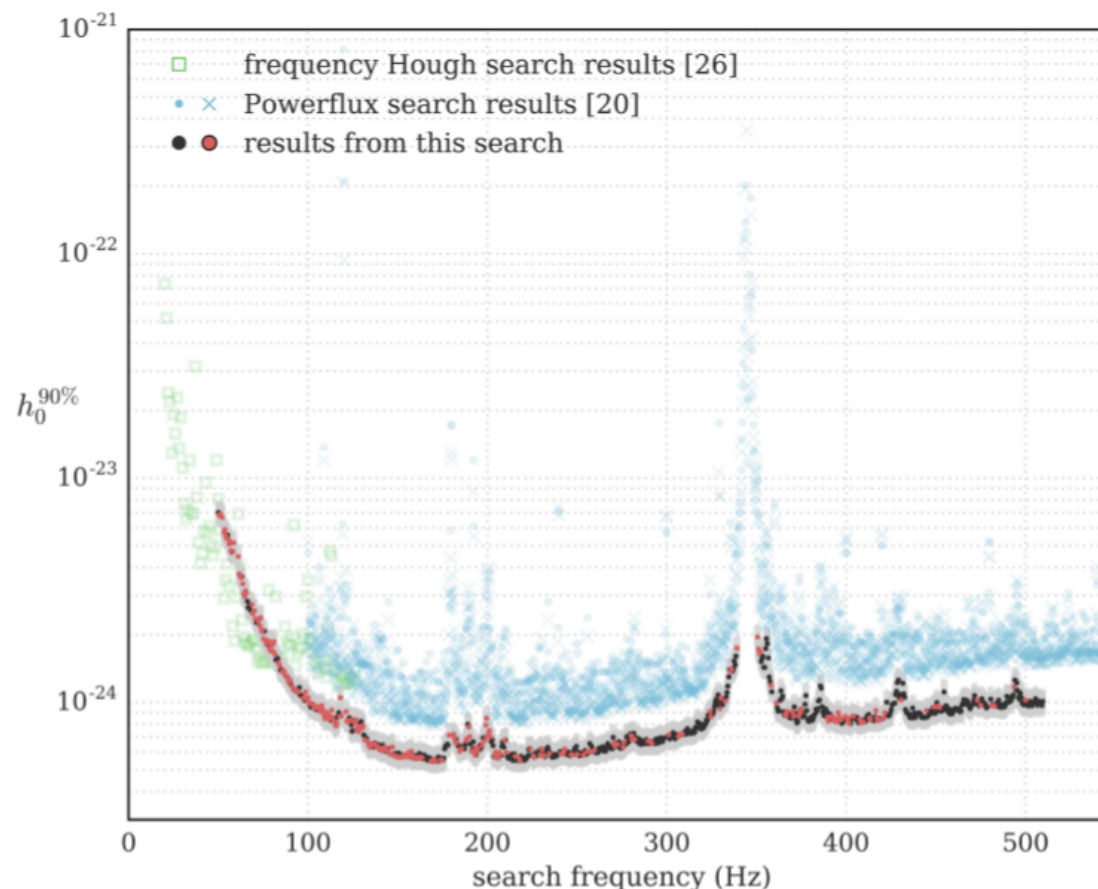
FIG. 9. 90% confidence upper limits on the gravitational wave amplitude of signals with frequency within half-Hz bands, from the entire sky and within the spin-down range of the search. The light red markers denote half-Hz bands where the upper limit value does not hold for all frequencies in that interval. A list of the excluded frequencies is given in the Appendix. Although not obvious from the figure, due to the quality of the data we were not able to analyze the data in some half-Hz bands, so there are some points missing in the plot. For reference we also plot the upper limit results from two searches: one on the same data (Powerflux) [2] and on contemporary data from the Virgo detector (frequency Hough) [4]. The Powerflux points are obtained by rescaling the best (crosses) and worst-case (dots) upper limit values as explained in the text. It should be noted that the Powerflux upper limits are set at 95% rather than 90% but refer to 0.25 Hz bands rather than half-Hz.

Neutron star distributions,
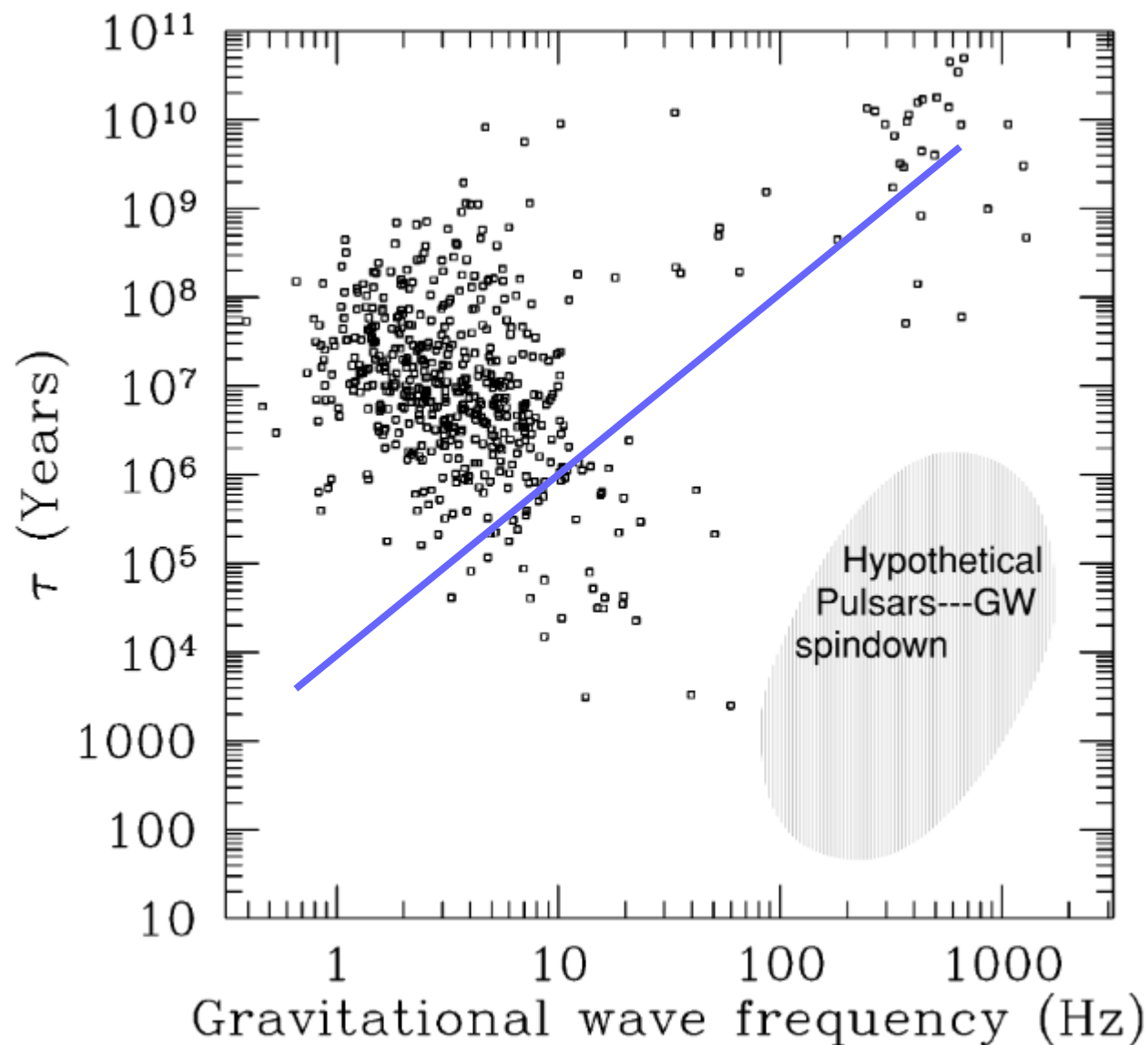
- Brady, PRD 57, 2101
  - Binary neutron stars and young neutron stars have higher spins,
  - Young neutron stars with high spin-down: possible high loss due to gravitational waves

Blind searches, binary systems

- e.g. Scorpio-X1 : neutron star observed in x-ray in binary system. Not observed as pulsar: frequency unknown

  - Search over possible spins and spin-downs with parameter space limited in orbital position and period

    - Binary system, period 0.787 days.
    - Different strategies pursued

- General searches for neutron stars in binary systems:

  - 5 extra parameters introducing Doppler shifts

  - Template bank would increase with a factor $\sim 10^{12}$ for a coherent search over 2 days, accommodating for reasonable binary-system parameters

  - For orbital periods of 2 hours, the coherent time of the peak in an FFT would be as short as 20 s.

  - Twospect method: periodigrams of FFT slices, to see periodic shifts in power

  - Polynomial search: enlarge the coherence of the filter by fitting polynomial phase

    $$\phi(t) = \phi_0 + f(t)t + \dot{f}\,t^2/2 + \ddot{f}\,t^3/6$$

    - Coherently combine neighboring stretches.

Two-spect pipeline search

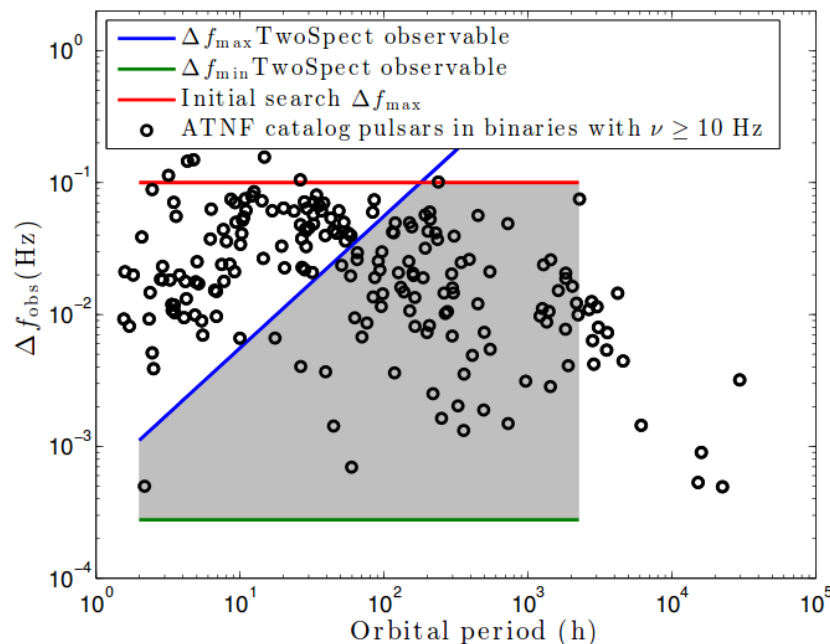- arXiv.org > gr-qc > arXiv:1405.7904

FIG. 1. Nominal parameter space that is analyzed using the TwoSpect algorithm (shaded region). The bounding curves given by Δf_{max} and Δf_{min} are limitations of the analysis, while the initial search boundary of $\Delta f_{max} = 0.1$ Hz is a choice. Data marked by circles are ATNF catalog pulsars found in binary systems with rotation frequencies ≥ 10 Hz (using Eq. (8) and assuming $f_0 = 2\nu$).
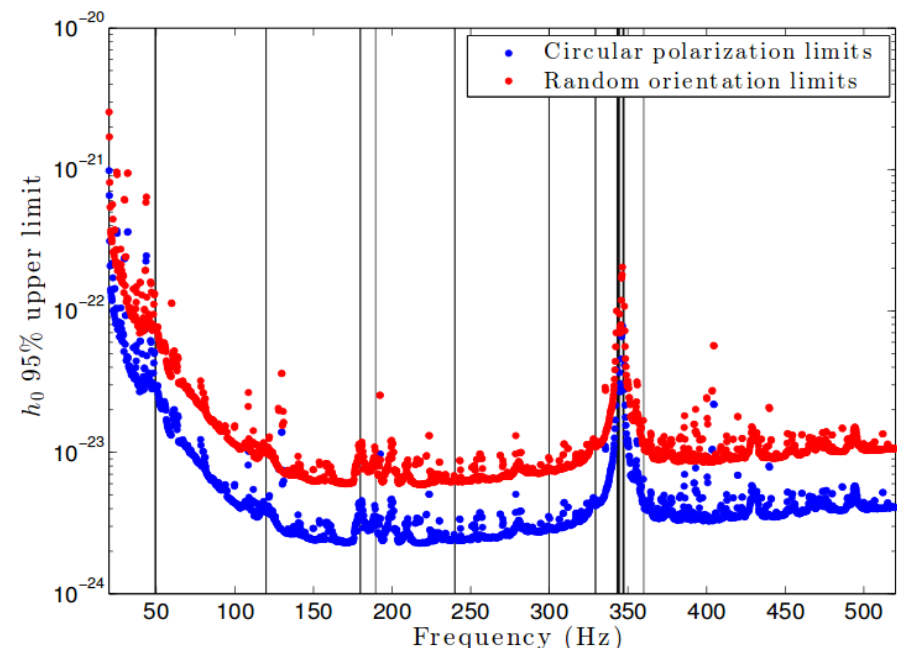
FIG. 2. All-sky strain upper limit results of S6/VSR2-3 for continuous gravitational waves assuming the source waves are circularly polarized (blue points) or randomly polarized from randomly oriented sources (red points). The vertical black lines indicate 0.25 Hz frequency bands in which no upper limits have been placed. The smoothness of the curve is interrupted due to various instrumental artifacts, such as the violin resonances of the mirror suspensions near 350 Hz.

# Example of filtering, polynomial search

- Polynomial search, try to match continuous signal with frequency dependence

- Correlations: compare data with shape filter

- Optimal filter : $$Z(f) = \frac{|S(f)|^2}{|S(f)|^2 + |N(f)|^2}$$

    - For small signals: divide the filter with the PSD of the data (i.e the noise)

- Heterodyne : compare only a small frequency band to the filter

    – Calculate the filter in a small band (e.g. 0.1 Hz)

    – Adapt the filter to all bands in the data (e.g. data going from 0-10 kHz – filter from 0-0.1 Hz, 0.0001-0.10001 Hz etc.

    – Example given for polynomial search (2 different neutron stars in a binary system injected on top of flat Gaussian noise)

Kepler orbitals

- ## ellipse

$$T^2 \propto a^3 \qquad v \propto T^{-1/3}$$

$$v_{ph} = \frac{1+\epsilon}{1-\epsilon} v_{ap}$$


εa

a
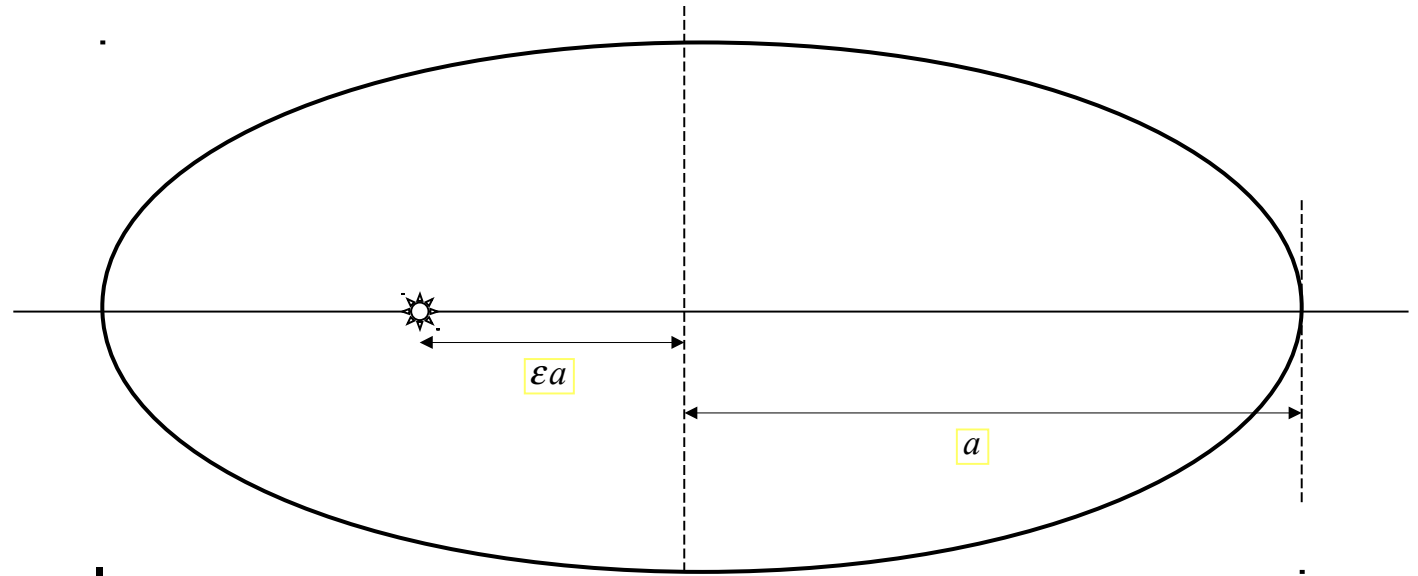
- ## We want to analyze:
  - orbital periods from 2 hours – infinite
  - masses companion star up to 15 solar masses
  - eccentricities up to about 0.7
  - frequency shifts up to 0.3%, frequency changes df/dt up to 10^{-6} s^{-2}
    - 1 mHz shift in 1 second, at f=1000Hz

Frequency shifts

- Binary system, companion 8 Msun, T=2.3 h,

- 56 deg. Declination, 0 deg ascension, 0.6 eccentricity.

  Doppler frequency shifts (right top) and frequency derivatives (right bottom). Zoom with annual variation (bottom)
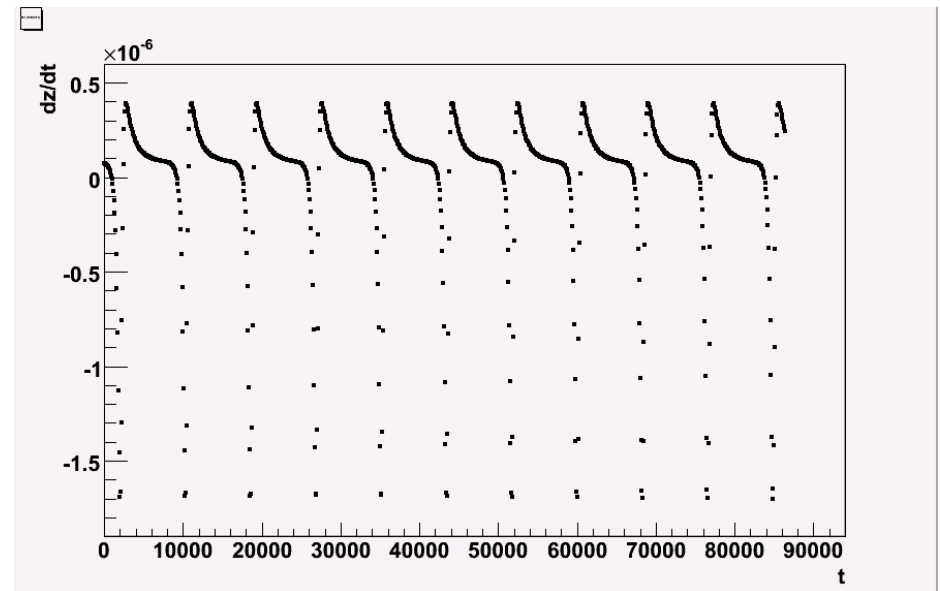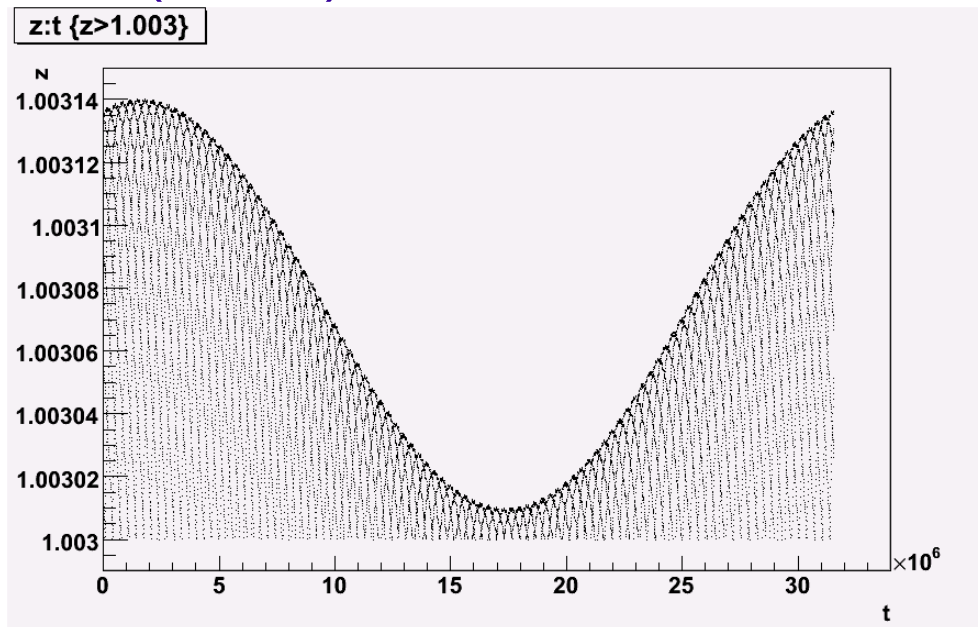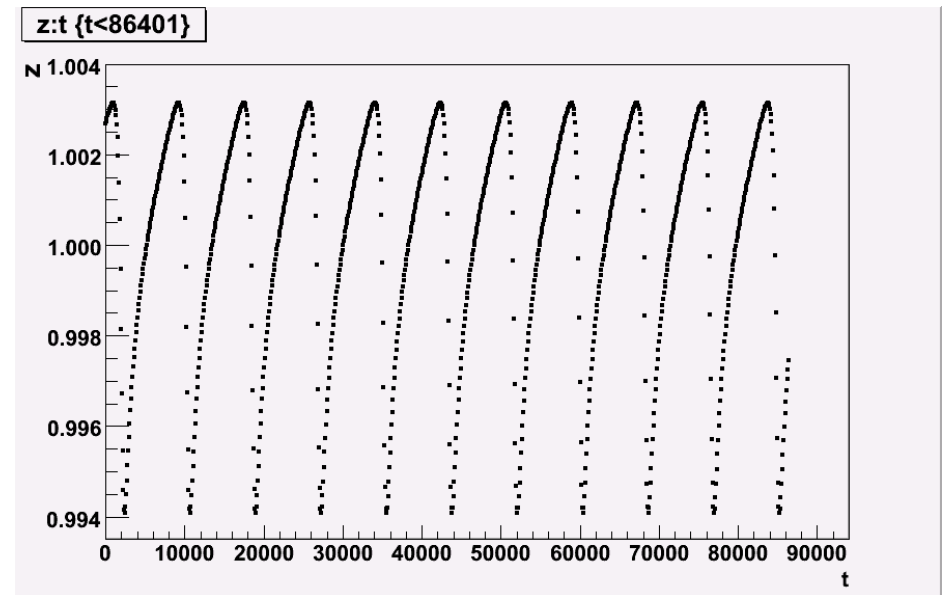
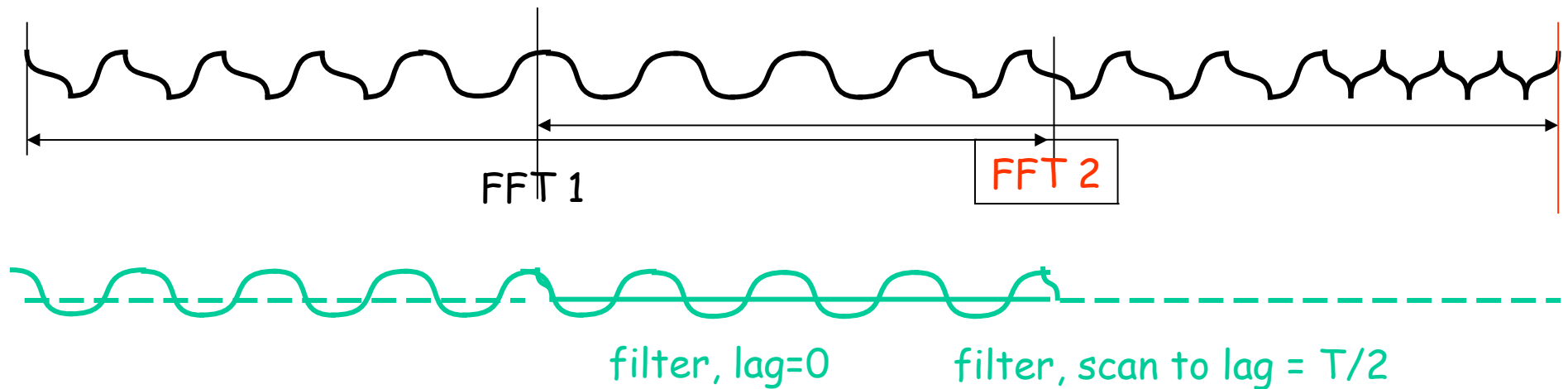# Correlation

- ## Correlation is given by

$$Corr(g,h)_\tau = \int_{-\infty}^{\infty} g(t)h(t+\tau)dt \Leftrightarrow G(f)H^*(f)$$

$FFT(G(f)H^*(f))$ gives array of correlation as a function of the lag τ

- presence of signal defined by overlap with filter.
- data is not periodic: make filter equal to zero for last N/2 samples and shift it maximally N/2 samples to the right
- FFT: interleave, to cover full dataset
  - about a billion filters needed per 1-Hz band. But much longer coherence time.

Filter search

data, split in overlapping periods

FFT 1

FFT 2

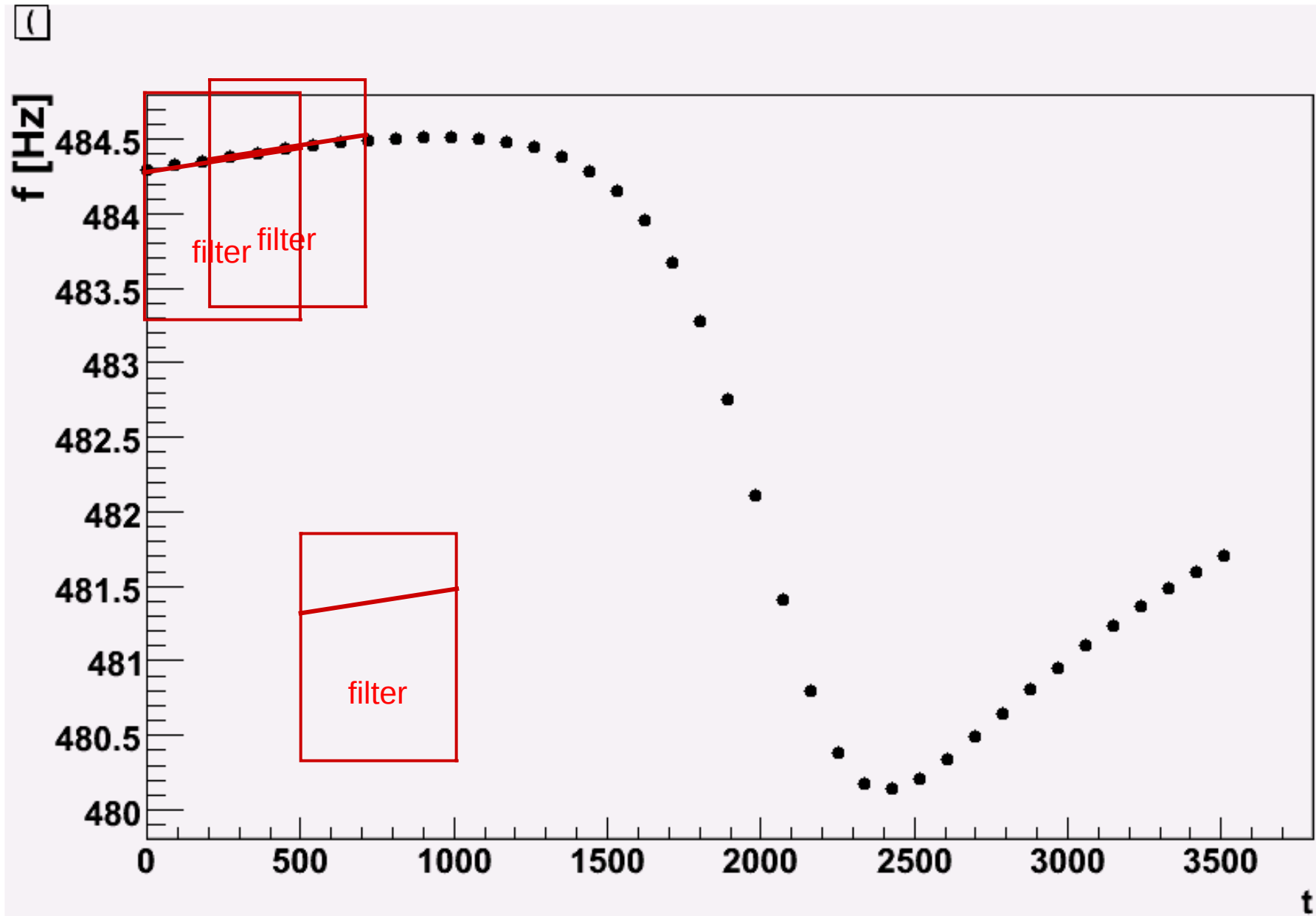filter, lag=0   filter, scan to lag = T/2

Filter: zero-padded for half length
check correlations from t=0 to t= ½T (FFT1)
check correlations from t= ½T to t=1T (FFT2)
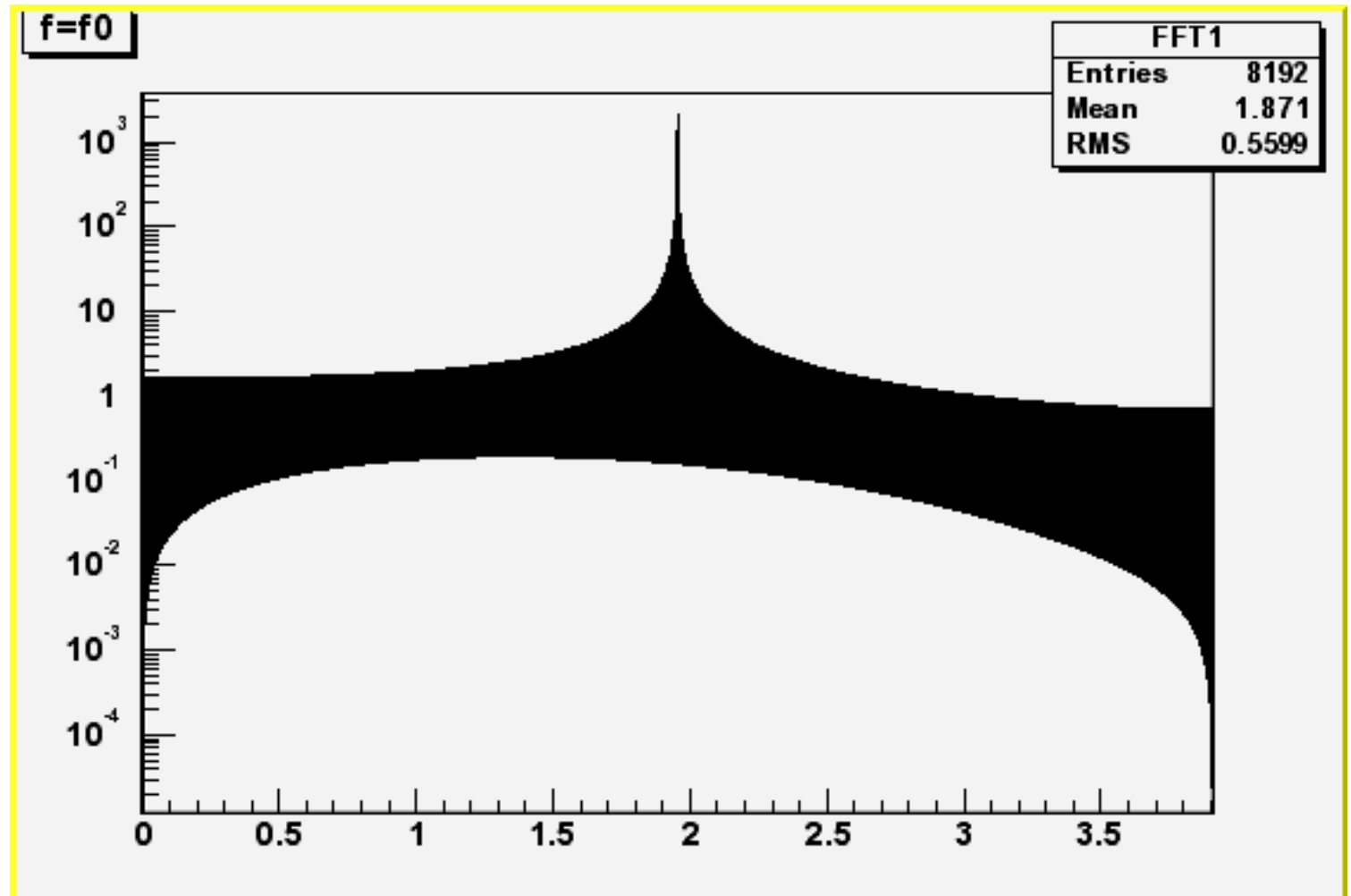check correlations from t=1T to t= 1½T (FFT3)
 maximum overlap: amplitude and time known
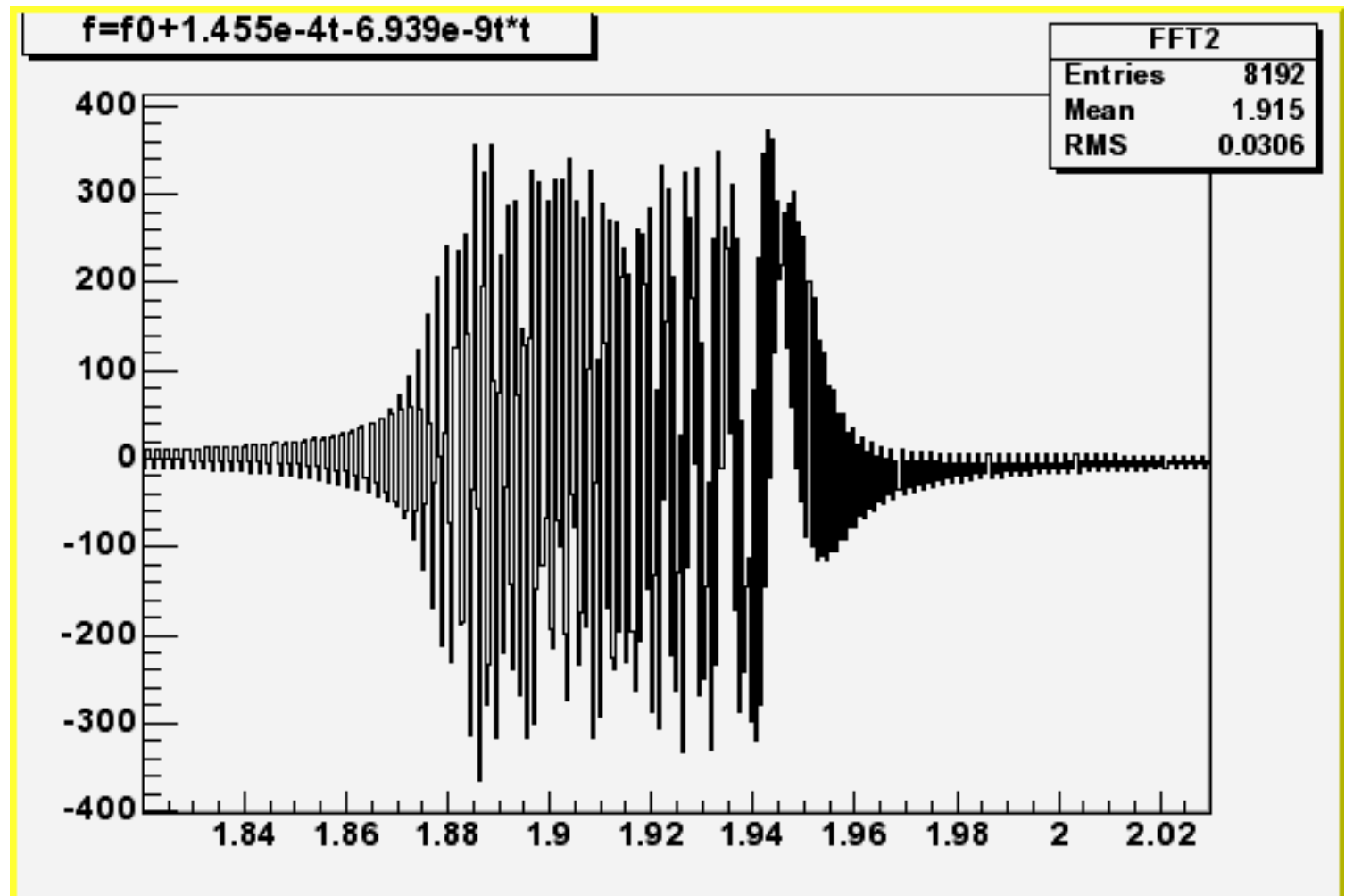
Polynomial Filter search, example

Example Filters
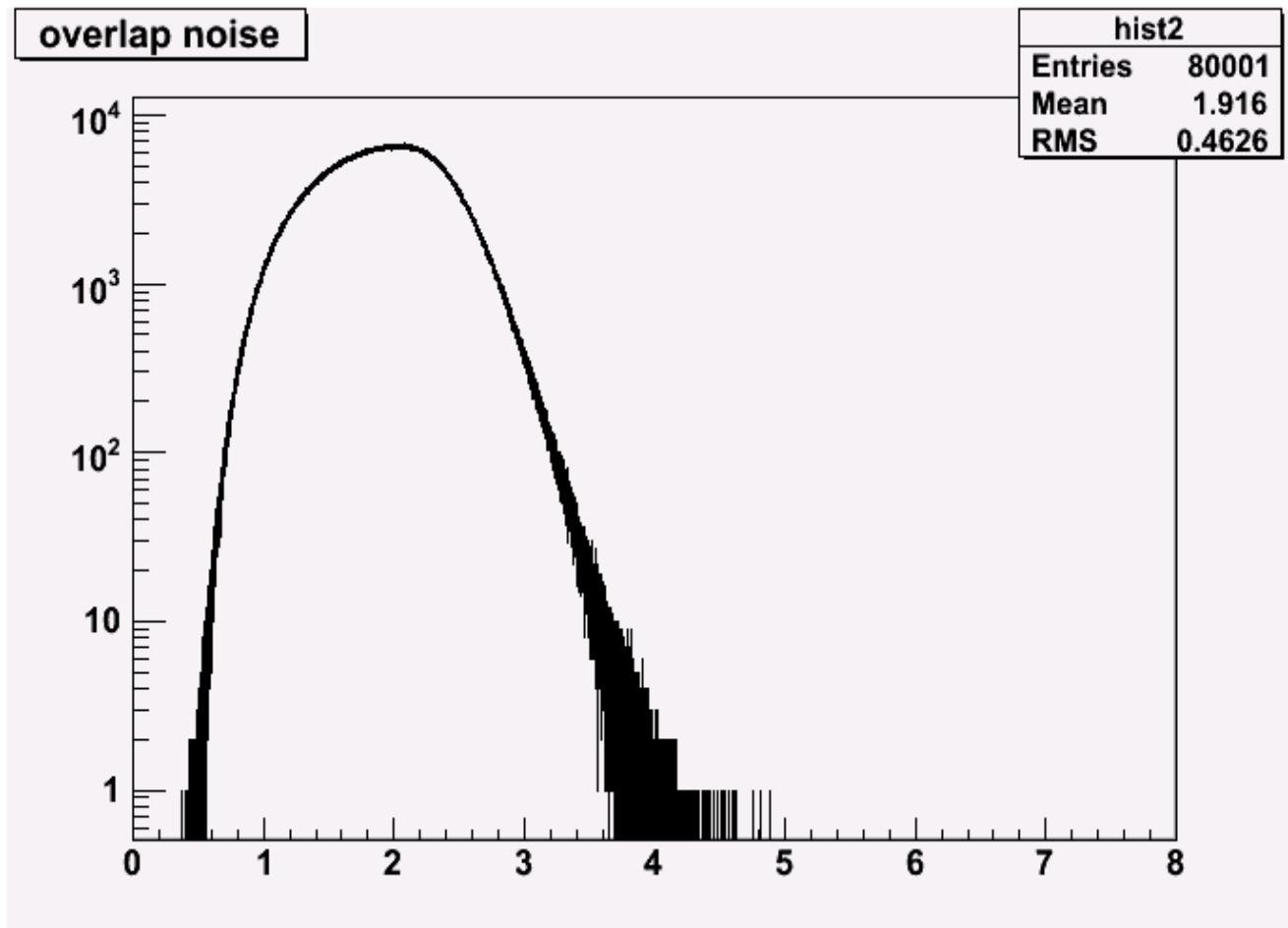
$$f(t) = \sin\phi, \quad \phi(t) = 2\pi f_0 t$$

$$filt(f) = \sin \phi, \quad \phi(t) = 2\pi f_0 t + \frac{\alpha}{2} t^2 + \frac{\beta}{6} t^3$$

$$f(t) = d\frac{\phi}{dt} = \alpha t + \frac{\beta}{2} t^{2,} \quad \alpha = 1.46 \times 10^{-4}\, Hz^{2,} \quad \beta = 1.39 \times 10^{-8}\, Hz^3$$

f=f0+1.455e-4t-6.939e-9t*t

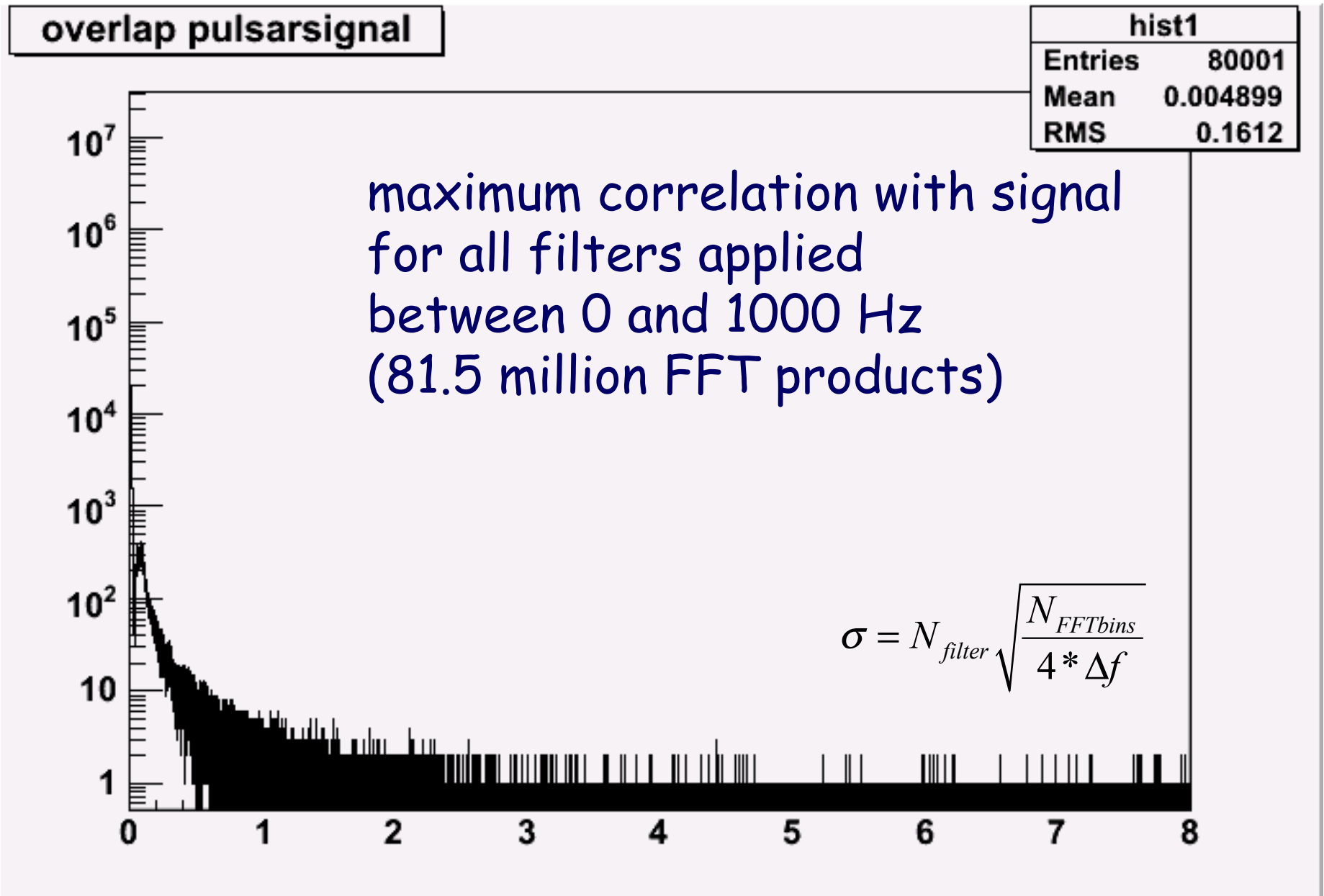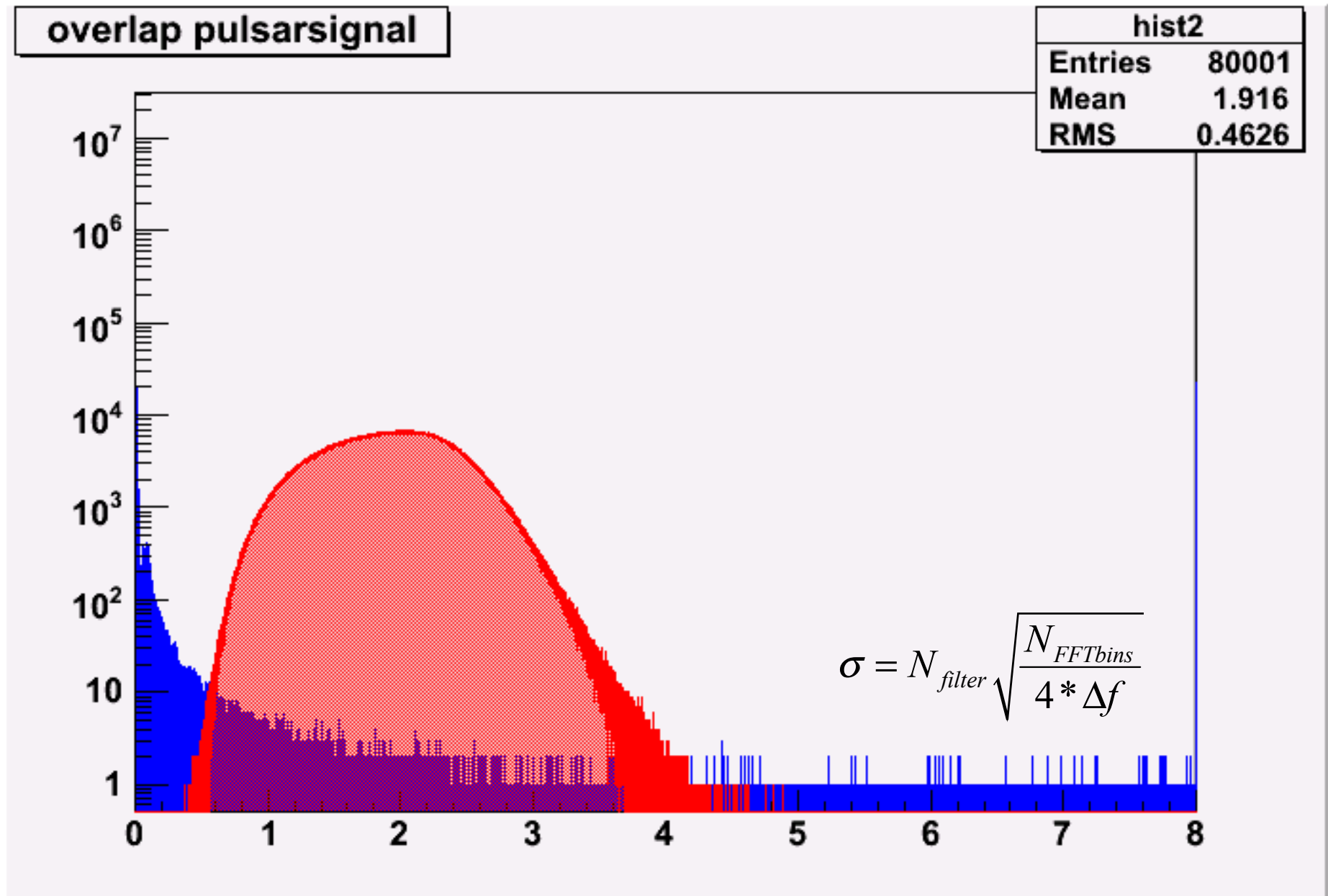| FFT2	
Entries	8192
Mean	1.915
RMS	0.0306

Test: overlap of filters, only noise

maximum correlation for all filters applied between 0 and 1000 Hz (81.5 million FFT products, 4096 lags per filter)
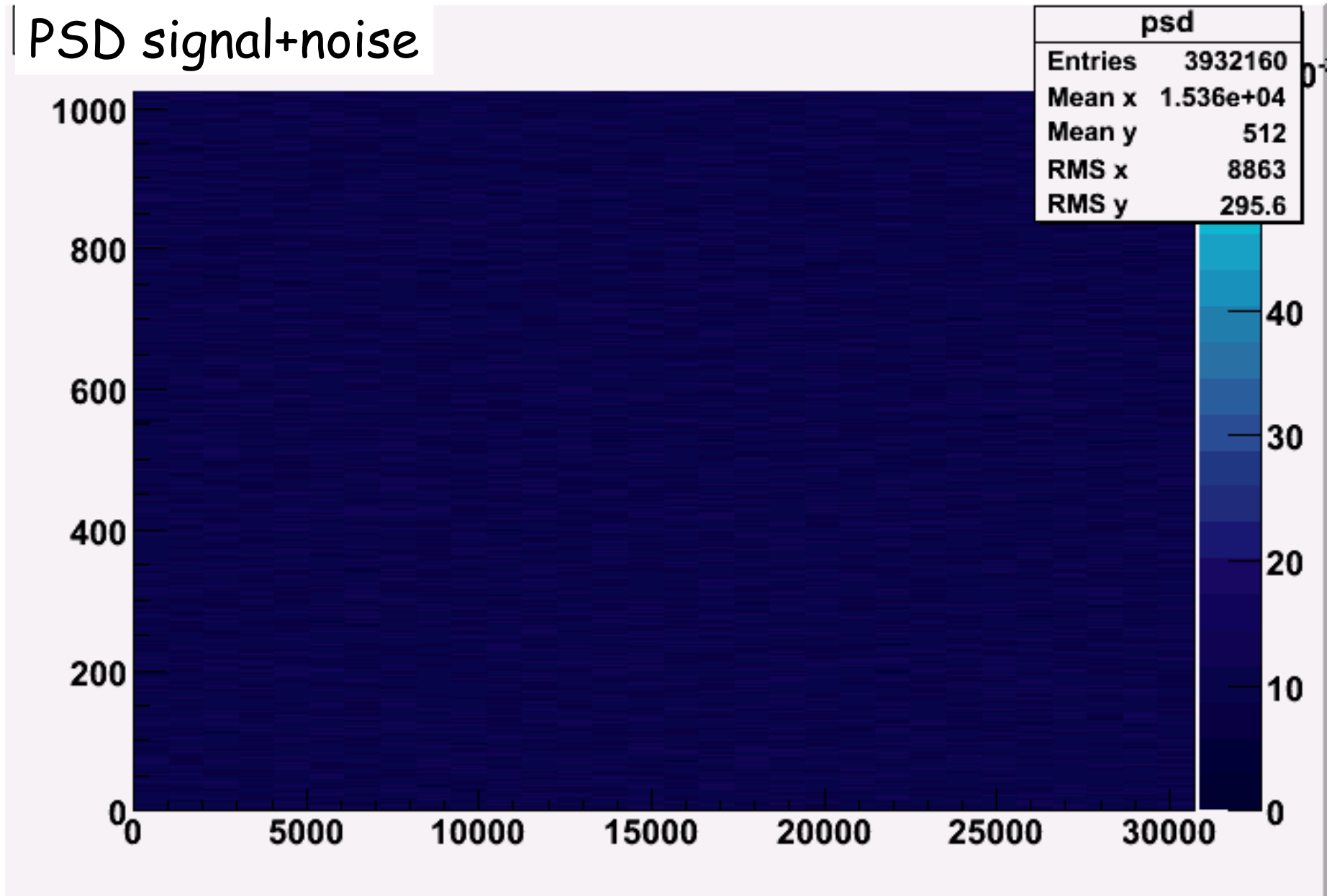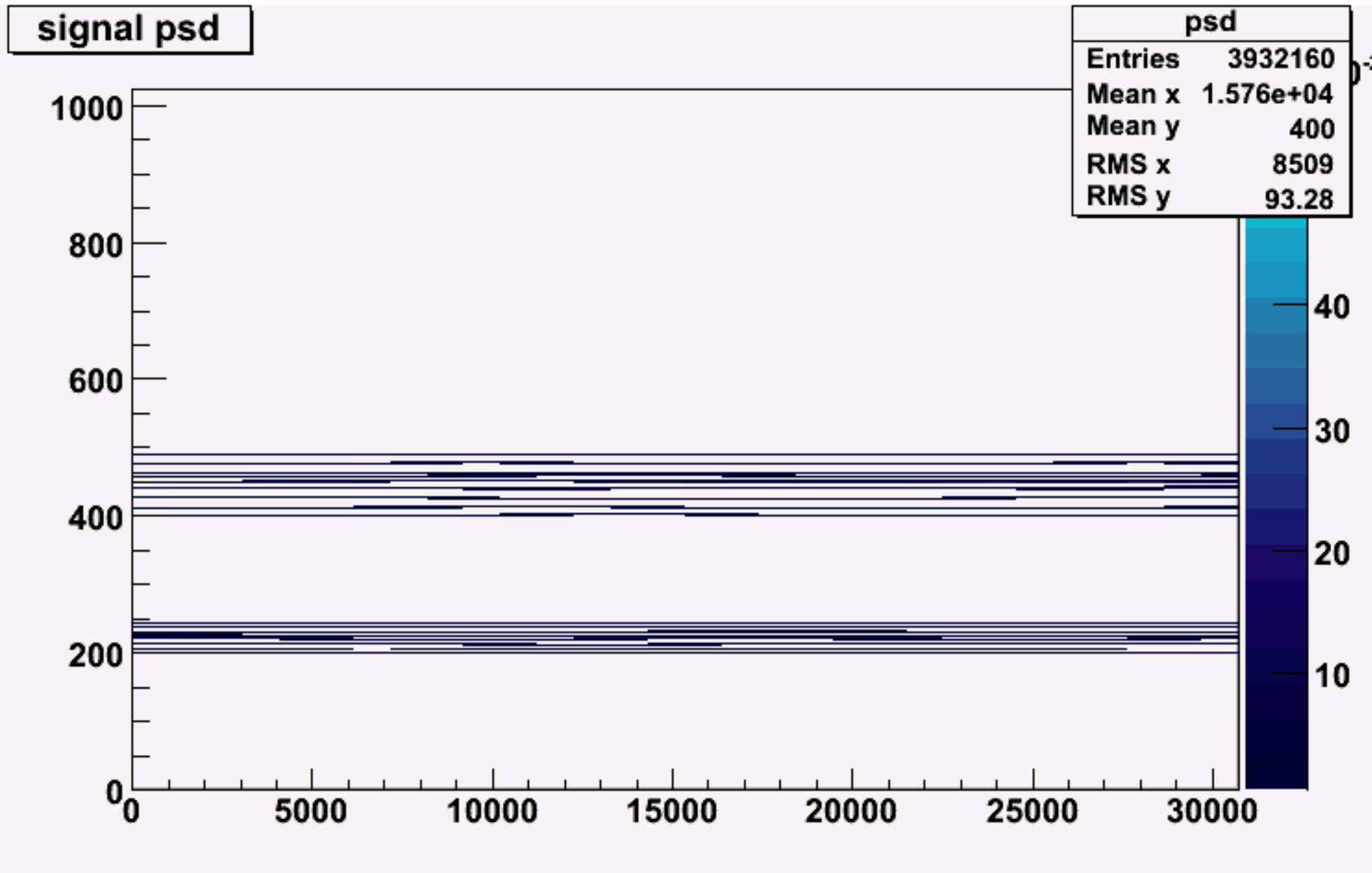
Overlap of filters with test signal

overlap pulsarsignal

hist1	
Entries	80001
Mean	0.004899
RMS	0.1612

maximum correlation with signal
for all filters applied
between 0 and 1000 Hz
(81.5 million FFT products)

$$\sigma = N_{filter} \sqrt{\frac{N_{FFTbins}}{4 * \Delta f}}$$

signal-to-noise

overlap pulsarsignal

hist2	
Entries	80001
Mean	1.916
RMS	0.4626

$$\sigma = N_{filter} \sqrt{\frac{N_{FFTbins}}{4 * \Delta f}}$$

Power spectral density

PSD signal+noise

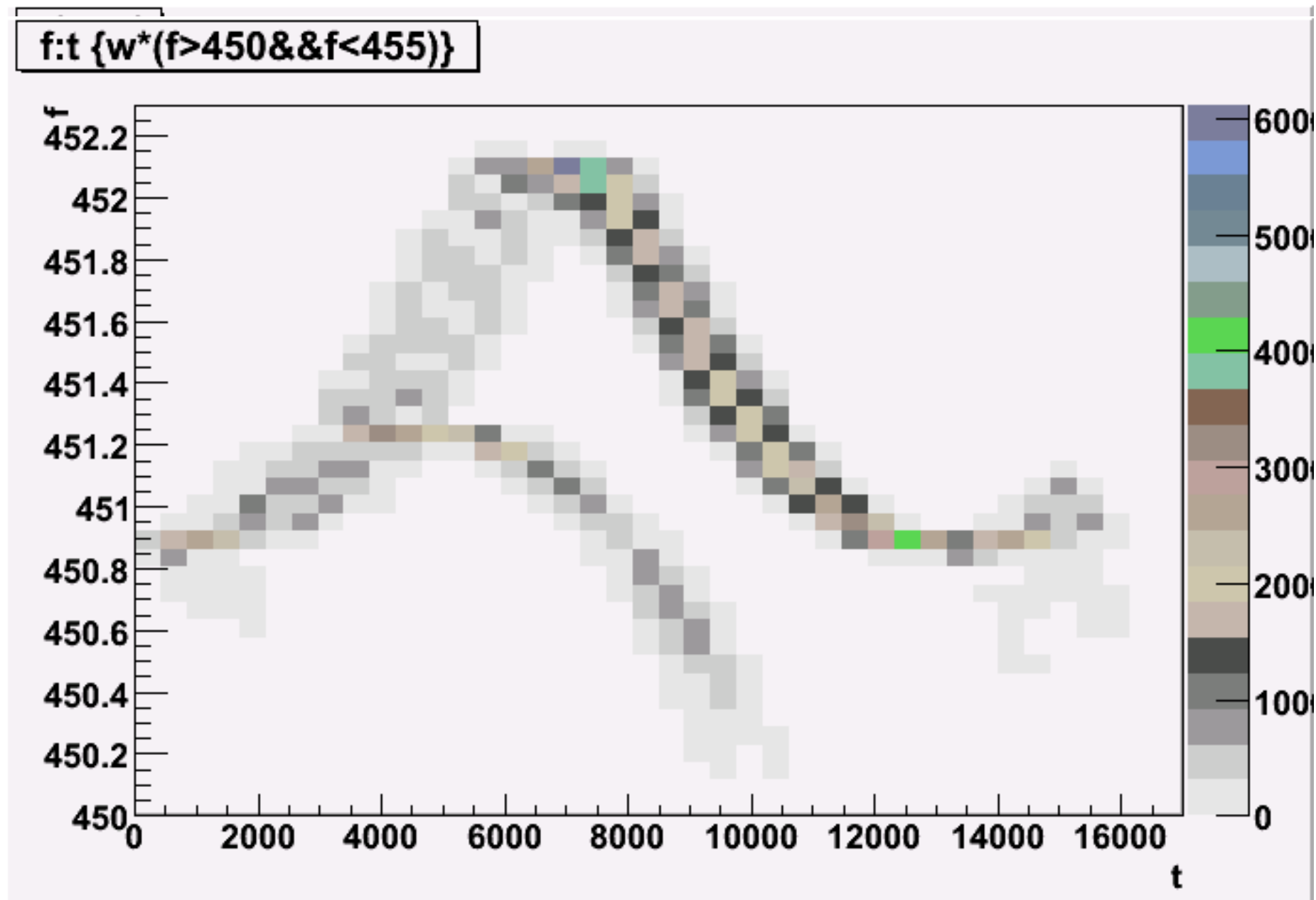| psd	
Entries	3932160
Mean x	1.536e+04
Mean y	512
RMS x	8863
RMS y	295.6

# PSD, signal only

PSD, signal only

Search results

- 30 FFTs, about 5h of data
  - Inserted 2 pulsars

- analyzed between 100 and 500 Hz
  - 2405 different filters

- about 1.3 billion filter multiplications, 28731 hits above threshold
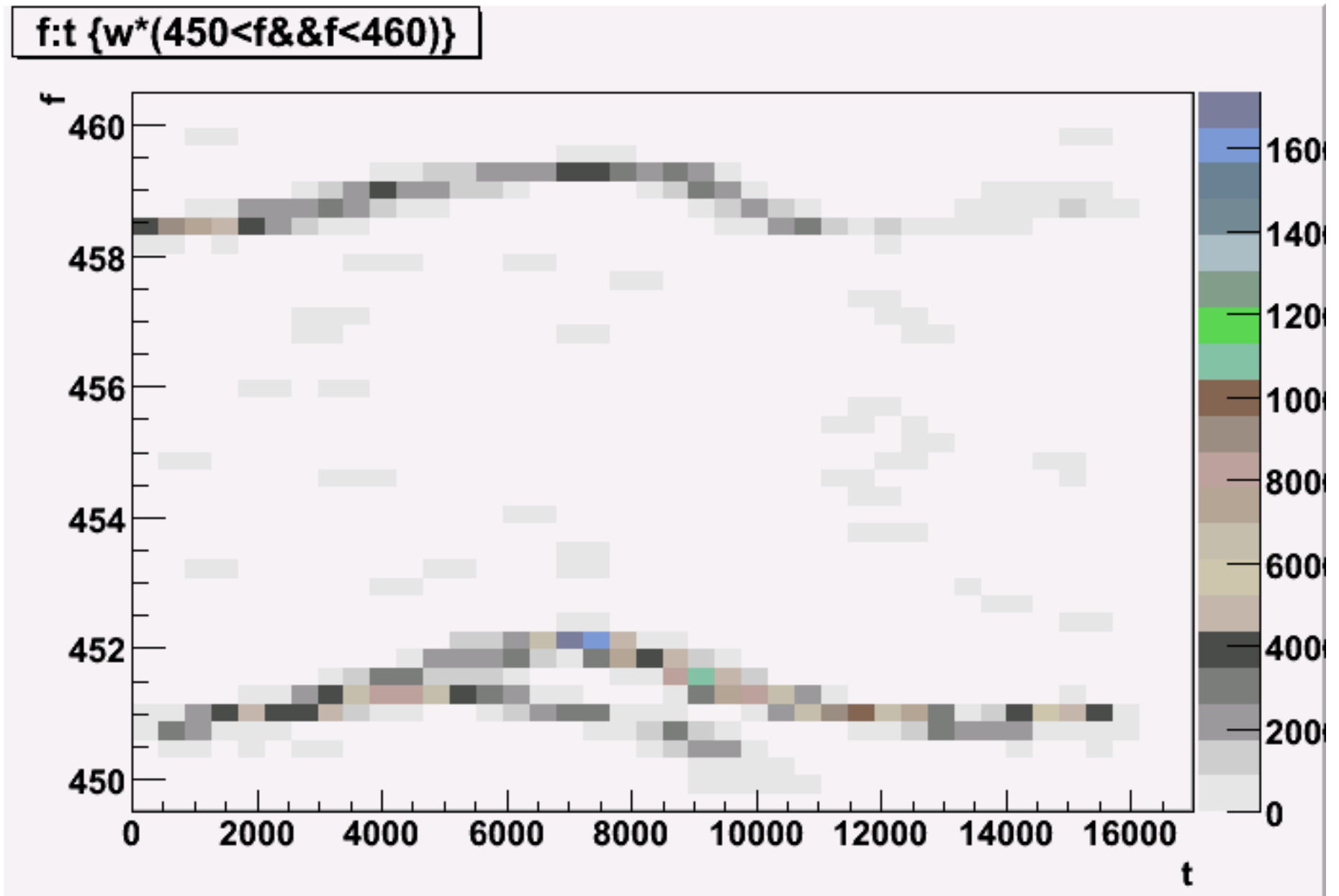
- No noise, pulsars only: 14972 hits

Search results

- 30 FFTs, about 5h of data
  - Inserted 2 pulsars

- analyzed between 100 and 500 Hz
  - 2405 different filters

- about 1.3 billion filter multiplications, 28731 hits above threshold
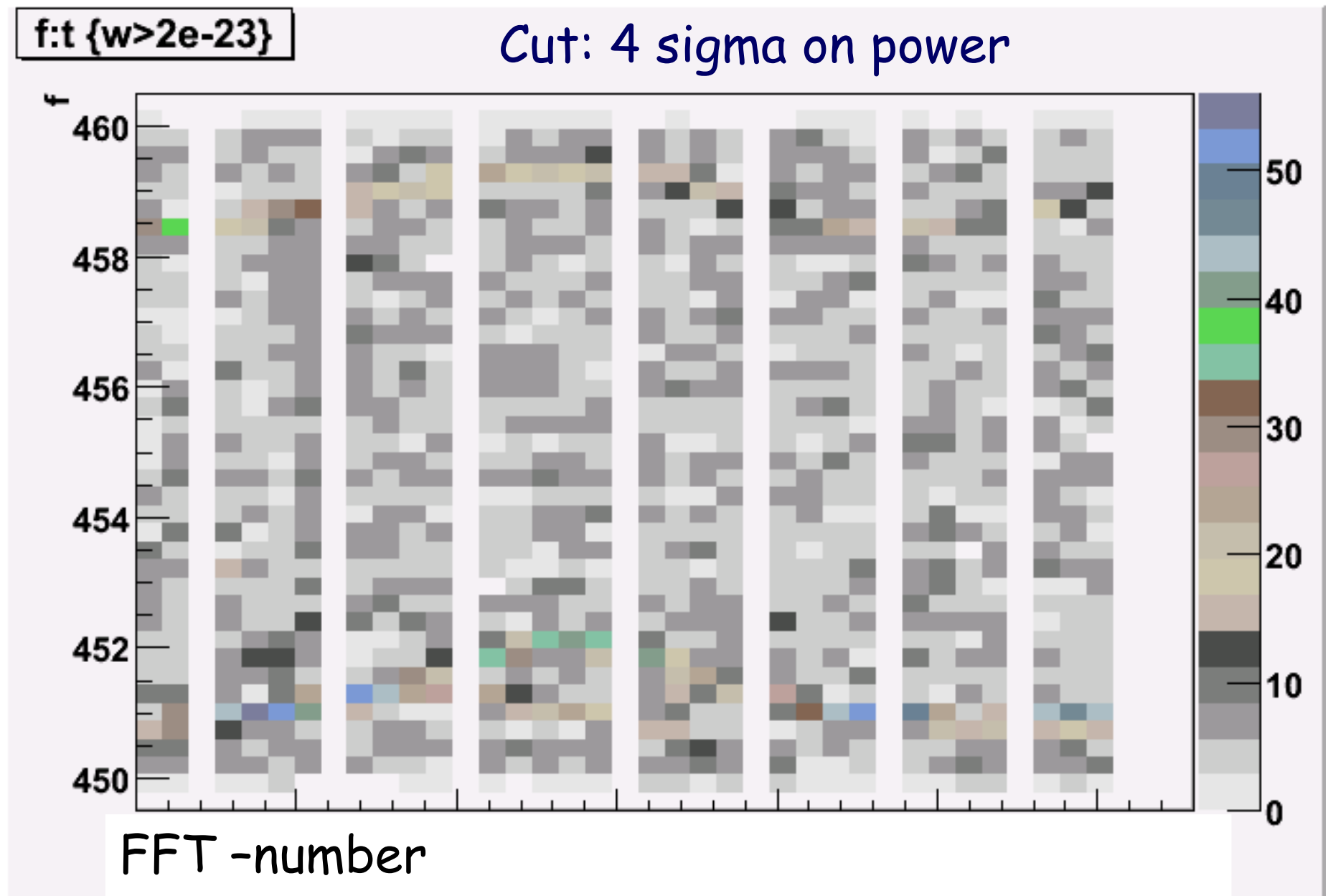
- No noise, pulsars only: 14972 hits

Search results, all hits
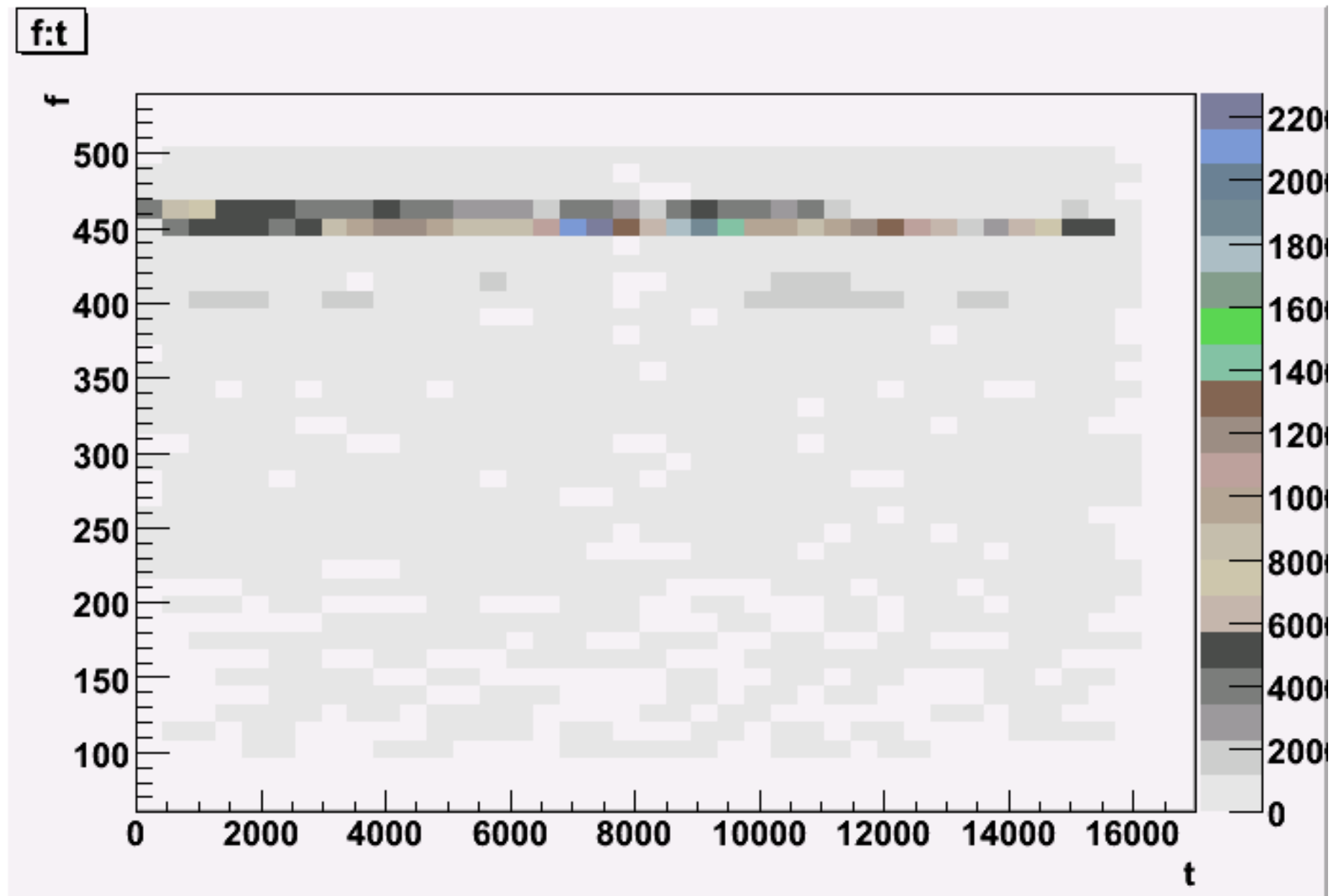
Search results

Alternative: cut on power (no filters, just FFT)

FFT –number

Summary

- Currently, no continuous gravitational waves have been observed
  - Targeted searches give upper limit for ~200 known neutron stars
  - There are about 1 billion neutron stars in our galaxy, some close-by neutron star may be detected in a blind search
  - Blind searches for solitary neutron stars show upper limits
  - For neutron stars in binary systems, or with high frequency derivatives, the parameter space is too large for the blind search pipelines
- Polynomial search algorithm increases sensitivity for continuous gravitational waves in blind searches in these cases
  - Where coherent F-statistic search cannot be applied ($>10^{30}$ filters needed)
  - Where power flux density cannot be applied due to high frequency derivatives.
  - Polynomial filters increase the coherence time with about a factor of 20, leading to 5 times higher sensitivity. However, when a continuous signal is found, the underlying physics is not known yet and a follow-up search is needed.