Pictures, Paths, Particles, Processes

Feynman Diagrams and All That and the Standard Model

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'...The Ancients were wont to draw Diagrams & thus divine Predictions for future Happenings, by Arts magickal or conjectural... likewise the Savants of the Future will learn to employ Diagrams ; yet not by Arts magickal, rather by Arts arithmetickal, algebraickal & by Geometrie and the Quadrature will they study to foretell the Events of Nature...

Simon Partlic (Třešť,1590- ?, 1649)
astronomer, mathematician and physician

Contents

Chapter 0

Introductory remarks

0.1 Preface

In what follows, whatever is correct I owe to many other people ; that which is wrong I managed on my own. I am perpetually in need of, and grateful to, those pointing out typing or thinking errors in these notes[1].

About exercises : these are intended to make you sufficiently proficient in actually computing things. Theoretical knowledge without dirty hands is not very good. The exercises apposite to a topic are indicated by a box like $\boxed{\mathbf{E}\ \aleph}$ in the margin, where '$\aleph$' is the number of the exercise, situated at the end of the chapter. $\boxed{\mathbf{E}\,0}$

[1]I cordially invite all and sundry to do so. The P^4 Hall of Fame collects the names of friends who have helped me in learning about, formulating, contemplating, or execrating one or several issues.

0.2 Basic tools

0.2.1 Units and fundamental units

The fundamental constants[2] of relativistic quantum field theory are the speed of light *in vacuo* :

$$c = 299792458\,\frac{\mathrm{m}}{\mathrm{sec}}\quad ,$$

and Planck's (or rather Dirac's) constant

$$\hbar = 1.054571726(47)\ \times\ 10^{-34}\ \mathrm{Joule\,sec}\quad .$$

Compared to the scales of our everyday experiences, $\hbar$ is miniscule and c is huge : in the world of elementary particles, they are just about right. We can see this as follows. It is customary to replace our human-scale meters, kilograms and seconds by what may be called *fundamental units* of mass, length and time :

$$\begin{aligned} M_f &= 1.7826618\ \ 10^{-27}\ \mathrm{kg}\quad ,\\ L_f &= 1.9732696\ \ 10^{-16}\ \mathrm{m}\quad ,\\ T_f &= 6.5821190\ \ 10^{-25}\ \mathrm{sec}\quad . \end{aligned}$$

In terms of these units, we have precisely

$$\hbar = \frac{M_f\,{L_f}^2}{T_f}\quad ,\quad c = \frac{L_f}{T_f}\quad ,$$

so that both $\hbar$ and c have the *numerical* value one ; and the unit of energy turns out to be

$$\frac{M_f\,{L_f}^2}{{T_f}^2} = 1.6021765\ \ 10^{-10}\ \mathrm{Joule} = 1\ \mathrm{GeV}\quad .$$

The mass and size of the proton are of the same order as M_f and L_f, respectively, and T_f is roughly the time scale of strong interactions. The use of fundamental units is attractive since you won't have to write factors of c and

[2]The values quoted here are taken from the 2014 Review of Particle Physics, K.A. Olive *et al.* (Particle Data Group), Chin. Phys. C, 38, 090001 (2014). The numbers in brackets denote the experimental error in the last digits. The speed of light is known *exactly* since it is the definition of the meter.

$\hbar$, and one then expresses both length and time in inverse GeV, and mass in GeV. But this usage obscures the dimensionality of the various objects[3], and I have decided to try to retain the $\hbar$'s and c's where they belong ; after all, it is much easier to erase them from formulæ than to put them back in.

0.2.2 Planck units

Along with $\hbar$ and c there exists a third[4] fundamental constant of nature, namely Newton's (or rather Cavendish's) gravitational constant :

$$G = 6.67384(80)\ \ 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg\,sec}^2}\ \ .$$

The truly, ultimately fundamental units of mass, length and time that can be recovered from c, $\hbar$ and G_N are then the *Planck units* :

$$\begin{aligned} M_P &= \sqrt{\frac{\hbar}{Gc}} = 2.17644\ \ 10^{-8}\,\mathrm{kg}\ \ , \\ L_P &= \sqrt{\frac{\hbar G}{c^3}} = 1.61625\ \ 10^{-35}\,\mathrm{m}\ \ , \\ T_P &= \sqrt{\frac{\hbar G}{c^5}} = 5.39124\ \ 10^{-44}\,\mathrm{sec}\ \ . \end{aligned}$$

These values are outrageously far removed from the typical scales of particle phenomenology. We may interpret this as an indication that in what follows the gravitational interaction will not play any part. In fact, in any case we do not (yet) have a satisfactory quantum theory of gravitation leading to specific and falsifiable predictions for particle phenomenology[5].

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[3]This has the unfortunate consequence of making it impossible to check the (at least) grammatical correctness of an expression by dimensional analysis, and even leads to erroneous statements about 'classical limits' and the like : see our discussion of particle masses in chapter 6.

[4]Consider that we have, basically, only the three measures of length, time, and mass to do physics with ; and there are precisely three natural constants. Suppose a *fourth* one is discovered : it would be only natural that we would try very hard to relate it to the other three in some way. Thus is fundamental physics formed.

[5]To bring the Planck units close to the fundamental units we need to increase the value of G, thus the strength of gravity, by a factor of about 10^{38}.

0.2.3 Charges

The electrostatic charge is adopted to the Gaussian system, so as to have no truck with the 'permeability of the vacuum' and suchlike : that is, two charges e_1 and e_2 separated by a distance r feel a mutual Coulomb force $\vec{F}$ characterized by

$$|\,\vec{F}\,| = \frac{1}{4\pi}\,\frac{|\,e_1 e_2\,|}{r^2} \quad .$$

This implies that the charge has the dimensionality of $\sqrt{\hbar c}$. It follows that, if we choose the proton charge as the unit charge e, the combination

$$\alpha_e = \frac{e^2}{4\pi\,\hbar\,c}$$

is a dimensionless number[6]. Experimentally,

$$\alpha_e = \frac{1}{137.035999074(44)} \quad ,$$

which yields the result

$$e = 0.30282212\,\sqrt{\hbar c} = 5.3843836\;\; 10^{-14}\,\frac{\mathrm{kg}^{1/2}\,\mathrm{m}^{3/2}}{\mathrm{sec}} \quad .$$

0.2.4 Conventions

Step functions

The Whittaker (or step) function is a function of a real number :

$$\theta(x) = \begin{cases} 1 & \text{if} \quad x \geq 1 \\ 0 & \text{if} \quad x < 0 \end{cases} \quad . \tag{1}$$

We extend this to a *logical* step function of a predicate $\mathcal{P}$:

$$\theta(\mathcal{P}) = \begin{cases} 1 & \text{if} \quad \mathcal{P} \text{ is true} \\ 0 & \text{if} \quad \mathcal{P} \text{ is false} \end{cases} \quad . \tag{2}$$

[6]Meaning that it has the same value in all possible systems of units ! An alien civilization in outer space will find the same value.

The metric

By convention, the Minkowski metric[7] has the form

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \quad .$$

In many textbooks the metric tensor is introduced as a diagonal *matrix*. This is of course misleading since the covariant metric tensor has only lower indices, whereas a matrix has one upper and one lower index. Unfortunately, the 'correct' matrix form of the metric, which would be $g^{\mu}{}_{\nu}$, equals the identity matrix whatever the metric !

Kronecker's symbol

The Kronecker symbol is defined by

$$\delta^{\alpha}{}_{\mu} = \begin{cases} 1 & \text{if } \alpha = \mu \\ 0 & \text{if } \alpha \neq \mu \end{cases} \quad .$$

Note : in Minkowski space, Kronecker symbols tend to carry one upper, and one lower index. Kronecker symbols with two upper or two lower indices are slightly suspect and to be treated with care.

The antisymmetrizer

Using Kronecker symbols we can build the following antisymmetric objects :

$$\left\{ \begin{array}{c} \mu_1 \\ \nu_1 \end{array} \right\} = \delta^{\mu_1}{}_{\nu_1} \quad , \quad \left\{ \begin{array}{cc} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{array} \right\} = \delta^{\mu_1}{}_{\nu_1}\delta^{\mu_2}{}_{\nu_2} - \delta^{\mu_1}{}_{\nu_2}\delta^{\mu_2}{}_{\nu_1} \quad ,$$

$$\begin{aligned} \left\{ \begin{array}{ccc} \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{array} \right\} = {} & \delta^{\mu_1}{}_{\nu_1}\delta^{\mu_2}{}_{\nu_2}\delta^{\mu_3}{}_{\nu_3} + \delta^{\mu_1}{}_{\nu_2}\delta^{\mu_2}{}_{\nu_3}\delta^{\mu_3}{}_{\nu_1} + \delta^{\mu_1}{}_{\nu_3}\delta^{\mu_2}{}_{\nu_1}\delta^{\mu_3}{}_{\nu_2} \\ & - \delta^{\mu_1}{}_{\nu_1}\delta^{\mu_2}{}_{\nu_3}\delta^{\mu_3}{}_{\nu_2} - \delta^{\mu_1}{}_{\nu_2}\delta^{\mu_2}{}_{\nu_1}\delta^{\mu_3}{}_{\nu_3} - \delta^{\mu_1}{}_{\nu_3}\delta^{\mu_2}{}_{\nu_2}\delta^{\mu_3}{}_{\nu_1} \end{aligned}$$

and so on. Here we encounter all *signed* permutations[8] of the lower indices. These are compuationally handy as we see below.

[7] In the usual Cartesian coordinate systems. Since we are not doing general relativity in these notes, we shall adhere to this simplest of coordinate system throughout, except when discussing phase space integration where polar coordinates often come in handy. But there we shall not refer to the metric.

[8] Even permutations occur with a +, and odd permutations with a – sign.

The Levi-Civita symbol

The totally antisymmetric Levi-Civita symbol is defined by

$$\epsilon_{0123} = +\,1 \quad \text{hence} \quad \epsilon^{0123} = -\,1 \quad .$$

This implies the following identities :

$$\begin{aligned}
\epsilon_{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\nu_1\nu_2\nu_3\nu_4} &= -\left\{\begin{array}{cccc} \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array}\right\} \quad , \\
\epsilon_{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\nu_1\nu_2\nu_3\mu_4} &= -\left\{\begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \\ \mu_1 & \mu_2 & \mu_3 \end{array}\right\} \quad , \\
\epsilon_{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\nu_1\nu_2\mu_3\mu_4} &= -2\left\{\begin{array}{cc} \nu_1 & \nu_2 \\ \mu_1 & \mu_2 \end{array}\right\} \quad , \\
\epsilon_{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\nu_1\mu_2\mu_3\mu_4} &= -6\left\{\begin{array}{c} \nu_1 \\ \mu_1 \end{array}\right\} \quad , \\
\epsilon_{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\mu_1\mu_2\mu_3\mu_4} &= -24 \quad .
\end{aligned}$$

In order not to lumber ourselves with too many explicit Lorentz indices, we shall use notations such as

$$\epsilon^{\mu}(a, b, c)$$

to stand for

$$\epsilon^{\mu\nu\rho\sigma} a_\nu b_\rho c_\sigma \quad .$$

Finally, although it falls outside the scope of these notes, it is useful to note that the Kronecker and antisymmetrizer symbols are fully-fledged *tensors* in a much wider class of spaces than Minkowski space, but the Levi-Civita symbols themselves are *not*. For instance, if we move from Cartesian to polar coordinates, say, the Kronecker symbol remains unaffected but the Levi-Civita is changed.

Minus in the momentum

There is a subtlety : the contravariant partial derivative contains a possibly surprising minus sign :

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\, -\vec{\nabla}\right) \quad . \tag{3}$$

This explains why in nonrelativistic quantum mechanics the momentum operator is $\vec{p} = -i\hbar\,\vec{\nabla}$ whereas in the relativistic theory we use $p^\mu = i\hbar\,\partial^\mu$.

The polyparenthetophobe rules

This deals with the notation for compound vector products. If $p^\mu_{1,2,3,4}$ are four-vectors, then the expression

$$(p_1 + p_2 \cdot p_3 + p_4)$$

must be understood to mean

$$(p_1 \cdot p_3) + (p_2 \cdot p_3) + (p_1 \cdot p_4) + (p_2 \cdot p_4)$$

The more rigorously correct form

$$\Big((p_1 + p_2) \cdot (p_3 + p_4)\Big)$$

is in my opinion less easily readable, unless by true parenthetophilomaniacs. In the same spirit, a slight tendency to periodophobia will lead us to write, *e.g.*, (pq) as shorthand for $(p \cdot q)$ where no risk of confusion is likely. Also, k^2 occurs when it is clear that $(k \cdot k)$ is intended rather than the second spacelike component of k^μ.

In favor of loose terminology

Among particle physicists there exist a tendency to be sloppy with some terms. In particular this holds for the usage of the words 'mass' and 'momentum'. Strictly speaking, in the Feynman rules to be discussed the 'mass' m and the 'momentum' k^μ have dimensions of *inverse length* and therefore cannot be the same as the notions of the *classical* mass M, expressed in kg, and the classical momentum p^μ, expressed in (kg m/sec) . As discussed in section 5.3.3, these various notions are related by

$$m = Mc/\hbar \quad , \quad k^\mu = p^\mu/\hbar \ \ .$$

Not wishing to succumb to pedantry, I shall use 'mass' and 'momentum' insouciantly. Experience shows that one easily gets used to it.

0.3 The P^4 Hall of Fame

Ernestos Argyres
Dima Bardin
Wim Beenakker
Frits Berends
Alain Blondel
Oscar Boher Luna
Stefan Brinck
Kristof de Bruyn
Sascha Caron
Chris Dams
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James Stirling
John Swain
Oleg Teryaev
Theodor Todorov
Tini Veltman
Rob Verheyen
Tai Tsun Wu
Sjoerd Ypma

Thank you for allowing me to think together with you.

0.4 Exercise

Excercise 0 Would-be exercise
This might have been an exercise.

Chapter 1

QFT in zero dimensions

1.1 Introduction

For the description of elementary particles, a theory including both relativity and quantum mechanics is necessary ; we shall introduce relativity further on, and concentrate in this chapter on the quantum-mechanical nature of nature. The fundamental object used for describing the particles is a *quantum field*. In many treatments quantum fields are considered to be *operator*-valued entities ; we shall rather adhere to Feynman's approach and use what are called *c-number fields*. Such a field assigns one or more numbers to every point in spacetime, and is hence a pretty complicated subject the behaviour of which is not to be characterized trivially, especially when it also undergoes quantum fluctuations. It is therefore useful to first build up expertise in the various necessary techniques in a more controllable situation. To this end we shall first simplify the whole four-dimensional spacetime arena of particle physics to a lower-dimensional system ; in fact, we shall reduce spacetime to a single point, hence a zero-dimensional arena. The quantum fields are then assignments of a single number ; the simplest quantum field is, in this case, a single stochastic, or random, number. Many of the techniques of quantum field theory *do* apply to this case : in particular the notion of path integrals, Green's functions, the Schwinger-Dyson equation, and Feynman diagrams come up naturally.

1.2 Probabilistic considerations

1.2.1 Quantum field and action

We shall consider a quantum field φ that takes its values on the whole real axis from $-\infty$ to $+\infty$. Since it is a random variable, the most we can specify about it is its probability density $P(\varphi)$, which we write, for now, as

$$P(\varphi) = N\ \exp\Big(- S(\varphi)\Big) \quad . \tag{1.1}$$

The function $S(\varphi)$ is called the *action* of the particular quantum field theory : in a sense, it *defines* the theory. For the probability density to be acceptable, $S(\varphi)$ must go to infinity sufficiently fast as $|\varphi| \to \infty$. The normalization factor N is defined by[1]

$$N^{-1} = \int \exp\Big(- S(\varphi)\Big)\ d\varphi \quad . \tag{1.2}$$

1.2.2 Green's functions, sources and the path integral

Since the quantum field is a random variable, the most that can be computed about it[2] is the collection of its moments, in the jargon called *Green's functions*[3] :

$$G_n \equiv \langle \varphi^n \rangle \equiv N \int \exp\Big(- S(\varphi)\Big)\ \varphi^n\ d\varphi \quad , \quad n = 0, 1, 2, 3, \ldots \quad . \tag{1.3}$$

[1]If not explicitly indicated otherwise, integrals run from $-\infty$ to $+\infty$.

[2]You are here approaching a career decision. You may decide simply to *measure* the value of φ : in that case you have decided to become an *experimentalist* rather than a *theorist*.

[3]A clarifying remark must be made here. In this text, the Green's functions are simply *defined* to be expectation values. This may appear to contrast with the use of Green's functions in the solution of inhomogeneous linear differential equations such as are encountered in classical electrodynamics where one uses them to compute the electromagnetic field configurations for given sources. The difference is only apparent since, as we shall recognize, the latter type of Green's functions are in our treatment simply the *two-point* Green's functions ; and for theories such as electrodynamics, where the electromagnetic fields do not undergo self-interaction, the two-point functions are in fact the *only* nonzero connected Green's functions. Be not, therefore, misled into thinking that there are somehow two sorts of Green's functions. The Green's function formulation of electrodynamics will in fact appear as the classical limit of the Schwinger-Dyson equation discussed below.

We shall assume that G_n exists for all n. By construction, we must always have

$$G_0 = \left\langle \varphi^0 \right\rangle = \langle 1 \rangle = 1 \quad . \tag{1.4}$$

The most fruitful way[4] of discussing the set of all Green's functions is in terms of their *generating function* :

$$Z(J) = \sum_{n\geq 0} \frac{1}{n!} J^n \, G_n \quad . \tag{1.5}$$

This is called the *path integral*, for reasons that will become clear later. It can be written as

$$Z(J) = N \int \exp\Big(- S(\varphi) + J\varphi \Big) \; d\varphi \quad . \tag{1.6}$$

The number J, which here serves purely as a device to distinguish the various Green's functions, is called a *source*, again for reasons that will become apparent later. Once $Z(J)$ is known, an individual Green's function is extracted by differentiation :

$$G_n = \left\lfloor \frac{\partial^n}{(\partial J)^n} Z(J) \right\rfloor_{J=0} \quad . \tag{1.7}$$

1.2.3 Connected Green's functions

The path integral $Z(J)$ contains all the information about the Green's functions, and hence about the probability density $P(\varphi)$. The same information is, therefore, *also* contained in its logarithm. We write

$$W(J) = \log Z(J) \equiv \sum_{n\geq 1} \frac{1}{n!} J^n \, C_n \quad , \tag{1.8}$$

where the sum starts at $n = 1$ since $Z(0) = 1$. The quantities C_n (with, obviously $C_0 = 0$ since $G_0 = 1$) are called the *connected Green's functions* of the theory, and will play an important rôle in what follows.

The connected Green's functions can be recognized to be the *cumulants* of the probability density:

$$\begin{array}{rcllcl} C_1 & = & \langle\varphi\rangle & & : & \text{the mean,} \\ C_2 & = & \langle(\varphi - \langle\varphi\rangle)^2\rangle & & : & \text{the variance,} \\ C_3 & = & \langle(\varphi - \langle\varphi\rangle)^3\rangle & & : & \text{the skewness,} \\ C_4 & = & \langle(\varphi - \langle\varphi\rangle)^4\rangle - 3{C_2}^2 & & : & \text{the kurtosis,} \end{array}$$

[4]Kids ! Do this at home. Whenever a infinite collection of objects with *some* kind of structure between them occurs, generating functions are *always* a good idea.

E 1

and so on.

Since $W(0) = C_0 = 0$, the same information about the probability density is *also* contained in the *field function*:

$$\phi(J) \equiv \frac{\partial}{\partial J} W(J) = \sum_{n \geq 0} \frac{1}{n!} J^n \, C_{n+1} \quad . \tag{1.9}$$

Since from its definition, we have

$$\phi(J) = \left[\int \exp\Big(-S(\varphi) + J\varphi \Big) \; \varphi \, d\varphi \right] \left[\int \exp\Big(-S(\varphi) + J\varphi \Big) \; d\varphi \right]^{-1} \quad , \tag{1.10}$$

we can say that $\phi(J)$ is the expectation value of the quantum field φ in the presence of sources: to denote this, we might write

$$\phi(J) = \langle \varphi \rangle_J \quad , \tag{1.11}$$

which explains the similar typographies for the quantum field and the field function. We should not, however, forget the difference in status of these objects : φ is the *physical* entity, an unknowable, fluctuating *random* field ; but $\phi(J)$ is an eminently well-defined *function* that contains all the information about the *probability density* of φ, and is[5] *computable* once the action is given.

1.2.4 The free theory

The simplest probability density is probably[6] the Gaussian one, given by the action

$$S(\varphi) = \frac{1}{2} \mu \varphi^2 \quad , \tag{1.12}$$

with μ a positive real number. For any action, we shall call the part quadratic in the fields (or bilinear in the case of several fields) the *kinetic part*. This action, called the *free action*, consists of *only* a kinetic part. The path integral

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[5] In principle, if not in practice completely.

[6] A uniform density may be thought even simpler, but then it cannot run from $\varphi = -\infty$ to $\varphi = +\infty$. As a matter of fact, ask any mathematician or physicist to name you a nice proability density over the whole real line, and she will almost without fail quote the Gaussian.

is now simply computed by

$$\begin{aligned} Z(J) &= N \int \exp\left(-\frac{1}{2}\mu\varphi^2 + J\varphi\right)\, d\varphi \\ &= N \int \exp\left(-\frac{1}{2}\mu\left(\varphi - \frac{J}{\mu}\right)^2 + \frac{J^2}{2\mu}\right)\, d\varphi \\ &= \exp\left(\frac{J^2}{2\mu}\right) \quad . \end{aligned} \tag{1.13}$$

It is not even necessary[7] to actually calculate the value of N. By Taylor expansion of the exponential, we immediately find that

$$G_{2n} = \frac{(2n)!}{2^n n!}\frac{1}{\mu^n} \quad , \quad G_{2n+1} = 0 \quad , \quad n = 0, 1, 2, \ldots, \quad . \tag{1.14}$$

The connected Green's functions follow from

$$W(J) = \log Z(J) = \frac{J^2}{2\mu} \quad , \quad \phi(J) = \frac{J}{\mu} \quad , \tag{1.15}$$

so that the only nonvanishing connected Green's function is

$$C_2 = \frac{1}{\mu} \quad . \tag{1.16}$$

The fact that here only the two-point connected Green's function is nonvanishing is the reason for calling this model the free theory (again, things will become clearer later on, in a more realistic spacetime).

1.2.5 The φ^4 model and perturbation theory

An action $S(\varphi)$ may contain other terms than just the quadratic one. Such terms are called *interaction terms* : they may be linear, but more usually they are of higher power in the field φ. The simplest acceptable interacting theory is therefore given by the action

$$S(\varphi) = \frac{1}{2}\mu\varphi^2 + \frac{1}{4!}\lambda_4\varphi^4 \quad . \tag{1.17}$$

The (nonnegative !) real number λ_4 is called a *coupling constant* : this model is called the φ^4 theory[8]. **E 2**

[7]Because we must always have $Z(0) = 1$.

Computing the path integral is now a much less trivial matter. A possible approach is to assume that, in *some* sense, the φ^4 theory is close to a free theory, that is, in the same *some* sense, λ_4 is a small number. We can then expand the probability density in powers of λ_4 :

$$\exp(-S(\varphi)) = \exp\left(-\frac{1}{2}\mu\varphi^2\right) \sum_{k\geq 0} \frac{1}{k!}\left(-\frac{\lambda_4}{24}\right)^k \varphi^{4k} \quad . \tag{1.18}$$

This procedure is called *perturbation theory*. Having thus reduced the problem to the previous case of the free theory, we cavalierly[9] interchange the series expansion in λ_4 with the integration over φ and arrive at the following expression for the Green's functions :

$$\begin{aligned} G_{2n} &= H_{2n}/H_0 \ , \\ H_{2n} &= \frac{1}{\mu^n}\sum_{k\geq 0} \frac{(4k+2n)!}{2^{2k+n}(2k+n)!k!}\left(-\frac{\lambda_4}{24\mu^2}\right)^k \quad . \end{aligned} \tag{1.19}$$

For example, we have

$$\begin{aligned} H_0 &= 1 - \frac{1}{8}u + \frac{35}{384}u^2 - \frac{385}{3072}u^3 + \cdots \quad , \\ 1/H_0 &= 1 + \frac{1}{8}u - \frac{29}{384}u^2 + \frac{107}{1024}u^3 + \cdots \ , \end{aligned} \tag{1.20}$$

with $u \equiv \lambda_4/\mu^2$. Note that, in this theory, also the normalization N has to be treated perturbatively, which explains the expression for $1/H_0$. For the first few nonvanishing Green's functions we find

$$\begin{aligned} G_0 &= 1 \ , \\ G_2 &= \frac{1}{\mu}\left(1 - \frac{1}{2}u + \frac{2}{3}u^2 - \frac{11}{8}u^3 + \cdots\right) \quad , \\ G_4 &= \frac{1}{\mu^2}\left(3 - 4u + \frac{33}{4}u^2 - \frac{68}{3}u^3 + \cdots\right) \quad , \\ G_6 &= \frac{1}{\mu^3}\left(15 - \frac{75}{2}u + \frac{445}{4}u^2 - \frac{1585}{4}u^3 + \cdots\right) \quad . \end{aligned} \tag{1.21}$$

[8] An action in which φ^3 is the highest power does not lead to a convergent integral over the real axis (see, however, Appendix 2). Of course, an action of the form $S(\varphi) = \mu\varphi^2/2 + \lambda_3\varphi^3/3! + \lambda_4\varphi^4/4!$ is perfectly acceptable, and we shall consider this '$\varphi^{3/4}$ model' later on.

[9] And not with impunity ! See Appendix 1.

The corresponding connected Green's functions are given by

$$\begin{aligned} C_2 &= \frac{1}{\mu}\left(1 - \frac{1}{2}u + \frac{2}{3}u^2 - \frac{11}{8}u^3 + \cdots\right) \quad , \\ C_4 &= \frac{1}{\mu^2}\left(-u + \frac{7}{2}u^2 - \frac{149}{12}u^3 + \cdots\right) \quad , \\ C_6 &= \frac{1}{\mu^3}\left(10u^2 - 80u^3 + \cdots\right) \quad . \end{aligned} \tag{1.22}$$

Note that, whereas the Green's functions all have a perturbation expansion starting with terms containing no λ_4, the connected Green's functions of increasing order are also of increasingly high order in λ_4 : the higher connected Green's functions need more interactions than the lower ones.

1.2.6 The Schwinger-Dyson equation for the path integral

Although the path integral is, generally, a very complicated function of J, it is nevertheless easy to find an equation describing it completely. This is the *Schwinger-Dyson equation* (SDe), which we construct as follows. Let the action be given by the general expression[10]

$$S(\varphi) = \sum_{k\geq 1} \frac{1}{k!}\,\lambda_k\,\varphi^k \quad , \tag{1.23}$$

where $\lambda_2 = \mu$. Now, from the observation that

$$\frac{\partial^p}{(\partial J)^p}Z(J) = N \int \exp\Big(-S(\varphi) + J\varphi\Big)\;\varphi^p\;d\varphi \quad , \quad p = 0,1,2,3,\ldots \tag{1.24}$$

we immedately deduce that

$$\begin{aligned} &\left[-J + \sum_{k\geq 0}\frac{\lambda_{k+1}}{k!}\frac{\partial^k}{(\partial J)^k}\right] Z(J) = \\ &= N\int \exp\Big(-S(\varphi)+J\varphi\Big)\;\left[-J + \sum_{k\geq 0}\frac{\lambda_{k+1}}{k!}\varphi^k\right]\;d\varphi \\ &= N\int \exp\Big(-S(\varphi)+J\varphi\Big)\;\Big[S'(\varphi) - J\Big]\;d\varphi \;=\; 0 \;, \end{aligned} \tag{1.25}$$

[10] A constant, φ-independent term in the action is always immediately swallowed up by the normalization factor N.

where in the last lemma we have used partial integration, and the fact that the integrand vanishes at the endpoints at infinity. Symbolically, we may write the SDe as

$$\left\lfloor \frac{\partial}{\partial\varphi} S(\varphi) \right\rfloor_{\varphi=\partial/\partial J} Z(J) = S'\left(\frac{\partial}{\partial J}\right) Z(J) = JZ(J) \ . \tag{1.26}$$

E 3 For our sample model, the φ^4 theory, the SDe reads[11]

$$\frac{1}{6}\lambda_4 Z'''(J) + \mu Z'(J) - JZ(J) = 0 \ . \tag{1.27}$$

Using the series expansion of the path integral we can express this as a relation between different Green's functions :

$$\frac{\lambda_4}{6} G_{n+3} + \mu G_{n+1} - nG_{n-1} = 0 \quad , \quad n \geq 1 \ . \tag{1.28}$$

This relation may usefully be rewritten as follows :

$$G_n = \frac{1}{\mu}\left((n-1)G_{n-2} - \frac{\lambda_4}{6} G_{n+2} \right) \quad , \quad n \geq 2 \ . \tag{1.29}$$

If we start by assigning to the Green's functions the values

$$G_0 = 1 \quad , \quad G_n = 0 \ , \ n \neq 0 \ , \tag{1.30}$$

then repeated applications of Eq.(1.29) will precisely reproduce the Green's functions of Eq.(1.21)[12].

1.2.7 The Schwinger-Dyson equation for the field function

From the definition of $\phi(J)$ as the logarithmic derivative of the path integral, we can infer that

$$\frac{\partial^p}{(\partial J)^p} Z(J) = Z(J) \left(\phi(J) + \frac{\partial}{\partial J} \right)^p e(J) \ . \tag{1.31}$$

[11] The SD equation is, in general, of higher than the first order. It therefore has several independent solutions, only *one* of which corresponds to the usual perturbative expansion. The nature of the other solutions is discussed in Appendix 2.

[12] The correct way to do this is to subsequently evaluate G_2, G_4, $G_6, \ldots$. On the first iteration, the lowest-order expressions are obtained. Each subsequent iteration gives one higher order in perturbation theory. Note that if we want to obtain the k^{th} order term in G_n, the $(k+1)^{\text{th}}$ order term in G_{n+2} is needed, and so on. It is therefore necessary to compute the lower-order terms for more G_n's.

Here, $e(J)$ is the unit function: $e(J) \equiv 1$. We immediately arrive at the form of the SDe for the field function: **E 4**

$$S'\left(\phi(J) + \frac{\partial}{\partial J}\right) e(J) = J \quad . \tag{1.32}$$

For the φ^4 theory, it reads

$$\phi(J) = \frac{J}{\mu} - \frac{\lambda_4}{6\mu}\left(\phi(J)^3 + 3\phi(J)\frac{\partial}{\partial J}\phi(J) + \frac{\partial^2}{(\partial J)^2}\phi(J)\right) \quad . \tag{1.33}$$

Although this leads to very nonlinear relations between the various connected Green's functions this form of the SD equation is actually even simpler to apply : with $\phi(J) = 0$ as a starting pont, iterating the assignment (1.33) then results[13] in the correct form of $\phi(J)$, giving the connected Green's functions of Eq.(1.22). For the $\varphi^{3/4}$ theory, the Schwinger-Dyson equation reads **E 5**

$$\begin{aligned} \phi(J) \;=\; & \frac{J}{\mu} - \frac{\lambda_3}{2\mu}\left(\phi(J)^2 + \frac{\partial}{\partial J}\phi(J)\right) \\ & - \frac{\lambda_4}{6\mu}\left(\phi(J)^3 + 3\phi(J)\frac{\partial}{\partial J}\phi(J) + \frac{\partial^2}{(\partial J)^2}\phi(J)\right) \quad . \end{aligned} \tag{1.34}$$

1.3 Diagrammatic considerations

1.3.1 Feynman diagrams

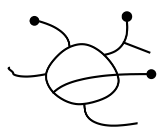
An extremely useful tool for computing Green's functions and connected Green's functions is at hand in the form of *Feynman diagrams.* In this section we shall first introduce these diagrams and their concomitant *Feynman rules.* Only after that shall we prove that these diagrams do, indeed, correctly describe Green's functions.

Feynman diagrams are constructs of lines and vertices. A vertex is a meeting point for one or more lines. Diagrams are allowed in which one or more lines do not end in a vertex but, in a sense *wandern ins Blaue hinein* : such lines are called *external lines.* Lines that are not external lines, and

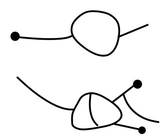
---

[13]For this approach to work in practice, it turns out to be useful to truncate $\phi(J)$ as a power series in J, the truncation order increasing by one with each iteration. If you don't do this, each iteration *triples* the highest power in J, leading to very unwieldy expressions with only the first few terms being actually correct.

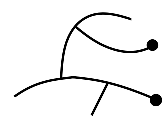
end up at vertices at both ends, are called *internal lines*. Diagrams may be *connected*, in which case one can move between any two points in the diagram following lines of that diagram ; or they may be *disconnected*, in which case it consists of two or more disjoint pieces that are themselves connected. Any graph[14] consists of a finite number of connected subgraphs. The 'empty' graph, containing no lines or vertices whatsoever, also exists ; it does not count as connected[15]. Diagrams containing one or more closed loops are perfectly allowed. Diagrams with no closed loops are called *tree diagrams*. Some examples of Feynman diagrams are



a connected graph



a disconnected graph



a connected tree graph

Note that the precise *shape* of the lines and the precise *position* of the vertices are irrelevant. The important thing is the way in which the lines are connected to the vertices[16].

1.3.2 Feynman rules

The noteworthy thing about Feynman diagrams is that they have an algebraic interpretation; that is, they correspond to *numbers* that may be added and multiplied. The assignment of a number to a Feynman diagram is governed by the *Feynman rules*, which postulate a numerical object for every ingredient of a Feynman graph. In the simple zero-dimensional theories that

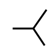
[14] The terms 'diagram' and 'graph' are interchangeable.

[15] Casuistically, it has no points between which one might wish to move.

[16] As you will discover, I have endeavoured in these notes to avoid drawing straight lines, or to draw blobs or closed loops as circles. Many texts *do* employ only straight lines and circles. This not only leads to awfully unæsthetic-looking pictures, but is also deeply misleading. Readers will often look at Feynman diagrams with the idea that the lines represent 'particles moving freely through space' so that the lines 'ought' to be straight according to Newton's first law. That this is completely wrong becomes immediately clear if we realize that, in the zero-dimensional world we are dealing with for now, there cannot be any notion of movement yet, let alone any Newton to pronounce on it. In fact, Newton's first law ought to be *derived* from our theory, and we shall do so in due course.

we consider here the Feynman rules are just numbers. We may use, for instance, the following rules :

 $\leftrightarrow\ 1/\mu$

 $\leftrightarrow\ -\lambda_3$

 $\leftrightarrow\ -\lambda_4$

 $\leftrightarrow\ +J$

$\boxed{\text{Feynman rules, version 1.1}}$ (1.35)

A vertex at which a single line ends (and which carries a Feynman rule factor $+J$) is called a *source vertex*.

A disconnected diagram evaluates to the product of the values of its disjunct connected pieces. Because of this multiplicative rule, the value of the empty diagram is taken to be unity.

In addition, we assign to every Feynman diagram a *symmetry factor*. The symmetry factor is the single most nontrivial ingredient of the diagrammatic approach. We shall therefore devote a separate section to this issue.

1.3.3 Symmetries and multiplicities

Feynman diagrams have, in general, an 'inner' and an 'outer' part. The 'inner' part consists of the various vertices and internal lines : the 'outer' part is made up from the external lines (if any). The inner part concomitates with the *symmetry factor* of the diagram, and for the outer part we have what may be called the *multiplicity*, to be discussed below. Let us first turn to the symmetry factor.

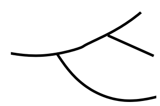
For the symmetry factor, the rules are the following :

- for every set of k lines that may be permuted without changing the diagram, there will be a factor $1/k!$;

- for every set of m vertices that may be permuted without changing the diagram, there will be a factor $1/m!$;
- for every set of p disjunct connected pieces that maybe interchanged without changing the diagram, there will be a factor $1/p!$;
- a factor $1/k$ for every k-fold rotational symmetry[17];
- a factor 1/2 for every mirror symmetry.

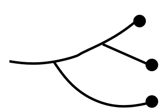
External lines *cannot* be permuted without changing the diagram. Since external lines cannot be permuted, only *vacuum diagrams*, that is diagrams without any external lines, can have a rotational symmetry. It is important to note that the symmetry factor cannot be read off from the individual components of the diagram, but depends on the topology of the whole diagram[18]. As our universe grows from zero to more dimensions, and as the particles considered acquire more properties, the Feynman rules will grow in complication ; but the symmetry factors remain the same[19].

A few examples of diagram values are presented here. First, consider the diagram



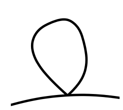
$$= \frac{{\lambda_3}^2}{\mu^5} \quad . \tag{1.36}$$

In this case, the symmetry factor is 1, since for a tree diagram no internal lines or vertices can be interchanged with impunity. The similar-looking diagram



$$= \frac{1}{2}\frac{{\lambda_3}^2}{\mu^5} J^3 \quad . \tag{1.37}$$

has a symmetry factor $1/2!$ since the upper two one-point vertices are interchangeable. Then, there is the graph



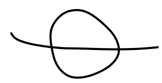
$$= -\frac{1}{2}\frac{\lambda_4}{\mu^3}$$

[17] Note: $1/k$, *not* $1/(k!)$.

[18] This is what makes the automated evaluation of diagrams a nontrivial task : component factors of diagrams can be easily assigned, but working out the symmetry factor of a diagram calls for for very complicated computer algorithms indeed.

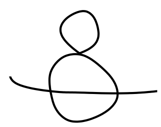
[19] This is only modified if we include lines of different types, or oriented lines. Then again, the more-dimensional diagrams have the same symmetry factors as their zero-dimensional siblings.

Here, there is a symmetry factor 1/2 because the 'leaf' can be flipped over without changing the diagram[20]. The diagram



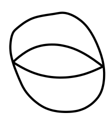
$$= \frac{1}{6}\frac{{\lambda_4}^2}{\mu^5}$$

carries a symmetry factor of 1/3! because the three internal lines are interchangeable. The graph



$$= -\frac{1}{4}\frac{{\lambda_4}^3}{\mu^7}$$

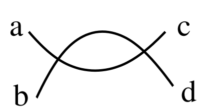
carries a symmetry factor (1/2!)(1/2!) since there are now only *two* interchangeable internal lines, and a single 'leaf'. Finally, the diagram



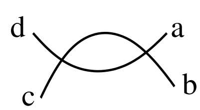
$$= \frac{1}{48}\frac{{\lambda_4}^2}{\mu^4}$$

has a symmetry factor (1/4!)(1/2!) since there are 4 equivalent internal lines, and moreover the diagram can be 'flipped over' without changing it. **E 6**

Next, we address the multiplicity. This is the number of different ways the external lines (that each have their own 'individuality') can be attached. To determine the multiplicity we must imagine that the whole diagram, or a part of it, can be 'flipped over' while retaining the same attachement of the external lines. To illustrate this, we temporarily denote the external lines with a letter, and then notice that the two diagrams



and



are, in fact, identical ; the multiplicity of this graph is therefore 3, since there are 3 ways to group four letters into two groups of two without regard

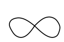
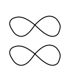
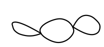
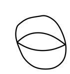
[20]This is due to the fact that the line in the loop is not oriented: for oriented lines it will no longer hold. The discussion of symmetry factors of Feynman diagrams goes, in practice, with a lot of remarks like '... so you flip over this leaf, you wriggle this set of internal lines, you shove these vertices back and forth ... see ?' Although the symmetry factor is totally unambiguous, the *arguments* for a symmetry factor often come with a lot of prestidigitatorial hand-waving and finger-wriggling in front of a blackboard.

to ordering. We see that the diagram of Eq.(1.36) has, also, multiplicity 3, while that of Eq.(1.37) has multiplicity 1. We see that, if we include the multiplicity, the replacing of p external lines with p one-point source vertices induces a factor of $1/p!$, which will become important later on.

The determination of symmetry factors may appear somewhat fanciful, calling for finger-wriggling and such, but of course it has a solid and unambiguous basis ; the symmetry factor (and the multiplicity) can always be computed. The procedure is somewhat involved, and will be outlined in appendix 2.

1.3.4 Vacuum bubbles

Feynman diagrams exist that contain neither external lines nor source vertices. These are called *vacuum bubbles.* The empty graph (which we shall denote by the symbol $\mathcal{E}$) is, obviously, a vacuum bubble. We may consider the set of *all* vacuum bubbles, which we denote by $\mathcal{H}_0$. Let us assume that only four-point vertices occur. Then, $\mathcal{H}_0$, given by

$\mathcal{H}_0 \;=\; \mathcal{E} \;+$  $+$  $+$  $+$  $+\;\cdots$ (1.38)

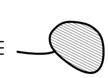
(where the ellipsis denotes diagrams with more four-vertices) evaluates to

$$\begin{aligned} \mathcal{H}_0 &= 1 - \frac{1}{8}\frac{\lambda_4}{\mu^2} + \frac{1}{2}\left(\frac{1}{8}\frac{\lambda_4}{\mu^2}\right)^2 + \frac{1}{16}\frac{{\lambda_4}^2}{\mu^4} + \frac{1}{48}\frac{{\lambda_4}^2}{\mu^4} + \cdots \\ &= 1 - \frac{1}{8}\frac{\lambda_4}{\mu^2} + \frac{35}{384}\frac{{\lambda_4}^2}{\mu^4} + \cdots \quad , \end{aligned} \tag{1.39}$$

which, indeed, looks suspiciously like H_0 for the φ^4 theory.

1.3.5 An equation for connected graphs

We shall now construct an equation for a special set of diagrams. We do this for the set of Feynman rules of section 1.3.2. First, let us denote by $\mathcal{C}_n$ the set of *all connected* graphs with *no* source vertices and precisely n external lines. Clearly this is a enumerably infinite set. Next, we define the object $\Psi(J)$, denoted by the symbol

$\Psi(J) \equiv$  (1.40) to be the set of *all connected* diagrams with precisely *one* external line, and any number of source vertices. The shading indicates that *all* the diagrams in the blob must be *connected*. Clearly, then, we have

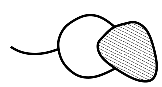
$$\Psi(J) \;=\; \sum_{n\geq 0} \frac{1}{n!} J^n\, \mathcal{C}_{n+1} \;\;, \tag{1.41}$$

where the extra factor $1/n!$ is the additional symmetry factor for n source vertices.

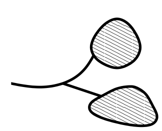
Let us now consider what can happen if we enter the blob of Eq.(1.40) along the single external line. In the first place, we can simply encounter a source vertex, so that the diagram is just

$$= \;\frac{J}{\mu}\;\;. \tag{1.42}$$

Alternatively, we may encounter a vertex. If this is a three-point vertex, the line splits into two. Taking one of these branches, we *may* be able to come back to the vertex via the other branch. In that case, the diagram has the form

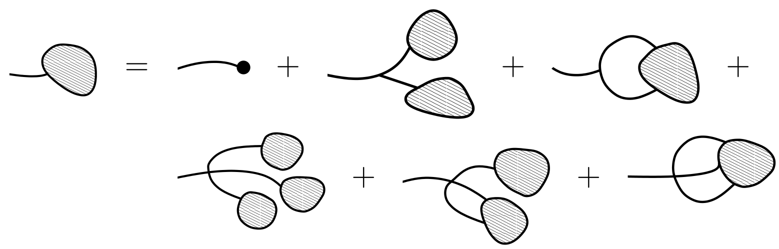


On the other hand, it may happen that the two branches end up in disjunct connected pieces of the diagram, which then looks like



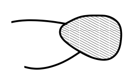
Note that these two alternative cases can be unambiguously distinguished because we have restricted ourselves to using only *connected* graphs. Another important insight is that, in the above diagram, the two final blobs (with their attached lines) are both *exactly identical* to the original $\Psi(J)$ of Eq.(1.40), and therefore also to each other : a situation that is of course only possible because the blobs represent *infinite* sets of diagrams. In contrast, the closed-loop blob of the first alternative is *not* equal to $\Psi(J)$ since it has not one but two lines sticking out ; but then again these two lines are *completely* equivalent.

If we encounter a four-point rather than a three-point vertex, the line splits into three, with three alternatives : no branches meeting again further on, all three meeting again, or only two out of the three. We find the diagrammatic equation



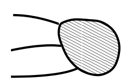
(1.43)

Now, realize that



$$= \sum_{n\geq 0} \frac{1}{n!} J^n\, \mathcal{C}_{n+2} = \frac{\partial}{\partial J}\Psi(J) \tag{1.44}$$

and



$$= \sum_{n\geq 0} \frac{1}{n!} J^n\, \mathcal{C}_{n+3} = \frac{\partial^2}{(\partial J)^2}\Psi(J) \ \ , \tag{1.45}$$

so that we can translate the diagrammatic equation (1.43) into an algebraic equation for $\Psi(J)$ by carefully implementing the correct Feynman rules, including nontrivial symmetry factors for equivalent blobs and lines:

$$\begin{aligned}\Psi(J) \;\; = \;\; \frac{J}{\mu} \;\; - \;\; \frac{\lambda_3}{\mu}\left(\frac{1}{2}\Psi(J)^2 \;\; + \;\; \frac{1}{2}\frac{\partial}{\partial J}\Psi(J)\right) \\ -\frac{\lambda_4}{\mu}\left(\frac{1}{6}\Psi(J)^3 \;\; + \;\; \frac{1}{2}\Psi(J)\frac{\partial}{\partial J}\Psi(J) \;\; + \;\; \frac{1}{6}\frac{\partial^2}{(\partial J)^2}\Psi(J)\right) \ \ . \end{aligned} \tag{1.46}$$

Now Eq.(1.46), obtained from the Feynman diagrams via the Feynman rules, has exactly the same form as Eq.(1.34), valid for the field function $\phi(J)$ – note the importance of the symmetry factors ! Moreover, the iterative solution for $\phi(J)$ starts with $\phi(J) = J/\mu$, also identical to the diagrammatic starting point . We therefore conclude that

$$\Psi(J) \;\; = \;\; \phi(J) \ \ , \tag{1.47}$$

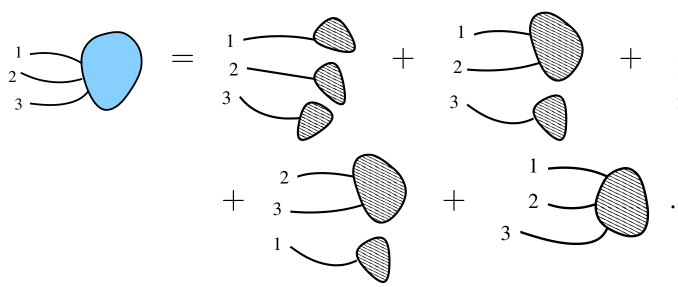
in other words

$$\mathcal{C}_n \;\; = \;\; C_n \quad , \quad n \geq 1 \ \ . \tag{1.48}$$

This proves that connected Green's functions can be obtained by the following recipe: **to obtain C_n ($n \geq 1$), write out *all* connected Feynman diagrams with no source vertices and precisely n external lines. Evaluate the diagrams using the Feynman rules, and sum them**. $\boxed{\textbf{E 7}}$

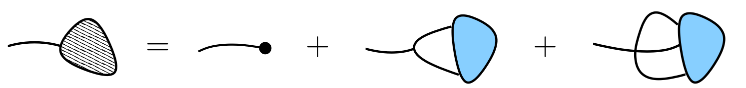
1.3.6 Semi-connected graphs and the SDe

A useful notion, which allows us to write SDe's more compactly, is that of *semi-connected graphs.* We shall denote these with a lightly shaded blob, and they are defined as follows : a semi-connected graph with $n \geq 1$ lines at the left is a general unconnected graph with n lines on the left (and *any* number of other external lines), with the constraint that each connected piece of the semi-connected graph is attached to at least one of the lines indicated on the left. This may sound more intimidating that is actually is : an example is



(1.49)

A single semi-connected graph with n indicated lines stands for $B(n)$ diagrams with explicit connected graphs, where $B(n)$ is the so-called Bell number : the number of ways to divide n distinct objects into non-empty groups[21]. For $\varphi^{3/4}$ theory, the SDe then becomes simply



(1.50)

We shall use semi-connected diagrams to good effect in later chapters. Note that the sum of the symmetry factors of all connected diagrams arising from

[21]For small n we have $B(0) = 1$, $B(1) = 1$, $B(2) = 2$, $B(3) = 5$, $B(4) = 15$, and $B(5) = 52$; more general values can be obtained from the identity

$$\sum_{n \geq 0} B(n) \frac{x^n}{n!} = \exp\left(e^x - 1\right) \quad .$$

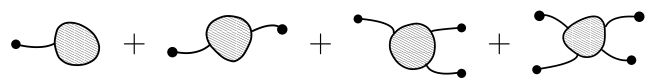
which is derived in Appendix 13.15.

a φ^p vertex must be equal to $B(p-1)/(p-1)!$, which may serve as a check on your SDe's.

1.3.7 The path integral as a set of diagrams

By affixing a source vertex to the single external line of $\Psi(J)$, we immediately have the result that **the generating function $W(J)$ is the sum of all connected Feynman diagrams without external lines and at least one source vertex**. If we explicitly indicate the source vertices, and recall that n source vertices in a diagram imply a factor $1/n!$, we can write

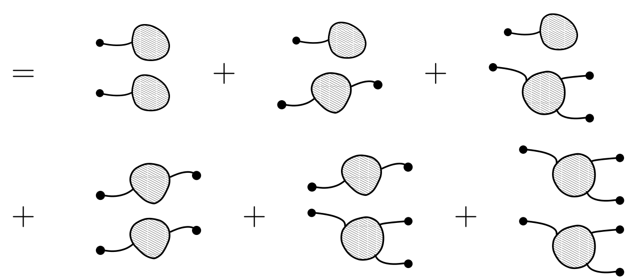
$W(J) =$



$+ \cdots$, (1.51)

where the ellipsis contains connected contributions with more source vertices. Vacuum bubbles do *not* contribute to $W(J)$. By taking careful account of the symmetry factor assigned to identical connected parts of a disconnected diagram, we can see that

$\frac{1}{2!}W(J)^2 =$



$+$ (lots of other diagrams) . (1.52)

Similar arguments hold for higher powers of $W(J)$. In addition, $W(J)^0 = 1$ is represented by the empty diagram. From this it easy to see that **the path integral $Z(J)$ consists of all Feynman diagrams without external lines, and without vacuum bubbles, but including the empty diagram**.

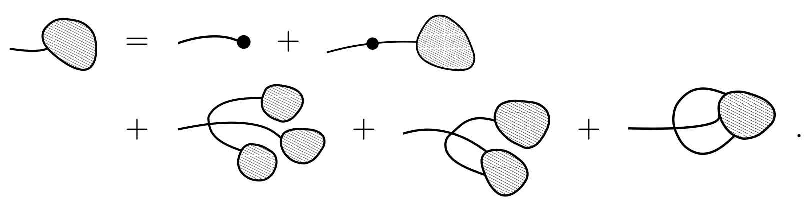
We might wonder why the vacuum bubbles are so conspicously absent. Suppose that we would allow the inclusion of arbitrary numbers of vacuum bubbles in $Z(J)$. Then the Green's function $G_0 = 1$ would be represented not by the single empty graph but by the whole set $\mathcal{H}_0$ discussed before: indeed, $\mathcal{H}_0$ is proportional to H_0. In fact, *any* Green's function G_n would acquire exactly the same additional factor $\mathcal{H}_0$. The normalization factor N, that must be chosen such as to make G_0 equal to unity, therefore extracts exactly the factor $\mathcal{H}_0$ from any Green's function. In the jargon, the vacuum bubbles 'disappear into the normalization of the path integral'. This is not to say that vacuum diagrams are never important ; but in our approach to computing Green's functions and connected Green's functions they are indeed irrelevant. Another way of seeing this is very simple : if we take our diagrammatic prescription of $Z(J)$ and then take $J=0$, all diagrams disappear except the empty one, and we find $Z(0)=\mathcal{E}=1$, just as we must.

1.3.8 Dyson summation

Why is the Feynman rule for lines, stemming from the quadratic part of the action, so different from those for the vertices, that come from the non-quadratic terms ? To see that our treatment is actually a consistent one, let us consider an action is given by

$$S(\varphi)=\frac{1}{2}\mu\varphi^2+\frac{1}{2}\lambda_2\varphi^2+\frac{1}{4!}\lambda_4\varphi^4 \quad . \tag{1.53}$$

If we wish, we may treat the λ_2 term as an interaction, described by a vertex with two legs. the SDe is then seen to be



(1.54)

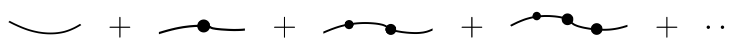
corresponding to

$$\phi(J)=\frac{J}{\mu}-\frac{\lambda_2}{\mu}\phi(J)-\frac{\lambda_4}{6\mu}\left(\phi(J)^3+3\phi(J)\frac{\partial}{\partial J}\phi(J)+\frac{\partial^2}{(\partial J)^2}\phi(J)\right) \quad . \tag{1.55}$$

Multiplying the equation by μ and transposing the λ_2 term to the left, we obtain

$$\phi(J)=\frac{J}{\mu+\lambda_2}-\frac{\lambda_4}{6(\mu+\lambda_2)}\left(\phi(J)^3+3\phi(J)\frac{\partial}{\partial J}\phi(J)+\frac{\partial^2}{(\partial J)^2}\phi(J)\right) \quad , \tag{1.56}$$

precisely what we woud have obtained by taking the combination $(\mu+\lambda_2)$ as the kinetic part from the start. This procedure, by which the effect of two-point (effective) vertices is subsumed in a redefinition of the kinetic part, is called *Dyson summation*. In the present example, the summation is of course trivial ; but we shall see that two-point interactions can also arise from more complicated Feynman diagrams corresponding to higher orders in perturbation theory. The manner in which Dyson summation is usually treated is by explicitly writing out the propagator, 'dressed' with two-point vertices in all possible ways :



$$
\begin{aligned}
&= \frac{1}{\mu} - \frac{1}{\mu}\lambda_2\frac{1}{\mu} + \frac{1}{\mu}\lambda_2\frac{1}{\mu}\lambda_2\frac{1}{\mu} - \frac{1}{\mu}\lambda_2\frac{1}{\mu}\lambda_2\frac{1}{\mu}\lambda_2\frac{1}{\mu} + \cdots \\
&= \frac{1}{\mu}\sum_{k\geq 0}\left(-\frac{\lambda_2}{\mu}\right)^k \\
&= \frac{1}{\mu}\,\frac{1}{1+\lambda_2/\mu} = \frac{1}{\mu+\lambda_2}\ ,
\end{aligned}
\tag{1.57}
$$

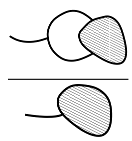
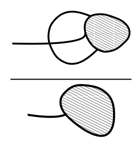
where it should come as no surprise that we cheerfully ignore all issues about convergence, in the spirit of perturbation theory. Every propagator line can (and must !) be dressed in this way once any two-point vertex (elementary of effective, that is, as the result of a collection of closed loops with two legs sticking out) is at hand.

1.4 Planck's constant

1.4.1 The loop expansion

As we have seen, Green's functions can be computed in a perturbative expansion in which the coupling constant λ_4 is in *some sense* a small number. Now consider doing perturbation theory in the $\varphi^{3/4}$ theory. We then have to decide on the *relative order of magnitude* of the two coupling constants λ_3 and λ_4 : are they of the same order, or should we take, say, λ_4 to be of the same order as ${\lambda_3}^2$? And what if even more coupling constants are involved ? We shall adopt the approach that the order of magnitude of the various diagrams should depend not on their coupling-constant content but, rather, on their *complexity*, in particular on the *number of closed loops*. That is, the more closed loops a diagram contains, the smaller it is considered to be ; and perturbation theory then prescribes the perturbation expansion to be truncated at a given number of closed loops.

To quantify these ideas we shall assign to every closed loop a factor $\hbar$, where $\hbar$ is a (small) number[22]. That is, we define the following ratios :

 $\sim \hbar$,  $\sim \hbar^2$,

etcetera. This implies, of course, a modification of the Schwinger-Dyson equation from the form (1.34) into

$$\begin{aligned} \phi(J) &= \frac{J}{\mu} - \frac{\lambda_3}{2\mu}\left(\phi(J)^2 + \hbar\phi(J)\frac{\partial}{\partial J}\phi(J)\right) \\ &\quad - \frac{\lambda_4}{6\mu}\left(\phi(J)^3 + 3\hbar\phi(J)\frac{\partial}{\partial J}\phi(J) + \hbar^2\frac{\partial^2}{(\partial J)^2}\phi(J)\right) \quad . \end{aligned} \tag{1.58}$$

In turn, we shall have to modify everything else as well : we must re-define

$$\phi(J) = \hbar\frac{\partial}{\partial J}\log Z(J) \quad , \tag{1.59}$$

so that the SDe for the path integral must read

$$S'\left(\hbar\frac{\partial}{\partial J}\right) Z(J) = J Z(J) \quad . \tag{1.60}$$

The path integral must therefore be re-defined with inclusion of $\hbar$:

$$Z(J) = N \int \exp\left(-\frac{1}{\hbar}\left(S(\varphi) - J\varphi\right)\right)\, d\varphi \quad , \tag{1.61}$$

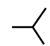
and for the Green's functions we have

$$G_n = \left\lfloor \hbar^n \frac{\partial^n}{(\partial J)^n} Z(J) \right\rfloor_{J=0} \quad , \quad C_n = \left\lfloor \hbar^n \frac{\partial^n}{(\partial J)^n} \log Z(J) \right\rfloor_{J=0} \quad . \tag{1.62}$$

The Feynman rules must, therefore, take the form

[22]As the notation suggests, it will develop into Planck's (or Dirac's) constant as our universe increases in complexity.

 $\leftrightarrow \dfrac{\hbar}{\mu}$

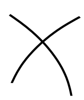
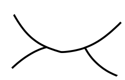
 $\leftrightarrow -\dfrac{\lambda_3}{\hbar}$

 $\leftrightarrow -\dfrac{\lambda_4}{\hbar}$

 $\leftrightarrow +\dfrac{J}{\hbar}$

$\boxed{\text{Feynman rules, version 1.2}}$ (1.63)

The introduction of $\hbar$ as the perturbation expansion parameter allows us to determine the relative orders of magnitude of coupling constants. Since with our definition all tree diagrams are of the same order, the two graphs

 and 

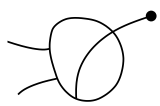
tell us that λ_4 is of the same order as ${\lambda_3}^2$. Similarly, a k-point coupling constant λ_k is of the same order as ${\lambda_3}^{k-2}$. As a last point, you may note that the including $\hbar$ does not influence the Dyson summation of sec.1.3.8, since every extra two-point vertex (with $1/\hbar$) also gives an extra propagator (with
E 8 $\hbar$).

1.4.2 Diagrammatic sum rules

Since in the Feynman rules $\hbar$ appears all over the place, it is advisable to check that the $\hbar$-behaviour of the Feynman graphs is indeed as desired. To this end, we shall first determine *diagrammatic sum rules*, valid for all non-trivial Feynman diagrams. For an arbitrary given unconnected diagram let us define the characteristics

$$\begin{array}{rcl} E & = & \text{number of external lines,} \\ I & = & \text{number of internal lines,} \\ V_q & = & \text{number of vertices of } q\text{-point type,} \\ L & = & \text{number of closed loops,} \\ P & = & \text{number of disjunct connected pieces.} \end{array}$$

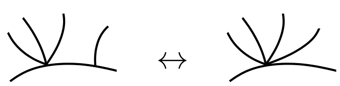
An example is



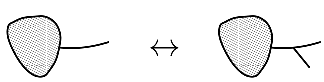
$$E = 2 \ , \ I = 6 \ , \ V_1 = 1 \ , \ V_3 = 3 \ ,$$
$$V_4 = 1 \ , \ P = 1 \ , \ L = 2 \ \ .$$

We now look for linear combinations T of these numbers that are the same for all diagrams. That is, whatever we do to a diagram, the value of T must remain unchanged. It is easy to see that *any* diagram can be transformed into *any other* diagram by application of the following four basic transformations, or their inverse :

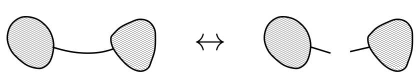
(i) coalescing a q-vertex and a 3-vertex :



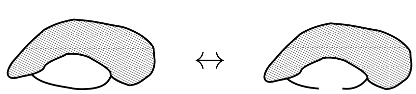
(ii) adding an external line onto any other line :



(iii) cutting through a line such that the graph falls apart :



(iv) cutting through a line which is part of a loop :



These four operations modify the characteristics as follows :

$$(i) \ : \ V_3 \ \to \ V_3 - 1 \quad , \quad V_q \ \to \ V_q - 1 \quad , \quad V_{q+1} \ \to \ V_{q+1} + 1 \quad , \quad I \ \to \ I - 1 \ \ ;$$

$$
\begin{array}{rclclcl}
(ii) & : & E \to E+1 & , & I \to I+1 & , & V_3 \to V_3+1 \ ; \\
(iii) & : & I \to I-1 & , & E \to E+2 & , & P \to P+1 \ ; \\
(iv) & : & I \to I-1 & , & E \to E+2 & , & L \to L-1 \ .
\end{array}
$$

If the combination

$$
T = \alpha_E E + \alpha_I I + \sum_q \alpha_q V_q + \alpha_L L + \alpha_P P \tag{1.64}
$$

is to be invariant under the four basic transformations, then the coefficients α must obey

$$
\begin{array}{rcl}
(i) & : & -\alpha_q + \alpha_{q+1} - \alpha_3 - \alpha_I \ = \ 0 \ , \\
(ii) & : & \alpha_E + \alpha_I + \alpha_3 \ = \ 0 \ , \\
(iii) & : & \alpha_I - 2\alpha_E - \alpha_P \ = \ 0 \ , \\
(iv) & : & \alpha_I - 2\alpha_E + \alpha_L \ = \ 0 \ .
\end{array} \tag{1.65}
$$

Adding (i) and (ii) we find

$$
-\alpha_q + \alpha_{q+1} + \alpha_E = 0 \ , \tag{1.66}
$$

with the general solution

$$
\alpha_q = \beta - q\alpha_E \ . \tag{1.67}
$$

(ii) then gives us

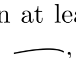
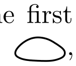
$$
\alpha_I = 2\alpha_E - \beta \tag{1.68}
$$

and (iii) and (iv) yield $\alpha_P = -\alpha_L = -\beta$. The invariant T can therefore be written as

$$
T = \alpha_E \left(-\sum_q qV_q + E + 2I \right) - \beta \left(-\sum_q V_q + I + P - L \right) \ , \tag{1.69}
$$

where α_E and β are undetermined. We see that we have precisely *two* diagrammatic sum rules. By inspection of an arbitrary[23] diagram we see that $T = 0$, so that the sum rules are

$$
\sum_q V_q = I + P - L \quad , \quad \sum_q qV_q = 2I + E \ . \tag{1.70}
$$

[23]Arbitrary, except that it must contain at least one vertex. There are two connected diagrams without vertices: the first one, , conforms to the sum rules by choosing $I = -1$, and the second one, , fits in if we choose $I = 0$. But these choices are obviously somewhat forced.

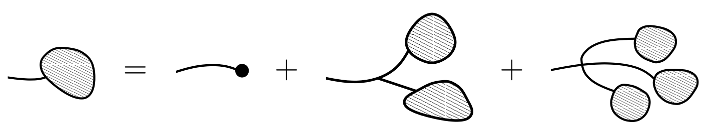
We are now able to read off the power of $\hbar$ associated with an arbitrary connected diagram (with $P = 1$). From the Feynman rules, we infer that every line contributes a factor $\hbar$ and every vertex a factor $1/\hbar$. The total power of $\hbar$ is, therefore

$$E + I - \sum_q V_q \quad = \quad E + L - 1 \quad .$$

Independently of its precise form, the power of $\hbar$ of any connected diagram depends *only* on the number of its external lines and the number of loops, and indeed each extra loop leads to an additional factor $\hbar$, as advertised.

1.4.3 The classical limit

Since in perturbation theory $\hbar$ is taken to be an infinitesimally small quantity, the limit $\hbar \to 0$ is of automatic interest. This limit has to be taken with some care since $\hbar = 0$ strictly would imply that only Green's functions with $E + L = 1$ would survive[24]. Instead, the *classical limit* $\hbar \to 0$ is meant to be the result of leaving out diagrams containing closed loops. The diagrammatic SDe will, for the $\varphi^{3/4}$ theory, then take the form



(1.71)

The corresponding solution will be denoted by $\phi_c(J)$ (with c for 'classical'), and the classical SDe is written as

$$\phi_c(J) = \frac{J}{\mu} - \frac{\lambda_3}{2\mu}\phi_c(J)^2 - \frac{\lambda_4}{6\mu}\phi_c(J)^3 \quad . \tag{1.72}$$

The classical field function is exclusively built up from tree diagrams : this is called the *tree approximation*. Note that it obeys an algebraic, rather than a differential, equation, that can be written as

$$S'(\phi_c(J)) = J \quad . \tag{1.73}$$

This is called the *classical field equation*. This is not to be confused with equations from classical, nonquantum physics. In fact, the classical field

[24]Later on, the discussion about *truncation* will clarify how this is not inconsistent.

equations will turn out to be the Klein-Gordon, Dirac, Proca and Maxwell equations. Of these, only the Maxwell equations can be considered classical, since they do not contain a particle mass.. Note that such equations have, in general, more than a single solution. Here, however, we are interested in that solution that vanishes as $J \to 0$, which may be written out using Lagrange expansion :

$$\phi_c(J) = \frac{J}{\mu} + \sum_{n\geq 1} \frac{1}{n!}\mu^{n-1}\frac{\partial^{n-1}}{(\partial J)^{n-1}}\left[\left(S'\left(\frac{J}{\mu}\right)\right)^n\right] \quad . \tag{1.74}$$

Let us now look at the path-integral picture of the classical limit. When $\hbar$ becomes small, the fluctuations in the path integrand

$$\exp\left(-\frac{1}{\hbar}\left(S(\varphi) - J\varphi\right)\right)$$

become extremely exaggerated. The main contribution to $\langle\varphi\rangle$ therefore comes from that value where the probability distribution attains its maximum, that is,

$$\langle\varphi\rangle_J \approx \varphi_c \quad , \text{ where } \quad S'(\varphi_c) = J \quad , \quad S''(\varphi_c) > 0 \quad . \tag{1.75}$$

Also in the classical limit, we therefore have $\phi_c(J) = \varphi_c$.

1.4.4 On second quantisation

The 'classical' approximations of our quantum field theory are[25] quantum equations. In fact, this is not so very surprising. In ordinary quantum mechanics, the *classical* variables such as position, momentum, etcetera are identified with the *expectation values* of their quantum-mechanical counterparts, and considered a useful approximation of reality as long as they are reasonably well-defined[26]. So it is here again : the field generating function $\phi(J)$ is considered as the *expectation value* of the quantum field φ, and it is identified with the quantum-mechanical *wave function* of whatever object it is we are studying. In this sense, to go from φ to a classical observable

[25] Will be found to be ; see the later chapters of these notes.

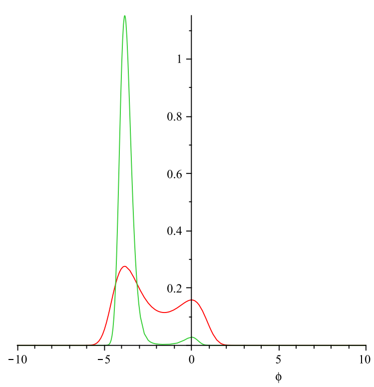
[26] With small *uncertainty*, that is, the variance of their statistical distribution around the expectation value.

we have to 'classicify' *two* times. The transition from ordinary quantum mechanics to what we are doing here is therefore dubbed 'second quantization'. Of course, from the point of view we have taken here, this is simply a matter of taking limits (expectation value upon expectation value), but if one comes in from the classical side it may look quite mysterious. This is another reminder that one *should not* try to build a more fundamental theory from a limiting case. Limiting cases are only *hints*.

1.4.5 Instanton contributions

As mentioned, for a non-free action $S(\varphi)$, the equation (1.73) has, of couse, more than a single solution[27]. Suppose that we have several such solutions, denoted by $\varphi_c^{(0)}$, $\varphi_c^{(1)}$, $\varphi_c^{(2)}$,..., and that the minimal value of $S(\varphi) - J\varphi$ is attained for $\varphi_c^{(0)}$. Then, the other classical solutions will give contributions that, relative to the dominant one, are suppressed by exponential factors of order

$$\exp\left(-\frac{1}{\hbar}\left(S(\varphi_c^{(k)}) - S(\varphi_c^{(0)}) - J\varphi_c^{(k)} + J\varphi_c^{(0)}\right)\right) \quad , \quad k = 1, 2, \ldots \quad .$$



Here we plot the (normalized) form of $\exp(-S(\phi)/\hbar)$ for the $\varphi^{3/4}$ model with $\mu = \lambda_4 = 1$ and $\lambda_3 = 1.8$, for $\hbar = 1$ and $\hbar = 0.15$. It is seen how the lowest minimum of $S(\varphi)$ starts to dominate the integral as $\hbar$ becomes small ; the contribution from the subleading maximum decreases nonperturbatively fast.

Such subdominant solutions to the classical field equations are called *instantons*. Their contribution to Green's functions do, as we see, *not* have a series expansion around $\hbar = 0$. Such *nonpertubative* effects are therefore not accessible using Feynman diagrams. This is not to say that they are irrelevant.

[27]Since the action is at least of order φ^3, the classical field equation is at least quadratic.

Indeed, we usually have a finite value for $\hbar$; more dramatically, if we let J vary as a parameter, $\varphi_c^{(1)}$, say, may for some value of J take over from $\varphi_c^{(0)}$ as the true maximum position of the probability density, causing a sudden shift in the value of $\phi_c(J)$ from $\varphi_c^{(0)}$ to $\varphi_c^{(1)}$.

1.5 The effective action

1.5.1 The effective action as a Legendre transform

Since perturbation theory presumes that higher orders in the loop expansion are small compared to lower orders, the following question suggests itself : is it possible to find, for a given action $S(\varphi)$, *another* action, called the *effective action*, with the property that *its* tree approximation reproduces the full field function of the original action S ? If such an effective action, denoted by $\Gamma(\phi)$, exists, we must have

$$\Gamma'(\phi) = J \quad , \tag{1.76}$$

where $\phi(J)$ is the full solution to the SDe belonging with $S(\varphi)$. We can use partial integration to find

$$\Gamma(\phi) = \int J \, d\phi = J\,\phi - \int \phi \, dJ = J\,\phi - \hbar W \quad , \tag{1.77}$$

where J is now to be interpreted as a function of ϕ. The transition from $W(J)$ to $\Gamma(\phi)$ is called the *Legrendre transform*. In classical mechanics, we have the same situation : there, $\hbar W$ would be the Lagrangian with J as the velocity and ϕ as the momentum, and then the effective action would turn out to be the Hamiltonian.

An important fact to be noted about the effective action can be inferred as follows. Let us consider the derivative of $\phi(J)$. If we denote the probability density (including the sources) of the quantum field φ by $P_J(\varphi)$, that is,

$$P_J(\varphi) = \frac{A(\varphi)}{\int d\varphi A(\varphi)} \quad , \quad A(\varphi) = \exp\left(-\frac{1}{\hbar}\left(S(\varphi) - J\varphi\right)\right) \quad , \tag{1.78}$$

we can write this derivative as

$$\frac{1}{\hbar}\phi'(J) \quad = \quad \frac{1}{\hbar}\,\frac{d}{dJ}\left(\frac{\int P_J(\varphi)\;\varphi\;d\varphi}{\int P_J(\varphi)\;d\varphi}\right)$$

$$
\begin{aligned}
&= \frac{\int P_J(\varphi)\,\varphi^2\,d\varphi}{\int P_J(\varphi)\,d\varphi} - \frac{\left(\int P_J(\varphi)\,\varphi\,d\varphi\right)^2}{\left(\int P_J(\varphi)\,d\varphi\right)^2} \\
&= \frac{\int P_J(\varphi_1)P_J(\varphi_2)\,({\varphi_1}^2 - \varphi_1\varphi_2)\,d\varphi_1\,d\varphi_2}{\left(\int P_J(\varphi)\,d\varphi\right)^2} \quad . \qquad (1.79)
\end{aligned}
$$

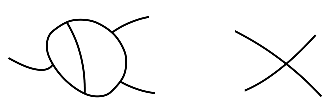
By symmetry, we can replace the factor $({\varphi_1}^2 - \varphi_1\varphi_2)$ by $(\varphi_1 - \varphi_2)^2/2$, so as to see that $d\phi(J)/dJ$ is positive. This implies that

$$
\frac{\partial^2}{(\partial\phi)^2}\Gamma(\phi) = \frac{dJ}{d\phi} > 0 \quad . \qquad (1.80)
$$

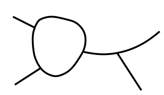
In other words, the effective action is *concave* everywhere[28]. Whereas one would assume that the effective action Γ would differ only slightly from the original action S, this can obviously no longer hold in situations where the action S is not concave.

1.5.2 Diagrams for the effective action

A tree approximation consists of tree diagrams only. To see how the loop effects of the action S end up in Γ, we define a new concept, that of a *one-particle irreducible* (1PI) diagram. A connected Feynman graph is 1PI if it contains no internal line such that cutting that line makes the diagram disconnected.

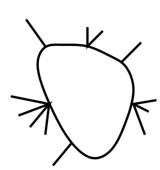


1PI diagrams



a non-1PI diagram

External lines, of course, do not enter in the 1PI criterion[29]. Note that a diagram consisting in only external lines and a single vertex also counts as 1PI, since it does not have any internal lines to be cut whatsoever. A typical one-loop 1PI diagram looks like this :

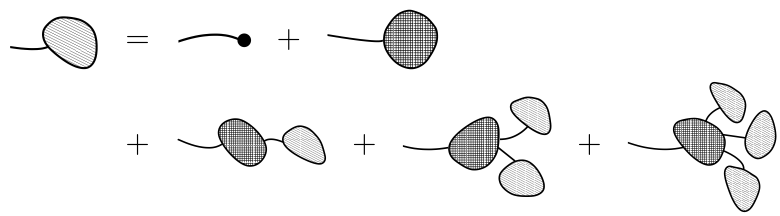


(1.81)

[28]This concavity persists in case there are more than just a single field involved. By extension, it also holds for Euclidean theories in more dimensions ; see also Appendix 3.

[29]Including them would be silly, since any diagram falls apart if we chop through an external line.

Let us denote the set of all 1PI graphs with precisely n external lines by $-\gamma_n/\hbar$, where the convention is that the Feynman factors for the external lines are *not* included. Consider, now, what happens if we enter the field function by way of its single external leg, as in the SDe. If we encounter a vertex, that vertex is part of a 1PI subdiagram (possibly consisting of only the vertex itself). Indicating the 1PI property with cross-hatches, we therefore obtain the diagrammatic equation



$\cdots$. (1.82)

Algebraically, it reads

$$\phi(J) = \frac{J}{\mu} - \frac{1}{\mu}\left(\gamma_1 + \gamma_2\phi(J) + \frac{1}{2!}\gamma_3\phi(J)^2 + \frac{1}{3!}\gamma_4\phi(J)^3 + \cdots\right), \tag{1.83}$$

in other words

$$\Gamma'(\phi) = J \quad , \tag{1.84}$$

where

$$\Gamma(\varphi) = \gamma_1\varphi + \frac{1}{2!}(\gamma_2 + \mu)\varphi^2 + \frac{1}{3!}\gamma_3\varphi^3 + \frac{1}{4!}\gamma_4\varphi^4 + \cdots \quad . \tag{1.85}$$

We conclude that **the vertices of the effective action are determined by the 1PI diagrams**. It must be noted that, in general, the effective action contains vertices with arbitrarily large numbers of legs, even if the original
E 10 action S goes up only to φ^3 or φ^4, say.

1.5.3 Computing the effective action

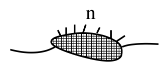
We shall now describe a computation of the effective action

$$\Gamma(\phi) = \Gamma_0(\phi) + \hbar\Gamma_1(\phi) + \hbar^2\Gamma_2(\phi) + \cdots \quad , \tag{1.86}$$

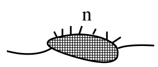
from its Feynman diagrams, for a theory with arbitrary couplings :

$$S(\varphi) = \frac{1}{2}\mu\,\varphi^2 + \sum_{k\geq 3}\frac{\lambda_k}{k!}\varphi^k \quad . \tag{1.87}$$

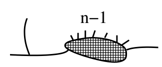
We start by considering a general one-loop 1PI diagram such as that of Eq.(1.81), and cutting through the loop at some arbitrary place. We then have a propagator 'dressed' with zero or more vertices where external lines are 'radiated off'. If there are precisely n external lines we can denote this by

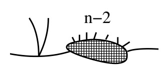


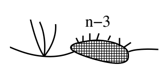
Such an object has, of course, its own SDe. Taking careful account of all possibilities to attach external lines, we can write it as



$= \theta(n=0)$

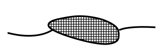
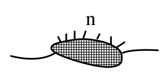
$+ \begin{pmatrix} n \\ 1 \end{pmatrix}$ 

$+ \begin{pmatrix} n \\ 2 \end{pmatrix}$ 

$+ \begin{pmatrix} n \\ 3 \end{pmatrix}$ 

$+ \cdots$ (1.88)

We define the generating function for such dressed propagators as

$P(z) =$  $= \sum_{n\geq 0} \frac{z^n}{n!}$ 

, (1.89)

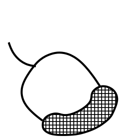
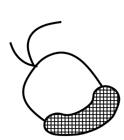
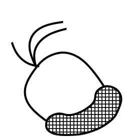
and see that the SDe reads

$$P(z) = \frac{\hbar}{\mu} - z\frac{\lambda_3}{\mu}P(z) - \frac{z^2}{2!}\frac{\lambda_4}{\mu}P(z) - \frac{z^3}{3!}\frac{\lambda_5}{\mu}P(z) - \cdots, \tag{1.90}$$

in other words,

$$P(z) = \frac{\hbar}{S''(z)} \quad . \tag{1.91}$$

We now close the loop again with an arbitrary vertex, at which vertex *at least* one other external line is included. By the same combinatorial arguments as above we can find the generating function $L(z)$ for such loops :

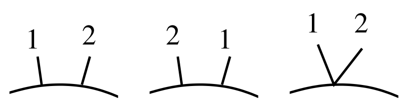
$L(z) =$  $+$  $+$  $+ \cdots$

$$= \frac{1}{2}\left\{-\frac{\lambda_3}{\hbar}P(z) - z\frac{\lambda_4}{\hbar}P(z) - \frac{z^2}{2!}\frac{\lambda_5}{\hbar}P(z) - \cdots\right\}$$
$$= -\frac{S'''(z)}{2S''(z)} \quad . \tag{1.92}$$

The symmetry factor 1/2 arises from the fact that the propagator is not oriented and thus we have to avoid double-counting. Considering that a propagator with n external legs leads to a closed loop with at least $n+1$ external legs, we see that the one-loop effective action is given by

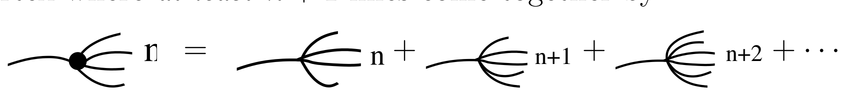
$$\Gamma_1(\phi) = -\hbar \int d\phi\ L(\phi) = \frac{\hbar}{2}\log(S''(\phi)) \quad . \tag{1.93}$$

A few remarks are in order here. In the first place, we see that the effective action obtained in this way is only well-defined where the action itself is concave, in agreement with the discussion in 1.5. In the second place, the trick of closing the loop with an extra vertex, rather than just trying to 'glue' the endpoints of $P(z)$ together, is technically useful since it avoids enormous problems with the symmetry factors. To see this, consider the three possibilities for $n = 2$:



If we glue the endpoints of the propagator, the first two diagrams result in *the same* loop graph, so that these three propagator diagrams end up as *two* loop diagrams. With more external legs attached, this becomes ever so much more complicated : assigning a special rôle to one external line avoids this. In the last place, the above calculation is possible since all external lines are, so to speak, identical. In more dimensions, where external lines can carry momentum, this is no longer true. However, the *effective potential*, that is the effective action at zero momentum, *does* lend itself to such a calculation in higher dimensions[30].

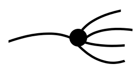
We can extend this treatment to higher loop orders as well. Let us denote a vertex where *at least* $n+1$ lines come together by



(1.94)

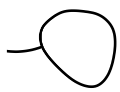
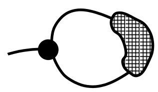
[30] For simple scalar theories. Of course external lines may carry more than just momentum information, that is, they can also carry spin/charge/colour$\cdots$ information. Then the calculation is again more difficult.

and assign to this dressed vertex the Feynman rule



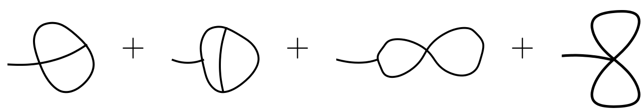
$$n \;=\; -\frac{1}{\hbar}\, S^{(n+1)}(z) \quad . \tag{1.95}$$

Now, we introduce the notion of a *tadpole diagram* : this is a connected diagram with precisely *one* external line and no source vertices. The effective action as given above then follows from writing out the 1PI tadpole diagrams, replacing propagators by dressed propagators and vertices by dressed vertices ; we can then simply read off the result.

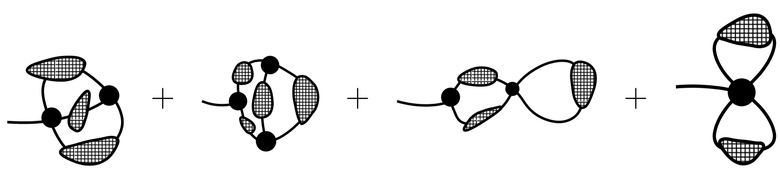
 $\rightarrow$ 

$$= -\frac{S^{(3)}(z)}{2S^{(2)}(z)} \quad \Rightarrow \quad \Gamma_1'(\phi) = \frac{S^{(3)}(\phi)}{2S^{(2)}(\phi)} \quad , \tag{1.96}$$

as before. In two loops, the 1PI tadpole is given by the diagrams



Dressing these tadpole diagrams gives us



$$= \quad \hbar\left\{\frac{S^{(3)}(z)\,S^{(4)}(z)}{6\;S^{(2)}(z)^3} - \frac{S^{(3)}(z)^3}{4\;S^{(2)}(z)^4} + \frac{S^{(3)}(z)\,S^{(4)}(z)}{4\;S^{(2)}(z)^3} - \frac{S^{(5)}(z)}{8\;S^{(2)}(z)^2}\right\} \tag{1.97}$$

The two-loop contribution to the effective action is therefore

$$\frac{d}{d\phi}\Gamma_2(\phi) = \frac{S^{(5)}(\phi)}{8\;S^{(2)}(\phi)^2} - \frac{5\;S^{(3)}(\phi)\,S^{(4)}(\phi)}{12\;S^{(2)}(\phi)^4} + \frac{S^{(3)}(\phi)^3}{4\;S^{(2)}(\phi)^4} \quad . \tag{1.98}$$

The effective action itself, the integral over the above experession, has no nice simple form as in Eq.(1.93), but is of course calculable as soon as $S(\phi)$ is explicitly given ; moreover, we see that it will become undefined where $S''(\phi)$ vanishes. From our diagrammatic approach we see that this will persist in all loop orders[31].

[31] Because in all 1PI diagrams we have to dress the propagators, which implies lots of $S''(\phi)$ in the denominators.

1.6 Exercises for Chapter 1

Excercise 1 Green's functions and connected Green's functions
We have

$$Z(J) = \sum_{n\geq 0} G_n \frac{J^n}{n!} \quad , \quad W(J) = \sum_{n\geq 0} C_n \frac{J^n}{n!}$$

and

$$W(J) = \log(Z(J)) \ \ .$$

Using that $G_0 = 1$, and the expansion

$$-\log(1-x) = \sum_{n\geq 1} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \cdots \quad ,$$

express $C_{0,\ldots,5}$ in terms of the G's.

Excercise 2 The problem with φ^4 theory
By examining the case where λ is infinitesimally small, either positive and negative, argue that the point $\lambda = 0$ establishes, in fact, an extremely singular theory.

Excercise 3 The SD equation for Z in another action
For the action

$$S(\varphi) = \frac{\mu}{2!}\varphi^2 + \sum_{k=3}^{6} \frac{\lambda_k}{k!}\varphi^k \quad ,$$

Find the SDe for the path integral $Z(J)$.

Excercise 4 Writing the SDE for ϕ
Let the action be $S(\varphi)$. The SDe for the path integral obeys

$$S'\left(\hbar \frac{d}{dJ}\right) Z(J) = J Z(J)$$

as we have seen in the course. Show that the SDe for the field function $\phi(J)$ can be written as

$$S'\left(\phi(J) + \hbar\frac{d}{dJ}\right) e(J) = J$$

where $e(J) \equiv 1$ is the unit function. Do this using the fact that

$$Z(J) = \exp\left(\frac{1}{\hbar}\int dJ\ \phi(J)\right) \quad ,$$

and then considering Z', Z'' and so on.

Excercise 5 The SD equation for ϕ in another action
For the action of Exercis 3, derive the SDe for $\phi(J)$.

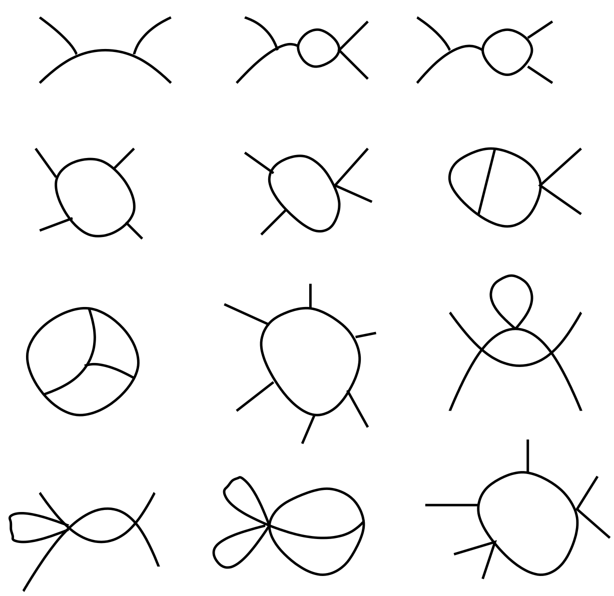
Excercise 6 The symmetry factor of life, the universe, and everything
Devise a diagram (or set of diagrams) that has a symmetry factor of 1/42.

Excercise 7 Diagrammatic SDE for φ^6 theory
Give the diagrammatic SDE for φ^6 theory, and write it out algebraically.

Excercise 8 Vacuum diagrams
Consider a theory with couplings φ^n, $n = 3, 4, 5, 6, \ldots$.

1. At one loop, there is a single vacuum diagram. Write it out, and prove that its symmetry factor is actually zero.
2. At two loops, there are three vacuum diagrams. Write them down and determine the symmetry factors.
3. A three loops, there are fifteen vacuum diagrams. Write them down and determine the symmetry factors.

Excercise 9 Some diagrams to consider
Of the following 12 diagrams, determine the multiplicity factor, the symmetry factor, and the number of loops :



Excercise 10 SDe for the effective action
The effective action is defined such that $\Gamma'(\phi(J)) = J$. Show that the SDe for pure φ^4 theory can be written as

$$\mu\phi = \Gamma'(\phi) - \frac{\lambda_4}{6}\left(\phi^3 + 3\hbar\frac{\phi}{\Gamma''(\phi)} - \hbar^2\frac{\Gamma'''(\phi)}{\Gamma''(\phi)^3}\right)$$

Let the effective action have a perturbation expansion :

$$\Gamma(\phi) = \Gamma_0(\phi) + \hbar\Gamma_1(\phi) + \hbar^2\Gamma_2(\phi) + \cdots$$

Use the SDe to find Γ_0, Γ_1 and Γ_2. Compare the results with the corresponding 1PI diagrams with up to 6 external legs.

Excercise 11 A wonderful action ?
Consider the action

$$S_L(\varphi) = -a\log(1 - b\varphi) - c\varphi \quad ,$$

with

$$ab = c \quad , \quad ab^2 = \mu \quad , \quad ab^3 = \frac{\lambda}{2} \quad .$$

1. The domain of φ is now no longer $(-\infty, \infty)$. Determine which domain is appropriate, and prove that the path integrand indeed vanishes at the endpoints of that domain.

2. Prove that this theory has φ^n interactions for all $n = 3, 4, 5, \ldots$ and that the vertex for a φ^n interaction reads $-(n-1)!ab^n$.

3. Prove that the SDe for this theory reads

$$\frac{\mu\hbar\frac{\partial}{\partial J}}{1 - b\hbar\frac{\partial}{\partial J}} Z(J) = JZ(J) \quad .$$

4. Prove that the field function is given by

$$\phi(J) = \frac{J - b\hbar}{\mu + bJ} \quad .$$

5. Prove that the field function has *only* one-loop corrections : all higher-order corrections vanish. Check this by writing out all 2-loop corrections to the two-point function and evaluating them.

6. Using $\Gamma'(\phi) = J$, compute the effective action. Show that it, also, if free from higher-order corrections beyond one loop.

7. For $J \to -\mu/b = -c$, the field function diverges. Show that this corresponds precisely to the limit in which the path integrand no longer vanishes at one of the endpoints.

Chapter 2

Renormalization : the principles

2.1 Doing physics : mentality against reality

In this chapter, we digress a bit into a discussion of what it is that particle physicists claim to be doing : confronting theory with reality. This leads to some interesting subtleties.

2.1.1 Physics *vs.* Mathematics

If we were mathematicians, the subject matter in the previous chapter might be formulated as the following task : given the parameters μ, λ_3 and λ_4 of the action, to compute the connected Green's functions. This 'mathematician's scheme' may be depicted as follows :

$$\boxed{\mu\ ,\ \lambda_3\ ,\ \lambda_4} \quad \longrightarrow \quad \boxed{C_1\ ,\ C_2\ ,\ C_3\ ,\ C_4\ ,\ C_5\ ,\ C_6\ ,\ C_7\ ,\ \ldots}$$

In this set-up, the parameters are supplied from *outside* the computational and experimental context. Since, however, we are physicists[1] the situation is somewhat different : we first have to *measure* the values of the parameters from *inside* the experimental context, using some of the connected Green's functions as *measurement processes*, and then predict some *other* connected Green's functions, which we shall call *prediction processes*. That, rather

[1] I hope.

different, situation may be depicted by the scheme

$$\boxed{E_k = C_k \ , \ k = 1 \ldots 4} \quad \longrightarrow \quad \boxed{\mu \ , \ \lambda_3 \ , \ \lambda_4} \quad \longrightarrow \quad \boxed{C_5 \ , \ C_6 \ , \ C_7 \ , \ \ldots}$$

Here, the quantities $E_{1,2,3,\ldots}$ stand for the *experimentally observed* values of the connected Green's functions : barring experimental errors, these *numerical* values do not change under any improvement of the theory. Now consider the fact that we are doing perturbation theory. That is, both the measurement and the prediction processes are known only as truncated series in $\hbar$. Let us suppose that by stolidity and perseverance a next higher order in perturbation theory for the prediction processes has become available. Is this any good ? Obviously not, unless a similar increased level of precision has been attained for the measurement processes. Only in that case a new 'fit' of the parameters of the action can be made, and improved values of the 'prediction' connected Green's functions can usefully be obtained. This order-by-order improvement is called *renormalization*. Let us denote by a superscript the order to which the connected Green's functions have been computed. The 'physicist's scheme' can then be envisaged as follows :

$$\begin{array}{ccccc}
\boxed{E_k = C_k^{(0)} \ , \ k = 1 \ldots 4} & \longrightarrow & \boxed{\mu^{(0)} \ , \ \lambda_3^{(0)} \ , \ \lambda_4^{(0)}} & \longrightarrow & \boxed{C_5^{(0)} \ , \ C_6^{(0)} \ , \ \ldots} \\
\boxed{E_k = C_k^{(1)} \ , \ k = 1 \ldots 4} & \longrightarrow & \boxed{\mu^{(1)} \ , \ \lambda_3^{(1)} \ , \ \lambda_4^{(1)}} & \longrightarrow & \boxed{C_5^{(1)} \ , \ C_6^{(1)} \ , \ \ldots} \\
\boxed{E_k = C_k^{(2)} \ , \ k = 1 \ldots 4} & \longrightarrow & \boxed{\mu^{(2)} \ , \ \lambda_3^{(2)} \ , \ \lambda_4^{(2)}} & \longrightarrow & \boxed{C_5^{(2)} \ , \ C_6^{(2)} \ , \ \ldots} \\
\boxed{E_k = C_k^{(3)} \ , \ k = 1 \ldots 4} & \longrightarrow & \boxed{\mu^{(3)} \ , \ \lambda_3^{(3)} \ , \ \lambda_4^{(3)}} & \longrightarrow & \boxed{C_5^{(3)} \ , \ C_6^{(3)} \ , \ \ldots} \\
\vdots & & \vdots & & \vdots
\end{array}$$

Order by order, the parameters keep getting updated, but in the overall picture they are just *bookkeeping devices* that allow one to go from measurements to predictions of the more physically interesting connected Green's functions. It should not come as a surprise that in the measurement-parameter-prediction protocol, a higher-order correction in the parameters due to an improved measurement expression is cancelled again, to some extent, in the prediction. In fact, for certain classes of theories, which are called *renormalizable*, these cancellations may be quite extreme.

2.1.2 The renormalization program : an example

As an example of the renormalization program, we shall investigate $\varphi^{3/4}$ theory. To order $\mathcal{O}(\hbar)$ in perturbation theory, the first few connected Green's functions are given by

$$
\begin{aligned}
C_1 &= \hbar\left(-\frac{\lambda_3}{2\mu^3}\right) + \mathcal{O}\left(\hbar^2\right) \quad , \\
C_2 &= \hbar\left(\frac{1}{\mu}\right) + \hbar^2\left(-\frac{\lambda_4}{2\mu^3} + \frac{{\lambda_3}^2}{\mu^4}\right) + \mathcal{O}\left(\hbar^3\right) \quad , \\
C_3 &= \hbar^2\left(-\frac{\lambda_3}{\mu^3}\right) + \hbar^3\left(-\frac{4{\lambda_3}^2}{\mu^6} + \frac{7\lambda_3\lambda_4}{\mu^5}\right) + \mathcal{O}\left(\hbar^4\right) \quad , \\
C_4 &= \hbar^3\left(-\frac{\lambda_4}{\mu^4} + \frac{3{\lambda_3}^2}{\mu^5}\right) + \hbar^4\left(\frac{24{\lambda_3}^4}{\mu^8} + \frac{7{\lambda_4}^2}{2\mu^6} - \frac{59{\lambda_3}^2\lambda_4}{2\mu^7}\right) + \mathcal{O}\left(\hbar^5\right) \quad , \\
C_5 &= \hbar^4\left(\frac{10\lambda_3\lambda_4}{\mu^6} - \frac{15{\lambda_3}^3}{\mu^7}\right) \\
&+ \hbar^5\left(\frac{605\lambda_4{\lambda_3}^3}{2\mu^9} - \frac{192{\lambda_3}^5}{\mu^{10}} - \frac{80{\lambda_4}^2\lambda_3}{\mu^8}\right) + \mathcal{O}\left(\hbar^6\right) \quad , \qquad (2.1)
\end{aligned}
$$

and of course the next-order corrections and connected Green's functions are readily computable. Let us assume that the *experimental* values of the connected Green's functions $C_{2,3,4}$ have been measured, with negligible experimental error for simplicity. We shall denote these values by $E_{2,3,4}$, respectively. For purposes of illustration, we shall assume that these values are

$$E_2 = \hbar \quad , \quad E_3 = -\hbar^2 \quad , \quad E_4 = 2\hbar^3 \quad . \qquad (2.2)$$

In lowest order of perturbation theory, we can then find the action's parameters to be

$$\mu = 1 \quad , \quad \lambda_3 = 1 \quad , \quad \lambda_4 = 1 \quad . \qquad (2.3)$$

If this were all, we could then compute the connected Green's functions. This 'naive' treatment would give the following results up to two loops :

$$
\begin{aligned}
{C_1}^{\text{naive}} &= -\frac{1}{2}\hbar + \frac{1}{24}\hbar^2 + \mathcal{O}\left(\hbar^3\right) \quad , \\
{C_2}^{\text{naive}} &= \hbar + \frac{1}{2}\hbar^2 - \frac{3}{4}\hbar^3 + \mathcal{O}\left(\hbar^4\right) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
C_3{}^{\text{naive}} &= -\hbar^2 - \frac{1}{2}\hbar^3 - \frac{131}{24}\hbar^4 + \mathcal{O}\left(\hbar^5\right) \quad , \\
C_4{}^{\text{naive}} &= 2\hbar^3 - 2\hbar^4 - \frac{147}{4}\hbar^5 + \mathcal{O}\left(\hbar^6\right) \quad , \\
C_5{}^{\text{naive}} &= -5\hbar^4 + \frac{61}{2}\hbar^5 + \frac{5665}{24}\hbar^6 + \mathcal{O}\left(\hbar^7\right) \quad , \\
C_6{}^{\text{naive}} &= 10\hbar^5 - 295\hbar^6 - \frac{5105}{4}\hbar^7 + \mathcal{O}\left(\hbar^8\right) \quad , \\
C_7{}^{\text{naive}} &= 35\hbar^6 - \frac{5195}{2}\hbar^7 - \frac{47075}{24}\hbar^8 + \mathcal{O}\left(\hbar^9\right) \quad ,
\end{aligned}
\tag{2.4}
$$

However, we see that now $C_{2,3,4} = E_{2,3,4}$ no longer hold, and therefore we must re-tune the parameters order by order in perturbation theory. In the present case, we find up to two-loop accuracy :

$$
\begin{aligned}
\mu &= 1 + \frac{1}{2}\hbar + \hbar^2 + \mathcal{O}\left(\hbar^3\right) \quad , \\
\lambda_3 &= 1 + \hbar - \frac{49}{24}\hbar^2 + \mathcal{O}\left(\hbar^3\right) \quad , \\
\lambda_4 &= 1 - \frac{3}{2}\hbar + \frac{5}{4}\hbar^2 + \mathcal{O}\left(\hbar^3\right) \quad ,
\end{aligned}
\tag{2.5}
$$

and the *renormalized* connected Green's functions, suitably truncated to the correct order in $\hbar$, read

$$
\begin{aligned}
C_1 &= -\frac{1}{2}\hbar + \frac{1}{24}\hbar^2 + \mathcal{O}\left(\hbar^3\right) \quad , \\
C_2 &= \hbar \quad , \\
C_3 &= -\hbar^2 \quad , \\
C_4 &= 2\hbar^3 \quad , \\
C_5 &= -5\hbar^4 + 3\hbar^5 - \frac{5}{2}\hbar^6 + \mathcal{O}\left(\hbar^7\right) \quad , \\
C_6 &= 10\hbar^5 - 45\hbar^6 + 90\hbar^7 + \mathcal{O}\left(\hbar^8\right) \quad , \\
C_7 &= 35\hbar^6 + 480\hbar^7 - 2065\hbar^8 + \mathcal{O}\left(\hbar^9\right) \quad .
\end{aligned}
\tag{2.6}
$$

The difference between the 'naive' and the renormalized connected Green's functions is quite evident. In particular $C_{2,3,4}$ are completely free of higher-order corrections. For the other connected Green's functions the coefficients in the perturbation expansion tend to be smaller in absolute value than in the 'naive' expressions.

The above discussion is obviously only a drastically simplified example of a phenomenological situation that is usually much more complicated. For instance, one does not, usually, renormalize connected Green's functions but rather quantities extracted from *scattering matrix elements*, that are themselves not identical to, but extracted from connected Green's functions. The experimental observables E therefore do not take the simple form given here. The higher-order corrections themselves are usually much more complicated, and not completely free from ambiguities, nor necessarily finite. Nevertheless, the *operational scheme* outlined above is essentially the same as those that are employed in real-life physics. In particular, it cannot be stressed often enough that the renormalization procedure is necessary simply because one does perturbation theory, *not* because loop corrections may contain infinities[2].

2.2 A handle on loop divergences

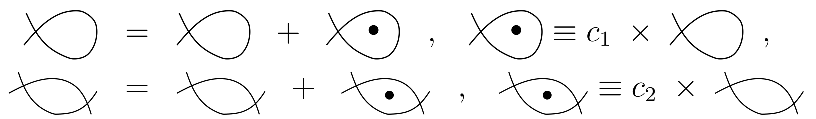
2.2.1 A toy : the dot model

Notwithstanding the above remarks on the *per se* necessity of renormalization, the fact that, in nontrivial theories, loop diagrams often contain infinities makes the need to do something about them all the more urgent. Loop divergences arise from summation over internal degrees of freedom of Feynman diagrams. In zero dimensions there are no such internal degrees of freedom, and all diagrams are finite. We can, however, introduce the following toy model. Consider, as before, our working-horse $\varphi^{3/4}$ theory. Let us assume that we introduce yet another Feynman rule : we shall apply a factor $1 + c_1$ to every closed loop that contains precisely *one* vertex, and a factor $1 + c_2$ to every closed loop that contains precisely *two* vertices. Loops with more vertices remain unaffected[3]. The numbers c_1 and c_2 may depend on the parameters of the theory, or on other parameters. In the spirit of 'loop divergences' we shall envisage that $c_{1,2} \to \infty$ at some stage. In terms of

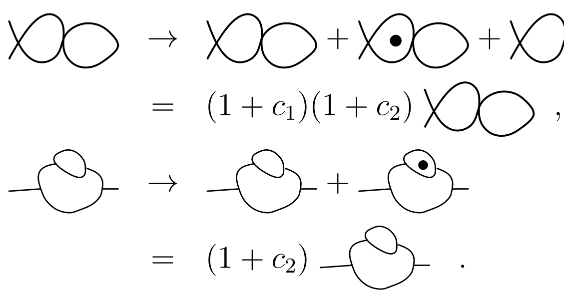
[2] This insight is, even at present, not as endemic as one might wish.

[3] This rule accords with 'naive power counting' for four-dimensional scalar theories without derivative couplings, the most direct four-dimensional extension of the zero-dimensional theories we are discussing in this chapter.

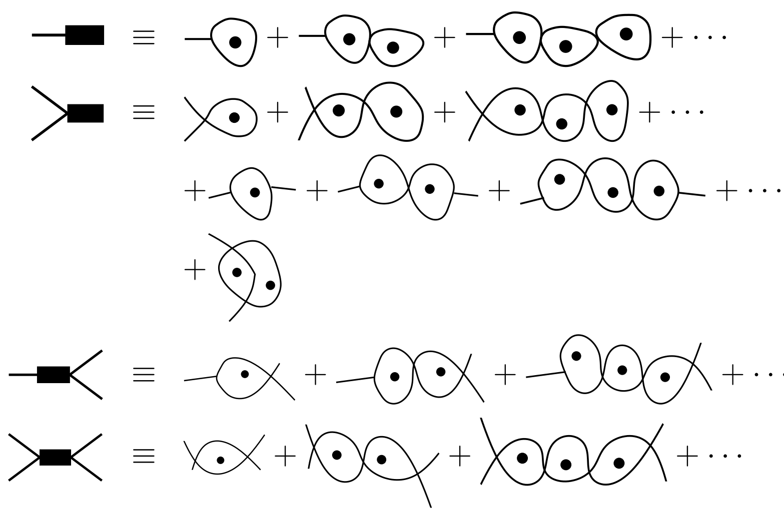
Feynman diagrams, this rule amounts to duplicating each one- or two-vertex loop with a 'dotted' loop :


(2.7)

For example, under this rule the following two-loop diagrams are modified accordingly :

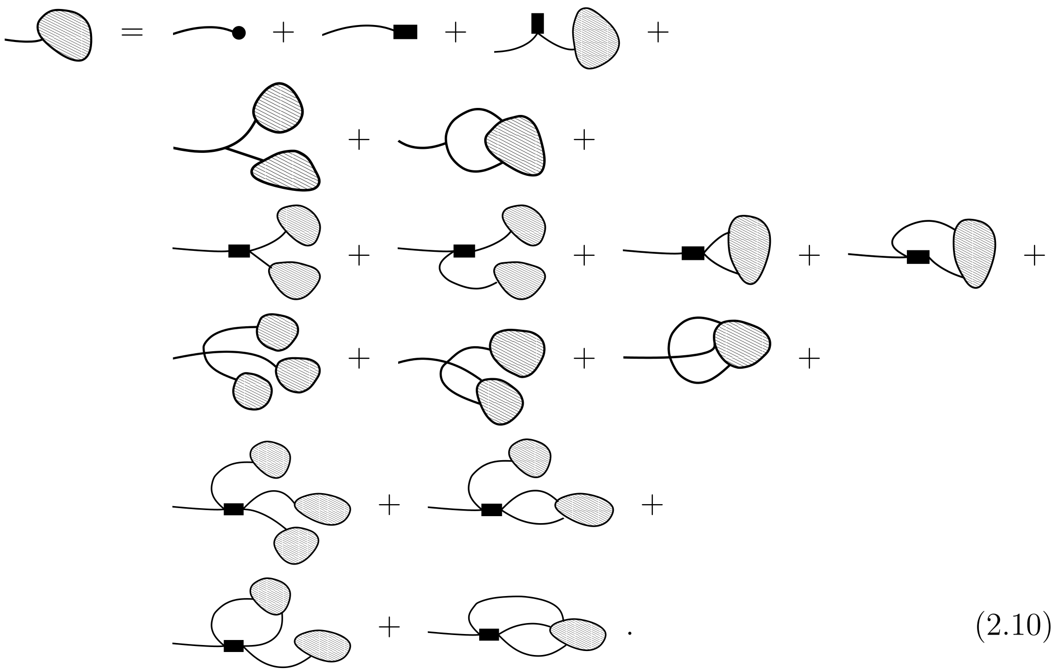

(2.8)

The Feynman diagrams are governed by the Schwinger-Dyson equation. Our new rule must therefore be implemented, somehow, into a modified SDe. Some reflection tells us that the necessary new ingredients are made up out of those Feynman diagrams that contain *only* dotted loops. Fortunately, these form a manageable set, where we differentiate between 1PI diagrams with up to 4 legs[4] :


(2.9)

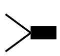
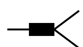
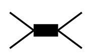
[4]With 5 or more legs our rule does not allow for diagrams with *only* dotted loops.

The only diagram that does not carry a 'tower' of loops on its back is the last diagram in the two-point dotted series. Using these artefacts, we can now rewrite the appropriate SDe for our $\varphi^{3/4}$ theory with the added dotting rule :



$$(2.10)$$

We can readily translate this SDe into algebraic form. If we take out the external propagators from the 'black box' graphs, we can write

 $= B_1$,  $= B_2$,  $= B_3$,  $= B_4$. (2.11)

We shall leave the actual evaluation of these sets of graphs for later: at this point, we shall simply treat them as effective vertices. The 'dotted-loop'-modified SDe then reads, when we work out the graphs one after the other, in the order in which they are displayed above :

$$\begin{aligned}
\phi &= \frac{J}{\mu} - \frac{B_1}{\mu} - \frac{B_2}{\mu}\phi - \frac{\lambda_3}{2\mu}\phi^2 - \frac{\hbar\lambda_3}{2\mu}\phi' \\
&\quad - \frac{B_3}{2\mu}\phi^2 - \frac{B_3}{\mu}\phi^2 - \frac{\hbar B_3}{2\mu}\phi' - \frac{\hbar B_3}{\mu}\phi' \\
&\quad - \frac{\lambda_4}{6\mu}\phi^3 - \frac{\hbar\lambda_4}{2\mu}\phi\phi' - \frac{\hbar^2\lambda_4}{6\mu}\phi''
\end{aligned}$$

$$-\frac{B_4}{2\mu}\phi^3 - \frac{\hbar B_4}{2\mu}\phi\phi' - \frac{\hbar B_4}{\mu}\phi\phi' - \frac{\hbar^2 B_4}{2\mu}\phi'' \ . \tag{2.12}$$

We can simply rewrite this SDe as

$$\begin{aligned}(\mu + B_2)\phi &= (J - B_1) - (\lambda_3 + 3B_3)(\phi^2 + \hbar\phi') \\ &\quad -(\lambda_4 + 3B_4)(\phi^3 \ + \ 3\hbar\phi\phi' \ + \ \hbar^2\phi'')\end{aligned} \tag{2.13}$$

But, this is exactly the SDe equation belonging to the action

$$S(\varphi) = B_1\varphi + \frac{1}{2}(\mu + B_2)\varphi^2 + \frac{1}{6}(\lambda_3 + 3B_3)\varphi^3 + \frac{1}{24}(\lambda_4 + 3B_4)\varphi^4 \ . \tag{2.14}$$

Therefore, the spirit of renormalization tells us that *in every application* the *bare* parameters μ, λ_3 and λ_4 will never occur on their own, but always only in the combinations $\mu + B_2$, $\lambda_3 + 3B_3$, and $\lambda_4 + 3B_4$; and that therefore, *whatever the values of* $B_{2,3,4}$, the combination will automatically be finite if the experimental quantities in which they enter are finite. We can therefore choose the action's parameters such that all Green's functions come out finite ; and the remaining B_1 can always be completely absorbed into a linear term in the bare action. Indeed, this is the way in which the notorious 'loop divergences' are absorbed into the bare action : infinite loop corrections are compensated for by infinite bare parameters.

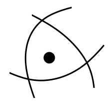
2.2.2 Nonrenormalizeable theories

The significant point in the discussion above is the fact that all dotted-loop contributions can be absorbed into a *finite* number of terms of the bare action. We may formulate the requirement of a *renormalizeable theory* as that which states that **a finite number of measured quantities[5] suffice to make all other predictions of the theory well-defined**. If an *infinite* number of measured quantities would be necessary, the theory would be called *non-renormalizeable* : but, worse, from the operational point of view it would be worthless[6].

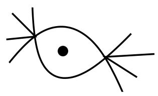
[5]Think of $E_{2,3,4,}$.

[6]In modern thought, this train of thought tends to be relaxed. If the necessary additional experimental values are only relevant at some very high energy scale, the theory would be *effectively* renormalizeable. It is a matter of taste whether you feel comfortable with this, or not.

As an example of a non-renormalizeable situation, let us consider a Feynman rule in which a loop with *three* vertices acquires a dotted counterpart : that is, we would have a (potentially infinite) contribution of the form



This can, of course, be repaired by introducing into the bare action a φ^6 term ; but in that case there would arise dotted loops with eight external legs :



which would necessitate a φ^8 term in the bare action — and so on. A theory would arise in which an infinite number of measured quantities would be needed before any consistent[7] prediction could be made : a non-renormalizeable situation ! The same problem occurs in a theory with a bare φ^6 interaction. It is seen that the requirement of renormalizeability puts constraints on the bare action[8].

2.3 Scale dependence and such

2.3.1 The scale, and its divergence ?

As mentioned above, the parameters of the action have to be determined by comparison to experimentally measured quantities. Such measurement experiments do not take place in some abstract realm, but rather in a concrete physical situation. This experimental context partially determines the measurement result. A very concrete example is the measurement of the coupling constant using a scattering process : in that case, one of the determining factors is the energy at which the scattering takes place. Also choices made in the theoretical computation of the measured quantities play their rôle : for example, in *dimensional regularization*[9] an *energy scale* must be introduced,

[7] *i.e.* finite in high orders of perturbation theory.

[8] It must come as no surprise that the Higgs potential of the Standard Model has no interaction terms for the Higgs field (which is scalar) more complicated than the four-point coupling.

[9] To be discussed later on.

and this scale can to a large extent be chosen arbitrarily. We shall lump all these effects together into a quantity s, which we shall call the *scale*. It must be stressed that the scale also contains the (regularized) loop divergences, and may be expected to become infinite at some stage.

Let us consider a theory with only one parameter : an example of such a theory is massless QCD, that is the theory of massless quarks and gluons and their interactions. The single parameter is then the coupling constant. Let the bare parameter, as it occurs in the action, be denoted by v. The renormalized parameter, extracted from experiment, will be denoted by w. The renormalized coupling is then given by the bare coupling and the experimental context, embodied by the scale s :

$$w = F(s; v) \ . \tag{2.15}$$

This relation ought to be invertible, so that we can find v given w :

$$v = G(s; w) \ . \tag{2.16}$$

Obviously we have

$$w = F(s; G(s; w)) \quad , \quad v = G(s; F(s; v)) \ . \tag{2.17}$$

By differentiation we find the following relations between the derivatives of F and G :

$$F_1 G_1 = 1 \quad , \quad F_0 + F_1 G_0 = 0 \ , \tag{2.18}$$

where the subscript 0 denotes partial derivatives with respect to s, and the subscript 1 stands for a partial derivative with respect to the other argument. Let us now consider an infinitesimal change in the scale. Since v is independent[10] of s, the value of w has to adapt itself in a manner prescribed by F :

$$\frac{d}{ds} w = F_0(s; v) \ . \tag{2.19}$$

This expression contains the (divergent) scale s and the (divergent) value of the bare coupling v. We can, of course, express everything in terms of w :

$$\frac{d}{ds} w = F_0(s; G(s; w)) \ . \tag{2.20}$$

[10]After all, the action doesn't know which experiment is going to be used to measure the parameter.

This expression now contains the finite number w and the divergent scale s. It can, therefore, only make sense[11] if s actually drops out. We therefore require $F(s,;v)$ to be such that

$$\frac{\partial}{\partial s} F_0(s;G(s;w)) = 0 \quad . \tag{2.21}$$

In other words,

$$F_{00} + F_{01}G_0 = 0 \quad . \tag{2.22}$$

Using Eq.(2.18) and dividing once by F_1 we can write this as

$$\frac{F_{00}}{F_1} - \frac{F_0 F_{10}}{{F_1}^2} = \frac{\partial}{\partial s}\left(\frac{F_0}{F_1}\right) = 0 \quad . \tag{2.23}$$

There must, therefore, exist a function $h(v)$ of v only, such that

$$\frac{\partial}{\partial s} F(s;v) = h(v)\ \frac{\partial}{\partial v} F(s;v) \quad . \tag{2.24}$$

By separation of variables, Eq.(2.24) is easily solved, and we find

$$w = F(s;v) = f\left(s + \rho(v)\right) \quad , \quad \frac{d}{dv}\rho(v) = \frac{1}{h(v)} \quad , \tag{2.25}$$

for some function f. There must exist some value s_0 for s, such that w and v precisely coincide. This value can, of course, depend on v, so we write it as $s_0(v)$. We therefore have

$$f\left(s_0(v) + \rho(v)\right) = v \quad , \quad s_0(v) + \rho(v) \equiv j(v) \quad , \tag{2.26}$$

so that f and j are each other's inverse : $f(j(v)) = v$. Applying j to Eq.(2.25) we see that

$$s + \rho(v) = j(w) = s_0(w) + \rho(w) \quad . \tag{2.27}$$

Note that in this equation, both terms on the left-hand side are divergent, but on the right-hand side they are finite. We can now determine the scale dependence of w. This dependence, called the *beta function* of the theory, is given by

$$\begin{aligned} \beta(w) \ \equiv\ \frac{d}{ds}w \ &=\ \frac{\partial}{\partial s} f\left(s + \rho(v)\right) \\ =\ f'\left(j(w)\right) \ &=\ \frac{1}{j'(w)} \ =\ \frac{h(w)}{1 + h(w)s_0'(w)} \quad . \end{aligned} \tag{2.28}$$

[11]Note that this is not a *proof*, it is a *requirement* that we make on the theory.

All reference to the bare coupling has been removed : we see that the renormalized coupling has a definite, predictable dependence on the energy scale of the measuring experiment[12]. Note, also, that whereas we introduced h as a function of the *bare* parameter, it enters in the beta function as a function of the *renormalized* parameter. Finally, as we shall see in the next section, the h functions usually starts at second order ($\mathcal{O}\left(v^2\right)$) in perturbation theory, so that the two first terms in the beta function are independent of the form of $s_0(v)$ as long as this is non-singular for $v = 0$.

2.3.2 Low-order approximation to the renormalized coupling

Let us examine the possible shape of the function $F(v, s)$ in some more detail. In the spirit of perturbation theory, it will be given by a series expansion like

$$F(v, s) = v + v^2\alpha_1(s) + v^3\alpha_2(s) + v^4\alpha_3(s) + \cdots \quad . \tag{2.29}$$

Let us assume that the functions $\alpha_j(s)$ vanish at $s = 0$, so that $s_0(v) = 0$ and $h(v) = \beta(v)$. The h function is given by

$$h(v) = \frac{F_2(v, s)}{F_1(v, s)} = \frac{v^2\alpha_1'(s) + v^3\alpha_2'(s) + v^4\alpha_3'(s) + \cdots}{1 + 2v\alpha_1(s) + 3v^2\alpha_2(s) + \cdots} \quad , \tag{2.30}$$

so that we see that the beta function must start with v^2 :

$$\beta(v) = \beta_0 v^2 + \beta_1 v^3 + \beta_2 v^4 + \cdots \tag{2.31}$$

The requirement that the beta function depend not on s governs the form of the functions $\alpha_j(s)$: to low order in v we have from Eq.(2.30)

$$\beta(v) = v^2\alpha_1'(s) + v^3\left(\alpha_2'(s) - 2\alpha_1(s)\alpha_1'(s)\right) + \cdots \quad , \tag{2.32}$$

---

[12] A remark is in order here. What, in these notes, is called the *scale* is usually understood to be the *logarithm* of the actual energy scale : indeed, whereas the energy scale has the dimension of energy (obviously), the number s is, strictly speaking, dimensionless. If we denote the scale by the conventional symbol μ, the derivative dw/ds should then be rewritten as

$$\frac{d}{ds}w \quad \to \quad \mu\frac{d}{d\mu}w$$

.

so that we can derive

$$\alpha_1(s) = \beta_0 s \quad , \quad \alpha_2(s) = (\beta_0 s)^2 + \beta_1 s \quad , \quad \ldots \tag{2.33}$$

It is easily derived that the leading term in $\alpha_n(s)$ is $(\beta_0 s)^n$.

Let us assume that the beta function is dominated by its lowest-order term, that is, $\beta(v) = \beta_0 v^2$. In that case, $h(v) = -1/(\beta_0 v)$, and we find

$$\frac{1}{w(s)} = \frac{1}{v} - \beta_0 s \quad . \tag{2.34}$$

We can exchange the bare parameter v for the measured value of w at some fixed scale s_0, and then the running is given by

$$\frac{1}{w(s)} = \frac{1}{w(s_0)} - \beta_0(s - s_0) \quad , \tag{2.35}$$

or

$$w(s) = \frac{w(s_0)}{1 - \beta_0 w(s_0)(s - s_0)} \quad . \tag{2.36}$$

At this point we may start to distinguish between different theories. The renormalized, *physical* parameter w is *a priori* unknown, and has to be determined by experiment ; but the number β_0 is perfectly computable from inside the theory[13]. The running of the coupling is therefore determined as soon as the action has been sufficiently specified. Now, it may happen that β_0 is *positive* : in that case, the effective coupling $w(s)$ *increases* with increasing s, and will eventually become infinite at some high scale. On the other hand, when β_0 is *negative*, the effective coupling *decreases* with increasing energy scale. This is called *asymptotic freedom*. It is the phenomenon that has saved the theory of strong interactions : in the 1960's when the typical energy scales of experiments where low, the effective coupling was so high (of order 10) as to cast doubts on the usefulness of perturbation theory, whereas at the high energies current from around 1975[14] the effective coupling has become small enough (of the order of 0.1) to warrant the use of perturbation techniques.

---

[13]The number β_0 is a combinatorial factor with the addition of some powers of π, and simple numbers depending on the ingredients and quantum numbers of the particles pertaining to the theory.

[14]I take the commissioning of the PETRA (Hamburg, BRD) and PEP (Stanford, USA) colliders as the definitive starting point of the relevance of perturbative QCD.

2.3.3 Scheme dependence

We must recognize that not only the scale of a given measurement process is important, but of course also the *nature* of the measurement process. That is, we may *define* the measured coupling constant w in two different ways, on the basis of two different measurement processes[15] : let us denote the two results by w and $\tilde{w}$. We say that such different values have been obtained using different *renormalization schemes*. In all cases I have encountered, two such schemes agree at the tree level[16], and the results are therefore perturbatively related :

$$\tilde{w} = w + t_1 w^2 + t_2 w^3 + t_3 w^4 + \cdots \quad , \tag{2.37}$$

with $t_{1,2,3,\ldots}$ computable numbers ; and conversely

$$w = \tilde{w} - t_1\tilde{w}^2 + \left(2{t_1}^2 - t_2\right)\tilde{w}^3 - \left(5{t_1}^3 - 5t_1t_2 + t_3\right)\tilde{w}^4 + \cdots \tag{2.38}$$

Having computed the beta function for w, we can now simply obtain it for $\tilde{w}$:

$$\begin{aligned}
\beta(\tilde{w}) &= \frac{d\tilde{w}}{ds} = \frac{d\tilde{w}}{dw}\,\frac{dw}{ds} \\
&= \left(1 + 2t_1w + 3t_2w^2 + 4t_3w^3 + \cdots\right)\left(\beta_0w^2 + \beta_1w^3 + \beta_2w^4 + \cdots\right) \\
&= \beta_0w^2 + \left(\beta_1 - 2t_1\beta_0\right)w^3 + \left(\beta_2 - 2t_1\beta_1 + 6{t_1}^2\beta_0 - 3t_2\beta_0\right)w^4 + \cdots \\
&= \beta_0\tilde{w}^2 + \beta_1\tilde{w}^2 + \left(\beta_0{t_1}^2 - \beta_0t_2 + t_1\beta_1 + \beta_2\right)\tilde{w}^2 + \cdots
\end{aligned} \tag{2.39}$$

The two beta functions can be transformed from one scheme to another ; for any scheme dependence for which Eq.(2.37) holds, the first two coefficients, β_0 and β_1, are seen to be independent of the actual scheme, as was to be expected : the two schemes correspond to different functions $s_0(v)$ as defined in section 2.3.

[15]In practice, this difference can be quite small, as between the so-called MS and $\overline{\mathrm{MS}}$ schemes. With 'different measurement processes', we here mean two different, complete operational schemes that both lead to a well-defined value for coupling constants.

[16]This rules out possible but, for a practicing physicist useless and/or irrelevant, differences such as for instance obtained by defining $\tilde{w} = 2w$. Get a life !

Chapter 3

More fields in zero dimensions

3.1 Enlarging the one-field picture

We now have established an overview of the quantum-field theoretic behaviour of a single field φ in zero dimensions. The generalization to a theory with more than one field is fairly straightforward, and we shall make it in this chapter.

3.2 The action and the path integral

We shall assume that there are K distinct fields, labelled φ_j, $j = 1, 2, \ldots, K$. The number K can be taken as large as we please, and even infinite provided that the fields form a countable set[1]. These fields have a combined probability density given by

$$P(\varphi_1, \varphi_2, \ldots, \varphi_K) = N\ \exp\left(-\frac{1}{\hbar} S(\varphi_1, \varphi_2, \ldots, \varphi_K) \right) \quad , \tag{3.1}$$

where we have immediately introduced $\hbar$ since we are now familiar with it. In the special case where the action is separable, that is,

$$S(\varphi_1, \varphi_2, \ldots, \varphi_K) = S_1(\varphi_1) + S_2(\varphi_2) + \cdots + S_K(\varphi_K) \quad ,$$

the fields are actually *independent* random variables ; the theory is just so many copies of the single-field one, and in the following we shall disregard

[1] This notion will have to be relaxed later on.

such an uninteresting situation. A remark is in order here : in concert with our convention of having a coupling constant λ_n accompanied by a factor $1/n!$, we shall let the coupling of several fields be accompanied by combinatorial factors for each field separately, so the action may contain a term

$$\frac{\lambda_{1,3,7}}{(2!)(2!)(4!)}{\varphi_1}^2{\varphi_3}^2{\varphi_7}^4 \quad ,$$

which gives the Feynman rule $-\lambda_{1,3,7}/\hbar$ for this vertex.

The Green's functions are defined by

$$G_{n_1,n_2,\ldots,n_K} = \langle {\varphi_1}^{n_1}\,{\varphi_2}^{n_2}\cdots{\varphi_K}^{n_K}\rangle \tag{3.2}$$

In order to be able to keep the various fields apart, we have to assign to each of them its own source J_j, $j = 1, 2, \ldots, K$. The path integral therefore reads[2]

$$\begin{aligned} Z(J_1,\ldots,J_K) &= \sum_{n_1,\ldots,K\geq 0} \frac{J_1^{n_1}\cdots J_K^{n_K}}{n_1!\cdots n_K!}G_{n_1,\ldots,n_K} \\ &= N\int \exp\left(-\frac{1}{\hbar}\left(S(\varphi_1,\ldots,\varphi_K) - \sum_{j=1}^{K} J_j\varphi_j\right)\right)\, d\varphi_1\cdots d\varphi_K \quad . \end{aligned} \tag{3.3}$$

The extraction of the Green's functions is then performed as

$$G_{n_1,\ldots,n_K} = \hbar^{n_1+\cdots n_K}\left\lfloor \frac{\partial^{n_1}}{(\partial J_1)^{n_1}}\cdots\frac{\partial^{n_K}}{(\partial J_K)^{n_K}}Z(J_1,\ldots,J_K)\right\rfloor_{J_1=\cdots=J_K=0} \quad . \tag{3.4}$$

3.3 Connected Green's functions and field functions

The relation between the Green's functions and their connected counterparts is again given by straightforward generalization:

$$\begin{aligned} W(J_1,\ldots,J_K) &= \log\Big(Z(J_1,\ldots,J_K)\Big) \\ &= \sum_{n_1,\ldots,K\geq 0}\frac{J_1^{n_1}\cdots J_K^{n_K}}{n_1!\cdots n_K!}C_{n_1,\ldots,n_K} \end{aligned} \tag{3.5}$$

[2]Where possible, we denote multiple integral by a single integration sign. This usually does not lead to confusion.

The precise expression of the G's in terms of the C's is of course now somewhat more involved : for instance, for $K = 3$ we have

$$\begin{aligned}
G_{1,0,0} &= C_{1,0,0} \ , \\
G_{1,1,0} &= C_{1,0,0}C_{0,1,0} + C_{1,1,0} \ , \\
G_{1,1,1} &= C_{1,0,0}C_{0,1,0}C_{0,0,1} + C_{1,1,0}C_{0,0,1} \\
&\quad + C_{1,0,1}C_{0,1,0} + C_{0,1,1}C_{1,0,0} + C_{1,1,1} \ .
\end{aligned} \tag{3.6}$$

We now have K field functions, one for each field ; they are given by

$$\phi_j(J_1, \ldots, J_K) = \hbar \frac{\partial}{\partial J_j} W(J_1, \ldots, J_K) \ . \tag{3.7}$$

An important thing to note is that, since the field functions are derivatives,

$$\frac{\partial}{\partial J_k}\phi_j(J_1, \ldots, J_K) = \frac{\partial}{\partial J_j}\phi_k(J_1, \ldots, J_K) \ . \tag{3.8}$$

3.4 The Schwinger-Dyson equation

The SDe for the path integral can be summarized as follows :

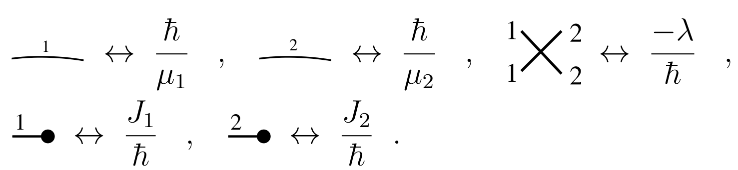
$$\left\lfloor \frac{\partial}{\partial \varphi_k} S(\varphi_1, \ldots, \varphi_K) \right\rfloor_{\varphi_j = \hbar \frac{\partial}{\partial J_j}} Z(J_1, \ldots, J_K) = J_k Z(J_1, \ldots, J_K) \ , \tag{3.9}$$

as can easily be verified.

For the field functions, the SDe is best illustrated with an example. Suppose that we have the following action for $K = 2$:

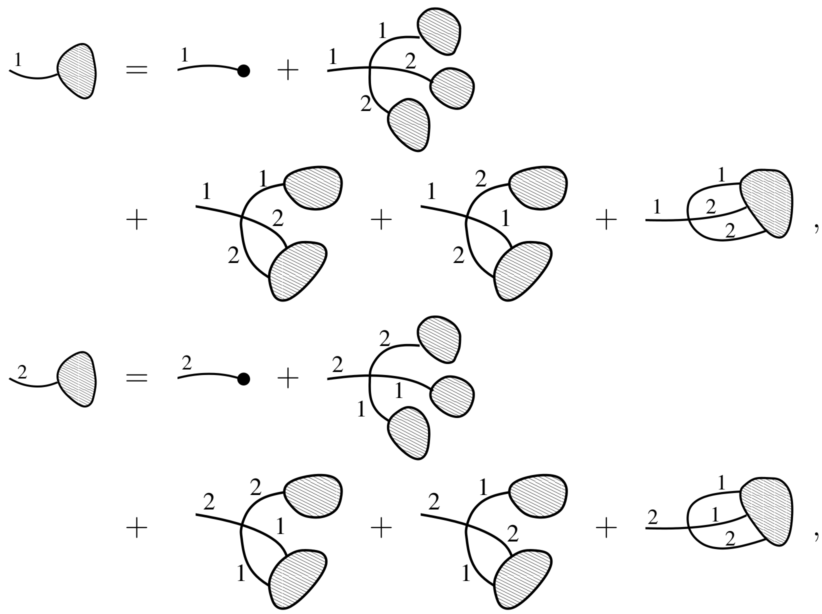
$$S(\varphi_1, \varphi_2) = \frac{1}{2}\mu_1 {\varphi_1}^2 + \frac{1}{2}\mu_2 {\varphi_2}^2 + \frac{\lambda}{4}{\varphi_1}^2{\varphi_2}^2 \ . \tag{3.10}$$

This time, the coupling constant λ carries a factor $1/(2!)/(2!)$ since there are not four identical fields 'meeting' at the vertex, but rather two pairs of identical fields, as mentioned above. We indicate the field type with either '1' or '2'. The Feynman rules for this case are



(3.11)

There are two coupled Schwinger-Dyson equations, one for each field :



(3.12)

E 12 with the following analytical representation for the field functions $\phi_j = \phi_j(J_1, J_2)$ $(j = 1, 2)$:

$$
\begin{aligned}
\phi_1 &= \frac{J_1}{\mu_1} - \frac{\lambda}{2\mu_1}\left(\phi_1\phi_2^2 + \hbar\phi_1\frac{\partial}{\partial J_2}\phi_2 + 2\hbar\phi_2\frac{\partial}{\partial J_2}\phi_1 + \hbar^2\frac{\partial^2}{(\partial J_2)^2}\phi_1\right) \quad , \\
\phi_2 &= \frac{J_2}{\mu_2} - \frac{\lambda}{2\mu_2}\left(\phi_2\phi_1^2 + \hbar\phi_2\frac{\partial}{\partial J_1}\phi_1 + 2\hbar\phi_1\frac{\partial}{\partial J_1}\phi_2 + \hbar^2\frac{\partial^2}{(\partial J_1)^2}\phi_2\right) \quad .
\end{aligned}
\tag{3.13}
$$

The effective action must of course be a two-variable function $\Gamma(\phi_1, \phi_2)$ such that

$$
\frac{\partial}{\partial\phi_j}\Gamma(\phi_1, \phi_2) = J_j \quad , \quad j = 1, 2 \quad . \tag{3.14}
$$

This effective action is also concave. The two-field case can, obviously, be extended to the case of arbitrarily many fields, provided the couplings are
E 13 unambiguously defined.

3.5 A zero-dimensional template for QED

We consider the following action for three fields, including sources :

$$S(\varphi, \bar{\varphi}, B) = \frac{1}{2}\mu B^2 + m\varphi\bar{\varphi} + e\bar{\varphi}B\varphi - \bar{J}\varphi - \bar{\varphi}J - HB \quad . \tag{3.15}$$

Note the absence of symmetry factors since all the fields in the three-point vertex are distinct. Also the two-point interaction term $m\bar{\varphi}\varphi$ carries no factor of 1/2. Such an action can stand for an extremely primitive model for QED, the theory of electrons and photons. The action has three partial derivatives :

$$\begin{aligned} \frac{\partial}{\partial\varphi}S(\varphi, \bar{\varphi}, B) &= m\bar{\varphi} + e\bar{\varphi}B - \bar{J} \quad , \\ \frac{\partial}{\partial\bar{\varphi}}S(\varphi, \bar{\varphi}, B) &= m\varphi + eB\varphi - J \quad , \\ \frac{\partial}{\partial B}S(\varphi, \bar{\varphi}, B) &= \mu B + e\bar{\varphi}\varphi - H \quad . \end{aligned} \tag{3.16}$$

The SDe's for the path integral are therefore

$$\begin{aligned} \left(\hbar m\frac{\partial}{\partial J} + e\hbar^2\frac{\partial^2}{\partial J\partial H} - \bar{J}\right) Z(\bar{J}, J, H) &= 0 \quad , \\ \left(\hbar m\frac{\partial}{\partial \bar{J}} + e\hbar^2\frac{\partial^2}{\partial \bar{J}\partial H} - J\right) Z(\bar{J}, J, H) &= 0 \quad , \\ \left(\hbar\mu\frac{\partial}{\partial H} + e\frac{\partial^2}{\partial \bar{J}\partial J} - H\right) Z(\bar{J}, J, H) &= 0 \quad . \end{aligned} \tag{3.17}$$

The field-generating functions (the 'field functions') are, of course, each a function of J, $\bar{J}$ and H, and are given by

$$\psi = \hbar\frac{\partial}{\partial\bar{J}}\log Z \quad , \quad \bar{\psi} = \hbar\frac{\partial}{\partial J}\log Z \quad , \quad A = \hbar\frac{\partial}{\partial H}\log Z \quad , \tag{3.18}$$

so that

$$\hbar\frac{\partial}{\partial\bar{J}}Z = \psi\, Z \quad , \quad \hbar\frac{\partial}{\partial J}Z = \bar{\psi}\, Z \quad , \quad \hbar\frac{\partial}{\partial H}Z = A\, Z \quad , \tag{3.19}$$

and Eq.(3.16) can be written as

$$\psi = \frac{1}{m}J - \frac{e}{m}\left(A\,\psi + \hbar\frac{\partial}{\partial H}\psi\right) \quad ,$$

$$
\begin{aligned}
\bar{\psi} &= \frac{1}{m}\bar{J} - \frac{e}{m}\left(\bar{\psi}\,A + \hbar\frac{\partial}{\partial H}\bar{\psi}\right) \quad , \\
A &= \frac{1}{\mu}H - \frac{e}{\mu}\left(\bar{\psi}\,\psi + \hbar\frac{\partial}{\partial J}\psi\right) \quad .
\end{aligned}
\tag{3.20}
$$

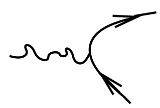
Incidentally, note that we could rewrite these SDe's since

$$
\frac{\partial}{\partial H}\psi = \frac{\partial}{\partial \bar{J}}A \quad , \quad \frac{\partial}{\partial H}\bar{\psi} = \frac{\partial}{\partial J}H \quad , \quad \frac{\partial}{\partial J}\psi = \frac{\partial}{\partial \bar{J}}\bar{\psi} \quad . \tag{3.21}
$$

The Feynman rules are, for this action, as follows :

$\bar{\psi}$  ψ $\leftrightarrow \dfrac{\hbar}{m}$,

 $\leftrightarrow \dfrac{\hbar}{\mu}$,

 $\leftrightarrow -\dfrac{e}{\hbar}$. (3.22)

A few things are of interest here. In the first place, we have here the first instance of an important concept : that of *oriented lines*. The propagator
E 14 is oriented, it runs from φ to $\bar{\varphi}$ (or the other way around). In the second place, in the action we find the two terms $\bar{J}\varphi$ and $\bar{\varphi}J$, which would *suggest* that J is the source in the SDe of $\bar{\psi}$, and $\bar{J}$ is the source in the SDe for ψ ; but it is actually the other way around ! What is the source for a given field function is seen by taking the derivative of the action, and inspecting which field then occurs as a linear term, and which source term is left by itself after
E 15 the differentiation.

3.6 Exercises for Chapter 3

Excercise 12 Two-field action
Consider the two-field action, now with the sources included :

$$
S(\varphi_1, \varphi_2) = \frac{\mu}{2}\left({\varphi_1}^2 + {\varphi_2}^2\right) + \frac{\lambda}{4}{\varphi_1}^2{\varphi_2}^2 - J_1\varphi_1 - J_2\varphi_2
$$

1. Determine the 2 SDe's for the path integral.

2. From these, determine the 2 SDe'd for the field functions ϕ_1 and ϕ_2.

3. Verify that this is agreement with the diagrammatically obtained result.

Excercise 13 A symmetric theory
Consider a theory containing N fields $\varphi_{1,2,\ldots,N}$, with the following action :

$$S(\vec{\varphi}) = \frac{\mu}{2}|\vec{\varphi}|^2 + \frac{\lambda}{24}|\vec{\varphi}|^4 \;\;,$$

and with sources $J_{1,2,\ldots,N}$.

1. Show by using diagrams that the SDe can be written as

$$\mu\phi_k = J_k - \frac{\lambda}{6}\left(\phi_k|\vec{\varphi}|^2 + \hbar\phi_k(\vec{\nabla}\cdot\vec{\phi}) + \hbar\frac{\partial}{\partial J_k}|\vec{\phi}|^2 + \hbar^2\vec{\nabla}^2\phi_k\right)$$

   Notice that to obtain this form one may have to use the fact that $\partial\phi_k/\partial J_m = \partial\phi_m/\partial J_k$.

2. Show that the same result can be obtained algebraically, by working out de SDe in the following form :

$$S_k\left(\vec{\phi} + \hbar\vec{\nabla}\right) e = J_k$$

   where S_k stands for the partial derivative of the action $S(\vec{\varphi})$ to φ_k, and ∇_k stands for $\partial/\partial J_k$.

3. By symmetry, W must depend on $|\vec{J}|$ alone. We can therefore write

$$\phi_k = J_k\ F\left(|\vec{J}|^2/2\right)$$

   for some function F. Show that this function obeys

$$\begin{aligned}\mu F(x) \;\;&=\;\; 1 - \frac{\lambda}{6}\Big(2xF(x)^3 + \hbar\left((N+2)F(x)^2 + 6xF(x)F'(x)\right)\\ &\qquad + \hbar^2\left((N+2)F'(x) + 2xF''(x)\right)\Big)\end{aligned}$$

Excercise 14 Symmetry factors in QED
For the model of section 3.5, write the three SDe's in diagrammatic form. Show that all symmetry factors are equal to unity for diagrams with at least one external leg. Show that this implies the same for actual Quantum Electrodynamics, that is, the same model but extended to Minkowski space and with more complicated propagators and vertices.

Excercise 15 The 123 theory
We consider a theory containing not 1 but 3 fields, labeld $\varphi_{1,2,3}$. The action, including the sourves, is given by

$$S(\varphi_1, \varphi_2, \varphi_3) = \frac{\mu}{2}\left({\varphi_1}^2 + {\varphi_2}^2 + {\varphi_3}^2\right) + g\varphi_1\varphi_2\varphi_3 - J_1\varphi_1 - J_2\varphi_2 - J_3\varphi_3$$

In the following we shall concentrate on Feynman diagrams and will not worry about the convergence of the path integral.

1. Prove that there are now 3 different SDe's, of the form

$$\mu\hbar\frac{\partial}{\partial J_i}Z + g\hbar^2\frac{\partial}{\partial J_j}\frac{\partial}{\partial J_k}Z = J_i Z$$

   where i, j, k is a permutation of 1,2,3.

2. There are now of course also 3 field functions $\phi_{1,2,3}$. Prove that

$$\frac{\partial}{\partial J_i}\phi_j = \frac{\partial}{\partial J_j}\phi_i \quad , \quad i, j \; \in \; (1, 2, 3)$$

3. Give the SDe for the field functions using diagrams.

4. Prove that for any diagram with n_j external lines of type j ($j = 1, 2, 3$) the following must hold: the n_j are either all even, or all odd.

5. By C_{ij} we denote the connected Green's function with two external legs, one of type i and one of type j. Prove that

$$C_{ij} = 0 \;\; , \;\; i \neq j \;\; ,$$

   and furthermore that there are no tadpole diagrams.

Chapter 4

QFT in Euclidean spaces

4.1 Introduction

The main characteristic of a space(-time) of more than zero dimensions is the fact that the quantum field is defined at more than one point ; in fact, at an infinity of points. The possibility of sending signals from one point to another one requires the existence of *correlations* between the field values at different points. The nature of this correlation, and its reflection in the appropriate Feynman rules, is our subject now.

4.2 One-dimensional discrete theory

4.2.1 An infinite number of fields

We shall consider a theory of a countably infinite set of fields in zero dimensions. We denote by $\{\varphi\}$ the set of all these fields :

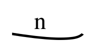
$$\{\varphi\} = \ldots,\ \varphi_{-3}\ ,\ \varphi_{-2}\ ,\ \varphi_{-1}\ ,\ \varphi_0\ ,\ \varphi_1\ ,\ \varphi_2\ ,\ \varphi_3\ ,\ldots$$

where the field labels run from $-\infty$ to $+\infty$. Similarly, there is the collection of all the corresponding sources, denoted by $\{J\}$. We shall, as a working example, consider a theory where the interaction consists of four fields with the same label meeting at one point. Moreover, we shall assume the kinetic terms to be uniform in the field labels. Thus, the action will be[1] :

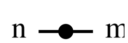
$$S(\{\varphi\},\{J\}) = \sum_n \left[\frac{1}{2}\mu{\varphi_n}^2 - \gamma\varphi_n\varphi_{n+1} + \frac{\lambda_4}{4!}{\varphi_n}^4 - J_n\varphi_n\right] \quad , \tag{4.1}$$

where we include the sources in the action[2]. If γ were zero, the action would be separable and the theory would be a rather uninteresting series of replicas of the zero-dimensional action for a single field. We shall consider positive values of γ ; in that case, the action tends to minimize if φ_n and φ_{n+1} carry the same sign : a *positive* correlation between neighbouring fields is the result. Note, moreover, that the action has be chosen such as to be invariant under the relabelling of n by $n + K$ with any fixed K : this is called *translation invariance*, in this case translation by a fixed increment in labelling[3]. The model is also invariant under the relabelling of n by $-n$: this is called *parity invariance*.

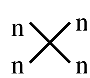
The Feynman rules are easily derived from the action of Eq.(4.1) :



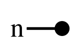
$$\leftrightarrow \quad \frac{\hbar}{\mu}$$



$$\leftrightarrow \quad +\frac{\gamma}{\hbar}\left(\delta_{m,n+1} + \delta_{m,n-1}\right)$$



$$\leftrightarrow \quad -\frac{\lambda_4}{\hbar}$$



$$\leftrightarrow \quad +\frac{J_n}{\hbar}$$

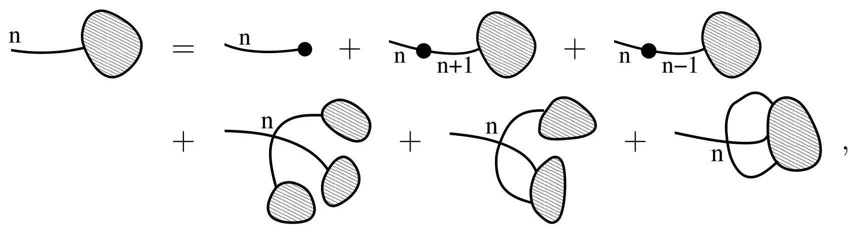
Feynman rules, version 4.1 (4.2)

[1]If not indicated explicitly otherwise, sums will run from $-\infty$ to $+\infty$.

[2]Take care to note that both μ and γ are independent of n simply *because we choose them so.*

[3]This will lead to momentum conservation later on. Note however that, as indicated above, momentum conservation is a consequence of our *choice*, or in practice of our *belief* in the translation invariance of our physical laws. Other models are possible and not *a priori* wrong : they are simply much more complicated.

The identity of the field is indicated by its label. Alternatively, the four-vertex and the source vertex may be labelled. The SDe now takes the following form, for any n :



(4.3)

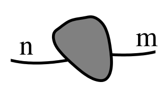
or, in terms of the field functions $\phi_n(\{J\})$, that depend on all sources :

$$\begin{aligned} \phi_n &= \frac{J_n}{\mu} + \frac{\gamma}{\mu}\left(\phi_{n-1} + \phi_{n+1}\right) \\ &\quad - \frac{\lambda_4}{6\mu}\left(\phi_n^3 + 3\hbar\phi_n \frac{\partial}{\partial J_n}\phi_n + \hbar^2 \frac{\partial^2}{\left(\partial J_n\right)^2}\phi_n\right) \quad . \end{aligned} \tag{4.4}$$

4.2.2 Introducing the propagator

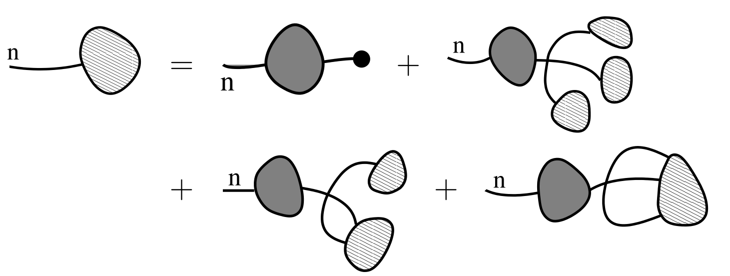
The Schwinger-Dyson equation (4.3) can be cast in another form, that will turn out to be more useful. Consider the fact that, upon entering the field function via its external leg, one must encounter either zero or more two-point functions before encountering a source vertex or a four-vertex. Let us denote by

$$\Pi_{m,n} \equiv$$



(4.5)

the total set of diagrams that contain *only* two-point vertices (or no vertices), and have fields n and m at its external legs[4]. The SDe can then be rewritten as follows :



(4.6)

[4]To go from φ_n to φ_m one needs, of course, at least $|n - m|$ vertices, but more vertices are also possible.

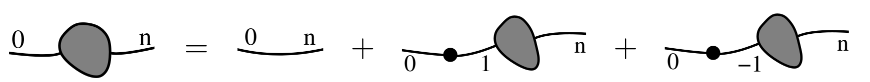
where a summation over the label of the field exiting the Π is implied. Therefore, we have

$$\begin{aligned}\phi_n &= \sum_m \Pi_{n,m} \times \\ &\left[\frac{J_m}{\mu} - \frac{\lambda}{6\mu}\left({\phi_m}^3 + 3\hbar\phi_m \frac{\partial}{\partial J_m}\phi_m + \hbar^2 \frac{\partial^2}{(\partial J_m)^2}\phi_m\right)\right] \quad . \end{aligned} \tag{4.7}$$

The object $\Pi_{m,n}$, which describes to what extent the field φ_n influences φ_m, will be called the *propagator* from now on.

4.2.3 Computing the propagator

From the translation and parity invariance of the model we have discussed, we can infer that $\Pi_{m,n}$ can actually only depend on $|m-n|$, so that we can restrict ourselves to $\Pi_{0,n}$; we denote this by $\Pi(n)$. For $\Pi(n)$, we have a very simple Schwinger-Dyson equation :



(4.8)

or

$$\Pi(n) = \frac{\hbar}{\mu}\delta_{0,n} + \frac{\gamma}{\mu}\Big(\Pi(n+1) + \Pi(n-1)\Big) \quad . \tag{4.9}$$

The easiest way to solve this set of equations is by Fourier transform. We define

$$R(z) = \sum_n \Pi(n)\; e^{-inz} \quad , \tag{4.10}$$

from which[5] the propagator may be recovered using

$$\Pi(n) = \frac{1}{2\pi}\int\limits_{-\pi}^{+\pi} e^{+inz}\; R(z)\; dz \quad . \tag{4.11}$$

[5]We choose e^{-inz} rather than e^{+inz} in Eq.(4.10) by convention. Although this may not be glaringly obvious at this point, our convention is ultimately related to the fact that, in nonrelativistic quantum mechanics, the Schrödinger equation has been chosen to read $i\hbar\partial|\psi\rangle/\partial t = \hat{H}|\psi\rangle$ rather than $-i\hbar\partial|\psi\rangle/\partial t = \hat{H}|\psi\rangle$.

Multiplying both sides of Eq.(4.9) by $\exp(-inz)$ and summing over n leads to

$$\begin{aligned} R(z) &= \frac{\hbar}{\mu} + \frac{\gamma}{\mu} R(z) \left(e^{iz} + e^{-iz} \right) \\ &= \frac{\hbar}{\mu - \gamma \left(e^{iz} + e^{-iz} \right)} = \frac{\hbar\, u}{\mu u - \gamma(u^2+1)} \end{aligned} \tag{4.12}$$

where we have introduced $u = e^{iz}$. This allows us to write the integral (4.11) as

$$\Pi_n = -\frac{\hbar}{2i\pi\gamma} \oint\limits_{|u|=1} du\, \frac{u^n}{(u-u_+)(u-u_-)} \quad , \tag{4.13}$$

where $u_\pm$ are the two roots of the quadratic form $\mu u - \gamma(u^2+1)$:

$$u_\pm = \frac{1}{2} \left(\frac{\mu}{\gamma} \pm \left(\frac{\mu^2}{\gamma^2} - 4 \right)^{1/2} \right) \quad . \tag{4.14}$$

Provided that μ exceeds 2γ, the two poles of the integrand are real, and $0 < u_- < 1 < u_+$. We can then contract the contour around the point $u = u_-$, upon which we find

$$\Pi(n) = \hbar \frac{{u_-}^n}{\gamma(u_+ - u_-)} \quad , \quad n \geq 0 \ . \tag{4.15}$$

The general solution for the propagator is therefore[6]

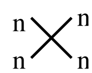
$$\Pi(n) = \frac{\hbar}{\sqrt{\mu^2 - 4\gamma^2}} {u_-}^{|n|} \quad . \tag{4.16}$$

Unsurprisingly, the propagator falls off exponentially with $|n|$. Some points are to be noted. In the first place, if γ were negative, then u_- would also be negative, and the propagator would oscillate between positive and negative correlations. In the second place, if μ were 2γ or smaller, the poles of the integrand would lie *on* the unit circle $|u| = 1$, making the integral ill-defined. **E 16**

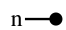
Having at hand the explicit form of the propagator, we can now switch to a new set of Feynman rules :

[6] This derivation is valid for $n \geq 0$. For negative n, Cauchy's theorem on which it is based does not hold immediately : but in that case we can perform the variable transform from u to $1/u$ and obtain the result.

$\leftrightarrow \; \Pi(m-n)$



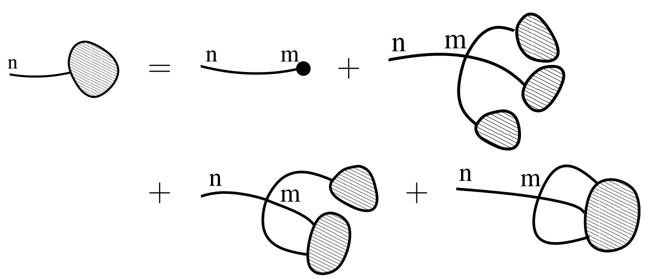
$\leftrightarrow \; -\dfrac{\lambda_4}{\hbar}$



$\leftrightarrow \; +\dfrac{J_n}{\hbar}$

$\boxed{\text{Feynman rules, version 4.2}}$ (4.17)

The difference with the previous set of rules is that now the line denotes a propagator running between n and m. The SDe is now very similar to that of the zero-dimensional φ^4 theory :

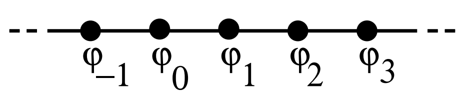


(4.18)

with the summation over m implied.

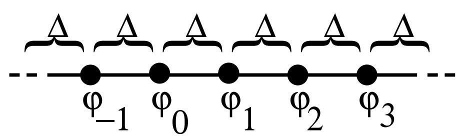
4.2.4 A figment of the imagination, and a sermon

The concept of an infinite number of fields all huddling together at a single point simply cries out for a better visualization. The most useful picture is that of each field occupying its own point. Indicating by a line those fields that have a direct coupling, we arrive at a picture like the following :



We now introduce a new notion, that of *distance*. In our sensorial experience, distances are, in their essence, measured by the sending and receiving of signals, and the weaker the signal from one point to another, the further those points are deemed to be apart ; in the language of these notes, the smaller

$\Pi(m-n)$, the larger the 'distance' between n and m. We can therefore dress up our picture by introducing a *fundamental distance* Δ, subsequent field locations being separated by this distance :



The distances between the points are all equal since the couplings γ are all equal. We have, as it were, *constructed* a one-dimensional universe. It may come as a surprise that the concept of space is here presented as a visualization device. If we reflect, however, on how someone who (like a new-born infant) has no *a priori* concept of spacelike separations would have to envisage the workings of the physical world, we shall conclude that that person had better invent space in order not to go insane pretty quickly. In its essence, space, like so much else in the world around us, is simply a mental construction that allows us to come to grips with, and control, our environment[7].

After all this has been said, we must acknowledge the emprical fact that to our knowledge space seems not to be made up from single points[8]. Therefore we have to assume that Δ must be much smaller than the smallest distances that can, at present, be resolved[9]. We therefore introduce the *continuum limit* : we assume that the theories we consider are such that the limit $\Delta \to 0$ can be taken in a sensible manner, yielding sensible results. This sidesteps the interesting question of whether Δ is really zero or not. Indeed, *we do not know*. Any theoretical result that depends sensitively on whether $\Delta = 0$ or $\Delta \neq 0$ would be extremely important since experimental information about it would allow us a look at the fundamental structure of space ; but for us it is safer to construct theories the predictions of which do not hinge on this unknown. As we shall see, this can be made to work. As an added bonus, we can feel free from misgivings about the mathematical rigour of taking the continuum limit : after all, we may not be at the limit after all.

[7]See also Peter L. Berger and Thomas Luckmann, *The Social Construction of Reality : A Treatise in the Sociology of Knowledge* (Garden City, New York: Anchor Books, 1966).

[8]Nor does it appear to be one-dimensional – but that is easily repaired, as we shall see.

[9]About 10^{-18} meter.

4.3 One-dimensional continuum theory

4.3.1 The continuum limit for the propagator

Having identified the positions occupied by the various fields with points in space (or time), we define the *distance* between points m and n by

$$x = (n - m)\Delta \ . \tag{4.19}$$

The *dimension* of x is that of Δ, that is, a length L. The continuum limit is, then, that where $\Delta \to 0$ and $|n - m| \to \infty$ while x remains fixed. The propagator is now a function of x, so we redefine it as

$$\Pi(x) \leftarrow \Pi(x/\Delta) \ .$$

This means that

$$\Pi(x) = \frac{\hbar}{2\pi} \int\limits_{-\pi}^{+\pi} dz \, \frac{\exp(ixz/\Delta)}{\mu - 2\gamma\cos(z)} \ , \tag{4.20}$$

where we have to assume $\mu > 2\gamma$ if we want to avoid singularities in the integrand. A corresponding change in the integration variable z is now in order : we write

$$z = k\Delta \ , \tag{4.21}$$

The dimension of k is therefore L^{-1}. The propagator becomes

$$\begin{aligned} \Pi(x) &= \frac{\hbar\Delta}{2\pi} \int\limits_{-\pi/\Delta}^{+\pi/\Delta} dk \, \frac{\exp(ixk)}{\mu - 2\gamma\cos(k\Delta)} \\ &\approx \frac{\hbar\Delta}{2\pi} \int dk \, \frac{\exp(ixk)}{(\mu - 2\gamma) + \gamma\Delta^2 k^2} \ . \end{aligned} \tag{4.22}$$

In the last line, we have taken Δ to be very small indeed. Note that the approximation $\cos(z) \approx 1 - k^2\Delta^2/2$ is, of course only justified as long as k is finite ; but for very large k the integrand is extremely oscillatory and contributes essentially nothing[10]. Now, in order to avoid a propagator that

[10] This handwaving argument is justified by the fact that we get the right propagator in the continuum limit.

either blows up or vanishes, we must define the Δ-dependence of μ and γ such that $\gamma \sim 1/\Delta$ and $\mu - 2\gamma \sim \Delta$. We shall take

$$\gamma \to \frac{1}{\Delta}\left(1 - \frac{m^2\Delta^2}{4}\right) \quad , \quad \mu \to \frac{2}{\Delta}\left(1 + \frac{m^2\Delta^2}{4}\right) \quad , \tag{4.23}$$

with m^2 a positive number (remember that we need $\mu > 2\gamma$). We shall also take m itself to be positive. We then find the exact results

$$\mu - 2\gamma = m^2\Delta \ \ , \ \ \sqrt{\mu^2 - 4\gamma^2} = 2m \ \ , \ \ u_- = \frac{1 - m\Delta/2}{1 + m\Delta/2} \ \ . \tag{4.24}$$

The propagator takes the form[11]

$$\Pi(x) = \frac{\hbar}{2\pi} \int dk \, \frac{e^{ixk}}{k^2 + m^2} = \frac{\hbar}{2m} \exp(-m|x|) \ \ . \tag{4.25}$$

To check that this result is indeed the correct one, we can consider the continuum limit directly for the propagator result (4.16) :

$$\Pi(n) \to \frac{\hbar}{2m}\left(\frac{1 - m\Delta/2}{1 + m\Delta/2}\right)^{|x/\Delta|} \to \frac{\hbar}{2m} \exp(-m|x|) \ \ , \tag{4.26}$$

as desired.

4.3.2 The continuum limit for the action

In the action (4.1), we shall want to replace the sum over n by an integral over x :

$$\sum_n \Delta \ \to \ \int dx \ \ .$$

It is therefore necessary that every term in the action acquires a factor Δ. Now, the action depends on the quantum fields φ_n. As we let the distance between the points shrink to zero, the collection of values $\{\varphi\}$ turns into a

[11]To obtain the last lemma of this expression, we can use the fact that the integrand has simple poles at $k = im$ and $k = -im$. For $x > 0$, the integral contour in the complex k-plane can be closed over the positive imaginary parts, and for $x < 0$ over the negative imaginary parts : the result then follows immediately by Cauchy integration.

function $\varphi(x)$. The precise correspondence between $\{\varphi\}$ and $\varphi(x)$ is something that, in the end, we have to decide for ourselves. Out of the several possibilities we shall adopt the following :

$$\varphi(x) = \frac{1}{2}\Big(\varphi_{n+1} + \varphi_n\Big) \quad , \quad \varphi'(x) = \frac{1}{\Delta}\Big(\varphi_{n+1} - \varphi_n\Big) \quad . \tag{4.27}$$

This assignment is called the *Weyl ordering*. Its converse reads, of course,

$$\varphi_n = \varphi(x) - \frac{\Delta}{2}\varphi'(x) \quad , \quad \varphi_{n+1} = \varphi(x) + \frac{\Delta}{2}\varphi'(x) \quad . \tag{4.28}$$

In a sense, the field value $\varphi(x)$ is sitting 'in between' the points φ_n and φ_{n+1}. Other assignments can be proposed, for instance $\varphi_n = \varphi(x)$. However, these are less attractive[12]. Upon careful application of Weyl ordering and the assumed continuum limits for μ and γ, the kinetic part of the action (4.1) has the following continuum limit :

$$\begin{aligned}
&\sum_n \left[\frac{\mu}{2}{\varphi_n}^2 - \gamma\varphi_n\varphi_{n+1}\right] = \\
&\quad = \sum_n \left[\frac{1}{2}(\mu - 2\gamma)\varphi(x)^2 + \frac{\Delta^2}{8}(\mu + 2\gamma)\varphi'(x)^2\right] \\
&\quad = \sum_n \left[\frac{1}{2}m^2\varphi(x)^2 + \frac{1}{2}\varphi'(x)^2\right]\;\Delta \\
&\quad = \int \left[\frac{1}{2}m^2\varphi(x)^2 + \frac{1}{2}\varphi'(x)^2\right]\;dx \quad .
\end{aligned} \tag{4.29}$$

The interaction and source terms in the path integral do not have a factor Δ coming out naturally, but we may simply define the continuum limits by *redefining* the objects in the action :

$$\lambda_4 \;\to\; \Delta\lambda_4 \quad , \quad J_n \;\to\; \Delta J(x) \quad , \tag{4.30}$$

[12] For example, consider a function $\varphi(x)$ that vanishes for $x \to \pm\infty$. The integral $\int 2\varphi(x)\varphi'(x)\;dx$ then vanishes upon partial integration. Weyl ordering tells us that $2\varphi(x)\varphi'(x) = ({\varphi_{n+1}}^2 - {\varphi_n}^2)/\Delta$, leading to the correspondence

$$\int 2\varphi(x)\varphi'(x)\;dx \quad \leftrightarrow \quad \sum_n \Delta\;({\varphi_{n+1}}^2 - {\varphi_n}^2) \quad ,$$

where the sum also vanishes explicitly after relabelling. For the alternative assignment $\varphi_n = \varphi(x)$ the vanishing cannot be proven.

so that the continuum limit of the full action, including this time also the sources, becomes[13]

$$S[\varphi, J] = \int \left[\frac{1}{2}m^2\varphi(x)^2 + \frac{1}{2}\varphi'(x)^2 + \frac{\lambda_4}{4!}\varphi(x)^4 - J(x)\varphi(x)\right] dx \quad . \tag{4.31}$$

Note the notation with square brackets: the action is now no longer a number depending on (a countably infinite set of) numbers, but rather on the *functions* $\varphi(x)$ and $J(x)$; this is called a *functional*.

4.3.3 The continuum limit of the classical equation

For the discrete action, there is an obvious classical equation :

$$\frac{\partial}{\partial\varphi_n} S(\{\varphi\}) = 0 \quad \forall n \ \ , \tag{4.32}$$

where, again, the source terms have been subsumed into the action. For the φ^4 model of Eq.(4.1), the classical equation is therefore

$$\mu\varphi_n - \gamma(\varphi_{n+1} + \varphi_{n-1}) + \frac{\Delta\lambda_4}{3!}\varphi_n^3 = \Delta J_n \tag{4.33}$$

for all n, and the extra factor Δ in the coupling constant and the sources have been taken into account. The Weyl prescription leads us to write

$$\mu\varphi_n - \gamma(\varphi_{n+1} + \varphi_{n-1}) \approx m^2\Delta\varphi(x) - \Delta\varphi''(x) \ \ , \tag{4.34}$$

so that the continuum limit of the classical field equation takes the form

$$m^2\varphi(x) - \varphi''(x) + \frac{\lambda_4}{3!}\varphi(x)^3 = J(x) \ \ . \tag{4.35}$$

This is precisely the *Euler-Lagrange equation*, that can also be obtained immediately from the continuum form of the action by taking functional derivatives. To see this, let us assume that the action of a theory can be written as

$$S[\varphi] = \int F\left(\varphi(x); \varphi'(x)\right)\, dx \ \ . \tag{4.36}$$

[13] Strictly speaking, the Weyl ordering requires the replacement of J_n not by $\Delta J(x)$ but by $\Delta J(x) + \Delta^2 J'(x)/2$. The additional term, however, vanishes in the continuum limit as $\Delta \to 0$, as do the higher powers of Δ involved in the ${\varphi_n}^4$ term.

Upon 'discretization' using the Weyl ordering, this becomes

$$
\begin{aligned}
S &= \sum_k \Delta\, F\left(\frac{1}{2}(\varphi_{k+1}+\varphi_k);\frac{1}{\Delta}(\varphi_{k+1}-\varphi_k)\right) \\
&= \quad \Delta\, F\left(\frac{1}{2}(\varphi_{n+1}+\varphi_n);\frac{1}{\Delta}(\varphi_{n+1}-\varphi_n)\right) \\
&\quad + \Delta\, F\left(\frac{1}{2}(\varphi_{n}+\varphi_{n-1});\frac{1}{\Delta}(\varphi_{n}-\varphi_{n-1})\right) \\
&\quad + \text{ terms not containing } \varphi_n \quad .
\end{aligned} \tag{4.37}
$$

The classical equation then reads

$$
\begin{aligned}
0 = \frac{1}{\Delta}\,\frac{\partial}{\partial\varphi_n}S &= \quad \frac{1}{2}F_1\left(\frac{1}{2}(\varphi_{n+1}+\varphi_n);\frac{1}{\Delta}(\varphi_{n+1}-\varphi_n)\right) \\
&\quad + \frac{1}{2}F_1\left(\frac{1}{2}(\varphi_{n}+\varphi_{n-1});\frac{1}{\Delta}(\varphi_{n}-\varphi_{n-1})\right) \\
&\quad - \frac{1}{\Delta}F_2\left(\frac{1}{2}(\varphi_{n+1}+\varphi_n);\frac{1}{\Delta}(\varphi_{n+1}-\varphi_n)\right) \\
&\quad + \frac{1}{\Delta}F_2\left(\frac{1}{2}(\varphi_{n}+\varphi_{n-1});\frac{1}{\Delta}(\varphi_{n}-\varphi_{n-1})\right) \quad ,
\end{aligned} \tag{4.38}
$$

where F_j denotes the partial derivative of F with respect to its j-th argument. Re-inserting the Weyl ordering, we can write this equation as

$$
\begin{aligned}
0 &= \quad \frac{1}{2}F_1\Big(\varphi(x);\varphi'(x)\Big) + \frac{1}{2}F_1\Big(\varphi(x-\Delta);\varphi'(x-\Delta)\Big) \\
&\quad - \frac{1}{\Delta}F_2\Big(\varphi(x);\varphi'(x)\Big) + \frac{1}{\Delta}F_2\Big(\varphi(x-\Delta);\varphi'(x-\Delta)\Big) \, .
\end{aligned} \tag{4.39}
$$

By Taylor expansion we get, for arbitrary f :

$$
\begin{aligned}
& f\Big(\varphi(x-\Delta);\varphi'(x-\Delta)\Big) \approx \\
&\quad \approx f\Big(\varphi(x)-\Delta\varphi'(x);\varphi'(x)-\Delta\varphi''(x)\Big) \\
&\quad \approx f\Big(\varphi(x);\varphi'(x)\Big) \\
&\qquad -\Delta\left\{\varphi'(x)f_1\Big(\varphi(x);\varphi'(x)\Big)+\varphi''(x)f_2\Big(\varphi(x);\varphi'(x)\Big)\right\} \\
&\quad = f\Big(\varphi(x);\varphi'(x)\Big) - \Delta\frac{d}{dx}f\Big(\varphi(x);\varphi'(x)\Big) \quad .
\end{aligned} \tag{4.40}
$$

The classical equation thus takes the form

$$F_1\Big(\varphi(x);\varphi'(x)\Big) - \frac{d}{dx}F_2\Big(\varphi(x);\varphi'(x)\Big) = 0 \quad . \tag{4.41}$$

we can cast this in the formal language of *functional derivatives* : we *define*, using the Dirac delta function,

$$\begin{aligned} \frac{\delta\varphi(y)}{\delta\varphi(x)} &= \delta(x-y) \quad , \quad \frac{\delta\varphi'(y)}{\delta\varphi(x)} = 0 \quad , \\ \frac{\delta\varphi'(y)}{\delta\varphi'(x)} &= \delta(x-y) \quad , \quad \frac{\delta\varphi(y)}{\delta\varphi'(x)} = 0 \quad , \end{aligned} \tag{4.42}$$

where, as we see, $\varphi(x)$ and $\varphi'(x)$ are treated as *independent* variables. Applying these rules to the continuum form of the action, we find that the formal form of the classical field equation is therefore that of the Euler-Lagrange equation. The language of functional derivatives is, in these notes, treated as an effective method, valid in the continuum limit, of writing the more fundamental discrete classical field equation. In the functional formalism, the Euler-Lagrange equation reads

$$\frac{\delta}{\delta\varphi(x)}S[\varphi,J] - \frac{d}{dx}\left(\frac{\delta}{\delta\varphi'(x)}S[\varphi,J]\right) = 0 \quad . \tag{4.43}$$

For φ^4 theory, the Euler-Lagrange equation takes precisely the form of Eq.(4.35).

4.3.4 The continuum Feynman rules and SDe

Let us have a look again at the SDe for the discrete model, for simplicity taking the φ^4 model again :

$$\begin{aligned} \phi_n &= \sum_m \Pi(n-m) \\ &\quad \times \left\{ J_m - \frac{\lambda}{6}\left(\phi_m^3 + 3\hbar\phi_m\frac{\partial}{\partial J_m}\phi_m + \hbar^2\frac{\partial^2}{(\partial J_m)^2}\phi_m\right)\right\} \quad . \end{aligned} \tag{4.44}$$

Going over to the continuum limit etails, as we have seen, the following substitutions :

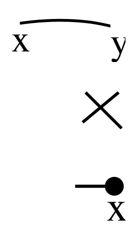
$$\phi_n, \phi_m \quad \to \quad \phi(x), \phi(y) \quad ,$$

$$
\begin{aligned}
\Pi(n-m) &\rightarrow \Pi(x-y) \ , \\
J_m &\rightarrow \Delta J(y) \ , \\
\lambda_4 &\rightarrow \Delta\,\lambda_4 \ , \\
\sum_m &\rightarrow \frac{1}{\Delta}\int dy \ , \\
\frac{\partial}{\partial J_m} &\rightarrow \frac{\delta}{\delta J(y)} \ .
\end{aligned}
\tag{4.45}
$$

With this, the SDe becomes

$$
\begin{aligned}
\phi(x) = \int dy\; \Pi(x-y)\ \times\ \Bigg\{ & J(y) \\
& -\frac{\lambda_4}{6}\left(\phi(y)^3 + 3\hbar\phi(y)\frac{\delta}{\delta J(y)}\phi(y) + \hbar^2\frac{\delta^2}{(\delta J(y))^2}\phi(y)\right)\Bigg\} \ .
\end{aligned}
\tag{4.46}
$$

On this basis, we can now formulate Feynman rules for the continuum limit :



$$
\begin{aligned}
&\leftrightarrow \Pi(x-y) \\
&\leftrightarrow -\frac{\lambda_4}{\hbar} \\
&\leftrightarrow +\frac{J(x)}{\hbar}
\end{aligned}
$$

Feynman rules, version 4.3 (4.47)

This comes with the understanding that the positions of all vertices are to be integrated over, and that the field function ϕ is now a functional of the source J. For a free theory there are no interactions, and we find

$$
\phi(x) = \int dy\; \Pi(x-y)\; J(y) \ . \tag{4.48}
$$

We see that the free field is the sum of its responses to the source, weighted by the correlation between the position where the field is measured and that of the strength of the source at *all* points. It is *this* property that establishes the propagator as the 'differential-equation' Green's function; but note that this correspondence is only valid for non-interacting theories.

4.3.5 Field configurations in one dimension

Before entering spaces of more dimensions, we may have a look at the field variables. The zero-dimensional variable φ, with its integration element, is in the discrete one-dimensional formulation replaced by the whole set φ, for which the path integration element reads, of course,

$$\mathcal{D}\varphi = \prod_n \ d\varphi_n$$

The continuum limit of this object is defined to be the continuum-formulation path integration element, *however badly defined this may be.* The assigning a functional value $S[\varphi]$ to a given field $\varphi(x)$ is not problematic ; rather it is the prescription of how all field configurations are to be summed over that makes it so hard to define path integrals rigorously[14]. It is instructive to consider the nature of the dominant contributions. Consider the part of the path integrand that governs the point-to-point variation of the paths: it is

$$\exp\left(-\frac{1}{2\hbar\Delta}(\varphi_{n+1}-\varphi_n)^2\right) \quad .$$

It is clear that the majority of values $(\varphi_{n+1}-\varphi_n)^2$ will be of order $\mathcal{O}\left(\hbar\Delta\right)$, as usual for Gaussian distributions. This means that φ_{n+1} and φ_n must approach each other as $\Delta \to 0$, so *the contributing fields are continuous.* On the other hand, the approach is not too fast, since by $\varphi_{n+1}-\varphi_n \approx \Delta\varphi'(x)$ we see that the derivative $\varphi'(x)$ diverges as $\Delta^{-1/2}$, hence *the contributing functions are nowhere differentiable.* This is not to say that differentiable fields are not allowed : rather, the nondifferentiable ones are the overwhelming majority. Two conclusions follow. In the first place, the use of continuum-formulation objects like $\varphi'(x)$ or $\varphi''(x)$ in the action are to be treated as highly symbolic, almost purely mnemonic, concepts. In the second place, the classical solution, which is typically almost everywhere differentiable, is itself *not* the dominant contribution to the path integral ; rather, it is the bundle of fields *close to* the classical one that constitutes the lowest-order approximation to the behaviour of the theory.

To gain some insight in the structure of a typical path (field configuration), let us consider the interrelation of three consecutive fields : it is given

[14]In fact, the mathematical definition of continuum path integrals relies on the discrete formulation !

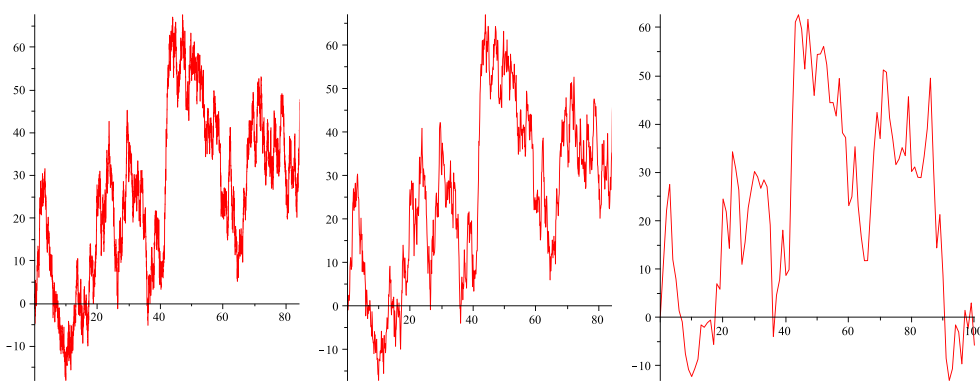
by

$$K_\Delta(\varphi_0,\varphi_1)K_\Delta(\varphi_1,\varphi_2) = \exp\left(-\frac{1}{2\hbar\Delta}\left((\varphi_0-\varphi_1)^2+(\varphi_1-\varphi_2)^2\right)\right) \quad . \tag{4.49}$$

For simplicity, we neglect the rest of the action. The positions of these three fields are separated by Δ. The 'typical' jumps in field values are of order $\sqrt{\Delta}$, as mentioned above. Now imagine 'zooming out', that is, disregarding the value of φ_1, and inspecting only φ_0 and φ_2, which are now separated by 2Δ. This is obtained by integrating over φ_1 in Eq.(4.49) :

$$\begin{aligned} \int d\varphi_1 K_\Delta(\varphi_0,\varphi_1)K_\Delta(\varphi_1,\varphi_2) &\propto \exp\left(-\frac{1}{4\hbar\Delta}(\varphi_0-\varphi_2)^2\right) \\ &= K_{2\Delta}(\varphi_0,\varphi_2) \quad , \end{aligned} \tag{4.50}$$

where the proportionality constant is absorbed in the normalization of the path integral. The typical jump from φ_0 to φ_2 is now of order $\sqrt{2\Delta}$. We conclude that, if we resolve the continuum path down to a scale Δ, the typical fluctuations over this scale will always be of order $\sqrt{\Delta}$. The typical path has a *fractal structure*. Such behaviour, with zigs and zags at every length scale, is encountered in Brownian motion – and in the behaviour of the stock market[15].



Here we plot a typical fractal path running over 10,000 points separated by a

[15]Note that this qualitative picture holds only for one-dimensional theories (and, luckily, the price of stocks, bonds, futures etc is expressed in one-dimensional currency). In more dimensions, the paths' behaviour is even more wild.

distance of 0.01, with $\Delta = 1$. The first plot shows all points ; in the second, only every 10$^{\text{th}}$ point is used, and in the third plot only every 100$^{\text{th}}$ point is used. The qualitative form of the three paths remains the same, as expected for a fractal path. The average absolute value of the point-to-point jumps are 0.80, 2.49, and 6.97, respectively : the ratios between these numbers are indeed roughly equal to $\sqrt{10}$.

4.4 The momentum representation

4.4.1 Fourier transforming the SDe

We are now ready to make an important technical change. So far, we have considered the fields and their expectation values as functions of *position*. It will turn out to more practical to consider them as functions of *momentum* or, in the one-dimensional case, of *wave number*[16]. There are a number of good reasons for doing so. In the first place, in the free theory the various momentum modes are independent of one another, in contrast to the fields at different space points[17]: propagators are *simpler* in momentum language than in position language. In the second place, there is a law of momentum conservation operative in the universe, and *not* a law of conservation of position. In the third place, momenta are more directly the physical characteristics that are controlled and measured in actual particle physics experiments. **E 17**

The transition from position to momentum is nothing but applying Fourier transforms : we already had, from Eq.(4.25),

$$\Pi(x-y) = \int \frac{dk}{2\pi}\, \frac{\hbar}{k^2+m^2}\; \exp(ikx) \quad , \tag{4.51}$$

and we now add[18]

$$\begin{aligned} \phi(x) &= \int \frac{dk}{2\pi}\, \phi(k)\; \exp(ikx) \quad , \\ J(x) &= \int \frac{dk}{2\pi}\, J(k)\; \exp(ikx) \quad . \end{aligned} \tag{4.52}$$

[16]Recall the discussion on loose terminology in Chapter 0.

[17]Indeed, the more-dimensional theories have been constructed expressly to make fields at different points correlate to one another!

[18]We use the same notation for the position-dependent quantities and their momentum-dependent Fourier transforms. This will not lead to confusion since we shall soon drop the position-dependent ones anyway.

We now have to figure out what the correct Feynman rules are in this new language. To do so we use (what else ?) the SDe. It suffices to restrict ourselves to φ^3 theory at the tree level, where it reads

$$\phi(x) = \int dy\; \Pi(x-y)\; \left[\; \frac{1}{\hbar}J(y) \;\; - \;\; \frac{\lambda_3}{2\hbar}\phi(y)^2 \;\right] \;\; . \tag{4.53}$$

Inserting the Fourier representation leads to

$$\begin{aligned} \int \frac{dk}{2\pi}\; \phi(k)\; e^{ikx} &= \int dy\; \int \frac{dk}{2\pi}\; e^{ik(x-y)}\; \frac{\hbar}{k^2+m^2} \times \\ &\left[\int \frac{dk_3}{2\pi}\; \frac{J(k_3)}{\hbar}\; e^{ik_3 y} - \frac{\lambda_3}{2\hbar} \int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\; \phi(k_1)\,\phi(k_2)\; e^{i(k_1+k_2)y} \right] \end{aligned} \tag{4.54}$$

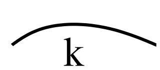
We find the following form of the SDe in momentum language :

$$\begin{aligned} \phi(k) \;\; &= \;\; \frac{\hbar}{k^2+m^2}\,\frac{J(k)}{\hbar} \\ & \quad -\frac{\hbar}{k^2+m^2}\,\frac{\lambda_3}{2\hbar} \int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\; \phi(k_1)\,\phi(k_2)\,(2\pi)\,\delta(k-k_1-k_2) \end{aligned} \tag{4.55}$$

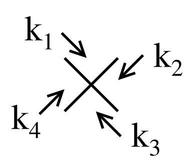
4.5 Doing it in momentum space

4.5.1 The Feynman rules

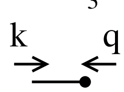
On the basis of the above we can now formulate the Feynman rules for our theory in momentum space (for the example of φ^4 theory) :



$\leftrightarrow \quad \dfrac{\hbar}{k^2+m^2}$



$\leftrightarrow \quad -\dfrac{\lambda}{\hbar}\,(2\pi)\,\delta\left(\sum_j k_j\right)$



$\leftrightarrow \quad +\dfrac{1}{\hbar}J(q)\,(2\pi)\,\delta\,(q+k)$

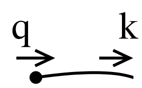
$\displaystyle\int_{-\infty}^{\infty} \frac{d\,k}{2\pi}$ for every momentum k

$\boxed{\text{Feynman rules, version 4.4}}$ (4.56)

Where before we had to integrate over the position of every vertex, we now have to integrate over every momentum. It is of course possible (and this is in fact the most common situation) that the source contains only a single momentum mode. In that case the external legs in a diagram carry a single momentum ; but all momenta of the internal lines have to be integrated over. Also note that the vertices now carry Dirac deltas imposing momentum conservation. This is a *direct* consequence of our *choosing* the vertices of the theory to be position-independent[19]. In addition, it has becomes necessary to indicate how the momenta involved in the vertices are to be counted. It is usual to count all the momenta either incoming or outgoing. The precise convention is unimportant, but it *is* important to use it consistently.

4.5.2 Some example diagrams

Here we present some diagrams, evaluated according to the rules we have formulated so far. The first one,

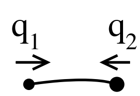


(4.57)

with a source on only one endpoint, evaluates to

$$\int \frac{dk}{2\pi}\,\frac{1}{\hbar}J(q)\,(2\pi)\,\delta(q-k)\,\frac{\hbar}{k^2+m^2} = J(q)\,\frac{1}{q^2+m^2} \quad . \tag{4.58}$$

When we add another source,

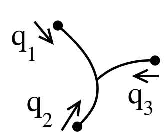


(4.59)

this gives us

$$\frac{1}{\hbar}\,J(q_1)\frac{1}{{q_1}^2+m^2}J(q_2)\,(2\pi)\,\delta(q_1+q_2) \quad . \tag{4.60}$$

we see here an important fact : every connected diagram contains one Dirac delta informing us that *overall* momentum must be conserved. The diagram



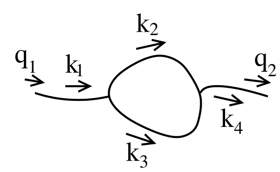
(4.61)

[19] This means that the homogeneity of space(-time) can be investigated by very carefully checking momentum(-energy) conservation in interactions. Of course, if vertices take on different values *very* far away in space or time these effects may be undetectable.

bears this out : it reads

$$\begin{aligned}
\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dk_3}{2\pi}\frac{J(q_1)}{\hbar}\frac{J(q_2)}{\hbar}\frac{J(q_3)}{\hbar} & \\
\times\frac{\hbar}{{k_1}^2+m^2}\frac{\hbar}{{k_2}^2+m^2}\frac{\hbar}{{k_3}^2+m^2} & \\
\times(2\pi)\delta(q_1-k_1)(2\pi)\delta(q_2-k_2)(2\pi)\delta(q_3-k_3) & \\
\times\frac{-\lambda_3}{\hbar}(2\pi)\delta(k_1+k_2+k_3) & \\
= \quad J(q_1)J(q_2)J(q_3)\frac{1}{{q_1}^2+m^2}\frac{1}{{q_2}^2+m^2}\frac{1}{{q_3}^2+m^2} & \\
\times\frac{-\lambda_3}{\hbar}(2\pi)\delta(q_1+q_2+q_3) \quad . & \qquad (4.62)
\end{aligned}$$

Next, we consider the one-loop diagram



(4.63)

for which we have to write down (including the symmetry factor !)

$$\begin{aligned}
\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dk_3}{2\pi}\frac{dk_4}{2\pi}\frac{J(q_1)}{\hbar}\frac{J(q_2)}{\hbar} & \\
\times\frac{\hbar}{{k_1}^2+m^2}\frac{\hbar}{{k_2}^2+m^2}\frac{\hbar}{{k_3}^2+m^2}\frac{\hbar}{{k_4}^2+m^2}\times\frac{1}{2} & \\
\times(2\pi)\,\delta(q_1-k_1)\,(2\pi)\,\delta(q_2-k_4) & \\
\times\left(\frac{-\lambda_3}{\hbar}\right)(2\pi)\,\delta(k_1-k_2-k_3)\left(\frac{-\lambda_3}{\hbar}\right)(2\pi)\,\delta(k_2+k_3-k_4) & \\
= \quad \frac{{\lambda_3}^2}{2}J(q_1)J(q_2)\frac{1}{{q_1}^2+m^2}\frac{1}{{q_2}^2+m^2}(2\pi)\,\delta(q_1-q_2) & \\
\times\int\frac{dk_2}{2\pi}\,\frac{1}{{k_2}^2+m^2}\,\frac{1}{(q_1-k_2)^2+m^2} & \qquad (4.64)
\end{aligned}$$

In addition to the overall momentum conservation delta, we see here the other significant fact : every closed loop involves a momentum that is not fixed by conservation, so an integral over that momentum.

E 18

4.6 More-dimensional theories

4.6.1 Continuum formulation

The choosing a labelling of fields with a single integer index is, of course, arbitrary. We can consider an alternative in which the fields are labelled by D integer indices :

$$\varphi_n \quad \to \quad \varphi_{\vec{n}} \quad , \quad \vec{n} = (n_1, n_2, \ldots, n_D) \ .$$

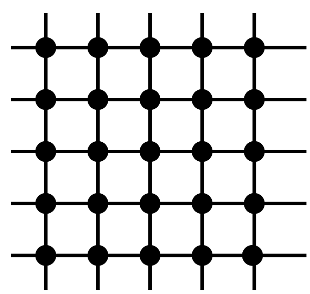
In the interest of brevity, we introduce the notation

$$\vec{n} \pm k = (n_1, \ldots, n_{k-1}, n_k \pm 1, n_{k+1}, \ldots, n_D) \ . \tag{4.65}$$

An appropriate action for this choice would be

$$S(\{f\}) = \sum_{\vec{n}} \left[\frac{1}{2}\mu {\varphi_{\vec{n}}}^2 - \gamma \sum_{k=1}^{D} \varphi_{\vec{n}}\varphi_{\vec{n}+k} + \frac{\lambda_4}{4!}{\varphi_{\vec{n}}}^4 - J_{\vec{n}}\varphi_{\vec{n}} \right] \ . \tag{4.66}$$

The obvious visualization for this choice is that of a space rather than a line, covered with a regular square grid of fields, each connected to $2D$ nearest neighbors: the corresponding continuum picture, therefore, is that of a theory in D equivalent dimensions. Here a part of the space for the case $D = 2$ is shown.



The propagator of this theory obeys, of course, the SDe

$$\Pi(\vec{n}) = \frac{\hbar}{\mu} \prod_{k=1}^{D} \delta_{n_k,0} + \frac{\gamma}{\mu} \sum_{k=1}^{D} \Big(\Pi(\vec{n}+k) + \Pi(\vec{n}-k) \Big) \ , \tag{4.67}$$

with the solution

$$\Pi(\vec{n}) = \frac{\hbar}{(2\pi)^D} \int\limits_{-\pi}^{+\pi} d^D z \ \frac{\exp(i(n_1 z_1 + \cdots + n_D z_D))}{\mu - 2\gamma\cos(z_1)\cdots - 2\gamma\cos(z_D)} \ , \tag{4.68}$$

so that, now, μ must exceed $2D\gamma$. The continuum limit takes a different form than in the one-dimensional case. We define

$$\begin{aligned} \vec{x} &= (x^1, x^2, \ldots, x^D) \quad , \quad x^j = n_j \Delta \quad , \\ \vec{k} &= (k^1, k^2, \ldots, k^D) \quad , \quad k^j = z_j/\Delta \quad , \end{aligned} \tag{4.69}$$

The simplest nontrivial choice is then to approach the continuum as follows :

$$\begin{aligned} &\gamma \to \Delta^{D-2} \quad , \quad \mu \to 2D\gamma + m^2\Delta^D \quad , \quad \lambda_4 \to \Delta^D \lambda_4 \quad , \\ &\varphi_{n_1,n_2,\ldots,n_D} \to \varphi(\vec{x}) \quad , \quad J_{n_1,n_2,\ldots,n_D} \to \Delta^D J(\vec{x}) \quad . \end{aligned} \tag{4.70}$$

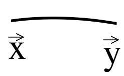
The propagator takes the continuum form

$$\Pi(\vec{x}) = \frac{\hbar}{(2\pi)^D} \int d^Dk \; \frac{\exp\left(i\vec{x}\cdot\vec{k}\right)}{\vec{k}\cdot\vec{k} \;+\; m^2} \quad . \tag{4.71}$$

The continuum form of the action is

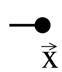
$$S[\varphi, J] = \int \left[\frac{1}{2}m^2\varphi(\vec{x})^2 + \frac{1}{2}(\vec{\nabla}\varphi(\vec{x}))^2 + \frac{\lambda_4}{4!}\varphi(\vec{x})^4 - J(\vec{x})\varphi(\vec{x})\right] d^Dx \quad , \tag{4.72}$$

The Feynman rules are seen to be



$\leftrightarrow \quad \Pi(\vec{x} - \vec{y})$


$\leftrightarrow \quad -\frac{\lambda_4}{\hbar}$



$\leftrightarrow \quad +\frac{J(\vec{x})}{\hbar}$

$\boxed{\text{Feynman rules, version 4.5}}$ (4.73)

and also the SDe is a straightforward generalization of the one-dimensional case :

$$\begin{aligned} \phi(\vec{x}) = \int d^Dy \; \Pi(\vec{x} - \vec{y}) \;\times\; &\Bigg\{ J(\vec{y}) \\ &- \frac{\lambda_4}{6}\left(\phi(\vec{y})^3 + 3\hbar\phi(\vec{y})\frac{\delta}{\delta J(\vec{y})}\phi(\vec{y}) + \hbar^2\frac{\delta^2}{(\delta J(\vec{y}))^2}\phi(\vec{y})\right)\Bigg\} \quad . \end{aligned} \tag{4.74}$$

The classical field equation for this case,

$$m^2\varphi(\vec{x}) - \vec{\nabla}^2\varphi(\vec{x}) + \frac{\lambda_4}{3!}\varphi(\vec{x})^3 = J(\vec{x}) \ \ , \tag{4.75}$$

can be obtained directly from the continuum action by the functional Euler-Lagrange equation

$$\frac{\delta}{\delta\varphi(\vec{x})}S[\varphi,J] - \vec{\nabla}\cdot\left(\frac{\delta}{\delta\vec{\nabla}\varphi(\vec{x})}S[\varphi,J]\right) = 0 \ \ . \tag{4.76}$$

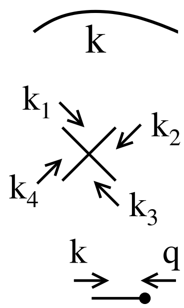
It should be noted that the propagator only depends on $|\vec{x}|$ and is therefore rotationally invariant : this is a larger symmetry[20] than that of the original lattice, that only allows rotations over multiples of $\pi/2$. The way in which the relation between field values at two points depends on the coordinates of these points *defines* the nature of the space. The 'real distance' between two points with coordinates x^j and y^j is in this case

$$|\vec{x}-\vec{y}|^2 = \sum_{j=1}^{D}(x^j - y^j)^2 \ \ , \tag{4.77}$$

the *Euclidean* distance between the points ; this type of quantum field theory is therefore said to be *Euclidean.*

As before, it will turn out to be more useful to go over to a momentum formulation of the theory. This is performed by a completely straightforward generalization of what we did in the one-dimensional case, and we can give the Feynman rules (in D dimensions) without more ado :

[20]The increase in symmetry depends on an interplay between the lattice action and the form of the continuum limit ; it is possible to construct actions in which the continuum symmetry is not larger than that of the lattice theory.



$$\leftrightarrow \quad \frac{\hbar}{|\vec{k}|^2+m^2}$$

$$\leftrightarrow \quad -\frac{\lambda}{\hbar}\,(2\pi)^D\,\delta\left(\sum_j \vec{k}_j\right)$$

$$\leftrightarrow \quad +\frac{1}{\hbar}J(q)\,(2\pi)^D\,\delta\left(\vec{q}+\vec{k}\right)$$

$$\int\limits_{-\infty}^{\infty}\frac{d^D\,k}{(2\pi)^D} \qquad \text{for every momentum } k$$

$\boxed{\text{Feynman rules, version 4.6}}$ (4.78)

4.6.2 The propagator, explicitly

It is possible to express the Euclidean propagator $\Pi(\vec{x})$ in terms of known functions, using a Gaussian representation :

$$\begin{aligned}
\Pi(\vec{x}) &= \frac{\hbar}{(2\pi)^D}\int\limits_0^\infty dt\,\int d^Dk\;\exp\Bigl(i\vec{x}\cdot\vec{k}-t\vec{k}\cdot\vec{k}-tm^2\Bigr)\\
&= \frac{\hbar}{(2\pi)^D}\int\limits_0^\infty dt\;e^{-m^2t}\prod_{j=1}^{D}\int dk^j\exp\Bigl(-z(k^j)^2+ik^jx^j\Bigr)\\
&= \frac{\hbar}{(2\pi)^D}\int\limits_0^\infty dt\;e^{-m^2t}\prod_{j=1}^{D}\left(\left(\frac{\pi}{t}\right)^{1/2}\exp\left(-\frac{(x^j)^2}{4t}\right)\right)\\
&= \frac{\hbar}{(4\pi)^{D/2}}\int\limits_0^\infty dt\;t^{-D/2}\;\exp\left(-m^2t-\frac{|\vec{x}|^2}{4t}\right)\\
&= \frac{\hbar}{2\pi}\left(\frac{2\pi|\vec{x}|}{m}\right)^{1-D/2}K_{1-D/2}(m|\vec{x}|)\quad .
\end{aligned}\tag{4.79}$$

The function K is the so-called modified Bessel function of the second kind, defined by the integral representation

$$K_\alpha(z)=K_{-\alpha}(z)=\frac{1}{2}\int\limits_0^\infty du\;u^{\alpha-1}\;\exp\left(-\frac{z}{2}\left(u+\frac{1}{u}\right)\right)\quad (z>0)\quad . \tag{4.80}$$

For very large values of z, the integrand is dominated by the region around $u = 1$, and we find

$$K_\alpha(z) \approx e^{-z}\sqrt{\frac{\pi}{2z}} \quad , \quad z \to \infty \ . \tag{4.81}$$

For very small (but positive) z, on the other hand, we may (for positive α) approximate the factor $u + 1/u$ in the exponent by just u, and

$$\begin{aligned} K_\alpha(z) &\approx \frac{1}{2}\left(\frac{2}{z}\right)^\alpha \Gamma(\alpha) \quad (\alpha > 0\ ,\ z \to 0)\ , \\ K_0(z) &\approx \log\left(\frac{1}{z}\right) \quad (z \to 0)\ . \end{aligned} \tag{4.82}$$

For large $m|\vec{x}|$, the propagator therefore decreases exponentially, while for small $m|\vec{x}|$, we have

$$\begin{aligned} \Pi(\vec{x}) &\approx \frac{\hbar}{2\pi}\log\left(\frac{1}{m|\vec{x}|}\right) \quad , \quad D = 2\ , \\ \Pi(\vec{x}) &\approx \frac{\hbar\Gamma\left(\frac{D}{2}-1\right)}{4\ \pi^{D/2}}\, x^{2-D} \quad , \quad D \geq 3\ . \end{aligned} \tag{4.83}$$

In every dimension, the propagator is normalized in the same way :

$$\begin{aligned} \int \Pi(\vec{x})\ d^Dx &= \frac{\hbar}{(2\pi)^D} \int d^Dx \int d^Dk\ \frac{\exp\left(i\vec{k}\cdot\vec{x}\right)}{|\vec{k}|^2 + m^2} \\ &= \frac{\hbar}{(2\pi)^D}\int d^Dk\ \frac{(2\pi)^D\delta^D(\vec{k})}{|\vec{k}|^2+m^2} = \frac{\hbar}{m^2}\ . \end{aligned} \tag{4.84}$$

4.6.3 Loop integrals : the principle

As stated above, diagrams with loops contain internal wave vectors that have to be integrated over, and many of these integrals are divergent. Therefore, we have two face two technical challenges. In the first place, we have to devise a way to quantify these divergences : this is called *regularization*. In the second place, regularizing these divergences does not make them go away, and therefore we shall have to arrive at a method of including these divergences into the theory in such a way as to yield finite and unambiguous answers for physically interesting quantities. This last procedure is called *renormalization*. In this section we shall only address regularization, for the case of one-loop integrals.

The idea of regularization is to let the theory depend on an arbitrarily introduced parameter, such that the divergences appear when that parameter takes on a certain value. Different regularization schemes are available, with different choices for the extra parameter, which may be particle masses, upper limits on momenta, etcetera. It must be kept in mind, however, that theories may depend sensitively on such parameters, and therefore it may be prudent to choose the parameter in such a way that the behaviour of the theory does not depend on it too sensitively. The most popular regularization scheme is that of *dimensional regularization* : in this approach the *number of dimensions*, D, is chosen as the freely varying parameter. Already anticipating that we shall study theories in four spacetime dimensions, we therefore write

$$D = 4 - 2\epsilon \ \ ,$$

with the implication that, *at the end of all calculations*, we shall take ϵ down to zero. Any divergences in the intermediate stages of the computation will then show up as singularities for $\epsilon \to 0$, and (with any luck) at the end all these singularities will have cancelled. If not, the theory is simply not very well defined.

4.6.4 Loop integrals : an example

As an example, we shall consider the loop part of Eq.(4.63), which in four dimensions reads

$$T(\vec{k}) \ = \ \int \frac{d^4k}{(2\pi)^4} \, \frac{1}{(|\vec{k}|^2 + m^2)(|\vec{k} - \vec{q}|^2 + m^2)} \ \ . \tag{4.85}$$

Dimensional regularization requests us to change the dimensionality of the integral in T from 4 to $D = 4 - 2\epsilon$. In doing so, however, we also change the *engineering dimension* of T, that is, its unit in powers of meters, seconds, and kilograms. This would make tree-level quantities and their loop corrections have different dimension, which is clearly unacceptable. We therefore introduce an *engineering scale* μ with the same dimension as $|\vec{q}|$, and write

$$T(\vec{k}) = \mu^{2\epsilon} \int \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \, \frac{1}{(|\vec{k}|^2 + m^2)(|\vec{k} - \vec{q}|^2 + m^2)} \ \ . \tag{4.86}$$

The 'Feynman trick' of sect.(13.9.1) allows us to write

$$\begin{aligned} & \frac{1}{(|\vec{k}|^2+m^2)(|\vec{k}-\vec{q}|^2+m^2)} \\ & \quad = \int_0^1 dx \frac{1}{\left(x|\vec{k}-\vec{q}|)^2+(1-x)|\vec{k}|^2+m^2\right)^2} \\ & \quad = \int_0^1 dx \; \frac{1}{\left(|\vec{k}-x\vec{q}|^2+x(1-x)s+m^2\right)^2} \quad , \end{aligned} \tag{4.87}$$

where $s=|\vec{q}|^2$. After shifting the integration variable[21] from $\vec{k}$ to $\vec{k}-x\vec{q}$, the general formula of sect.(13.9.2) then gives, up to terms of order $\mathcal{O}(\epsilon)$,

$$T(\vec{k}) = \frac{1}{(4\pi)^2}\int_0^1 dx \; \left(\frac{1}{\epsilon}-\gamma_E-\log(4\pi)+\log(\mu^2)-\log\left(sx(1-x)+m^2\right)\right) \quad . \tag{4.88}$$

Since

$$sx(1-x)+m^2 = s(x_+-x)(x-x_-) \quad , \quad x_\pm = \frac{1}{2}\left(1\pm\sqrt{1+\frac{4m^2}{s}}\right) \quad , \tag{4.89}$$

the integral is easily performed, and we find

$$\begin{aligned} T(\vec{k}) &= \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma_E-\log(4\pi)-F(|\vec{k}|^2)\right) \quad , \\ F(s) &= \log\left(\frac{s}{\mu^2}\right)+2x_+\log(x_+)-2|x_-|\log|x_-|-2 \quad . \end{aligned} \tag{4.90}$$

Two limits are of interest. In the first place, when m^2/s becomes very small, x_+ goes to 1 and x_- goes to $-m^2/s$ so that

$$F(s) \approx \log\left(\frac{s}{\mu^2}\right)-2 \quad , \quad s/m^2\to\infty \quad . \tag{4.91}$$

[21] The assumed convergence of the integral for suitably chosen ϵ jstifies this kind of shift, at least for the case we are considering here. This is not always the case : more tricky situations may lead to so-called *anomalies*.

On the other hand, when m^2 is very large compared to s, we have that $\log(sx(1-x)+m^2)$ approaches $\log(m^2)$, so that

$$F(s) \approx \log\left(\frac{m^2}{\mu^2}\right) \quad , \quad s/m^2 \to 0 \ . \tag{4.92}$$

A final remark is in order. One may wonder why we treat loop integrals in *Euclidean* space in such detail, since after all our known spacetime *may* be (approximately) *Minkowskian*, but is certainly not Euclidean. The reason is that, even in Minkowskian spacetime, loop integrals are invariably computed by transforming the Minkowskian theory into a Euclidean one, and then performing the integrals as described above. The precise relation between Euclidean and Minkowskian theories will be discussed in the next chapter.

4.7 Exercises for Chapter 4

Excercise 16 Guessing the correlator
For the 'one-dimensional discrete model' we found for the correlator the recursion relation

$$\Pi_n = \frac{\hbar}{\mu}\,\delta_{0,n} + \frac{\gamma}{\mu}\left(\Pi_{n+1} + \Pi_{n-1}\right)$$

Make the following Ansatz:

$$\Pi_n = A\ B^{|n|}$$

and compute A and B. Compare to the correct answers.

Excercise 17 Fourier transformation of the action and the rules
Consider the one-dimensional continuum action

$$S = \int dx\ \left(\frac{1}{2}\varphi'(x)^2 + \frac{m^2}{2}\varphi(x)^2 + \frac{\lambda}{4!}\varphi(x)^4 - \varphi(x)J(x)\right)$$

Here, $\varphi(x)$ and $J(x)$ are real fields.

1. We define the Fourier transformations by

$$\begin{aligned} \varphi(x) &= \frac{1}{2\pi}\int dk e^{-ikx}\varphi(k) \ , \\ J(x) &= \frac{1}{2\pi}\int dk e^{-ikx}J(k) \end{aligned}$$

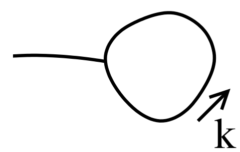
For simplicity we employ the same symbol for the transforms. Show that $\varphi(k) = \varphi(-k)$, and $J(k) = J(-k)$.

2. Compute the transformed form for the action.

3. Determine the Feynman rules in the transformed formulation.

Excercise 18 Being clever in one dimension
In one dimension, loop diagrams can be conveniently computed using Cauchy integration. We shall do this in the momentum representation.

1. Consider the tadpole diagram in $\varphi^{3/4}$ theory :



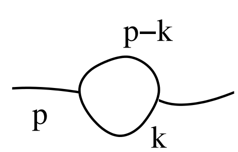
where the loop momentum is indicated. We drop the Feynman factors corresponding to the external line : this is called the *amputated* diagram.

(a) Show that the external line carries no momentum.

(b) Write down the expression for the tadpole diagram. By direct integration over k from $-\infty$ to $+\infty$, show that the result is $\lambda_3/(2m)$.

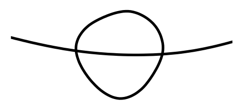
(c) Redo the above calculation in another way : the integrand has poles at $k = \pm im$. Close the integration contour in the complex k plane and contract it around a pole. Show that it does not matter which pole you choose.

2. Consider the following amputated diagram :

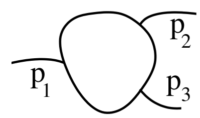


where the external momentum and the loop momenta are indicated. Compute this diagram using the same contour technique as above. Show that again the choice of the upper or the lower half-plane is irrelevant.

3. Apply the same technique *twice* to compute the amputated diagram



4. Compute also the amputated diagram



where all external momenta are counted ingoing, so that $p_1+p_2+p_3 = 0$.

Chapter 5

QFT in Minkowski space

5.1 Introduction

Since the known space in which particle physics takes place is not of a Euclidean, but rather of a Minkowskian nature[1], it behooves us to make the transition to this new type of space. Essentially, this involves singling out one of the coordinate directions in order to allow for *time*.

5.2 Moving into Minkowski space

5.2.1 Distance in Minkowski space

Whereas the 'real distance', that is, the distance measure that actually governs the relative influence of fields at different points, is given in Eulidean space by the Euclidean square distance of Eq.(4.77), we know that in the spacetime in which we actually live and do physics, the real distance is quite different. In particular, one of the coordinate directions represents *time*. That is, events in spacetime taking place at position $\vec{x} = (x^1, x^2, x^3)$ and time t relative to some freely chosen origin are denoted by four coordinates:

$$x^\mu = (x^0, x^1, x^2, x^3) \quad , \quad x^0 = ct \ , \tag{5.1}$$

[1]We shall not involve ourselves in the horrible complications that arise upon the use of *curved* space ; a consistent theory of quantum gravity is not, at present, relevant to particle physics.

where c is the universal constant providing the exchange rate between units of distance and units of time[2] ; it is the necessary velocity of massless particles[3], and the real distance between two events with coordinates x^μ and y^μ is given by

$$\begin{aligned} (x-y)^2 &= (x_0-y_0)^2 - \sum_{j=1}^{3} (x^j - y^j)^2 \\ &= g_{\mu\nu}\,(x-y)^\mu\,(x-y)^\nu \quad , \end{aligned} \tag{5.2}$$

(summation over repeated indices implied), where $g_{\mu\nu}$ is the covariant metric tensor[4]

$$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) \equiv \begin{cases} 1 & \text{if } \mu=\mu=0 \\ \text{-1} & \text{if } \mu=\nu \,\in\, \{1,2,3\} \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

We also have the contravariant metric tensor $g^{\mu\nu}$, defined by

$$g^{\mu\alpha}\,g_{\alpha\nu} = \delta^\mu{}_\nu \quad , \tag{5.4}$$

so that $g^{\mu\nu}$ is *numerically* equal[5] to $g_{\mu\nu}$. The metric tensors allow for the raising or lowering of indices : for instance,

$$x_\mu = g_{\mu\nu}\,x^\nu \quad : \quad x_0 = x^0 \ \ , \ \ x_j = -x^j \ \ (j=1,2,3) \ \ . \tag{5.5}$$

The special rôle of time in physics is evidenced by the relative minus sign in the metric tensor.

5.2.2 Farewell probability, hello SDe

Up to now, the action of our theory was real-valued, and the path integrand a real probability density. In de derivation of the SDe, however, and the consequent use of our Feynman diagrams, we have not used that fact anywhere ;

[2]See section 0.2.1.

[3]It is customary to add the provision *in vacuo* here, but since particles inside a medium with which they interact are no longer massless, this may not be necessary.

[4]See section 0.2.4.

[5]By coincidence. Even in the flat Minkowski space, another set of coordinates (spherical ones, for instance) would lead to a $g^{\mu\nu}$ quite different from $g_{\mu\nu}$. However, we shall always use the sensible (pseudo)Cartesian coordinates in these lectures.

the only requirement for the validity of the SDe is that the path integrand go to zero sufficiently fast at the endpoints. As long as this is guaranteed we may generalize the parameters of the action (including the sources) to complex values : indeed we shall take them to be purely imaginary, with one exception.

We can repeat the treatment of chapter 4 in this different setting. In what follows we shall use the notation $\vec{n} = (n_0, n_1, \ldots, n_{D-1})$. In addition we adopt the crazy-looking notation $\vec{n} \pm k$ to mean

$$\vec{n} = (n_0, n_1, \ldots, n_k, \ldots, n_{D-1}) \quad \Rightarrow \vec{n} \pm k = (n_0, n_1, \ldots, n_k \pm 1, \ldots, n_{D-1}) \ . \tag{5.6}$$

For the action we choose

$$S(\{\varphi\}) = \sum_{\vec{n}} \left[\frac{1}{2}\mu {\varphi_{\vec{n}}}^2 - \gamma_0 \varphi_{\vec{n}} \varphi_{\vec{n}+0} - \gamma \sum_{k=1}^{D-1} \varphi_{\vec{n}} \varphi_{\vec{n}+k} + \frac{\lambda_4}{4!} {\varphi_{\vec{n}}}^4 - J_{\vec{n}} \varphi_{\vec{n}} \right] \ , \tag{5.7}$$

where the special rôle of the 'time' component is evidenced. We ensure the validity of the Schwinger-Dyson equations by letting μ have a real part. This can be arbitrarily small, as long as it is *positive*. The SDe for the propagator now reads

$$\begin{aligned} \Pi(\vec{n}) \quad &= \quad \frac{\hbar}{\mu} \prod_{k=0}^{D-1} \delta_{n_k,0} + \frac{\gamma_0}{\mu} \Big(\Pi(\vec{n}+0) + \Pi(\vec{n}-0) \Big) \\ &\quad + \frac{\gamma}{\mu} \sum_{=1}^{D-1} \Big(\Pi(\vec{n}+k) + \Pi(\vec{n}-k) \Big) \quad . \end{aligned} \tag{5.8}$$

For the generating function we choose

$$R(\vec{z}) = \sum_{\vec{n}} \Pi(\vec{n}) \, \exp\Big(i n_0 z_0 - i(n_1 z_1 + \cdots n_{D-1} z_{D-1}) \Big) \quad . \tag{5.9}$$

Note again the special rôle of the zeroeth component, which we put in *by hand*. Hence $\Pi(\vec{n})$ is given by

$$\Pi(\vec{n}) = \frac{\hbar}{(2\pi)^D} \int d^D z \, \frac{\exp\Big(-i n_0 z_0 + i(n_1 z_1 + \cdots n_{D-1} z_{D-1}) \Big)}{\mu - 2\gamma_0 \cos(z_0) - 2\gamma \sum\limits_{k=1}^{D-1} \cos(z_k)} \quad . \tag{5.10}$$

We now proceed to the continuum limit ; we choose, as $\Delta \to 0$,

$$\vec{n} = \frac{1}{\Delta}x^\mu \;\;,\;\; x^\mu = (x^0, x^1, \ldots, x^{D-1}) \;\;,\;\; \vec{z} = \Delta\, k^\mu \;\;,\;\; k^\mu = (k^0, k^1, \ldots, k^{D-1}) \;\;, \tag{5.11}$$

and, in addition to $\varphi_{\vec{n}} \to \varphi(x)$,

$$\begin{aligned} &\gamma = -\gamma_0 = i\Delta^{D-2} \;\;,\;\; \mu = 2\gamma_0 + 2(D-1)\gamma + im^2\Delta^D + \epsilon \;\;, \\ &\lambda_4 \to i\Delta^D\lambda_4 \;\;,\;\; J_{\vec{n}} \to i\Delta^D J(x) \;\;. \end{aligned} \tag{5.12}$$

The continuum propagator now has the form

$$\Pi(x) = \frac{i\hbar}{(2\pi)^D}\int d^Dk\; \frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} \tag{5.13}$$

Here $k \cdot x = k^0x^0 - \vec{k}\cdot\vec{x}$ and $k^2 = (k^0)^2 - |\vec{k}|^2$. By a slight redefinition[6], the path integral can then be written as

$$Z[J] = N \int \mathcal{D}\varphi\; \exp\left(\frac{i}{\hbar}S[\varphi]\right) \;\;, \tag{5.14}$$

with

$$S[\varphi] = \int d^Dx\left[\frac{1}{2}\partial^\mu\varphi(x)\partial_\mu\varphi(x) - \frac{m^2 - i\epsilon}{2}\varphi(x)^2 - \frac{\lambda_4}{4!}\varphi(x)^4 + J(x)\varphi(x)\right] \;\;. \tag{5.15}$$

5.2.3 A closer look at almost nothing: $i\epsilon$ and $-$

In the above, the introduction of the quantity ϵ deserves some close attention. We have seen that it is necessary in order to maintain the validity of the SDe's by making the path integrand vanish at the endpoints. Another issue is the correctness of Eq.(5.10) ; in order to stay well away from singularities, we would rather have $|\mu| > 2(D-1)|\gamma| + 2|\gamma_0|$ rather than $|\mu| > 2|(D-1)\gamma + \gamma_0|$. On the other hand, the very presence of ϵ ensures that, in the integration over $\vec{z}$, the singularities are bypassed and the integral is well-defined. We may also notice that (as can be seen from our treatment of the one-dimensional case in chapter 4) the essential contributions to the integral come from $z \sim u_- \sim 1$. The choice of Eq.(5.11) is therefore

[6] We take a factor $-i$ out of S.

justified. But, it may be argued, a new parameter ϵ appearing in the theory, even if it becomes infinitesimal, must surely have a physical meaning ? Indeed, and we shall come back to this point later on. In the meantime, the ϵ is understood and always never written explicitly in the action ; indeed 'real' physics assumes that we take $\epsilon \to 0$ in the end of any discussion anyway.

As we have seen, the Minkowskian nature of the theory has been imposed, rather than derived, at two separate moments. First, the distinguishing of γ_0 and γ in Eq.(5.7), and secondly in the definition (5.9). Of course, there is no *a priori* reason why the universe would have Minkowskian rather than Euclidean symmetry[7]. So the difference between γ and γ_0 can be understood from a phenomenological point of view. It is somewhat surprising, then, that a second *'by hand'* intervention is necessary to have position x and momentum k have the right Minkowskian product.

5.2.4 The need for quantum transition amplitudes

We now find ourselves in a new interpretational situation. Since the exponent in the path integrand is now no longer real but complex-valued a straightforward probabilistic picture of the path integral is no longer possible. Indeed, every path gives a contribution which is a complex phase factor, with the same *absolute value*, namely precisely one[8]. In fact, all possible dynamics must now arise from interference effects. The leading contribution still comes from the bundle of paths around the classical solution (that is still given by the Euler-Lagrange equation), because there the phases are to first order approximation constant. Further away from the classical solution the phases of nearby path fluctuate wildly as $\hbar \to 0$ and these paths contribute very little[9].

As mentioned before, the complex-valued character of the action does not prevent us from keeping the machinery of Green's functions, connected Green's functions and the Feynman diagrams to compute them. But we shall have to reinterpret them. In accordance with standard quantum me-

[7] Although you may wonder what the world would look like if time was indistinguishable from space. But, even in computer simulations of spacetime that attempt to develop a picture of quantum gravity, *some* kind of 'foliation' that preconceives something like a special rôle for time, has to be put in essentially by hand.

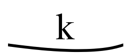
[8] In the limit $\epsilon \to 0$.

[9] The remarks about instantons remain valid also in Minkowski space.

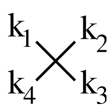
chanical practice, we postulate that **the (connected) Green's functions are *related to* the quantum-mechanical transition amplitudes**. The squared modulus of such an amplitude is the transition *probability*, to be used in the computation of cross sections and decay rates. The precise nature of the Green's function-amplitude relation will be elucidated later.

5.2.5 Feynman rules for Minkowskian theories

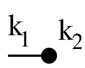
Having deduced the propagator in four-dimensional Minkowski space, we can now formulate the provisional Feynman rules for Green's functions with fixed external wave vectors :



$$\leftrightarrow\ i\hbar\,\frac{1}{k\cdot k - m^2 + i\epsilon}$$



$$\leftrightarrow\ -\frac{i}{\hbar}\lambda_4(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)$$



$$\leftrightarrow\ +\frac{i}{\hbar}J(k_2)(2\pi)^4\delta^4(k_1+k_2)$$

In the wavevector conservation at the vertices, the wavevectors must be counted either all incoming or al outgoing.
Each internal wave vector k^μ is to be integrated over, with integration element $d^4k/(2\pi)^4$.

$$\boxed{\text{Feynman rules, version 5.1}} \tag{5.16}$$

The vertices also pick up an additional factor i, and all vectors from now on are assumed to be Minkowskian four-vectors.

5.2.6 The Klein-Gordon equation

For a free theory, with vanishing interaction vertices, the SDe is again quite simple. In position, rather than wave vector, representation, we have

$$\begin{aligned} \phi(x) &= \frac{i}{\hbar}\int d^4y\;\Pi(x-y)\;J(y) \\ &= -\frac{1}{(2\pi)^4}\int d^4y\;d^4k\;\frac{\exp(-ik\cdot(x-y))}{k\cdot k-m^2+i\epsilon}\;J(y)\quad . \end{aligned} \tag{5.17}$$

The classical equation is immediately seen to be

$$\left(\partial^\mu\partial_\mu+m^2\right)\phi(x)=J(x)\quad , \tag{5.18}$$

and this is known as the *Klein-Gordon* equation. In more conventional treatments, this equation is the *starting point* for a relativistic quantum field theory, being introduced as a direct relativistic adaptation of the nonrelativistic Schrödinger equation ; for us, it is a fairly unimportant[10] result following from the Feynman rules. What *is* important, however, is the light it sheds on the source J : the natural interpretation is, indeed, for J to be a physical source, generating the field ϕ via Huygens' principle. The propagator takes the rôle of the Green's function as used in the solution of inhomogeneous differential equations.

5.3 Particles and sources

5.3.1 Unstable particles, $i\epsilon$ and the flow of time

We are now in a position to investigate the physical meaning of the $i\epsilon$ prescription. In order to so so, let us assume that ϵ is *not* infinitesimal, but rather of fixed value γ. That is, we shall use a propagator

$$\Pi_\gamma(x-y)=\frac{i\hbar}{(2\pi)^4}\int d^4k\;\frac{\exp(-ik\cdot(x-y))}{k^2-m^2+i\gamma}\quad,\quad \gamma>0\quad . \tag{5.19}$$

[10]Unimportant in the sense that we shall not derive any consequences from it. The same will be seen to hold for the Dirac, Proca and Maxwell equations.

Moreover, let us choose a source that emits particles simultaneously[11] at time $t = 0$, all over space : we take[12]

$$J(x) \propto \delta(x^0) \ . \tag{5.20}$$

The response of the field can be written as

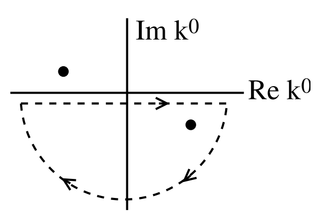
$$\begin{aligned}\phi(x) &= \frac{i}{\hbar}\int d^4y\ \Pi_\gamma(x-y)\ J(y^0)\\ &= -\frac{1}{(2\pi)^4}\int d^4y\ d^4k\ \frac{\exp\Big(-ik\cdot x + ik^0y^0 - i\vec{k}\cdot\vec{y}\Big)\ \delta(y^0)}{(k^0)^2 - |\vec{k}|^2 - m^2 + i\gamma}\\ &= \frac{-1}{2\pi}\int dk^0\ \frac{\exp(-ix^0k^0)}{(k^0)^2 - m^2 + i\gamma} \ . \end{aligned}\tag{5.21}$$

The integrand has poles in the complex k^0 plane at

$$k^0 \ = \ \pm\sqrt{m^2 - i\gamma} \ \approx \ \pm\left(m - i\frac{\gamma}{2m}\right) \ ,$$

where we have assumed that γ is small compared to m^2. For times later than $t = 0$, the integration contour can be closed along the lower half complex plane, and we find

$$\phi(x) \propto \exp\Big(-imx^0 - \frac{\gamma}{2m}x^0\Big) \ . \tag{5.22}$$



The integration contour used in Eq.(5.21). The two poles are indicated. The contour must be closed in the lower half plane in order to make the exponent $\exp(-ix^0k^0)$ vanish at infinity.

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[11] Simultaneity is an ambiguous concept in Minkowski space : here, we mean simultaneous *in our frame*.

[12] We do not worry about normalization issues here.

In accordance with the quantum-mechanical interpretation of our theory, $|\phi(x)|^2$ must be (related to) the probability of finding particles. In the present case, we have

$$|\phi(x)|^2 \propto \exp\left(-\frac{\gamma}{m}x^0\right) = \exp\left(-\frac{t}{\tau}\right) \quad , \quad \tau \equiv \frac{\gamma c}{m} \quad . \tag{5.23}$$

That is, the probability of finding particles anywhere decreases exponentially as time goes on. This is what one expects for *unstable* particles with a mean lifetime equal to τ. We shall write $\gamma = m\Gamma$, where Γ is called the *total decay width* of the particle. We see that a Feynman rule is now available for *unstable* particles :

$$\underset{\rule{4em}{0.4pt}}{\mathrm{k}} \quad \leftrightarrow \quad i\hbar\,\frac{1}{k\cdot k - m^2 + im\Gamma}$$

The propagator for an unstable particle with mean lifetime Γ/c.

$\boxed{\text{Feynman rules, version 5.1 (}\textit{addendum}\text{)}}$ (5.24)

The $i\epsilon$ prescription is seen to just mean that we should treat *stable* particles as the infinitely-long-lifetime limit of *unstable* particles.

One point to note is that Eq.(5.22) describes particles *at rest* since there is no space dependence in the wave function. The lifetime/width is therefore that of particles *at rest*, which is indeed the usual definition. For particles in motion, time dilatation indicates that the lifetime be increased by a factor p^0/m ; we shall encounter this situation in the next chapter.

Another issue that appears resolved is the *direction* of time flow. Whereas Minkowski space itself, being essentially static, does not assign any preferred direction associated with the time coordinate, the direction of time flow is now defined to be that direction in which unstable particles *disappear*, rather than *appear*[13].

[13] Attractive as the above argument appears, a drawback comes from the case $x^0 < 0$. In that case, the contour integral must be closed along the upper half plane, so that the pole $k^0 = -m + i\gamma/(2m)$ becomes the significant one. We find $\phi(x) \propto \exp(-|t|/\tau)$, which is to be interpreted as a particle density that starts out as zero at $t = -\infty$, and grows to a crescendo at $t = 0$; this lacks an obvious interpretation. We ascribe this to the use of the simple form (5.20). A better source, needed for a more rigorous treatment, can be simply constructed. Notice that this really means that the direction of time is governed by the *sources* !

Another point to be noted is the following. The unstable propagator by itself is seen to lead to a decreasing overall probability, in contradiction to the normal unitary evolution of quantum mechanics. This, however, is not the whole story : for a particle to be unstable it must be able to go over into other particles, that is, there must be interactions. These have been left out of our discussion. In a more complete treatment, we shall of course see that, as the unstable particles disappear, the density of other particles will increase, and total probability will be preserved. In other words, the decay width must be consistently computable from the interactions present in the theory.

The assumption that γ is considerably smaller than m^2 implies that Γ is small compared to m. Indeed, if we assume that Γ becomes nonzero due to interactions, the very spirit of perturbation theory argues that Γ is relatively small. Rigorous upper limits on the width of any given particle cannot easily be given ; but let us imagine a particle of mass M (in kilograms, not inverse meters !). Its natural 'size' is given by its Compton wavelength $\lambda_c = \hbar/(Mc)$. If Γ (a quantity with the dimension of inverse length) were larger than $1/\lambda_c$, this would mean that such a particle would, upon production, decay even before a lightlike signal could have crossed its diameter : it is as if the particle would vanish before it was even aware that it existed. In general, the situation $\Gamma > m$ is held to signal a breakdown of the concept of a particle as a more or less identifiable entity.

5.3.2 The Yukawa potential

As another illustration, we can consider a static pointlike source :

$$J(x) \quad \propto \quad \delta^3(\vec{x}) \quad . \tag{5.25}$$

The response of the field is then

$$\begin{aligned}\phi(x) &= \int d^4y \; \frac{i\hbar}{(2\pi)^4} \int d^4k \frac{e^{-ik\cdot x}}{k\cdot k - m^2 + i\epsilon} \; \frac{i}{\hbar}\delta^3(\vec{y}) \\ &= \frac{1}{(2\pi)^3} \int d^3\vec{k} \; \frac{e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2 + m^2} \quad .\end{aligned} \tag{5.26}$$

The $i\epsilon$ term in the denominator can safely be neglected here. Writing $|\vec{x}| \equiv r \geq 0$ and $k \equiv |\vec{k}|$, and going over to polar coordinates for $\vec{k}$, we have

$$\begin{aligned}
\phi(x) &= \frac{1}{(2\pi)^3} \int\limits_0^\infty dk\, k^2 \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{1} d\cos\theta\, \frac{e^{ikr\cos\theta}}{k^2+m^2} \\
&= \frac{1}{4i\pi^2\, r} \int\limits_0^\infty dk\; k \left[\frac{e^{ikr}}{k^2+m^2} - \frac{e^{-ikr}}{k^2+m^2}\right] \\
&= \frac{1}{4i\pi^2\, r} \int dk\; \frac{k}{k^2+m^2} e^{ikr} \quad . \qquad (5.27)
\end{aligned}$$

For $r > 0$ we can close the integration contour in the upper half of the complex-k plane, and we find

$$\phi(x) = \frac{1}{4\pi}\, \frac{\exp(-mr)}{r} \quad . \qquad (5.28)$$

This is the so-called *Yukawa* potential, introduced in the 1930's as a model for the strong nucleon-nucleon force, with m the mass of the pion. The Compton wavelength of the pion is, indeed, roughly the range of the nuclear forces. If we take $m \to 0$ we find the Coulomb potential of a static electric source ; the real propagator of the photon field, responsible for the Coulomb interaction, is however more complicated, so that the above derivation is more or less just handwaving for the case of electromagnetism.

5.3.3 Kinematics and Newton's First Law

Let us see to what extent the picture of the source as an object that, in a sense, emits particles can be reconciled with standard ideas in classical relativistic mechanics. That is, we want to measure positions and times, as well as energies, velocities and momenta, as well as possible. To this end, we shall choose the source to be

$$J(x) \propto \exp\!\left(-\frac{|x^0|}{\sigma_0} - \frac{|\vec{x}|^2}{4\sigma^2} - \frac{i}{\hbar}\left(p^0 x^0 - \vec{x}\cdot\vec{p}\right)\right) \quad . \qquad (5.29)$$

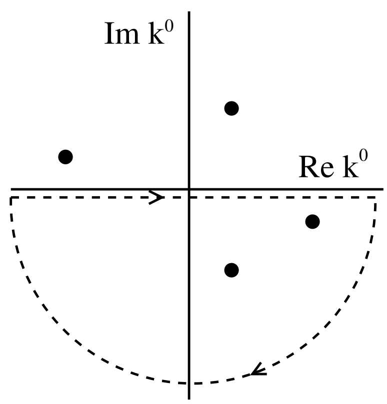
That is, the source is active for a period σ_0/c around $t = 0$, and in a region of volume σ^3 around the spatial origin. Its Fourier transform,

$$J(k) \propto \left[\frac{1}{{\sigma_0}^2} + \left(k^0 - \frac{p^0}{\hbar}\right)^2\right]^{-1} \exp\!\left(-\sigma^2\left(\vec{k} - \frac{\vec{p}}{\hbar}\right)^2\right) \quad , \qquad (5.30)$$

shows that it emits particles with all kinds of wave vectors $k^\mu = (k^0, \vec{k})$, centered around values $p^\mu/\hbar$, with $p^\mu = (p^0, \vec{p})$. For a bridge to non-quantum physics to be built, both the position and wave representation of the source should be adequately localized ; σ_0 and σ should be neither too large nor too small. For now, we do not assume any particular relation between p^0 and $\vec{p}$.

Let us now study the response of the field to this source for positive times. We have

$$\phi(x) \propto \int d^4k \, \frac{\exp\left(-ik^0x^0 + i\vec{k}\cdot\vec{x}\right)}{(k^0)^2 - |\vec{k}|^2 - m^2 + i\epsilon} \, J(k) \quad . \tag{5.31}$$



We exhibit the k^0 integral in the complex plane. For $x^0 > 0$, the contour is to be closed in the lower half complex-k^0 plane. The integrand displays simple poles at the loci

$$k^0 = \omega(\vec{k}) - i\epsilon \ , \quad k^0 = \frac{p^0}{\hbar} - \frac{i}{\sigma_0} \ ,$$

$$k^0 = -\omega(\vec{k}) + i\epsilon \ , \quad k^0 = \frac{p^0}{\hbar} + \frac{i}{\sigma_0} \ ,$$

the latter two lying outside the contour. The k^0 integral therefore leads to the following expression for $\phi(x)$:

$$\begin{aligned}
\phi(x) \;\propto\; & \int d^3\vec{k} \, \exp\left(i\vec{x}\cdot\vec{k} - \sigma^2\left(\vec{k} - \frac{\vec{p}}{\hbar}\right)^2\right) \\
\times \; & \left[\frac{1}{2\omega(\vec{k})} \frac{\exp\left(-ix^0\omega(\vec{k})\right)}{\left(p^0/\hbar - \omega(\vec{k})\right)^2 + 1/{\sigma_0}^2} \right. \\
& \left. + \frac{i\sigma_0}{2} \frac{\exp(-ix^0p^0/\hbar \; - \; x^0/\sigma_0)}{(p^0/\hbar - i/\sigma_0)^2 - \omega(\vec{k})^2 + i\epsilon}\right] \quad .
\end{aligned} \tag{5.32}$$

The second term in the square brackets decays exponentially at the same rate as the source. Since we are interested in the behaviour of the field when it is free, *i.e.* unaffected by any interactions, we can only study that behaviour once the source has died out, and then so has this term[14]. The first

[14]This is comparable with what you would do classically: studying the trajectory of a

term describes Fourier modes of the field that obey the dispersion relation $k^0 = \omega(\vec{k})$, together with the resonance condition that tells us that the field *can* only be appreciable if both $p^0/\hbar \approx \omega(\vec{k})$ and $\vec{p}/\hbar \approx \vec{k}$. We therefore expect any fruitful resonance in the field, which *can* allow for the transmission of signals over macroscopic distances, if

$$\frac{p^0}{\hbar} \approx \omega\left(\frac{\vec{p}}{\hbar}\right) \quad . \tag{5.33}$$

If we relate the zero component p^0 (with dimension kg m/s) to an energy E by writing

$$p^0 = E/c \quad , \tag{5.34}$$

we find that the only particle modes emitted by the source that have a chance of propagating over distances much further than σ must satisfy

$$E \approx \sqrt{|\vec{p}|^2c^2 + M^2c^4} \quad , \quad m = \frac{Mc}{\hbar} \quad . \tag{5.35}$$

This is the *mass shell condition*, which prescribes the relation between the energy E (in Joule), momentum $\vec{p}$ (in kg m/s), and mechanical mass M (in kg) of a particle moving freely through spacetime. We recognize the quantity m that we have been using so far as the *inverse Compton wavelength* of the particle[15]. Note the occurrence of $\hbar$ in the denominator : interpreted strictly, this implies that a truly classical limit leads to $m \to \infty$. The Euler-Lagrange equations are therefore not classical at all except for the Maxwell equations holding for massless photons.

Given that the particle is emitted on its mass shell, the integral $\phi(x)$ is not yet automatically large. The complex phase in Eq.(5.31) will lead to extremely rapid oscillatory behaviour of the integrand, and an essentially vanishing result, except for those regions where the phase of the integrand is

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thrown ball to see whether Newton's laws are obeyed only makes sense once the ball has definitively left your hand.

[15]A particle is called *on-shell* if its momentum p^μ satisfies Eq.(5.35) ; if not, it is called *off-shell*. Off-shell particles are not exotic or improbable ; they are just not visible as the result of any experiment since they cannot propagate well. In popular literature, off-shell particles are often dicussed with a lot of mumbling about 'uncertainty relations', 'borrowing energy from the vacuum', and so on. Do not allow yourself to be misled ! When a theorist starts invoking the uncertainty principle as a *reason* for something, keep your hand on your wallet. The 'uncertainty principle' is not a *reason* but a *result*.

stationary. This happens if

$$\frac{\partial}{\partial\vec{k}}\left(x^0k^0 - \vec{x}\cdot\vec{k}\right) = \frac{\partial}{\partial\vec{k}}\left(x^0\omega(\vec{k}) - \vec{x}\cdot\vec{k}\right) = \frac{\vec{k}}{\omega(\vec{k})}x^0 - \vec{x} = 0 \ \ . \tag{5.36}$$

That is, $\phi(x)$ is appreciable on a line in spacetime given by

$$\vec{x} = t\,\frac{c\,\vec{p}}{p^0} \ \ : \tag{5.37}$$

the particle moves along a straight line, with constant velocity $c\vec{p}/p^0$. This is Newton's First Law.

One might envisage other time-dependences of the source. Two additional cases are discussed in Appendix 13.8 ; the conclusions (although based on slightly different mathematics) are the same.

From this simple investigation we may conclude that **(a)** motion of free particles over macroscopic distances follows Newton's first law; and **(b)** that we can *effectively* assume that the Fourier modes of the fields obey the dispersion relation $k^0 = \omega(\vec{k})$ for positive times large enough for sources to have died out.

5.3.4 Antimatter

We again consider the free SDe :

$$\begin{aligned}
\phi(x^0,\vec{x}) &= -\int\frac{dk^0}{2\pi}\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{\exp(-ik\cdot x)}{(k^0)^2-\omega(\vec{k})^2+i\epsilon}\,J(k^0,\vec{k}) \ \ , \\
\omega(\vec{k}) &= \sqrt{|\vec{k}|^2+m^2} \ \ .
\end{aligned} \tag{5.38}$$

If $x^0 > 0$, the integration contour can be closed through the lower half of the complex k^0 plane :

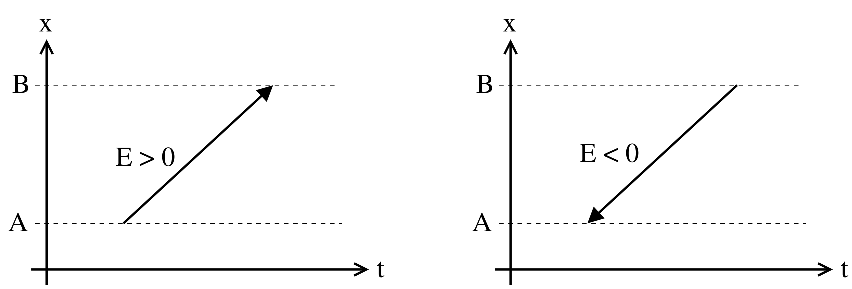
$$\phi(x^0,\vec{x}) = i\int\frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k})}\ \exp\Big(-i(x^0\omega(\vec{k})-\vec{x}\cdot\vec{k})\Big)\ J(\omega(\vec{k}),\vec{k}) \ \ . \tag{5.39}$$

If, on the other hand, $x^0 < 0$, the closure must be over the upper half of the plane, and then

$$\phi(x^0,\vec{x}) = i\int\frac{d^3\vec{k}}{(2\pi)^3 2\omega(\vec{k})}\ \exp\Big(-i(-x^0\omega(\vec{k})-\vec{x}\cdot\vec{k})\Big)\ J(-\omega(\vec{k}),\vec{k}) \ \ . \tag{5.40}$$

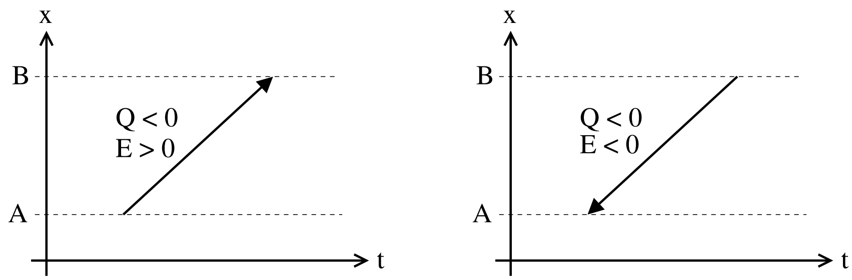
We see that the propagator essentially describes *plane waves*, with the following characteristic: *positive energies travel towards the future, and negative energies travel towards the past.*

While the concept of particles with positive energy, moving from past to future, conforms to our everyday experience, the idea of negative (kinetic) energies and movement backwards in time is not only æsthetically repellent but may lead to splitting headaches in the verbal description of physical processes. When, however, we consider more closely how such a situation will appear, it becomes clear that negative energies moving backwards in time are *indistinguishable* from positive energies moving forward.



$E(A) \downarrow \quad , \quad E(B) \uparrow \qquad\qquad E(A) \downarrow \quad , \quad E(B) \uparrow$

Some bookkeeping will easily convince you of this, with the help of the above two diagrams. Consider two loci in space, denoted by A and B. In the first diagram a particle moves *forward* in time, with *positive* energy, from A to B. As a result the energy at A *decreases*, and that at B *increases*. In the second diagram, a particle with *negative* kinetic energy starts at B, and moves *backwards* in time to A. The net effect on the energies at A and B is exactly the same ! The two situations are indistinguishable from the point of view of the energy balance.



$$E(A)\downarrow \;,\quad E(B)\uparrow \qquad\qquad E(A)\downarrow \;,\quad E(B)\uparrow$$
$$Q(A)\uparrow \;,\quad Q(B)\downarrow \qquad\qquad Q(A)\downarrow \;,\quad Q(B)\uparrow$$

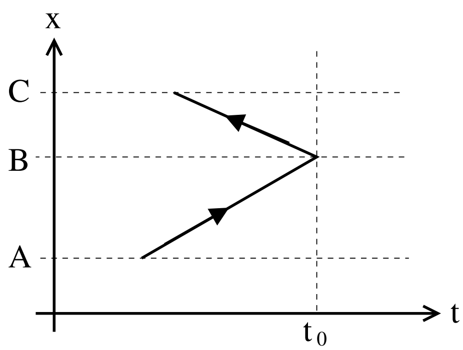
There may still be a difference, of course ; if the particles have additional properties such as electric charge, the backwards-moving particles will appear with the *opposite* charge. For instance, a negatively charged *electron* moving backwards will appear as a positively charged *positron* moving forward, as can be seen from the two diagrams above. Such re-interpreted time-reversed particles are called *antiparticles*. Every particular object whose propagator contains the denominator of Eq.(5.38) is seen to contain both the regular particles and their antiparticles. Moreover, we find the fundamental result that *particles and their antiparticles must have exactly the same mass and lifetime*. Particles and their antiparticles *may* be identical, the photon being an example. Such particles must, of course, be electrically neutral. On the other hand, not all neutral particles are their own antiparticles ; neutrons and antineutrons are distinct from one another[16]. We have thus found the following result for *free* particles : if we (a) replace all particles by their antiparticles and *vice versa*, the so-called *charge conjugation* operation **C**, (b) inverse all space directions[17], the so-called *parity transformation* **P**, and (c) invert the direction of time, the so-called *time reversal operation* **T**, then the world will look exactly the same ! This is (a restricted form of) the CPT theorem, valid for the propagation of free particles. The more interesting,

[16]Once the neutron is seen to be a collection of charged quarks, the distinction becomes obvious. So, in some sense, the realization that the neutron and the antineutron are distinct is an argument for their compositeness ! On the other hand, neutrinos, while electrically neutral, are not equal to antineutrinos, and are yet believed to be elementary.

[17]Since, as can be seen from our diagrams, inverting the direction of the motion through time will simultaneously change motion towards the left (say) into motion towards the right, and so on.

real CPT theorem, valid also for interacting particles, needs more tools than we have at our disposal right now : its proof is referred to Appendix 13.14.

Let us consider the (classically depicted) path a particle tracks out in spacetime, as given by the space-time diagram given below. In one description, the particle starts at A and moves to B, where at time t_0 it reverses its time direction, and moves backwards in time to C. In the alternative description, a particle starts at A and its antiparticle starts at C, and the pair collides at B at time t_0. For times later than t_0, the particle and/or its antiparticle have disappeared ; but because of momentum conservation their combined energy has to be transferred onto one or more other particles (not depicted). The two descriptions are completely equivalent, but the second one conforms much better to the way we tend to view the world[18]. At the 'collision/reversal-point' B the particle coming from A must dump its energy, and even an additional amount since its energy must become negative for it to start moving backwards to C. Therefore, particle-antiparticle collisions release energy, often in the form of photons[19]. For instance, when positrons meet electrons, the usual result[20]



[18]Note that the antiparticle interpretation is just the way we surrender to a prejudice about motion in time. Physicists from some alien civilization might have less problems with the other interpretation.

[19]It is sometimes stated that particles can *only* annihilate with their *own* antiparticle. This is a somewhat restricted point of view, since for instance electrons can annihilate with anti-neutrinos into W particles, as we shall see. It may be more appropriate to say that it needs particles with their *own* antiparticles to annihilate into something that has quantum numbers (electric charge, fermion number, etcetera) equal to those of the vacuum. Neutrinos and their antineutrinos cannot easily annihilate into photons, being electrically neutral : but they can annihilate into one or more Z bosons.

[20]Note that the simpler-seeming process $e^- \ e^+ \ \rightarrow \ \gamma$ is kinematically impossible if

is $e^- \, e^+ \;\to\; \gamma\,\gamma$. We also see that nothing forbids the opposite process, in which available energy turns into particle-antiparticle pairs : $\gamma\,\gamma \;\to\; e^- \, e^+$.

5.3.5 Counting states : the phase-space integration element

The treatment of the previous section is also useful in that it provides a hint on how to count the wave-vector states. For on-shell particles of mass m we use the integration element

$$\frac{1}{(2\pi)^3}\,\frac{d^3\vec{k}}{\omega(\vec{k})} \quad , \quad \omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2} \ .$$

This object has dimension L^{-2}. It is not *explicitly* Lorentz-covariant, but we can write it also in the more attractive form

$$\frac{1}{(2\pi)^3}\,\frac{d^3\vec{k}}{\omega(\vec{k})} \;=\; \frac{1}{(2\pi)^3}\,d^4k\;\delta\left(k^2 - m^2\right)\;\theta(k^0) \ . \tag{5.41}$$

Note that if k^0 is positive for an on-shell particle in any given inertial frame, it is positive in all intertial frames that can be reached by Lorentz transformations from the first one. This ensures that the step function $\theta(k^0)$ always has the same value, irrespective of any Lorentz boosts we may care to make. Lorentz covariance of the phase space integration element is thus guaranteed. We shall use the density of states (5.41) for all on-shell particles in the calculation of cross sections and lifetimes.

If, for a given scattering process, the final state contains N particles with masses m_j, $j = 1, 2, \ldots, N$, and wavevectors $p_1^\mu, p_2^\mu, \ldots, p_N^\mu$, the combined phase-space integration element is

$$dV(P; p_1, p_2, \ldots, p_N) \equiv \left(\prod_{j=1}^{N} \frac{1}{(2\pi)^3}\;d^4p_j\;\delta({p_j}^2 - {m_j}^2)\right)(2\pi)^4\delta^4\left(P - \sum_{j=1}^{N} p_j\right) \quad , \tag{5.42}$$

where P^μ is the total wavevector of the scattering system. The four-dimensional Dirac delta forces the overall conservation of wavevectors[21]. The condition

the resulting photon is to be on its mass shell. On the other hand, an single *off-shell* photon can be produced, but such a photon must immediately decay again, in for instance a particle-antiparticle pair of some kind.

[21] Conservation of total energy and momentum.

$\theta(p^0 > 0)$ imposing positive energy for the outgoing particles is, here and in the following, always understood.

5.4 Excercises for chapter 5

Excercise 19 Dangerous ϵ
Explain why, in Eq.(5.11), it is important that $\epsilon > 0$.

Excercise 20 Short-range weak forces
From our discussion of the Yukawa potential, estimate the effective range of the static *weak*-interaction potential, mediated by particles of masses 80 to 90 GeV/c^2.

Excercise 21 Counting states
Prove Eq.(5.41).

Chapter 6

Scattering processes

6.1 Introduction

In this chapter we turn our attention to the bread-and-butter subject of particle phenomenology : the description of scattering processes. We shall discuss the way in which Feynman diagrams and their evaluation are postulated to predict the probability for finding specified final states given specified initial states. We also investigate the consequences of the claim that our approach describes quantum physics and is therefore of a probabilistic nature : that is, we can only compute *probabilities*, which are necessarily bounded[1]. This leads to the notion of *unitarity* and the use (and usefulness) of *cutting rules*.

6.2 Incursion into the scattering process

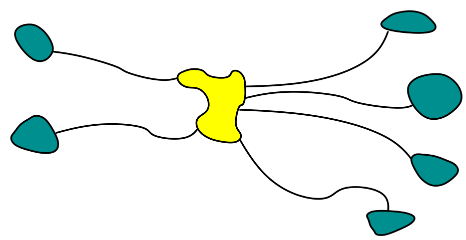
6.2.1 Diagrammatic picture of scattering

To a large extent, particle phenomenology can be viewed as the study of scattering processes, in which some *initial state* is prepared and allowed to *time-evolve*, and finally an observation is made in which the system is seen to have resulted in some *final state*. A useful example is provided by the current practice in high-energy colliders : here the initial state is prepared by machine physicists operating the collider, and it consists of two (beams of) particles with more or less well-defined momenta coming out of the beam pipes. The interesting part of the time-evolution of the system is that during

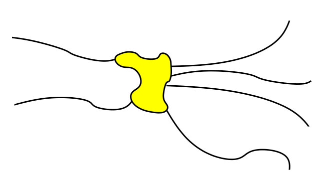
[1]After all, the probability of a certain scattering process occurring cannot exceed 100%.

which the initial-state particles approach one another and meet (hopefully[2] !) in the interaction point, where the dynamics takes place. The final state is observed by the detector operated by the particle physicists.

Since not only the scattering itself but also the initial-state preparation and the final-state observation are quantum processes, all these parts of the process must, according to our assumptions, be described by Feynman diagrams in a manner still to be established. The diagrammatic form of the complete process will then look as follows :



Here and in the following we adopt the convention that the initial state appears on the left-hand side of the diagrams, and the final state on the right-hand side. This does *not* imply any spatial or timelike relation between any of the vertices in the diagram: indeed, they are supposed to be integrated over all of spacetime[3]. Another observation on the above diagram is also relevant : the initial-state preparation and the final-state observation should contain physics that is better understood than the scattering part, and there should be a clear notion of precisely which particles constitute the initial and final states. This is indicated by the identifiable propagators connecting the various ingredients of the process. We therefore adopt the idealization that the only relevant part of the scattering should reside in the central, or *scattering* part, in this case

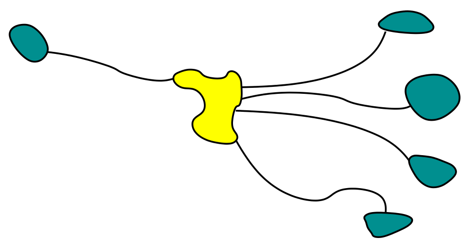


[2]In the sense that particles with perfectly well-defined momenta form plane waves of infinite spatial extent, they can hardly avoid meeting. In practice, the momenta and spatial extensions of the particles' wave packets are of course more limited.

[3]Of course, if there is any justice the *contribution* from paths in which a vertex is very far out ought to be small.

We now have to confront the two following questions. In the first place, which Feynman diagrams should occur in the scattering part ? And secondly, in actual experiments the initial- and final-state particles travel over many meters between preparation, scattering, and detection. These particles should therefore be on their mass shell, but isn't this precisely the case in which their propagators blow up ? The situation obviously calls for some reinterpretation and additional Feynman rules, to which we shall come.

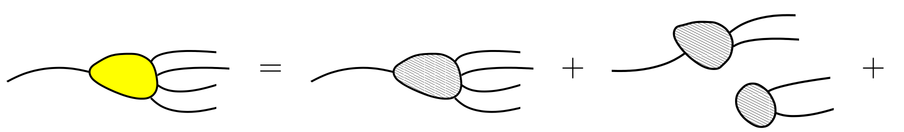
Before finishing this section, let us remark that also initial states consisting of only a single particle occur :



In this case, we simply study the decay properties of the particle, such as its total or partial decay width.

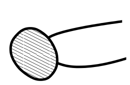
6.2.2 The argument for connectedness

Let us consider the set of all Feynman diagrams describing a decay process. As discussed before, we omit any vacuum bubbles that do not contain external lines. The set can then be split up into its connected pieces, for instance



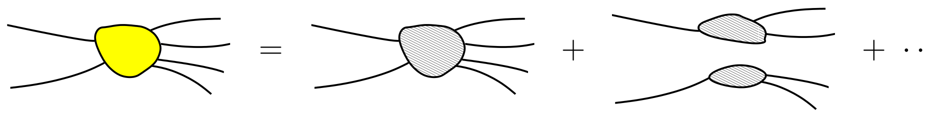
(6.1)

where as before the shading indicates *connected* diagrams. Now, recall that every vertex in any diagram contributes a Dirac delta imposing energy-momentum conservation. Therefore, every connected diagram has an overall Dirac delta imposing overall energy conservation. That, however, implies that a diagram like



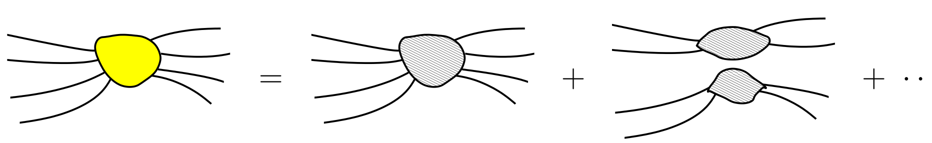
(6.2)

asks for particles carrying positive energy to originate (by some interactions) from the *vacuum*. Such contributions therefore vanish by energy conservation, and the *only* contributing diagrams are contained in the totally connected blob. Next, consider two-particle scattering. If we forbid (for the same reason as above) connected parts where particles are created from the vacuum, the only possible contributions are given by



Now, the second term here is in principle possible but *only* if **a)** the two incoming particles are inherently unstable[4] and **b)** the outgoing particles arrange themselves in precisely two groups according to the indicated decay patterns. Leaving aside such special cases, we conclude that **the scattering amplitude is given by the *connected* Feynman diagrams**. Note that the restriction to connected diagrams only arises here from simple energy considerations, and not from any deep inherent superiority of connected diagrams over disconnected ones : in essentially all cases of interest, the result of the disconnected diagrams vanish anyway.

In fact, we may conceive of situations where particles *can* be created from the vacuum. This is the cases in 'field theories at high temperature' where processes take place in a heat bath which can deliver energy to create particles. In such a picture the heat bath is the 'vacuum' of the theory, and diagrams such as that of Eq.(6.2) are not automatically zero. Another more delicate situation is that of more incoming particles : for instance, we might consider four particles scattering into four, in which we might recognize two groups of two particles scattering into two :



In this case, the only argument to disregard the disconnected diagrams is an appeal to the special kinematics.

[4]This makes the notion of particles 'coming in from infinity' conceptually dubious in this scattering.

6.3 Building predictions

6.3.1 General formulæ for decay widhts and cross sections

Consider a 'slightly unstable' particle of mass m at rest, with momentum P^μ. We shall adopt the following prescription for its differential decay width into n particles with momenta $p_1^\mu, p_2^\mu, \ldots, p_n^\mu$:

$$d\Gamma = \Phi_\Gamma \; \langle |\mathcal{M}|^2 \rangle \; dV(P; p_1, p_2, \ldots, p_n) \; F_{\text{symm}} \quad . \tag{6.3}$$

Here, $\mathcal{M}$ stands for the transition amplitude, which we still have to establish. The symbol $\langle |\mathcal{M}|^2 \rangle$ indicates that in accordance with quantum-mechanical practice we have to square the absolute value of $\mathcal{M}$ in order to arrive at a probability, and the brackets indicate summation and/or averaging over degrees of freedom other than the momenta : at present such degrees of freedom are not in our theory yet, but they will come ! The symbol Φ_Γ denotes the collection of factors that must be included to account for the density of states for the incoming particle, etcetera. The momentum P^μ is that of the incoming particle at rest. The *symmetry factor* F_{symm} is included to handle identical particles in the final state. In quantum mechanics, the statement that two particles are identical means that an interchange of these particles leads to the *physically identical* final state, so that an unconstrained summation over their momenta (and other quantum numbers) would lead to over-counting. We therefore prescribe that F_{symm} contains a factor $1/k!$ for every group of precisely k indentical particles in the final state[5]. For example, a final state containing precisely 2 photons, 3 electrons and 1 positron leads to $F_{\text{symm}} = 1/(2!)(3!)(1!) = 1/12$.

Note that the decay width is inversely proportional to the particle's lifetime. This means that for a *moving* particle the decay width must decrease by a factor m/P^0 to account for time dilatation.

In the case of two stable incoming particles with momenta p_a^μ and p_b^μ, we rather talk about the transition rate per unit flux, that is, the *cross section* for their scattering. It has dimension L^2, and must be given by a formula of

[5] Some authors choose to include a factor $1/\sqrt{k!}$ in the transition amplitude $\mathcal{M}$. I am opposed to this since such a prescription introduces a distinction between particles in the initial and those in the final state, which may destroy the crossing symmetry of the amplitude.

the form

$$d\sigma = \Phi_\sigma \, \langle |\mathcal{M}|^2 \rangle \, dV(p_a + p_b; p_1, p_2, \ldots, p_n) \, F_{\text{symm}} \quad . \tag{6.4}$$

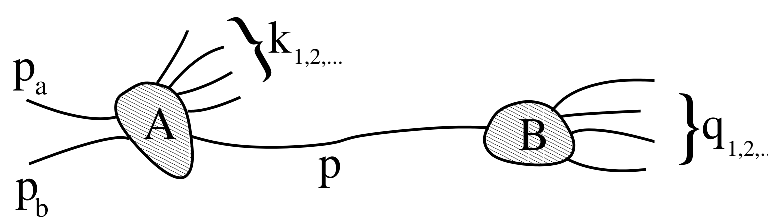
We see that, in order to get the formulae (6.3) and (6.4) to actually work, we have to establish

- the *flux factors* Φ_Γ and Φ_σ ;
- the algorithm to derive from the connected Green's function the amplitude. In particular this calls for a special treatment of the external lines.

We shall solve these issues in the next section.

6.3.2 The truncation bootstrap

We have come to one of the centrally important steps in our treatment of scattering. Consider the process in which two particles with momenta p_a and p_b scatter and yield $j + n$ *stable* particles in the final state, whose momenta we label by $k_1, k_2, \ldots, k_j$ and $q_1, q_2, \ldots, q_n$. The distinction between these groups lies in the fact that, whereas the k's emerge 'directly' from the scattering, the q's are in fact the decay products of an unstable particle that was 'directly' produced together with the k's. Nevertheless, the complete final state consists of both the k's and the q's. The relevant diagrams are given here :



Note that the connected blobs may themselves contain many different individual diagrams. By separating the blobs A and B we indicate that the unstable particles is actually quite long-lived so that the place where it is produced and that where it decays tend to be clearly separated.

Now, we shall *assume* that we have somehow solved the problem of how to go from connected Green's function to amplitude, and that we have applied this procedure to the above process. We then have for the amplitude the form

$$\mathcal{M} = [A] \, \frac{i\hbar}{p^2 - m^2 + im\Gamma} \, [B] \quad , \tag{6.5}$$

where $p = q_1 + \cdots + q_n$ is the momentum of the (internal !) line corresponding to the unstable particle, and $p^2 = p\cdot p$. The unstable particle's mass is m, and its *total* decay width is Γ. The symbols $[A]$ and $[B]$ stand for the processed connected Green's functions for the 'production' process A and the 'decay' process B, but with the Feynman factors for the unstable particle removed. Assuming, for simplicity, that $F_{\text{symm}} = 1$, we then have for the differential cross section the form

$$d\sigma = \Phi_\sigma \, |[A]|^2 \, |[B]|^2 \, \frac{\hbar^2}{\left(p^2 - m^2\right)^2 + m^2\Gamma^2} dV(P; k_1, \ldots, k_j, q_1, \ldots, q_n) \quad , \tag{6.6}$$

where $P = p_a + p_b$. In order to emphasize that p is the sum of the q's, we may write this also as

$$\begin{aligned} d\sigma \;\; = \;\; & \Phi_\sigma \, |[A]|^2 \, |[B]|^2 \, dV(P; k_1, \ldots, k_j, q_1, \ldots, q_n) \\ & \frac{\hbar^2}{\left(p^2 - m^2\right)^2 + m^2\Gamma^2} \frac{d^4p}{(2\pi)^4} \, (2\pi)^4 \delta^4(p - \Sigma q) \quad , \end{aligned} \tag{6.7}$$

with obvious notation for the sum over the wavevectors q.

Now, we let the unstable particle approach stability, so that the location where it decays becomes widely separated from that where it is produced. That is, we examine the case that Γ becomes very, *very* small, and we may approximate[6]

$$\frac{1}{\left(p^2 - m^2\right)^2 + m^2\Gamma^2} \quad \to \quad \frac{\pi}{m\Gamma} \, \delta(p^2 - m^2) \quad . \tag{6.8}$$

We can then use this to rewrite

$$\frac{dV(P; k_1, \ldots, k_j, q_1, \ldots, q_n)}{\left(p^2 - m^2\right)^2 + m^2\Gamma^2} \frac{d^4p}{(2\pi)^4} \, (2\pi)^4 \delta^4(p - \Sigma q) \tag{6.9}$$

[6] This follows from the well-known representation of the Dirac delta function as

$$\delta(x) = \lim_{z \to 0} \frac{1}{\pi} \, \frac{z}{x^2 + z^2} \quad ,$$

which has unit integral and vanishes for every $x \neq 0$.

as

$$
\begin{aligned}
& \frac{1}{2m\Gamma}\;dV(P;k_1,\ldots,k_j,q_1,\ldots,q_n)\frac{d^4p\;\delta(p^2-m^2)}{(2\pi)^3}\;(2\pi)^4\delta^4(p-\Sigma q) \\
= \;& \frac{1}{2m\Gamma}\;dV(P;k_1,\ldots,k_j,p)\;dV(p;q_1,\ldots,q_n)\;\;. \qquad (6.10)
\end{aligned}
$$

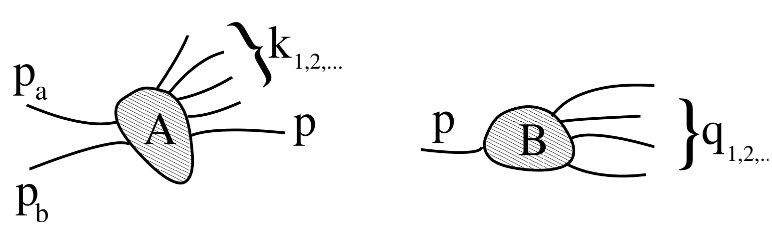
Inserting this in Eq.(6.7) we see that the cross section now takes the form

$$
\begin{aligned}
d\sigma \;=\; & \left(\hbar\;|[A]|^2\right)\;dV(P;k_1,\ldots,k_j,p) \\
& \frac{1}{\Gamma}\;\frac{1}{2m}\;\left(\hbar\;|[B]|^2\right)\;dV(p;q_1,\ldots,q_n)\;\;. \qquad (6.11)
\end{aligned}
$$

Let us now step back and consider what it is we are actually computing here : it is the cross section for producing an almost-stable particle p, together with the k's in a specified configuration, followed by the decay of the particle p into a specified configuration of q's. Under the usual ideas of conditional probability, this is the same as first computing the cross section for the production of p and the k's, followed by the *conditional* probability that, *given* p, we see it decay into the q's. This conditional probability, called the (differential) *branching ratio*, is the *partial* decay width for p to go into the q's (computed in the p rest frame !), divided by the *total* decay width, in this case Γ. We conclude that

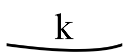
- $\hbar\;|[A]|^2$ is $\langle|\mathcal{M}|^2\rangle$ for the process $p_a+p_b\;\to\;k_1+\cdots+k_j+p\;\;;$
- $\hbar\;|[B]|^2$ is $\langle|\mathcal{M}|^2\rangle$ for the process $p\to q_1+\cdots+q_n\;\;;$
- Φ_Γ must be given by $1/(2m)$.

In a sense, we have managed to cut through the p line, and interpret the process rather as it would be given by the diagrams

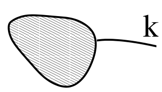


A point to be noted here has been somewhat hidden so far. The connected Green's functions contain overall factors $(2\pi)^4\delta^4()$ for overall wavevector conservation. This conservation has been imposed already, however, in our choice of the phase space integration elements dV. We therefore have to remove these factors as well in the transition from connected Green's function to $\mathcal{M}$.

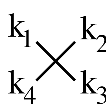
What about the treatment of the external lines ? In the above discussion we started with p as an internally ocurring unstable particle, carrying its own propagator. As we let it become stable, the propagator has disappeared into the phase space counting, leaving only a residue of a factor $\hbar^2$. At the end of the story the particle p has become a stable particle occuring as an external line in the blob A. This, therefore, must be the prescription for the external lines ! This is called *truncation* or *amputation* of external lines. An external line must apparently carry, instead of its undefined propagator, simply a factor $\sqrt{\hbar}$. We arrive at the following, expanded set of rules for the calculation of scattering amplitudes $\mathcal{M}$ (as opposed to Green's functions) :



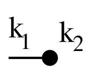
$\leftrightarrow \quad \dfrac{i\hbar}{k\cdot k - m^2 + i\epsilon}$ internal lines



$\leftrightarrow \quad \sqrt{\hbar}$ external lines



$\leftrightarrow \quad -\dfrac{i}{\hbar}\lambda_4(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)$



$\leftrightarrow \quad +\dfrac{i}{\hbar}J(k_2)(2\pi)^4\delta^4(k_1+k_2)$

ϵ is replaced by $m\Gamma$ for unstable particles.
In the wavevector conservation at the vertices, the wavevectors must be counted either all incoming or al outgoing.
Each internal wave vector k^μ is to be integrated over, with integration element $d^4k/(2\pi)^4$.

$\boxed{\text{Feynman rules, version 6.1}}$ (6.12)

The flux factor Φ_Γ for particle decay has been found to be $1/(2m)$. It is related to how we count the density of states of the incoming particle. We can directly translate to the case of two-particle scattering. Let us work in the Lorentz frame in which particle b is at rest while particle a impinges upon it. Keeping in mind the effect of Lorentz transformations on the density of states we see that whereas m_b remains, m_a has to be replaced by p_a^0, in accordance with the discussion in section 5.3.1. The density-of-states factor for the two-body initial state is therefore $1/4p_a^0 m_b$. Since, however, we are asking for a cross section rather than a transition rate, we have to divide this by the velocity of particle a in b's rest frame, that is, by a factor $|\vec{p}_a|/p_a^0$. The flux factor therefore becomes

$$\Phi_\sigma = \left(4m_b|\vec{p}_a|\right)^{-1} \quad .$$

This expression, being given in a specific Lorentz frame, is not very attractive. We can, however, write it in an explicitly Lorentz-invariant form :

$$\Phi_\sigma = \frac{1}{2\lambda\left((p_a+p_b)^2, p_a^2, p_b^2\right)^{1/2}} \quad , \tag{6.13}$$

where we have introduced the Källén function

$$\lambda(x,y,z) \equiv x^2+y^2+z^2-2xy-2xz-2yz = (x-y-z)^2-4yz \quad . \tag{6.14}$$

It often happens that the colliding particles have masses that are negligible compared to their combined invariant mass, the square of which is commonly denoted by the Mandelstam variable s. In that case, we may write

$$\Phi_\sigma \approx \frac{1}{2s} \quad , \quad s \equiv (p_a+p_b)^2 \quad . \tag{6.15}$$

This finishes our bootstrap treatment of the relation between connected Green's functions and scattering amplitudes, or *matrix elements*.

6.3.3 A check on dimensionalities

It is instructive to check that the widths and cross section expressions that we have derived do, indeed, have the correct dimensionality. By **dim**[ ] we shall denote the dimensionality of objects. In the first place, from the fact that the action S must have the same dimension as $\hbar$, we can immediately derive the dimensionality of the fields[7] :

$$\mathbf{dim}\Big[\varphi\Big] = \mathbf{dim}\Big[\phi\Big] = \mathbf{dim}\left[\frac{\hbar^{1/2}}{L}\right] \quad , \tag{6.16}$$

where, as before, L denotes a length. Therefore, a connected Green's function with n external lines (being nothing much more than the expectation value of φ^n) has dimension[8]

$$\mathbf{dim}\Big[C_n\Big] = \mathbf{dim}\left[\frac{\hbar^{n/2}}{L^n}\right] \quad . \tag{6.17}$$

The Dirac delta function imposing wavevector conservation has dimensionality

$$\mathbf{dim}\Big[\delta^4(k)\Big] = \mathbf{dim}\Big[k^{-4}\Big] = \mathbf{dim}\Big[L^4\Big] \quad . \tag{6.18}$$

To go from the connected Green's function C_n to the n-point matrix element $\mathcal{M}_n$, we have to extract the external propagators as well as the overall wavevector conservation delta function, and assign a factor $\hbar^{1/2}$ to each external line: therefore,

$$\mathbf{dim}\Big[\mathcal{M}_n\Big] = \mathbf{dim}\left[\frac{C_n}{(C_2)^n\ \delta^4(k)}\ \hbar^{n/2}\right] = \mathbf{dim}\Big[L^{n-4}\Big] \quad . \tag{6.19}$$

The n-particle phase-space integration element dV_n has dimensionality L^{4-2n} as we have seen. Taking into account that the flux factor $\Phi_\Gamma = 1/2m$ must have the dimensionality of $1/m$, that is, L, the dimensionality of the decay width of a single particle into n particles is given by

$$\mathbf{dim}\Big[\Gamma(1\to n)\Big] = \mathbf{dim}\left[\frac{1}{m}(\mathcal{M}_{n+1})^2\ dV_n\right] = \mathbf{dim}\Big[L^{-1}\Big] \quad , \tag{6.20}$$

as required. Similarly, for the cross section of two particles going into n particles we have

$$\mathbf{dim}\Big[\sigma(2\to n)\Big] = \mathbf{dim}\left[\left(\frac{1}{m}\right)^2 (\mathcal{M}_{n+2})^2\ dV_n\right] = \mathbf{dim}\Big[L^2\Big] \quad , \tag{6.21}$$

[7] In four spacetime dimensions! In d dimensions it would read $\mathbf{dim}[\varphi] = \mathbf{dim}\Big[\hbar^{1/2}L^{1-d/2}\Big]$.

[8] Higher-order contributions to Green's functions contain, of course, additional powers of $\hbar$: but these must occur only in dimensionless combinations with the coupling constants of the theory.

again as required. Note that the above analysis is kept simple because we have restricted ourselves to the use of wavevectors rather than mechanical momenta, which would introduce additional factors of $\hbar$ in the calculation. The other natural constant, c, need not enter here.

6.3.4 Crossing symmetry

In our treatment of antimatter in the previous chapter we have seen that the production (absorption) of a particle is, in a sense, æquivalent to the absorption (production) of its antiparticle. We can make this even more specific as a relation between various scattering amplitudes : this goes by the name of *crossing symmetry*. Consider a generic $2 \to 2$ scattering process :

$$a(p_1) + b(p_2) \to c(q_1) + d(q_2)$$

where we have indicated the momenta of the particles. Let us write the corresponding amplitude as $\mathcal{M}(p_1, p_2, q_1, q_2)$. By moving particles from the initial to the final state[9], or *vice versa*, we can then find the amplitudes for the crossing-related processes, for example :

$$\begin{aligned} a + b \to c + d &\;:\; \mathcal{M}(p_1, p_2, q_1, q_2) \;\;, \\ a + \bar{c} \to \bar{b} + d &\;:\; \mathcal{M}(p_1, -p_2, -q_1, q_2) \;\;, \\ a + \bar{d} \to \bar{b} + c &\;:\; \mathcal{M}(p_1, -p_2, q_1, -q_2) \;\;, \\ \bar{c} + \bar{d} \to \bar{a} + \bar{b} &\;:\; \mathcal{M}(-p_1, -p_2, -q_1, -q_2) \;\;. \end{aligned} \tag{6.22}$$

Since the momenta of all (anti)particles have positive energy, the minus signs yield momenta with negative energy. Depending on the type of the particle[10], this may involve an analytic continuation of the amplitude function $\mathcal{M}$.

6.4 Unitarity issues

6.4.1 Unitarity of the S matrix

If $\mathcal{M}$ is to be a correct form of the scattering amplitude for a given initial state to be observed, after time evolution, as a given final state, it must obey

[9] You can visualize this by taking an outgoing particle, say, and dragging its external leg from the final to the initial state.

[10] Especially for Dirac particles.

the constraints of *unitarity* which we shall now discuss. In a more traditional quantum-mechanical parlance, the initial state is given to us at some time in the far past, where the incoming particles are supposed to be so widely separated that they are essentially free : the state of the system is then

$$|\mathrm{in}, t = -\infty\rangle$$

We now let nature take its course : the incoming particles approach one another, the interaction is 'switched on', and the system evolves into some, possibly very complicated, superposition of free-particle states :

$$|\mathrm{in}, t = -\infty\rangle \quad \to \quad |\mathrm{in}, t = +\infty\rangle$$

Finally, the final state is observed to be a particular free-particle state (assuming the final-state particles have been able to move very far away from one another), that is,

$$|\mathrm{out}, t = +\infty\rangle \ \ .$$

The probability amplitude for this to happen is of course

$$\mathcal{M} = \langle \mathrm{out}, t = +\infty | \mathrm{ins}, t = +\infty\rangle \ \equiv \ \langle \mathrm{out}, t = +\infty | \ S \ | \mathrm{in}, t = -\infty\rangle \ \ , \tag{6.23}$$

where S is the matrix describing the time evolution of the incoming state from $t = -\infty$ to $t = +\infty$.

An important consideration here is the *conservation of probabiity.* That is, *any* initial state $|i\rangle$ must go to *some* final state $|f\rangle$ with 100% probability[11] ; of course $|f\rangle = |i\rangle$ may also be one of the possibilities. Writing this using the S matrix we have

$$1 = \sum_f |\left\langle f \left|S\right| i\right\rangle|^2 = \sum_f \left\langle i \left|S^\dagger\right| f\right\rangle \left\langle f \left|S\right| i\right\rangle = \left\langle i \left|S^\dagger S\right| i\right\rangle \ \ . \tag{6.24}$$

Conversely, *any* final state $|f\rangle$ must have come from *some* initial state $|i\rangle$ with 100% probability, so that

$$1 = \sum_i |\left\langle f \left|S\right| i\right\rangle|^2 = \sum_i \left\langle f \left|S\right| i\right\rangle \left\langle i \left|S^\dagger\right| f\right\rangle = \left\langle f \left|S S^\dagger\right| f\right\rangle \ \ . \tag{6.25}$$

[11] Try to imagine a world in which this does not hold ! Conservation of probability is a *dogma* — but a reasonable one.

Since this must hold for all states $|i, f\rangle$ we have

$$S^\dagger S = SS\dagger = 1 \quad , \tag{6.26}$$

and the S matrix is unitary. Note that we have had to assume that the set of initial states $|i\rangle$ and final states $|f\rangle$ are *complete*[12]. The free-particle states are natural choices for complete orthonormal bases, and we see that $\mathcal{M}$ is simply a matrix element of the S matrix. We shall investigate this in some more detail.

For simplicity, let us assume that we can label the initial states with a discrete label i, and the final states by a similar discrete label f. We can then write the S matrix element as

$$S_{fi} = \delta_{fi} + \mathcal{M}_{fi} \quad , \tag{6.27}$$

where the Kronecker delta embodies what would happen if there were *no* interactions : the only possible observed final state would in that case be identical to the initial state (two particles, say, continuing on their way without having interacted). The remainder $\mathcal{M}_{fi}$ is the object described by Eq.(6.23) ; it is the result of the interactions of the theory, and is described by the Feynman diagrams. Note that $\mathcal{M}_{ii} \neq 0$ is quite possible ; it corresponds to the case where the final state happens to reproduce the initial state, so to speak *in spite of* the interactions. This is called the *forward scattering amplitude*. Now, the unitarity of the S matrix is expressed[13] as $SS^\dagger = S^\dagger S = 1$, or

$$\sum_k S^*_{kf} S_{ki} = \delta_{fi} \quad , \tag{6.28}$$

or, in terms of $\mathcal{M}$:

$$\mathcal{M}_{fi} + \mathcal{M}^*_{if} + \sum_k \mathcal{M}^*_{kf}\mathcal{M}_{ki} = 0 \quad . \tag{6.29}$$

As a special case, we can consider $f = i$: we then have the *optical theorem*,

$$2\;\mathrm{Re}\,(\mathcal{M}_{ii}) + \sum_k |\mathcal{M}_{ki}|^2 = 0 \quad , \tag{6.30}$$

[12]Again, try to imagine what things would look like if that were not the case.

[13]Since S may be an infinite matrix, both conditions are necessary, whereas for a finite matrix one would suffice.

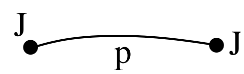
which immediately shows that the forward scattering amplitude must have *negative* real part[14]. Another simple result is the well-known property of unitarity matrices : by putting $f = i$ in Eq.(6.28) we see that for every S-matrix element we have

$$|S_{fi}| \leq 1 \quad \forall \quad i, f \tag{6.31}$$

which implies that $\mathcal{M}_{fi}$ can not be arbitrarily large. We shall employ this idea extensively later on.

6.4.2 An elementary illustration of the optical theorem

We consider the following physical process. We start with an empty initial state i (that is, a state containing no particles). At some moment a source kicks in, producing an unstable particle with wavevector p, mass m and total width Γ. Sometime later, the same source absorbs the particle, and at the end the final state f is empty again. The simple Feynman diagram describing this is



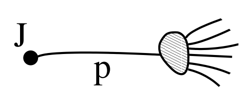
Since the initial and final state coincide, $f = i$ and this is a forward scattering amplitude ; it must obey the optical theorem. We shall now verify this. The matrix element is given by

$$\mathcal{M}_{ii} = \left(i\frac{J}{\hbar}\right) \frac{i\hbar}{p^2 - m^2 + im\Gamma} \left(i\frac{J}{\hbar}\right) \quad , \tag{6.32}$$

so that

$$\mathrm{Re}\,(\mathcal{M}_{ii}) = -\frac{J^2}{\hbar}\frac{m\Gamma}{\left(p^2 - m^2\right)^2 + m^2\Gamma^2} \quad , \tag{6.33}$$

which is indeed negative. Now, we consider the matrix elements $\mathcal{M}_{ki}$ as used in Eq.(6.30). These describe the initial state i going over in any final state k, that is, they describe the decay of the particle after it has been produced by the source :



[14]A word of caution : in much of the literature, the statement reads that the amplitude must have *positive imaginary part*. This is simply due to the fact that in those texts, the S matrix element is written not $\delta + \mathcal{M}$ but $\delta + i\mathcal{M}$. I do not see any particular virtue in this.

and we shall denote them by

$$\mathcal{M}_{ki} = -i\frac{J}{\hbar}\,\frac{\hbar}{p^2 - m^2 + im\Gamma}D \quad , \tag{6.34}$$

where iD is the contribution of the 'decay blob'. We then have

$$\sum_k |\mathcal{M}_{ki}|^2 = \frac{J^2}{\left(p^2 - m^2\right)^2 + m^2\Gamma^2}\sum_k |D|^2 \quad . \tag{6.35}$$

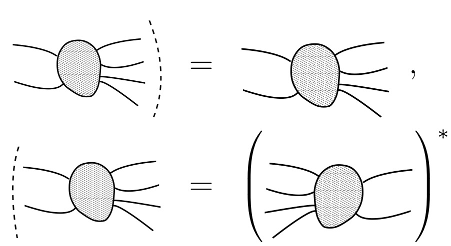
The optical theorem (6.30) will therefore be satisfied if

$$\Gamma = \frac{1}{2m}\sum_k \,\hbar\,|D|^2 \quad . \tag{6.36}$$

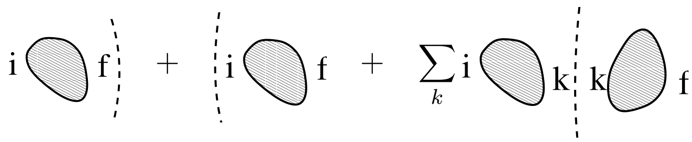
But this is, of course, precisely the prescription for the decay width of the particle, if we realize that the final state k indicates not only all possible final states, but also that the summation over k should include the phase-space integration. This short excercise illustrates both the optical theorem and the computational prescriptions arrived at before. Note that the factor $\hbar$ corresponds precisely with the Feynman rule that an external line should carry a factor $\sqrt{\hbar}$.

6.4.3 The cutting rules

We shall now consider how the unitarity relation (6.29) can be made useful in the language of Feynman diagrams. To start, we realize that this equation contains, in addition to the 'standard' matrix element $\mathcal{M}_{fi}$ for initial state i and final state f, also $\mathcal{M}^*_{if}$ which describes the (complex conjugate) matrix element for initial state f going over into final state i, that is, the *time-reversed* process. We shall embody this in a useful manner by introducing a *cutting line*. A cutting line cuts across diagrams separating them into a 'left' and 'right' piece. Any diagram to the left of a cutting line is interpreted in the usual manner ; any diagram to the right of a cutting line is interpreted to be *the complex conjugate of the time-reversed version of the diagram.* That is,

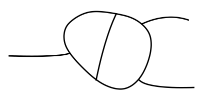


If the diagram contains *oriented* lines, the time-reversal also inverts the orientation of those lines (if the orientation is indicated by an arrow, we reverse the arrow). We can write Eq.(6.29) diagrammatically as



$= 0$. (6.37)

It is possible to sharpen this equation to make it more useful. In the first place, Eq.(6.37) holds for whole matrix elements, evaluated to all orders in perturbation theory. This implies that it must also hold for each order separately[15]. However, even at some fixed order, $\mathcal{M}_{fi}$ can contain very many diagrams. Consider a somewhat-complicated Feynman diagram in φ^3 theory :

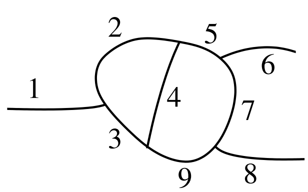


(6.38)

The corresponding Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\varphi)(\partial_\mu\varphi) - \frac{1}{2}m^2\varphi^2 - \frac{1}{6}\lambda\varphi^3 \quad . \tag{6.39}$$

The unitarity structure of the above Feynman diagram is not immediately obvious since there are, at this order of perturbation theory, quite a few diagrams that contribute to this amplitude (57, in fact). We can, however, employ the following trick. Let us assign a different label to each line in the diagram, in an arbitrary manner, for instance



(6.40)

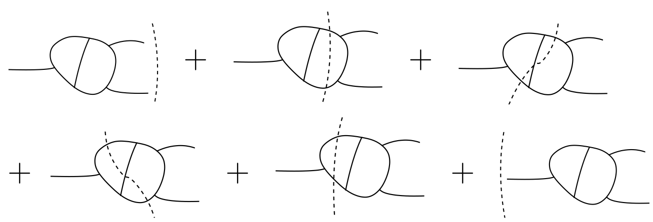
and let us pretend that each line corresponds to a different field. This diagram can then be interpreted as coming from a theory with 9 distinct fields (with

[15]If Eq.(6.37) were *not* to hold order-by-order, this would imply subtle relations between coupling constants, $\hbar$, and the like. We would then be in a position to actually *compute* coupling constants from first principles, which would be good — too good to be true, in fact.

identical mass) and Lagrangian

$$\begin{aligned}\mathcal{L} &= \sum_{n=1}^{9}\left(\frac{1}{2}(\partial^\mu\varphi_n)(\partial_\mu\varphi_n) - \frac{1}{2}m^2\varphi_n^2\right) - V \ , \\ V &= \lambda_{123}\varphi_1\varphi_2\varphi_3 + \lambda_{245}\varphi_2\varphi_4\varphi_5 + \lambda_{349}\varphi_3\varphi_4\varphi_9 \\ &\quad +\lambda_{567}\varphi_5\varphi_6\varphi_7 + \lambda_{789}\varphi_7\varphi_8\varphi_9 \ \ .\end{aligned} \tag{6.41}$$

Nothing forbids us to assign to the various $\varphi\varphi\varphi$ couplings precisely the value λ. Now, it is easily seen that, *in order* $\lambda_{123}\lambda_{245}\lambda_{349}\lambda_{567}\lambda_{789}$, the diagram (6.40) is the *only* diagram that can contribute in this theory[16] ! We can do even more : by inspection of all possibilities, we can simply realize that the only final states k in the unitarity condition (6.37) must be precisely $k = \{2,3\}$, $\{5,9\}$, $\{2,4,9\}$ or $\{3,4,5\}$, if we want to end up with the right order in perturbation theory[17]. In other words,



$= \ 0$, (6.42)

where we have omitted the line labellings : indeed, the same identity must hold for the original diagram (6.38) ! This establishes the so-called cutting rules (also called the Cutkosky rules), which can be most simply expressed in words : take a diagram and move the cutting line through it from right to left in all possible manners, making sure that the two halves in which the diagram is cut remain connected and that neither the inital state or the final state is dissected. The particles described by internal lines through which the cut runs must be assumed to be on their mass shell[18]. The sum of all the possible contributions then vanishes[19].

[16]The secret resides in the fact that in V the external fields 1,6 and 8 occur precisely *once*, and the other fields precisely *twice*.

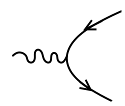
[17]Note that, for instance, the choice $k = \{5,7,8\}$ would result in the right-hand half of the diagram being disconnected ; the choice $k = \{2,4,7\}$ is inconsistent since both 6 and 8 are in the final state.

[18]This *may* mean that the situation thus described fails to meet the restrictions of momentum/energy conservation ; then, that contribution vanishes.

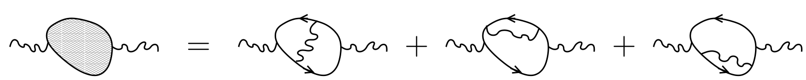
[19]You might object that in a theory with many different fields the *symmetry factors* of the diagrams will, in general, be different from those of a theory with only a single field,

6.4.4 Infrared cancellations in QED

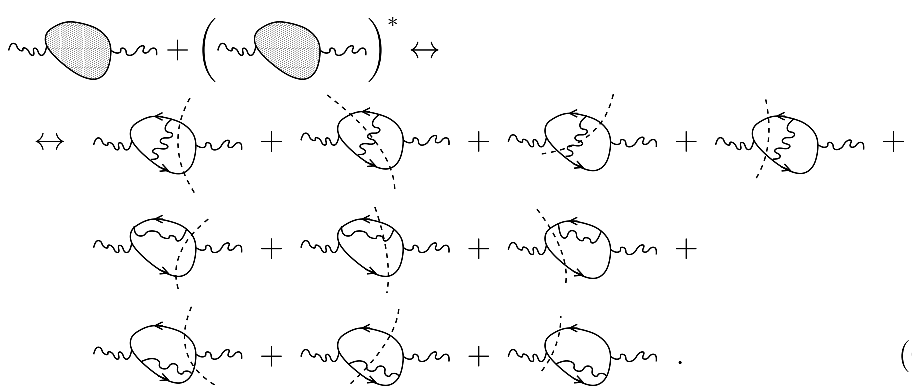
As an illustration of how the cutting rules may be applied we shall make a slight jump ahead and consider quantum electrodynamics, that is the theory of photons and electrons. Their Feynman rules will be dicussed later ; for now it is sufficient to know that the only interaction vertex in the theory is the three-point vertex



where the *oriented* lines stand for electrons and positrons, and the wavy line denotes the photon. Let us consider the 1PI two-loop corrections to the photon propagator. These are given by

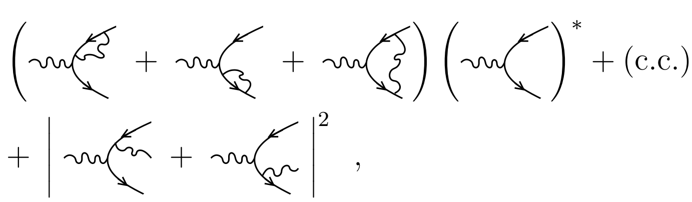


By applying the cutting rules we can investigate the real part of this two-loop contribution:



(6.43)

This set of 10 cut diagrams is, as we can see, equal to



(6.44)

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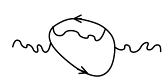
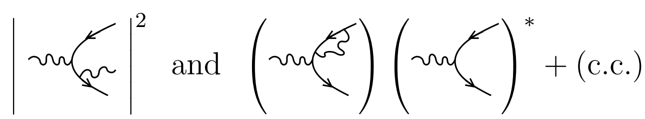
and this is true : however, in the summation over the 'intermediate states' k we must of course also include the 'indentical-particle' symmetry factor F_{symm}, which precisely repairs the correspondence — another illustration of the crucial rôle of the symmetry factors !

where integration over the final state is implied. As we shall see, the presence of a photon in the final state leads to a so-called *infrared* (IR) divergence arising from the fact that the probability of emitting an on-shell photon goes to infinity as the photon energy goes to zero. The process described by the last two diagrams has therefore an infrared divergence. This divergence is neatly cancelled by a compensating divergence in the diagrams with a virtual photon in the first line. This is a well-known fact[20] ; but it is instructive to see that the statement about the cancellation of the infrared divergences can be replaced by the simpler statement that the photon propagator is free from infrared divergences[21]. This is one example of a useful rule of thumb : when you encounter loop diagrams, try to envisage the physics that is described by cutting them. In fact, the cancellation can be pinpointed further ; the single statement that the single diagram



is IR-finite means that the IR divergences in



must cancel between them.

6.5 Some example calculations

6.5.1 The FEE model

As an example of an application of what we have learned so far, we shall investigate at theory that contains two particle types, one of mass m, denoted

[20] And a fortunate one.

[21] Two remarks are in order here. In the first place, the virtual-photon diagrams do contain divergences related to the loop momentum going to infinity : these are *ultraviolet* (UV) divergences. The photon propagator is therefore still ultraviolet divegrent, and this is cured in the usual manner by renormalization. In the second place, the cancellation of IR divergences takes place even when we restrict the phase space for the outgoing particles, provided that zero-energy photons are admitted.

by E, and another denoted by F, of mass M. The Lagrangian density of this theory is given by

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\partial^\mu\varphi_E\right)\left(\partial_\mu\varphi_E\right) - \frac{m^2}{2}{\varphi_E}^2 \\ &\quad + \frac{1}{2}\left(\partial^\mu\varphi_F\right)\left(\partial_\mu\varphi_F\right) - \frac{M^2}{2}{\varphi_F}^2 - \frac{m\lambda}{2}\varphi_F{\varphi_E}^2 \quad . \end{aligned} \tag{6.45}$$

There exists a single coupling between two E's and one F. Note that the Feynman rule for the vertex is given[22] by $-im\lambda/\hbar$; we have introduced a factor m in order to ensure that

$$\mathbf{dim}[\lambda] = \mathbf{dim}\left[\frac{1}{\hbar^{1/2}}\right]$$

with no length scale.

6.5.2 Two-body phase space

Since we shall consider processes ending in a two-body final state, it is expedient first to work out the corresponding two-body phase space. For the sake of generality we shall do this for a final state containing two momenta ${q_{1,2}}^\mu$ with general masses $m_{1,2}$. Furthermore we shall write

$$P^\mu = {q_1}^\mu + {q_2}^\mu \quad , \quad s = P^\mu P_\mu \quad . \tag{6.46}$$

The phase space (and with it widths and cross sections) is often most easily evaluated in the *rest frame* of P^μ, in which $\vec{q}_1 = -\vec{q}_2$. The phase space integration element is given by[23]

$$dV(P;q_1,q_2) = \frac{1}{(2\pi)^2}\; d^4q_1\;\delta({q_1}^2-{m_1}^2)\; d^4q_2\;\delta({q_2}^2-{m_2}^2)\;\delta^4(P-q_1-q_2) \quad . \tag{6.47}$$

As a first step, we cancel d^4q_2 against the four-dimensional Dirac delta, and write the q_1 integration in its not-explicitly-covariant form :

$$dV(P;q_1,q_2) = \frac{1}{(2\pi)^2}\,\frac{d^3\vec{q}_1}{2{q_1}^0}\,\delta\left((P-q_1)^2-{m_2}^2\right) \quad . \tag{6.48}$$

[22] It is customary to leave out the $(2\pi)^4\delta^4()$ of momentum conservation, since it is present in *all* vertex Feynman rules for translation-invariant interactions.

[23] It is usual not to include the step functions that require the energies to be positive.

Now, the q_1 integration element can be expressed in polar coordinates as

$$\frac{d^3\vec{q}_1}{2{q_1}^0} = \frac{|\vec{q}_1|^2\ d|\vec{q}_1|\ d\Omega}{2{q_1}^0} = \frac{1}{2}|\vec{q}_1|\ d{q_1}^0\ d\Omega \quad , \tag{6.49}$$

where we denote the $\vec{q}_1$ solid angle by

$$d\Omega = d\cos\theta\ d\phi \tag{6.50}$$

with a polar angle θ running from 0 to π and an azimuthal angle ϕ running from 0 to 2π, and use the fact that

$$|\vec{q}_1|\ d|\vec{q}_1| = {q_1}^0\ d{q_1}^0 \quad . \tag{6.51}$$

The Dirac delta imposing the mass shell condition on q_2 can be written as

$$\begin{aligned} \delta\left((P-q_1)^2 - {m_2}^2\right) &= \delta\left(s + {m_1}^2 - {m_2}^2 - 2{q_1}^0\sqrt{s}\right) \\ &= \frac{1}{2\sqrt{s}}\ \delta\left(\frac{s+{m_1}^2-{m_2}^2}{2\sqrt{s}} - {q_1}^0\right) \quad , \end{aligned} \tag{6.52}$$

where the rest frame of P has been used. We immediately find that

$${q_1}^0 = \frac{s+{m_1}^2-{m_2}^2}{2\sqrt{s}} \quad , \quad {q_2}^0 = \frac{s+{m_2}^2-{m_1}^2}{2\sqrt{s}} \quad , \tag{6.53}$$

and

$$|\vec{q}_1| = |\vec{q}_2| = \frac{1}{2\sqrt{s}}\ \lambda(s,{m_1}^2,{m_2}^2)^{1/2} \quad , \tag{6.54}$$

where the Källén function crops up again. In the P^μ rest frame, the phase space integration element is therefore given by

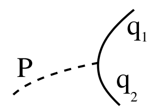
$$dV(P;q_1,q_2) = \frac{1}{32\pi^2}\ \lambda\left(1,\frac{{m_1}^2}{s},\frac{{m_2}^2}{s}\right)^{1/2}\ d\Omega \quad . \tag{6.55}$$

6.5.3 A decay process

As a first application, we shall assume that $M > 2m$ so that the F particle can decay into a pair of E's:

$$F(P) \quad\to\quad E(q_1)\ E(q_2) \quad .$$

In lowest order, its single Feynman graph is given by



The corresponding matrix element is quite simple :

$$\mathcal{M} = -i\frac{m\lambda}{\hbar}\,\left(\sqrt{\hbar}\right)^3 = -im\lambda\sqrt{\hbar}\ \ , \tag{6.56}$$

so that it has dimensionality $\mathbf{dim}[1/L]$ as it should. The decay width is therefore

$$\begin{aligned} d\Gamma(F \to EE) &= \frac{1}{2M}\,|\mathcal{M}|^2\,dV(P;q_1,q_2)\,\frac{1}{2!} \\ &= \frac{m^2\lambda^2\hbar}{128\pi^2 M}\,\sqrt{1-\frac{4m^2}{M^2}}\,d\Omega\ \ . \end{aligned} \tag{6.57}$$

Note the occurrence of the symmetry factor $1/2!$ arising from the fact that the two final-state E particles are indistinguishable. The angular integration is of course trivial in this simple case, and we immediately find the total width

$$\Gamma(F \to EE) = \frac{m^2\lambda^2\hbar}{32\pi M}\,\sqrt{1-\frac{4m^2}{M^2}}\ \ , \tag{6.58}$$

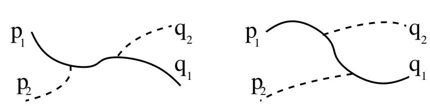
with the correct dimensionality $\mathbf{dim}[\Gamma] = \mathbf{dim}[1/L]$.

6.5.4 A scattering process

As a second application, we take the mass M of the F particle to be zero. We now have an extremely primitive picture of the electron-photon system, where E is the electron and F the photon. We consider the process of 'Compton scattering' :

$$E(p_1)\;F(p_2) \;\;\to\;\; E(q_1)\;F(q_2)$$

which, to lowest order, is given by two Feynman diagrams:



The total momentum involved is now

$$P^\mu = {p_1}^\mu + {p_2}^\mu = {q_1}^\mu + {q_2}^\mu \quad , \tag{6.59}$$

and we shall use the invariant products

$$s = (p_1 + p_2)^2 \quad , \quad u = (p_1 - q_2)^2 \quad . \tag{6.60}$$

Again applying the rules for the construction of the matrix element, we find

$$\mathcal{M} = i\lambda^2 m^2 \hbar \left(\frac{1}{s - m^2} + \frac{1}{u - m^2} \right) \quad . \tag{6.61}$$

We shall also introduce the quantity

$$K \equiv \lambda(s, m^2, 0)^{1/2} = s - m^2 \quad , \tag{6.62}$$

which allows us to write

$$u - m^2 = -2(p_1 \cdot q_2) = -\frac{K^2}{2s}(B + \cos\theta) \quad , \quad B = \frac{s + m^2}{s - m^2} \quad . \tag{6.63}$$

Here, θ is the angle between $\vec{p}_1$ and $\vec{q}_1$ in the centre-of-mass frame, that is, the angle over which the E particle is scattered in the collision. The differential cross section is now written as

$$d\sigma = \frac{\lambda^4\ m^4\ \hbar^2}{64\ \pi^2\ s} \left(\frac{1}{K^2} - \frac{4s}{K^3(B + \cos\theta)} + \frac{4s^2}{K^4(B + \cos\theta)^2} \right) d\Omega \quad . \tag{6.64}$$

At high energies, where $B \approx 1$, the cross section is strongly peaked in the backward direction. At low collision energy, where $s \approx m^2$, B is very large and the angular distribution is flat. The total cross section is found, after some straightforward algebra, to be

$$\sigma = \frac{\lambda^4\ m^4\ \hbar^2}{32\ \pi\ s} \left(\frac{2}{K^2} + \frac{2s}{K^2 m^2} - \frac{4s}{K^3} \log\left(1 + \frac{K}{m^2}\right) \right) \quad . \tag{6.65}$$

At first sight the cross section might appear to diverge at the very lowest energies, since K vanishes there. However, by carefully expanding the logarithmic term to third order we find that the poles in K cancel, and

$$\lim_{s \to m^2} \sigma(EF \to EF) = \frac{\lambda^4\ \hbar^2}{48\ \pi\ m^2} \quad . \tag{6.66}$$

A remark is in order here. In the first place, the factor λ^4 and consequently the factor $\hbar^2$ could have been foreseen from the start. The fact that the cross section must have $\mathbf{dim}[\sigma] = \mathbf{dim}[L^2]$ implies that at the threshold, where m is the only length scale in the problem, there must also be an overall factor $1/m^2$. Moreover, n body phase space contains a power π^{4-3n} from its definition ; and also it contains $n-1$ solid angles to be integrated over, each giving rise to[24] a factor π. This means that the total cross section for an n-body final state will contain a factor π^{3-2n}. In this way, almost the whole cross section formula is determined, and all the calculational effort is only used to find the numerical factor 1/48.

6.6 Excercises for Chapter 6

Excercise 22 Dirac is the limit
Consider the function

$$f_\epsilon(x) = \frac{\epsilon}{\pi}\,\frac{1}{x^2+\epsilon^2}$$

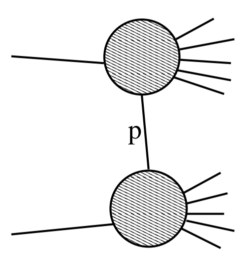
Show the following :

1. $\int\limits_{-\infty}^{\infty} f_\epsilon(x)\;dx = 1$

2. $\lim_{\epsilon\to 0} f_\epsilon(x) = 0$ if $x \neq 0$, and $\lim_{\epsilon\to 0} f_\epsilon(0) = \infty$.

This proves that $f_\epsilon(x)$ approaches the Dirac delta function $\delta(x)$ as $\epsilon \to 0$.

Excercise 23 Stability implies safety
Consider a process in which two stable particles collide and produce a number of final-state particles. One possible diagram is



[24] This is to say that the angular integral does not necessarily evaluate to π, but rather that a factor π invariably arises in the result of a solid-angle integral.

The propagator indicated with ‘p’ reads, of course,

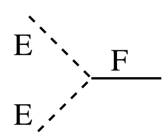
$$\frac{i\hbar}{p^2 - m^2 + i\epsilon}$$

and this would blow up if $p^2 = m^2$, making the cross section infinite. Show that this cannot happen.

Excercise 24 A real calculation ?
We shall now do a simple calculation, again using the FEE model, with the mass of the E particles equal to zero, and everything at the tree level. We make use of the formula for massless two-body phase space:

$$d^4p_1\ \delta({p_1}^2)\ d^4p_2\ \delta({p_2}^2)\ \delta^4(P - p_1 - p_2) = \frac{1}{32\pi^2} d\Omega$$

where Ω is the solid angle in the rest frame of P. There is one vertex :



$$\leftrightarrow \quad -i\frac{m\lambda}{\hbar}$$

Here λ is a dimensionless coupling constant.

1. Compute the decay width $\Gamma(F \to EE)$.

2. Give the three tree-level diagrams for $E(q_1)E(q_2) \to E(p_1)E(p_2)$.

3. Write the scattering matrix element $\mathcal{M}$. NB 1: The F partile is unstable, as we have just proven ! NB 2: do not square it, because that becomes ugly.

4. Show that, if we write $\mathcal{M} = iT$, then $\Im(T)$ is positive, as required by unitarity.

5. Assume λ to be a quite small number. For s (the centre-of-mass energy squared) equal to m^2, find the approximate form of $|\mathcal{M}|^2$, and compute the cross section.

Excercise 25 Yet another one
Compute the total cross section for the process $EE \to FF$ in the FEE model, again assuming the E particles to be massless.

Excercise 26 Infrared divergence
We consider again the FEE theory containing but now with massive E and massless F. This is a very crude example of QED (Quantum Electrodynamics), the theory of charged spin-1/2 particles and photons.

1. Consider a general process (perhaps involving other particles) in which an E with momentum p is emitted. Draw its blob.

2. Conider he same process but now with the additional emission of also an F of momentum k. Consider the diagram where the F comes from the external E line, and draw it.

3. Show that, if we neglect the $i\epsilon$ term, the resulting amplitude contains a factor $1/(p \cdot k)$.

4. Show that the cross section is proportional to $1/(k^0)^2$.

5. Show that the phase space integration element is proportional to $k^0\, dk^0$.

6. Show that the *total* cross section for this process of F emission must be divergent.

Chapter 7

Dirac particles

7.1 Pimp my propagator

7.1.1 Extension of the propagator and external lines

So far we have been studying particles that can carry only a limited amount of information : such a particle is completely specified once we have determined its identity and its momentum. In this chapter we shall start increasing the number of properties that particles can carry, by examining how the Feynman propagator can be modified. Since the pole structure of the propagator is closely connected with the causality of the theory, and must be used to derive Newton's first law in the approximation of propagation over macroscopic distances, we will not mess around with the *denominator* of the propagator. The generalizations we shall propose therefore concern themselves with the *numerator*, and are of the form

$$i\hbar\frac{1}{p^2 - m^2 + i\epsilon} \quad \to \quad i\hbar\frac{\mathcal{T}}{p^2 - m^2 + i\epsilon} \quad , \tag{7.1}$$

where $\mathcal{T}$ is *some* object that informs us that the particle propagating is not as simple as we have seen so far, but has additional properties. What those properties are depends, of course, on the choice of $\mathcal{T}$.

7.1.2 Down with dyads !

One very important observation is in order here. The particle propagator never occurs in isolation, but always between two vertices, where the particle is 'produced' and where it is 'absorbed'[1]. This implies that, as long as we have not committed ourselves to particular vertices, a change in the propagator may be compensated to some extent by a change in the vertices. For instance, suppose that $\mathcal{T}$ is a simple *number*: then the predictions of the theory will remain unchanged if we opt to multiply the vertices by $\mathcal{T}^{-1/2}$. Therefore, $\mathcal{T}$ must be more complicated than a single number, *i.e.* it must have some *matrix* form. In which kind of space such matrices live is at the moment of course not yet determined, we just assume that the matrix has some indices. Now, consider the case where the matrix is actually a *dyad*, that is, a tensor product of a column 'vector' and a row 'vector' : for instance, if the matrix space is three-dimensional, it might read

$$\mathcal{T} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix} \quad .$$

In such a case, the row 'vector' could be assigned to one of the vertices, and the column 'vector' to the other vertex, and the remaining propagator would again be trivial.

We therefore assume that

$$\mathcal{T}^a{}_b = \sum_n \left(\mathcal{U}(p)^{(n)}\right)^a \left(\mathcal{W}(p)^{(n)}\right)_b \quad , \tag{7.2}$$

where a, b are *some* indices living in *some* linear space. They may be Lorentz indices[2], but not necessarily. The sum over n must contain at least two terms. The vertices of the theory must, of course, contain corresponding indices a, b with which those of the propagator are contracted, otherwise the matrix element could not be a simple number. The objects $\mathcal{U}$ are column 'vectors' indicated by *upper* indices, and the $\mathcal{W}$ are row 'vectors' indicated by *lower* indices. It is tempting to think of the $\mathcal{W}$ as 'hermitean conjugates' of the $\mathcal{U}$ but this is not necessarily true[3].

If we now reappraise the truncation argument of the previous chapter, we see that we can redo it with the more complicated propagator. Again the

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[1]It may be realized that this statement holds true also in the case of *external* lines, if it is kept in mind that these are defined in the *square* of the matrix element.

[2]As in the case of spin-1 particles, see later on.

[3]For Dirac particles, the $\mathcal{W}$ are the *Dirac conjugates* of the $\mathcal{U}$. More about this later.

denominator contribution will end up in the phase space, but the numerator will be left. We can remedy this for instance by assigning the factor $\mathcal{W}_b$ to the *production* matrix element, and $\mathcal{U}^a$ to the *decay* amplitude, with the understanding that this only holds if the particle is *on-shell*[4]. We see that an extension of the propagator naturally leads to new Feynman rules for the external lines as well. In the following we shall investigate several such extensions.

7.1.3 The spin interpretation

As we have seen, particles with generalized propagators will carry factors $\mathcal{U}$ or $\mathcal{W}$ when they occur as external lines in Feynman diagrams. Such particles therefore carry, by definition, additional information which is somehow embodied in the label (n). Adhering to good quantum practice, we shall assume that particles with different values of (n) are physically distinct from one another even if their momentum is the same. That is, for $p^2 = m^2$ we require

$$\sum_a \left(\mathcal{W}^{(n)}\right)_a \left(\mathcal{U}^{(n')}\right)^a = K\ \delta_{n,n'} \quad , \tag{7.3}$$

with K some constant (that is, the external-line factors are (multiples of) the elements of an orthonormal set). This implies that

$$\mathcal{T}^2 \ \propto\ \mathcal{T} \quad , \tag{7.4}$$

In other words, $\mathcal{T}$ must have properties of a projection operator. Later on, we shall see that the 'additional property' can be interpreted as the *spin* of the particle. By simple counting arguments, it would seem reasonable to interpret the external factors $\mathcal{U}^{(n)}$ as members of a $(k-1)/2$-spin multiplet if the label (n) runs over k values : however, the more careful treatment is to first see how the $\mathcal{U}$ transform under rotations in the rest frame of p^μ, and only then to assign them a spin interpretation[5]. We shall do this explicitly for various particle types.

Another point of importance is the following. We have noted that the sum over dyads is actually a sum over *distinct additional properties* of the

[4]Where the $\mathcal{U}$ and the $\mathcal{W}$ are assigned depends, of course, on which vertex their respective indices are coupled to.

[5]This becomes particularly important in the case of massless particles.

particle that propagates. Now, we should like the different 'versions' of the particle to propagate *in the same manner*, otherwise by just sitting and waiting we would see the 'additional' properties of the particle change[6]. We shall therefore require that $\mathcal{T}$ *only* depends on the momentum (and possibly on the mass) of the particle :

$$\mathcal{T} = \mathcal{T}(p, m) \ \ .$$

Before closing this section, we point out that the particles we have studied in the previous chapter, whose propagator has the trivial numerator $\mathcal{T} = 1$, of course transform trivially (*i.e.* not at all) under rotations : such particles are therefore *scalars*, or spin-0 particles.

7.2 The Dirac algebra

7.2.1 The Dirac matrices

Probably the simplest nontrivial choice for $\mathcal{T}$ is to let it depend *linearly* on the momentum. At this point, an immediate objection may be raised ; for the momentum carries a Lorentz index. Now we do *not* want to contract this index with a corresponding index in one of the vertices since this would simply amount to a redefinition of the vertices. On the other hand, we cannot afford to have the Lorentz index floating loose, which would destroy the Lorentz invariance of the theory. We therefore choose

$$\mathcal{T}(p) = p^\mu \gamma_\mu + K\ m\ 1 \ \ . \tag{7.5}$$

Here, γ_μ ($\mu = 0, 1, 2, 3$) is a set of four *matrices* since as we have argued the propagator's numerator must be of matrix form[7]. The symbol 1 stands for the unit matrix of whatever space the γ matrices live in, and the term $Km1$ has been added since there is no clear reason to forbid it from the start. Of course, simply prescribing the γ matrices would again destroy the

[6] Suppose an electron with 'spin-up' propagates less than an electron with 'spin-down' ; after some time, we would see almost exclusively 'spin-down' electrons coming by. This, *apparently*, is not how nature works.

[7] Another argument against the γ's being simple numbers is that, in that case, they would define a preferential *vector* γ^μ. This would destroy the assumed isotropy of Minkowski space, and a frame in which $\vec{\gamma}$ vanishes would deserve to be equated with Newton's absolute reference frame.

Lorentz invariance of the theory since any matrix element would have γ's all over the place. We therefore require that, *in the final form of the matrix element*, all reference to the specific choice of these matrices can be removed in a Lorentz-invariance-respecting manner. That is, the γ matrices must be endowed with a property that allows us to remove them from the final answer. The momenta with which they are contracted should then end up in ordinary Minkowski products. That is, there must be a requirement of the form

$$Q(\gamma^\mu, \gamma^\nu, \gamma^\rho, \ldots, \gamma^\sigma) = (\text{some tensor})^{\mu\nu\rho\cdots\sigma} \quad , \tag{7.6}$$

where Q is some algebraic combination of Dirac matrices. This had better be a simple as possible, otherwise we might not be able to eliminate the Dirac matrices from very simple amplitudes. A moment's reflection will tell us that essentially the *only* possible such property is[8]

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}1 \quad . \tag{7.7}$$

Note that this is a matrix equation: in its full glory it would read

$$\sum_c \left\{ (\gamma^\mu)^a{}_c(\gamma^\nu)^c{}_b \quad + \quad (\gamma^\nu)^a{}_c(\gamma^\mu)^c{}_b \right\} = 2\; g^{\mu\nu}\; \delta^a{}_b \quad ,$$

but, as is conventional, we shall not explicitly write out the Dirac indices unless it is unavoidable. Note also that Eq.(7.7) immediately confirms that the Dirac objects γ cannot be simple numbers[9]. Dirac matrices with different indices *anti*commute, while

$$(\gamma^0)^2 = 1 \quad , \quad (\gamma^k)^2 = -1 \;\; (k = 1, 2, 3) \quad . \tag{7.8}$$

We also find immediately that[10]

$$\gamma^\mu\; \gamma_\mu = 4 \quad . \tag{7.9}$$

[8]The *anti*commutation is necessary because of the symmetry of $g^{\mu\nu}$ in its indices. Another possibility might read something like $\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = \epsilon^{\mu\nu\alpha\beta}1$ but this woud not allow us to remove fewer than 4 Dirac matrices in any matrix element. The factor 2 in Eq.(7.7) is simply conventional.

[9]Because in that case the fact that $g^{01} = 0$ would imply that γ^0 or γ^1, or both, vanish: and that would clash with $g^{00} = -g^{11} = 1$.

[10]At least in precisely four spacetime dimensions. In so-called *dimensional regularization* this may change.

From Eq.(7.8) we see that the eigenvalues of γ^0 are either 1 or -1 ; and those of $\gamma^{1,2,3}$ are either i or $-i$. We therefore have the following Hermiticity properties for the Dirac matrices :

$$\gamma^{0\dagger} = \gamma^0 \quad , \quad \gamma^{k\dagger} = -\gamma^k \ \ (k = 1,2,3) \ \ . \tag{7.10}$$

For the rest of these notes, the eigenvalues of the Dirac matrices are actually unimportant. Any choice of Dirac matrices satisfying Eqs.(7.7) is acceptable. Many possible choices have been proposed in the literature. That none of them possesses a physical advantage over the others follows from the 'fundamental theorem of Dirac matrices' which shows that any two representations of the Dirac algebra (7.7) can be transformed into each other[11]. This again strengthens our conviction that *any* result involving Dirac particles should be deriveable without any reference whatsoever to their particular form, and we shall endeavour to adhere to this. Note that, at this point, we have *not* specified the dimensionality of the Dirac matrices. In order to avoid confusion with Lorentz indices, the Dirac indices will be called *spinor indices*, and the objects $\mathcal{U}$ and $\mathcal{W}$ for this propagator will be called *spinors.* Spinors carry only a single spinor index.

Before finishing this section, let us introduce the Feynman 'slash' notation : if a^μ is a Lorentz vector, we shall mean by $\not{a}$ its contraction with Dirac matrices:

$$\not{a} \equiv a^\mu \ \gamma_\mu \ \ . \tag{7.11}$$

The Dirac equation (7.7) can therefore also be written as[12]

$$\not{a}\not{b} + \not{b}\not{a} = 2\ (a\cdot b) \ \ \forall \ \ a^\mu, b^\nu \ \ , \tag{7.12}$$

with the corollary that

$$\not{a}\not{a} = a^2 \ \ . \tag{7.13}$$

We stress that the vector object a^μ and the matrix $\not{a}$ encode *exactly the same information* ; further on we shall see how the vector can be recovered once the matrix is given. A few simple results, which can be checked by repeated application of the anticommutation rule, are[13]

$$\gamma^\mu \not{a}\, \gamma_\mu \ \ = \ \ -2\not{a} \ \ ,$$

[11] We defer the proof of this theorem to Appendix 13.10.

[12] It is customary to leave the unit matrix 1 out of the notation. Its presence can always be inferred where necessary.

[13] Again, in four spacetime dimensions.

$$\begin{aligned}\gamma^\mu \not{a} \not{b} \gamma_\mu &= 4(a \cdot b) \ , \\ \gamma^\mu \not{a} \not{b} \not{c} \gamma_\mu &= -2 \not{c} \not{b} \not{a} \ . \end{aligned} \tag{7.14}$$

E 27

7.2.2 The Clifford algebra

By the anticommutation relation (7.7), any product of more than four Dirac matrices can be reduced to a smaller number. Let us define the enormously useful object[14]

$$\gamma^5 \equiv i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 \ , \tag{7.15}$$

for which we can immediately derive that

$$\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5 \quad , \quad (\gamma^5)^2 = 1 \ . \tag{7.16}$$

Also we can define the *commutator* of Dirac matrices as **E 28**

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\,[\gamma^\mu, \gamma^\nu] = \frac{i}{2}\,(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \ , \tag{7.17}$$

whence

$$\gamma^\alpha\gamma^\beta = g^{\alpha\beta} - i\,\sigma^{\alpha\beta} \ . \tag{7.18}$$

Obviously there are 6 independent σ matrices. The most general object that can be constructed using Dirac matrices is therefore **E 29**

$$\Gamma = S\,1 \ + \ V_\mu\gamma^\mu \ + \ T_{\mu\nu}\sigma^{\mu\nu} \ + \ A_\mu\,\gamma^5\gamma^\mu \ + \ P\,\gamma^5 \ , \tag{7.19}$$

and these objects form the *Clifford algebra*. We see that $\mathcal{T}(p)$ must be an element of the Clifford algebra. The various coefficients are called, respectively, the scalar (S), vector (V_μ), tensor ($T_{\mu\nu}$), axial-vector (A_μ) and pseudo-scalar (P) coefficients. Since the tensor coefficient may be taken antisymmetric, there are in total 1+4+6+4+1=16 coefficients. This suggests (but does not prove) that the Dirac matrices are 4×4 matrices. Given an element Γ in the

[14] In some texts the definition of γ^5 is slightly different, for instance it may lack the factor i. Some care is necessary in comparing results between different texts. The reason why it is called γ^5 and not γ^4 is that in some older treatments the Minkowski indices were assumed to run from 1 to 4, with the 4^{th} index playing the rôle of our 0^{th} one.

Clifford algebra, we can recover its coefficients using the trace identities that we shall discuss below. Finally, we can define the two Clifford elements

$$\omega_\pm = \frac{1}{2}\left(1 \,\pm\, \gamma^5\right) \quad . \tag{7.20}$$

They are mutually exclusive projection operators ; that is,

$$\omega_\pm^2 = \omega_\pm \quad , \quad \omega_+\omega_- = \omega_-\omega_+ = 0 \quad , \quad \omega_+ + \omega_- = 1 \quad . \tag{7.21}$$

These operators are widely used.

7.2.3 Trace identities

A very important rôle is played by traces of Dirac matrices or Clifford elements. To start, we have of course

$$\mathrm{Tr}\,(1) = N \quad , \tag{7.22}$$

where N is the (as yet unknown) dimensionality of the Dirac matrices[15]. Using γ^5 and the cyclicity property of the trace operation, we see that

$$\mathrm{Tr}\,(\gamma^\mu) = \mathrm{Tr}\left(\gamma^\mu\gamma^5\gamma^5\right) = \mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^5\right) = -\mathrm{Tr}\left(\gamma^\mu\gamma^5\gamma^5\right) = -\mathrm{Tr}\,(\gamma^\mu) \tag{7.23}$$

so that the trace of a single Dirac matrix vanishes ; and by the same method we see that the trace of a product of an *odd* number of Dirac matrices is also zero, in particular

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\right) = 0 \quad . \tag{7.24}$$

For two matrices we have

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{Tr}\,(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = N\; g^{\mu\nu} \quad , \tag{7.25}$$

from which we see that the trace of a σ matrix must vanish[16]. To continue,

$$\mathrm{Tr}\left(\gamma^5\right) = \mathrm{Tr}\left(\gamma^5\gamma^0\gamma^0\right) = \mathrm{Tr}\left(\gamma^0\gamma^5\gamma^0\right) = -\mathrm{Tr}\left(\gamma^5\gamma^0\gamma^0\right) = -\mathrm{Tr}\left(\gamma^5\right) \quad , \tag{7.26}$$

[15] We shall prove later on that $N = 4$.

[16] Generally, *all commutators are always traceless*, by the cyclicity property of traces.

so that also this trace evaluates to zero. The trace of 4 Dirac matrices requires a bit more anticommutation :

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = \mathrm{Tr}\left(2g^{\mu\nu}\gamma^\alpha\gamma^\beta - 2g^{\mu\alpha}\gamma^\nu\gamma^\beta + 2g^{\mu\beta}\gamma^\nu\gamma^\alpha - \gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\mu\right) \quad , \tag{7.27}$$

so that, by cyclicity,

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = N\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right) \quad ; \tag{7.28}$$

and the same method may be used to arrive at the 15 terms for a trace of 6 Dirac matices, the 105 terms for a trace of 8 matrices, and so on[17]. Furthermore, since the anticommutation operations used in Eq.(7.27) might as well have moved to the left inside the trace instead of to the right, we immediately find that[18]

$$\mathrm{Tr}\left(\gamma^{\mu_1}\,\gamma^{\mu_2}\,\gamma^{\mu_3}\,\cdots\,\gamma^{\mu_{n-1}}\,\gamma^{\mu_n}\right) = \mathrm{Tr}\left(\gamma^{\mu_n}\,\gamma^{\mu_{n-1}}\,\cdots\,\gamma^{\mu_3}\,\gamma^{\mu_2}\,\gamma^{\mu_1}\right) \quad . \tag{7.29}$$

Since γ^5 is the product of all four different Dirac matrices, the product $\gamma^5\gamma^\mu\gamma^\nu$ (with $\mu \neq \nu$) is actually a product of two different Dirac matrices, and therefore

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) = 0 \quad . \tag{7.30}$$

Finally, it is immediately seen that

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = iN\epsilon^{\mu\nu\alpha\beta} \quad . \tag{7.31}$$

Returning to the general Clifford algebra element Γ, we can straightforwardly derive the following results :

$$\begin{aligned} \mathrm{Tr}\left(\Gamma\right) &= N\,S \quad , \\ \mathrm{Tr}\left(\Gamma\,\gamma^\mu\right) &= N\,V^\mu \quad , \\ \mathrm{Tr}\left(\Gamma\,\sigma^{\mu\nu}\right) &= 2N\,T^{\mu\nu} \quad , \\ \mathrm{Tr}\left(\Gamma\,\gamma^5\gamma^\mu\right) &= -N\,A^\mu \quad , \\ \mathrm{Tr}\left(\Gamma\,\gamma^5\right) &= N\,P \quad . \end{aligned} \tag{7.32}$$

This shows that we can indeed recover all coefficients from a given Γ. It also leads to the following useful insight : if all the above five traces vanish,

[17] It is clear that such trace evaluations are best performed by computer algebra.

[18] For even n, we have the proof here ; for odd n it is trivial since $0 = 0$.

then Γ itself must be identically zero. The above method of computing the Clifford coefficients from the algebra element is also called *Fierzing*.

A final, important remark : we have shown that the trace identities, which have been obtained using *only* Eq.(7.7), evaluate to expressions containing only the metric and the Levi-Civita symbol, which are Lorentz tensors. Therefore, *if* we can show that all matrix elements (or, at a pinch, their absolute squares) can be written as traces, we have realized our goal : the particular representation of the Dirac matrices is irrelevant, and all possible choices will lead unambiguously to a unique result.

E 30

Below, we shall prove that we may take $N = 4$: for this choice, we here summarize the more important trace identities.

$$
\begin{aligned}
&\mathrm{Tr}\,(1) = 4\\
&\mathrm{Tr}\,(\gamma^{\alpha_1}\,\gamma^{\alpha_2}\cdots\gamma^{\alpha_{2n+1}}) = 0\\
&\mathrm{Tr}\left(\gamma^{\alpha}\,\gamma^{\beta}\right) = 4\,g^{\alpha\beta}\\
&\mathrm{Tr}\left(\gamma^{\alpha}\,\gamma^{\beta}\,\gamma^{\mu}\,\gamma^{\nu}\right) = 4\,\left(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu}\right)\\
&\mathrm{Tr}\left(\gamma^{5}\right) = 0\\
&\mathrm{Tr}\left(\gamma^{5}\,\gamma^{\alpha}\,\gamma^{\beta}\right) = 0\\
&\mathrm{Tr}\left(\gamma^{5}\,\gamma^{\alpha}\,\gamma^{\beta}\,\gamma^{\mu}\,\gamma^{\nu}\right) = 4i\,\epsilon^{\alpha\beta\mu\nu}\\
&\mathrm{Tr}\,(\gamma^{\alpha_1}\,\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}) = \mathrm{Tr}\,(\gamma^{\alpha_n}\cdots\gamma^{\alpha_2}\,\gamma^{\alpha_1})
\end{aligned}
\tag{7.33}
$$

7.2.4 Dirac conjugation

The linear space in which the Dirac matrices operate can be endowed with an attractive notion of conjugation, called *Dirac conjugation*, which we shall now construct. Denoting the Dirac conjugation by an over-bar, we require that the Dirac matrices be all self-conjugate :

$$
\overline{\gamma^{\mu}} = \gamma^{\mu} \quad , \quad \mu = 0,1,2,3 \quad . \tag{7.34}
$$

Obviously, then, Dirac conjugation cannot be simple Hermitean conjugation, and we look for a definition of the form

$$\overline{\Gamma} = \Omega\,\Gamma^\dagger\,\Omega^{-1} \tag{7.35}$$

for any Clifford element Γ ; such a form ensures the reasonable property

$$\overline{\Gamma_1\Gamma_2} = \overline{\Gamma_2}\ \ \overline{\Gamma_1} \tag{7.36}$$

for two Clifford elements. Double conjugation should be equal to the identity :

$$\Gamma = \overline{\left(\ \overline{\Gamma}\ \right)} = \Omega\left(\Omega^{-1}\right)^\dagger \Gamma\ \Omega^\dagger\Omega^{-1} = B^{-1}\Gamma\ B \quad , \quad B = \Omega^\dagger\Omega^{-1}\ \ . \tag{7.37}$$

The element B must therefore commute with any Clifford element, which implies that B is a multiple of the unit element (this is a variant of Schur's lemma, see excercise **??**). Without loss of generality we may therefore take $B = 1$, so that Ω is Hermitean. The straightforward choice (in fact the only one, see excercise **??**) is therefore to take $\Omega = \gamma^0$, and the Dirac conjugate is then defined as

$$\overline{\Gamma} = \gamma^0\ \Gamma^\dagger\ \gamma^0\ \ . \tag{7.38}$$

For a spinor ξ (which carries an upper spinor index) we have

$$\overline{\xi} = \xi^\dagger\ \gamma^0\ \ , \tag{7.39}$$

which is seen to carry a *lower* spinor index. A *conjugate* spinor $\overline{\eta}$, which carries a lower index, obeys

$$\overline{\overline{\eta}} = \eta\ \ , \tag{7.40}$$

which has an upper index. A *spinor sandwich*[19] is an object of the form

$$\overline{\eta}\,\Gamma\,\xi\ \ ,$$

and it carries no spinor indices as can be seen ; reasonably, we have

$$\overline{\overline{\eta}\,\Gamma\,\xi} = \overline{\xi}\,\overline{\Gamma}\,\eta = \left(\overline{\eta}\,\Gamma\,\xi\right)^*\ \ . \tag{7.41}$$

Further conjugacy properties follow immediately from Eq.(7.34) :

$$\overline{\sigma^{\mu\nu}} = \sigma^{\mu\nu}\ \ ,\ \ \overline{\gamma^5\gamma^\mu} = \gamma^5\gamma^\mu\ \ ,\ \ \overline{\gamma^5} = -\gamma^5\ \ ,\ \ \overline{\omega_\pm} = \omega_\mp\ \ . \tag{7.42}$$

[19]Named after John Montagu, 4th Earl of Sandwich, PC, FRS (13 November 1718 - 30 April 1792).

In order for a general Clifford element of the form (7.19) to be self-conjugate, the coefficients S, V^μ, $T^{\mu\nu}$ and A^μ must be real, and P imaginary.

The standard Dirac spinors which we shall investigate are defined such that $\mathcal{W} = \overline{\mathcal{U}}$, although as we have already mentioned this is not an unavoidable choice to make. Note that the Dirac choice implies that

$$\overline{\mathcal{T}(p)} = \mathcal{T}(p) \quad . \tag{7.43}$$

7.2.5 Sandwiches as traces

Consider a spinor sandwich:

$$\overline{\eta}\,\Gamma\,\xi \quad .$$

In terms of explicit indices, this reads

$$\overline{\eta}\,\Gamma\,\xi = \sum_{a,b} \; (\overline{\eta})_a \; (\Gamma)^a{}_b \, \xi^b \quad . \tag{7.44}$$

Once we realize that the individual terms in this double sum are, in fact, simple numbers, it is clear that we may also write

$$\overline{\eta}\,\Gamma\,\xi = \sum_{a,b} \; \xi^b \; (\overline{\eta})_a \; (\Gamma)^a{}_b = \mathrm{Tr}\left(\xi\;\overline{\eta}\;\Gamma\right) \quad , \tag{7.45}$$

where $\xi\overline{\eta}$ is seen as a (dyadic) *matrix*. This 'mental flip', whereby we may suddenly interpret the combination spinor-conjugate spinor as a matrix, frequently turns out to be extremely useful in the evaluation of objects involving Dirac matrices.

7.2.6 A Fierz identity

As an application of what we have learned of the Clifford algebra, we shall prove the *Fierz identity*. This deals with the object

$$F(1,2,3,4) = \overline{\xi}_1\omega_+\gamma^\mu\xi_2 \;\; \overline{\xi}_3\omega_+\gamma_\mu\xi_4 \quad , \tag{7.46}$$

where the ξ's are arbitrary spinors. Obviously, $F(1,2,3,4) = F(3,4,1,2)$. Now, as F stands denoted above, it appears to be the (Minkowski) product of two spinor sandwiches, but we may also (by the 'mental flip' mentioned above) see it as the single sandwich

$$F(1,2,3,4) = \overline{\xi}_1\omega_+\gamma^\mu \left(\xi_2\overline{\xi}_3\right) \omega_+\gamma_\mu\xi_4 \quad , \tag{7.47}$$

since $\xi_2\overline{\xi}_3$ is an element of the Clifford algebra. We therefore have coefficients such that

$$\xi_2\overline{\xi}_3 = S + V_\alpha\gamma^\alpha + T_{\alpha\beta}\sigma^{\alpha\beta} + A_a\gamma^5\gamma^\alpha + P\gamma^5 \quad . \tag{7.48}$$

The contraction over the indices μ is then possible :

$$\begin{aligned}
&\omega_+\gamma^\mu\xi_2\overline{\xi}_3\omega_+\gamma^\mu = \\
&= \omega_+\gamma^\mu\left(V_\alpha\gamma^\alpha + A_\alpha\gamma^5\gamma^\alpha\right)\omega_+\gamma_\mu \\
&= \omega_+\gamma^\mu\left(V_\alpha\gamma^\alpha + A_\alpha\gamma^5\gamma^\alpha\right)\gamma_\mu \\
&= -2\omega_+\left(V_\alpha\gamma^\alpha - A_\alpha\gamma^\alpha\right) \quad ,
\end{aligned} \tag{7.49}$$

where we have used the fact that $\omega_+\Gamma\omega_+ = \omega_+\omega_-\Gamma = 0$ if Γ contains an *odd* number of Dirac matrices[20]. We can therefore write

$$F(1,2,3,4) = -2\overline{\xi}_1\omega_+\left(V_\alpha - A_\alpha\right)\gamma^\alpha\xi_4 \quad . \tag{7.50}$$

Now, we also know that, whatever the spinors ξ_2 and ξ_3 are,

$$\begin{aligned}
V_\alpha &= \frac{1}{N}\mathrm{Tr}\left(\xi_2\overline{\xi}_3\gamma_\alpha\right) = \frac{1}{N}\overline{\xi}_3\gamma_\alpha\xi_2 \quad , \\
A_\alpha &= -\frac{1}{N}\mathrm{Tr}\left(\xi_2\overline{\xi}_3\gamma^5\gamma_\alpha\right) = -\frac{1}{N}\overline{\xi}_3\gamma^5\gamma_\alpha\xi_2 \quad .
\end{aligned} \tag{7.51}$$

This leads us to the alternative form

$$\begin{aligned}
F(1,2,3,4) &= -\frac{2}{N}\overline{\xi}_1\omega_+\left(\overline{\xi}_3\left(1+\gamma^5\right)\gamma_\alpha\xi_3\right)\gamma^\alpha\xi_4 \\
F(1,2,3,4) &= -\frac{2}{N}\overline{\xi}_1\omega_+\gamma^\alpha\xi_4\;\;\overline{\xi}_3\left(1+\gamma^5\right)\gamma_\alpha\xi_3 \\
&= -\frac{4}{N}F(1,4,3,2) \quad .
\end{aligned} \tag{7.52}$$

As we have already mentioned, we shall show that $N = 4$ and the Fierz identity then becomes

$$F(1,2,3,4) = -F(1,4,3,2) \quad . \tag{7.53}$$

In words, the spinors ξ_2 and ξ_4 may be interchanged at the price of a minus sign[21].

---

[20] So that S, T, and P drop out.

[21] This is very suggestive, once we are convinced that the Dirac system describes fermions. However, the Fierz identity holds only for this particular sandwich, and relies heavily on the presence of the $\omega_\pm$. On the other hand again, it is eminently suited to resolve a potential problem in the Fermi model of muon decay, which we shall discuss later on.

7.2.7 The Chisholm identity

Consider a Clifford algebra element Γ that consists of only an *odd* number of γ matrices (that is, one or three). In that case it has the decomposition

$$\Gamma = V_\mu\,\gamma^\mu + A_\mu\,\gamma^5\gamma^\mu \quad . \tag{7.54}$$

Let us define the *reverse* Γ^R as the result of writing all the Dirac matrices involved in the reverse order[22]. By the reflection property of Eq.(7.29), this means that

$$\mathrm{Tr}\left(\Gamma^R\right) = \mathrm{Tr}\left(\Gamma\right) \quad , \tag{7.55}$$

for *all* elements of the Clifford algebra. In the present case, we have

$$\Gamma^R = V_\mu\,\gamma^\mu - A_\mu\,\gamma^5\gamma^\mu \quad . \tag{7.56}$$

Therefore,

$$\Gamma^R + \Gamma = 2V_\mu\,\gamma^\mu \quad . \tag{7.57}$$

We immediately arrive at the so-called *Chisholm identity* :

$$\gamma_\mu\,\mathrm{Tr}\left(\Gamma\,\gamma^\mu\right) = \frac{N}{2}\left(\Gamma + \Gamma^R\right) \quad . \tag{7.58}$$

This identity is quite be useful in the evaluation of spinor sandwiches that contain a free Lorentz index.

7.3 Dirac particles

7.3.1 Dirac spinors

The requirements on the object $\mathcal{T}(p)$ that we have gathered so far are that it be a member of the Clifford algebra, and that

$$\mathcal{T}(p)^2 = \mathcal{T}(p) \quad , \quad \overline{\mathcal{T}(p)} = \mathcal{T}(p) \quad , \tag{7.59}$$

although by a renormalization we may relax the first requirement into a proportionality. Now, it must be remembered that any modification of the

[22] Note that, fortunately, $(\gamma^5)^R = \gamma^5$, so that $(\gamma^5\gamma^\mu)^R = -\gamma^5\gamma^\mu$.

propagator *may* be compensated for by a transformation of the vertices : so, if there is a Clifford-algebra object Σ such that

$$\Sigma\overline{\Sigma} = \overline{\Sigma}\Sigma = 1 \quad ,$$

then, effectively, the propagator

$$\Sigma \;\; \mathcal{T}(p) \;\; \overline{\Sigma}$$

is equivalent to $\mathcal{T}(p)$ itself. We may then perform a search[23] through all inequivalent possibilities for $\mathcal{T}$. The upshot is that there are precisely four projection operators, for a choice of two Minkowski vectors k^μ and s^μ such that

$$k \cdot k = 1 \quad , \quad s \cdot s = -1 \quad , \quad k \cdot s = 0 \quad , \tag{7.60}$$

and they read

$$\Pi(\lambda_1, \lambda_2) = \frac{1}{4}\left(1 + \lambda_1 \not{k}\right)\left(1 + \lambda_2 \gamma^5 \not{s}\right) \quad , \tag{7.61}$$

where $\lambda_{1,2} = \pm 1$. We have

$$\overline{\Pi(\lambda_1, \lambda_2)} = \Pi(\lambda_1, \lambda_2) \tag{7.62}$$

and

$$\Pi(\lambda_1, \lambda_2)\Pi(\lambda_1', \lambda_2') = \;\delta_{\lambda_1,\lambda_1'}\;\delta_{\lambda_2,\lambda_2'}\;\Pi(\lambda_1, \lambda_2) \tag{7.63}$$

and also we conclude that, since there are precisely 4 projection operators, we can settle for $N = 4$ for the Dirac matrices[24]. Since for on-shell particles

[23] This is a quite tedious task, in particular the unearthing of the necessary Σ matrices. This is relegated to Appendix 8, based on the efforts of J. de Groot.

[24] This presupposes that a four-dimensional choice of dirac matrices *is* actually possible. This is the case, witness the so-called Pauli representation :

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad , \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad ,$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad . \tag{7.64}$$

Any other representation will do as well: that is the whole *point* of it !

$p^2 = m^2$ we can settle on $k^\mu = p^\mu/m$, and then the *new* degree of freedom is the choice of the vector s^μ which we shall call the *spin* vector. We are, then, naturally led to define two Dirac spinors, depending on momentum and spin vector, by

$$\begin{aligned} u(p,s)\overline{u}(p,s) &= \frac{1}{2}\left(\not{p}+m\right)\left(1+\gamma^5\not{s}\right) \quad , \\ v(p,s)\overline{v}(p,s) &= \frac{1}{2}\left(\not{p}-m\right)\left(1+\gamma^5\not{s}\right) \quad . \end{aligned} \tag{7.65}$$

These are defined for momenta p^μ that are on-shell, and have positive energy p^0. To see this last property, inspect

$$2p^0 = \mathrm{Tr}\left(u(p,s)\overline{u}(p,s)\gamma^0\right) = \overline{u}(p,s)\gamma^0 u(p,s) = u(p,s)^\dagger u(p,s) \quad , \tag{7.66}$$

which is cleary positive ; and the same goes for the spinor v. Spinors for negative-energy particles *can* be defined, but then they will not be Dirac spinors and the relation $\mathcal{W} = \overline{\mathcal{U}}$ does not hold. The following properties are easily ascertained :

$$\begin{aligned} &(\not{p} \pm m)^2 = \pm\, 2m(\not{p} \pm m) \quad , \quad (\not{p}+m)(\not{p}-m) = 0 \quad , \\ &(1 \pm \gamma^5\not{s})^2 = 2(1 \pm \gamma^5\not{s}) \quad , \quad (1+\gamma^5\not{s})(1-\gamma^5\not{s}) = 0 \quad , \\ &(\not{p} \pm m) \ \text{ and } \ (1 \pm \gamma^5\not{s}) \ \text{ commute} \quad , \end{aligned} \tag{7.67}$$

E 31 provided that $p \cdot p = m^2$, $s \cdot s = -1$ and $p \cdot s = 0$. We can immediately conclude that

$$\begin{aligned} &\overline{u}(p,s)u(p,s) = 2m \quad , \quad \overline{v}(p,s)v(p,s) = -2m \quad , \\ &\overline{u}(p,s)v(p,s') = 0 \quad , \quad \overline{u}(p,s)u(p,-s) = 0 \quad . \end{aligned} \tag{7.68}$$

E 32 Another point to be made here, and used later, is that the Dirac spinors contain all the information about their momentum and spin vectors. That is, if we are told that ξ is some Dirac spinor, then we can at once determine whether it is of the form $u(p,s)$ or $v(p,s)$ by computing $\bar{\xi}\xi$ and using Eq.(7.68) ; this will also tell us the value of m, and the sign will reveal whether we are dealing with a u or a v. If $\xi = u(p,s)$, we can recover p^μ and s^μ from

$$\bar{\xi}\,\gamma^\mu\,\xi = 2p^\mu \quad , \quad \bar{\xi}\,\gamma^5\gamma^\mu\,\xi = -2m\,s^\mu \quad ; \tag{7.69}$$

if, on the other hand $\xi = v(p,s)$ we use

$$\bar{\xi}\,\gamma^\mu\,\xi = 2p^\mu \quad , \quad \bar{\xi}\,\gamma^5\gamma^\mu\,\xi = +2m\,s^\mu \quad . \tag{7.70}$$

E 33

7.3.2 Example of the Casimir trick

In the last section we saw that u-spinors with the same momentum p and opposite spin vectors are orthogonal. Could there be other spin vector choices also yielding an orthogonal state ? To this end we can consider $\overline{u}(p,s)u(p,s')$ where s^μ and s'^μ are spin vectors. If the spinors refer to orthogonal quantum states, then the absolute square of the spinor product must vanish. We shall now compute this exactly, by turning the product into a trace using the so-called *Casimir trick*. It helps to write the Dirac indices explicitly for once :

$$
\begin{aligned}
&|\overline{u}(p,s)u(p,s')|^2 = \\
&= \sum_{a,b}\Big(\overline{u}(p,s)\Big)_a\Big(u(p,s')\Big)^a\Big(\overline{u}(p,s')\Big)_b\Big(u(p,s)\Big)^b \\
&= \sum_{a,b}\Big(u(p,s)\Big)^b\Big(\overline{u}(p,s)\Big)_a\Big(u(p,s')\Big)^a\Big(\overline{u}(p,s')\Big)_b \\
&= \sum_{a,b}\Big(u(p,s)\overline{u}(p,s)\Big)^b{}_a\Big(u(p,s')\overline{u}(p,s')\Big)^a{}_b \\
&= \sum_{b}\Big(u(p,s)\overline{u}(p,s)u(p,s')\overline{u}(p,s')\Big)^b{}_b \\
&= \mathrm{Tr}\Big(u(p,s)\overline{u}(p,s)\;\; u(p,s')\overline{u}(p,s')\Big)\quad .
\end{aligned}
\tag{7.71}
$$

For any correctly constructed amplitude involving Dirac particles, its absolute square is always amenable to the Casimir trick : traditionally, therefore, the evaluation of such amplitudes is done in this way[25]. This establishes the last requirement for the uniqueness (up to a phase) of matrix elements involving Dirac particles (*cf.* section 7.2.3). We can evaluate the trace by standard operations. For didactical purposes we give them here in excruciating detail :

$$
\begin{aligned}
&\mathrm{Tr}\Big(u(p,s)\overline{u}(p,s)\;\; u(p,s')\overline{u}(p,s')\Big) = \\
&= \frac{1}{4}\mathrm{Tr}\Big((\not{p}+m)(1+\gamma^5\not{s})(\not{p}+m)(1+\gamma^5\not{s}')\Big)
\end{aligned}
$$

[25] Note that there is a price: the *length* of the expressions is doubled by the squaring, and if the amplitude contains many diagrams the algebra can become very cumbersome indeed. A lot of computational shortcuts have been proposed, the most useful of which appears to be not to bother with squaring at all but rather to evaluate the spinor products themselves directly as complex numbers, by so-called *spinor techniques*. On the other hand, the *existence* of the Casimir trick ensures that, as required, one can completely get rid of the Dirac matrices in the prediction of cross sections using *only* their anticommutation properties.

$$
\begin{aligned}
&= \frac{1}{4}\mathrm{Tr}\left((\not{p}+m)^2(1+\gamma^5\not{s})(1+\gamma^5\not{s}')\right)\\
&= \frac{m}{2}\mathrm{Tr}\left((\not{p}+m)(1+\gamma^5\not{s})(1+\gamma^5\not{s}')\right)\\
&= \frac{m}{2}\mathrm{Tr}\left(\not{p}+m+\not{p}\gamma^5\not{s}+m\gamma^5\not{s}+\not{p}\gamma^5\not{s}'+m\gamma^5\not{s}'+\not{p}\gamma^5\not{s}\gamma^5\not{s}'+m\gamma^5\not{s}\gamma^5\not{s}'\right)\\
&= \frac{m}{2}\mathrm{Tr}\left(m+m\gamma^5\not{s}\gamma^5\not{s}'\right) = \frac{m}{2}\mathrm{Tr}\left(m-m\not{s}\not{s}'\right) = 2m^2\left(1-(s\cdot s')\right) \quad . \qquad (7.72)
\end{aligned}
$$

Note that only two out of the eight terms contain the right number of Dirac matrices to survive the trace. Since we can work in the p^μ rest frame, where the spin vectors must be spatial unit vectors, we conclude that, in that frame

$$
|\overline{u}(p,s)u(p,s')|^2 = 2m^2\left(1+\vec{s}\cdot\vec{s}'\right) \quad . \qquad (7.73)
$$

E 34 The states are only strictly orthogonal if $\vec{s}' = -\vec{s}$.

E 35

E 36

7.3.3 The Dirac propagator, and a convention

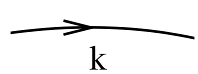
We have now arrived at a possible choice for the Dirac propagator. Since the two spin states described by $u\overline{u}$ should propagate in the same manner[26], we shall use the projection operator

$$
\mathcal{T}(p) = \not{p} + m \quad , \qquad (7.74)
$$

and adopt this choice also off the mass shell (where it is actually used). The Dirac propagator therefore takes the form

$$
i\hbar\,\frac{\not{p}+m}{p^2-m^2+i\epsilon} \quad .
$$

The fact that the numerator is linear in p means that the propagator is *oriented*, in contrast to what we have used so far. To indicate this we define the orientation with an arrow, and adhere to the convention that the momentum is counted *in the direction of the arrow*, irrespective of the sign of the energy component. The first Dirac Feynman rule therefore becomes



$$
\leftrightarrow \quad i\hbar\,\frac{\not{k}+m}{k\cdot k-m^2+i\epsilon}
$$

Feynman rules, version 7.1

(7.75)

[26]Otherwise we would not consider them to be states of the *same* particle

In writing out Feynman diagrams containing Dirac particles, we of course have to keep track of the Dirac indices resident in propagator and vertices. This may lead to incredibly cumbersome notation, that may however be greatly simplified if we adopt the following writing convention : **write out the Dirac-index carrying factors in order, moving *against* the orientation of the line**. Then, all these factors are contracted together using the usual rules for matrix multiplication, and one hardly ever needs to write the Dirac indices explicitly. This convention is really to be urged on anyone contemplating any calculation involving Dirac particles[27] !

A final word on notation : since

$$\left(\not{p}+m\right)\left(\not{p}-m\right) = p^2 - m^2 \quad , \tag{7.76}$$

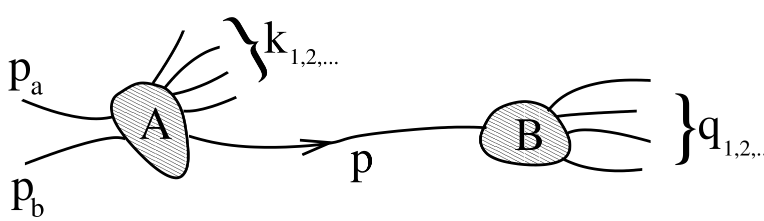
the Dirac propagator might be written as

$$i\hbar\frac{\not{p}+m}{p^2-m^2+i\epsilon} = \frac{i\hbar}{\not{p}-m+i\epsilon} \quad . \tag{7.77}$$

In instances were the $i\epsilon$ can be neglected, this is certainly allowed ; however in more delicate situations (such as inside loops) the first alternative is probably to be preferred. Nevertheless we shall occasionally also use Eq.(7.77).

7.3.4 Truncating Dirac particles : external Dirac lines

Let us now return to the truncation argument that gave us the Feynman rule for external lines in chapter 6. We shall redo this for Dirac particles moving between production and decay. As a first case, let the 'p'-line connecting production and decay be oriented from production to decay, as indicated in the following diagram :



According to the convention described above we then have for the amplitude

$$\mathcal{M} = [B]\frac{i\hbar(\not{p}+m)}{p^2-m^2+im\Gamma}[A] \quad . \tag{7.78}$$

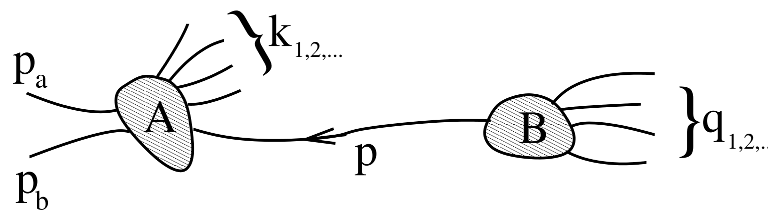
[27]Try it out for yourself ; after at most ten minutes you will be convinced.

Note that, in this amplitude, the factor $[A]$ must carry the upper Dirac index of a spinor, and $[B]$ the lower index of a conjugate spinor. p^μ, obviously, carries positive energy. As we let Γ vanish and p^μ approaches the mass shell, we may then write

$$\not{p} + m = \sum_s \; u(p,s)\,\overline{u}(p,s) \;\;, \tag{7.79}$$

where the sum over s runs over two values, s^μ and $-s^\mu$. Following the truncation argument, we readily see that the spinor $u(p,s)$ must then be included in the decay amplitude, and $\overline{u}(p,s)$ in the production amplitude.

In the alternative case, where the line is oriented against the flow of energy, the amplitude is given by



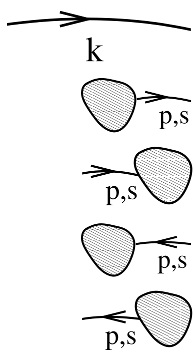
and reads (again with our convention !)

$$\mathcal{M} = [A]\frac{i\hbar(-\not{p} + m)}{p^2 - m^2 + im\Gamma}[B] \;\; . \tag{7.80}$$

Note that it is now $[A]$ that is the conjugate spinor, and $[B]$ the regular one. Of course, they describe a physical process different from the first case ! We are now forced by the negativity of the energy to write

$$-\not{p} + m = -\sum_s \; v(p,s)\,\overline{v}(p,s) \;\; . \tag{7.81}$$

The sign flip in the projection operator is of course precisely that which turns a particle description (with negative energy, moving backwards in time along the orientation of the propagator) into the *anti*particle description, with positive energy. The truncation argument then tells us that $v(p,s)$ must be the factor associated with the *production*, and $\overline{v}(p,s)$ must be associated with the *annihilation*, of the antiparticle. There remains the question of where to put the left-over *Fermi minus sign*. Consistently, we may decide to keep it with the $\overline{v}$, in which case we arrive at the following Dirac Feynman rules :



$$\leftrightarrow \quad i\hbar\,\frac{\not{k}+m}{k\cdot k-m^2+i\epsilon}$$

$$\leftrightarrow \quad \sqrt{\hbar}\;\overline{u}(p,s)$$

$$\leftrightarrow \quad \sqrt{\hbar}\;u(p,s)$$

$$\leftrightarrow \quad \sqrt{\hbar}\;v(p,s)$$

$$\leftrightarrow \quad -\sqrt{\hbar}\;\overline{v}(p,s)$$

Feynman rules, version 7.2

(7.82)

The awkward-looking minus sign is usually subjected to the argument that any matrix element containing an incoming antiparticle will have the factor $-\overline{v}$ in each of its diagrams, and since we are interested in absolute values squared anyway, there would appear to be little harm in deleting this overall minus sign from the Feynman rules : and this is what is commonly done. A little reflexion, though, will remind us that the *sign* of the amplitude's real part is fixed by unitarity, and now we have changed it ! Clearly, the minus sign will be back to haunt us later on.

7.3.5 The spin of Dirac particles

We shall now determine the spin of Dirac particles. Although the fact that they have two orthonormal spin states strongly suggests that they have spin-1/2, a real proof must rest on the way they form a representation of the rotation group. The rotation group is, of course, a subgroup of the Lorentz group. Now, we have argued that the vector p^μ and the matrix $\not{p}$ contain exactly the same information, for any vector p^μ. Therefore, we must be able to find how $\not{p}$ transforms under a Lorentz transformation. Let us define by $\Lambda(p;q)$ the *minimal* Lorentz transformation, that is it makes p^μ go over in q^μ while keeping *any* vector r^μ unchanged for which $p\cdot r = q\cdot r = 0$. Rotations are an example : in that case $p^0 = q^0 = 0$, $|\vec{p}| = |\vec{q}|$, and $\vec{r}\cdot\vec{p} = \vec{r}\cdot\vec{q} = 0$. Since $\not{p}$ is a matrix, the effect of a Lorentz transformation must be represented by a matrix transformation, that is

$$\Lambda(p;q): \quad \not{p} \;\to\; \Sigma_1\,\not{p}\,\Sigma_2 \quad . \tag{7.83}$$

Since we must ensure that Dirac conjugation commutes with Lorentz transformation, we must have $\Sigma_2 = \overline{\Sigma}_1$; and in order to have matrix multiplication commute with Lorentz transformations as well[28] we must have $\Sigma_2\Sigma_1 = 1$. We conlude that

$$\Lambda(p;q): \quad \not{p} \;\to\; \Sigma\,\not{p}\,\overline{\Sigma} \quad , \quad \Sigma\,\overline{\Sigma} = 1 \quad . \tag{7.84}$$

The explicit form of Σ reads[29]

$$\Sigma = C\left(1 + \frac{\not{q}\not{p}}{p^2}\right) \quad , \quad |C|^2 = \frac{p^2}{(p+q)^2} \quad . \tag{7.85}$$

You can simply check that this is indeed correct :

$$\begin{aligned} \Sigma\,\overline{\Sigma} &= |C|^2\left(1 + \frac{\not{q}\not{p} + \not{p}\not{q}}{p^2} + \frac{\not{q}\not{p}\not{p}\not{q}}{p^4}\right) \\ &= |C|^2\left(1 + \frac{2(pq)}{p^2} + \frac{p^2q^2}{p^4}\right) = 1 \quad , \end{aligned} \tag{7.86}$$

and

$$\begin{aligned} \Sigma\,\not{p}\overline{\Sigma} &= |C|^2\left(\not{p} + \frac{\not{q}\not{p}\not{p} + \not{p}\not{p}\not{q}}{p^2} + \frac{\not{q}\not{p}\not{p}\not{p}\not{q}}{p^4}\right) \\ &= |C|^2\left(\not{p} + 2\not{q} + \frac{\not{q}\not{p}\not{q}}{p^2}\right) = \not{q} \quad , \end{aligned} \tag{7.87}$$

where we have used the anticommutation result $\not{q}\not{p}\not{q} = 2(pq)\not{q} - \not{p}q^2$. The other requirements, $\overline{\Sigma}\not{q}\Sigma = \not{p}$ and $\Sigma\not{r}\overline{\Sigma} = \not{r}$, are proven trivially. For general Clifford elements Γ, we have now also ensured that

$$\Gamma \;\to\; \Sigma\,\Gamma\,\overline{\Sigma} \tag{7.88}$$

under Lorentz transformations. It is somewhat surprising to see that the form of the Lorentz transformation in Clifford space is quite simple. Since all spinorial dyads $\xi\overline{\eta}$ are Clifford elements, we find from the above that the transformation rules are

$$\xi \;\to\; \Sigma\,\xi \quad , \quad \overline{\xi} \;\to\; \overline{\xi}\,\overline{\Sigma} \quad . \tag{7.89}$$

---

[28]So that we can either first mutiply $\not{p}_1$ and $\not{p}_2$, and then Lorentz-transform them, or do the Lorentz transform first and the multiplication afterwards.

[29]This form tacitly assumes that under minimal Lorentz transforms the *sign* of p^2 and $(p+q)^2$ are the same. This is not obvious ; however, for boosts and spatial rotations it does hold.

Let us now select the spinor of a particle in its rest frame, and consider rotations of the space axes. By x^μ, y^μ and z^μ we shall mean the four-dimensional extensions of the spatial unit vectors in the x-, y- and z-directions, respectively. A rotation Σ_z over an infintesimal angle θ from x towards y around the z axis[30] is then determined by choosing

$$p^\mu = x^\mu \quad , \quad q^\mu = \cos(\theta)x^\mu + \sin(\theta)y^\mu \approx x^\mu + \theta y^\mu \ \ , \tag{7.90}$$

if we restrict ourselves to first order in θ. To this order, we find that $|C| = 1/2$, and so

$$\Sigma_z \approx \frac{1}{2}\left(1 - (\not{x} + \theta\not{y})\not{x}\right) = 1 + \frac{\theta}{2}\not{x}\not{y} \ \ . \tag{7.91}$$

(realize that $x^2 = y^2 = z^2 = -1$). The generators of the rotation group must therefore be[31]

$$T_x = \beta\not{y}\not{z} \quad , \quad T_y = \beta\not{z}\not{x} \quad , \quad T_z = \beta\not{x}\not{y} \ \ , \tag{7.92}$$

where we have used cyclicity, but not specified the constant β. This constant can be determined from the rotation group algebra requirement:

$$[T_x, T_y] = T_xT_y - T_yT_x = i\hbar T_z \ \ , \tag{7.93}$$

which for the Dirac system is seen to read

$$[T_x, T_y] = \beta^2\left(\not{y}\not{z}\not{z}\not{x} - \not{z}\not{x}\not{y}\not{z}\right) = 2\beta^2\not{x}\not{y} = 2\beta\ T_z \ \ , \tag{7.94}$$

from which we see that $\beta = i\hbar/2$. Noticing also that[32]

$$T_z{}^2 = \beta^2\not{x}\not{y}\not{x}\not{y} = -\beta^2x^2y^2 = \frac{\hbar^2}{4} = T_x{}^2 = T_y{}^2 \ \ , \tag{7.95}$$

we conclude that the total-spin operator comes to

$$\vec{T}^2 = T_x{}^2 + T_y{}^2 + T_z{}^2 = \frac{3}{4}\hbar^2 \ \ . \tag{7.96}$$

The spinors are, therefore, representatives of a spin-1/2 system.

[30]Here the confusing active-passive distinction rears its ugly head. We shall not worry about it since the rotation algebra is the same in each case.

[31]By inserting the Pauli representation of the Dirac matrices, one may figure out that these generators are nothing but the Pauli matrices in disguise. The present treatment aims at a more relativistic description.

[32]The fact that the square of any of the generators is proportional to the unit matrix is more or less a coincidence ; for systems with higher spins it no longer holds.

7.3.6 Full rotations in Dirac space

It is instructive to see how Dirac particles behave under certain *non*-infinitesimal rotations. To this end, consider the action of a rotation over $\pi/2$ in the $x-y$ plane ; we denote this by

$$\Sigma(\pi/2) = \frac{1}{\sqrt{2}}\left(1 - \not{y}\not{x}\right) \quad . \tag{7.97}$$

Taking powers of this rotation operator, we obtain, successively,

$$\begin{aligned} \Sigma(\pi) &= \Sigma(\pi/2)^2 = -\not{y}\not{x} \ , \\ \Sigma(2\pi) &= \Sigma(\pi/2)^4 = -1 \ , \\ \Sigma(4\pi) &= \Sigma(\pi/2)^8 = 1 \ . \end{aligned} \tag{7.98}$$

E 37 We see that a full rotation over 2π changes the sign of any spinor state ; to obtain the identically original state we have to rotate, instead, over 4π. In standard quantum-mechanical parlance, we say that the wave function for spin-1/2 particles is *two-valued.* Of course, under a rotation over just 2π any spinor *sandwich* is again transformed into itself.

7.3.7 Massless Dirac particles ; helicity states

In the projection operators $u(p,s)\overline{u}(p,s)$ and $v(p,s)\overline{v}(p,s)$ as we have defined them, the limit $m \to 0$ appears unproblematic. There is, however, a subtlety. Let us take a Dirac particle with definite *helicity* : in that case, the spin vector is parallel to the direction of motion[33]. Let us take $\vec{p}$ along the z axis for simplicity. Then, the requirements $s^2 = -1$, $(ps) = 0$ determine that

$$p^\mu = \begin{pmatrix} p^0 \\ 0 \\ 0 \\ p \end{pmatrix} \quad , \quad s^\mu = s_\parallel^\mu \equiv \begin{pmatrix} p/m \\ 0 \\ 0 \\ p^0/m \end{pmatrix} \quad , \tag{7.99}$$

where $p = |\vec{p}|$. As $m \to 0$, the spin vector diverges, and the massless limit is not so obvious. We may, however, write for this case

$$s_\parallel^\mu = \begin{pmatrix} p^0/m \\ 0 \\ 0 \\ p/m \end{pmatrix} + \frac{p^0 - p}{m}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{m}p^\mu + \mathcal{O}\left(m/p^0\right) \quad , \tag{7.100}$$

[33] This is, obviously, not a Lorentz-invariant notion. As the particle's velocity approaches c, however, it becomes Lorentz-invariant.

since $(p^0 - p)/m = m/(p^0 + p)$. The projection operator can then be evaluated by

$$
\begin{aligned}
u(p,s)\overline{u}(p,s) &= \frac{1}{2}\left(1+\gamma^5 \not{s}_{\parallel}\right)(\not{p}+m) \\
&= \frac{1}{2}\left(1+\frac{1}{m}\gamma^5\not{p}+\mathcal{O}\left(\frac{m}{p^0}\right)\right)(\not{p}+m) \\
&= \frac{1}{2}(1+\gamma^5)(\not{p}+m)+\mathcal{O}\left(\frac{m}{p^0}\right) \\
&\approx \omega_+\not{p} \quad , \qquad (7.101)
\end{aligned}
$$

which is well-defined. Of course, for s^μ antiparallel to the velocity, we find

$$
u(p,s)\overline{u}(p,s) \approx \omega_-\not{p} \quad . \qquad (7.102)
$$

These are the so-called *helicity states* for massless Dirac particles, which can also be written as[34]

$$
u_\lambda(p)\overline{u}_\lambda(p) = v_\lambda(p)\overline{v}_\lambda(p) = \omega_\lambda\not{p} \quad , \quad \lambda = \pm \quad . \qquad (7.103)
$$

Because of their simplicity, massless helicity states are very popular in high-energy calculations where fermion masses may be neglected ; but we should not forget that states without pure helicity are also possible. Indeed, we can consider the case where $\vec{p}$ and $\vec{s}$ make a fixed angle θ. In that case the spin vector reads

$$
s^\mu = \frac{m\cos\theta\ s_\parallel^\mu + p^0\sin\theta\ s_\perp^\mu}{\sqrt{(p^0)^2 - p^2\cos^2\theta}} \quad , \qquad (7.104)
$$

where

$$
s_\perp^\mu = \begin{pmatrix} 0 \\ \sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \quad . \qquad (7.105)
$$

Here ϕ denotes the azimuthal angle of $\vec{s}$ around $\vec{p}$. If we now let $m \to 0$ so that $p \to p^0$, then the limit of the projection operator becomes

$$
u(p,s)\overline{u}(p,s) \approx \frac{1}{2}(1+\gamma^5\not{s}_\perp)\not{p} \quad , \qquad (7.106)
$$

[34] Strictly speaking, the antiparticle of the right-handed particle is left-handed, whereas the above definition does not respect this. In practice this does not usually lead to confusion.

and we see that this limit is indistinguishable from a masless, *transversely* polarized Dirac particle. The message is that the massless limit is always defined, but must be taken with some care[35].

7.3.8 The parity transform

An interesting excercise is the following. Let ξ be an *arbitrary* spinor. The object

$$u(p,s) = C\ (\not{p}+m)(1+\gamma^5\not{s})\xi \tag{7.107}$$

is then *exactly* the spinor for a Dirac particle with momentum p^μ an spin vector s^μ, provided that C is chosen appropriately[36]. Now, let us consider

$$\gamma^0 u(p,s) = C\ \gamma^0(\not{p}+m)(1+\gamma^5\not{s})\xi\ \ . \tag{7.108}$$

By anticommuting the γ^0 to the right, we can arrive at

$$\gamma^0 u(p,s) = C\ (\hat{\not{p}}+m)(1+\gamma^5\hat{\not{s}})\ \gamma^0\xi\ \ . \tag{7.109}$$

Here, the vectors with and without hats are related as follows :

$$p^\mu = \begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix}\ \ ,\ \ \hat{p}^\mu = \begin{pmatrix} p^0 \\ -\vec{p} \end{pmatrix}\ \ \ ;\ \ \ s^\mu = \begin{pmatrix} s^0 \\ \vec{s} \end{pmatrix}\ \ ,\ \ \hat{s}^\mu = \begin{pmatrix} -s^0 \\ \vec{s} \end{pmatrix}\ \ . \tag{7.110}$$

Since $\gamma^0\xi$ is also an arbitrary spinor, the object $\gamma^0 u(p,s)$ is exactly the spinor $u(\hat{p},\hat{s})$ for a Dirac particle with momentum $\hat{p}^\mu$ and spin vector $\hat{s}^\mu$. What is this, precisely ? The spatial momentum of the particle has been reversed : this is called the *parity transform*. The spin vector, however, retains its spatial part while its time-part has now been flipped. The spin vector is, therefore, a four-vector of a different type from the more regular vector p^μ : such four-vectors are called *axial vectors*[37]. We conclude that multiplying a spinor by γ^0 induces its parity transform. For antiparticle spinors, as well as for the conjugate spinors, the treatment is completely identical.

[35] It is also clear that to produce, say, beams of ultrahigh-energy electrons with given helicity, one needs to be able to align the spin vector *very* precisely with the momentum, to angles of order m/p^0. Nevertheless, this is feasible in practice, as the LEP/SLC colliders have proven.

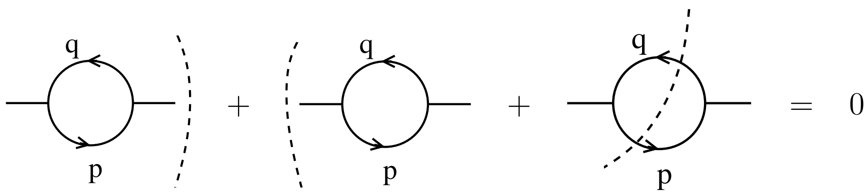
[36] This idea lies at the basis of the spinor techniques, to be discussed below.

[37] This explains the term 'axial-vector' coefficient we used in the Clifford algebra.

7.4 The Feynman rules for Dirac particles

7.4.1 Dirac loops...

As mentioned above, there is a natural tendency in formulating the Feynman rules to leave out the Fermi minus sign in the rules for external particles. Let us suppose that we choose to do that. Now, consider the following cutting rule :



Here, a scalar particle has a three-point coupling to a pair of Dirac particles[38]. We shall not evaluate the whole diagram, but rather concentrate on the two Dirac propagators. In the third, cut-through diagram, they occur as external lines, giving rise to a factor

$$\overline{u}(p)\Gamma_1 v(q) \quad \overline{v}(q)\Gamma_2 u(p) \ \ ,$$

where $\Gamma_{1,2}$ represent the rest of the diagrams. The momenta p and q are assumed to run from left to right. We have not indicated the spins since anyway we have to sum over them. Therefore we would have to evaluate the trace

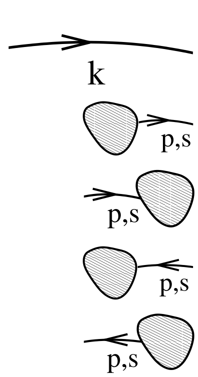
$$\mathrm{Tr}\left((\not{p} + m_p)\Gamma_1(\not{q} - m_q)\Gamma_2\right) \ \ ,$$

where we have indicated that the two Dirac particles are not necessarily of the same type. Let us now shift our attention to the first diagram, say. A closed loop of Dirac particles is automatically also a trace: this diagram, then, requires the analogous trace

$$\mathrm{Tr}\left((\not{p} + m_p)\Gamma_1(-\not{q} + m_q)\Gamma_2\right) \ \ ,$$

[38]The requirement that amplitudes do not contain uncontracted indices essentially forces us to use Feynman rules in which the orientation of Dirac lines is conserved at every vertex. For so-called *Majorana fermions* this is not true : Majorana fermions, therefore, have no distinction between particle and antiparticle.

since the momentum q is running against the orientation[39]. The second trace has the opposite sign of the first one ! To solve this problem (and save unitarity of the S matrix !) we therefore have to introduce an additional Feynman rule for Dirac particles :



$$\leftrightarrow \quad i\hbar\,\frac{\not{k}+m}{k\cdot k-m^2+i\epsilon}$$

$$\leftrightarrow \quad \sqrt{\hbar}\;\overline{u}(p,s)$$

$$\leftrightarrow \quad \sqrt{\hbar}\;u(p,s)$$

$$\leftrightarrow \quad \sqrt{\hbar}\;v(p,s)$$

$$\leftrightarrow \quad \sqrt{\hbar}\;\overline{v}(p,s)$$

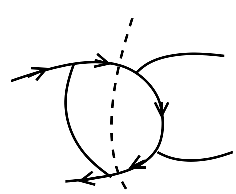
-1 for every closed Dirac loop

Feynman rules, version 7.3

(7.111)

7.4.2 . . . and Dirac loops only

In the above we have not yet explained why the minus sign must be assigned only to those closed loops that contain *only* Dirac particles. The reason for this is based on crossing symmetry. Consider a (cut) diagram like this one :



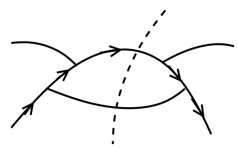
The lines without arrows have no Dirac propagators but just the 'original' ones[40]. The cut crosses two Dirac lines, and we might conclude that a minus

---

[39]We disregard the denominators of the Dirac propagators since they do not influence our argument.

[40]*i.e.* $i\hbar/(p^2-m^2+i\epsilon)$ for momentum p and mass m.

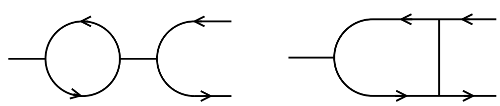
sign is called for. However, by crossing symmetry this diagram is related to



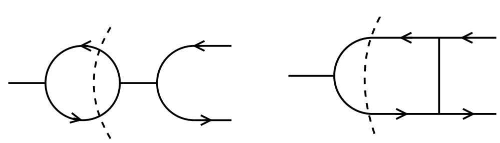
where now the cut crosses one Dirac line and one line without an arrow. Since the propagator in that line is even in its momentum, we can always choose the loop momentum to run in the 'correct' direction for the Dirac line, and no minus sign is needed. Therefore, the first diagram also takes no extra minus sign, since crossing symmetry forbids for an amplitude to suddenly pick up an extra minus sign under crossing. It is only when a closed loop consists of *only* Dirac particles that no crossing can be found for which the loop momentum can be chosen to run in the 'correct' direction. Therefore, only for such loops is a minus sign unavoidable[41].

7.4.3 Interchange signs

Consider the following two diagrams, that can both contribute to the decay of a scalar into a Dirac-antiDirac pair at the one-loop level :



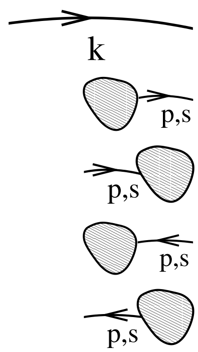
The first diagram contains a fermion loop and hence carries an overall minus sign ; the second one does not. Now consider the cut versions of these diagrams :



The left-hand sides of the cut-through diagrams are *identical*. The right-hand sides differ in the way that the in-going fermions are connected to the out-going ones ; the ingoing ones are interchanged in in the second diagram with respect to the first one. This, then, must correspond to a minus sign

[41]This holds true later on, where we also introduce vector particles, the propagator of which is also even in the momentum.

associated with the interchange of external lines in a diagram, and we arrive at the final form of the Feynman rules for Dirac particles :



$$\leftrightarrow \quad i\hbar \, \frac{\not{k} + m}{k \cdot k - m^2 + i\epsilon}$$

$$\leftrightarrow \quad \sqrt{\hbar}\, \overline{u}(p, s)$$

$$\leftrightarrow \quad \sqrt{\hbar}\, u(p, s)$$

$$\leftrightarrow \quad \sqrt{\hbar}\, v(p, s)$$

$$\leftrightarrow \quad \sqrt{\hbar}\, \overline{v}(p, s)$$

-1 for every closed Dirac loop

-1 for every Dirac particle interchange

Feynman rules, version 7.4

(7.112)

Note that the interchange rule only determines the *relative* sign between two Feynman diagrams. How the interchange sign can be determined is best illustrated by an example. Consider, for instance, a process with 6 external fermions. Three of them must then be oriented *out*ward from the diagram, carrying a $\overline{u}$ of $\overline{v}$, and the other three must be oriented *in*ward and carry a u or a v. Let us assume that there are three Feynman diagrams, schematically given by[42]

$$\begin{array}{ll} \text{diagram 1 :} & \overline{u}_1\Gamma_1 u_2 \;\; \overline{v}_3\Gamma_2 u_4 \;\; \overline{u}_5\Gamma_3 v_6 \\ \text{diagram 2 :} & \overline{u}_1\Gamma_4 u_2 \;\; \overline{v}_3\Gamma_5 v_6 \;\; \overline{u}_5\Gamma_6 u_4 \\ \text{diagram 3 :} & \overline{u}_1\Gamma_7 u_4 \;\; \overline{v}_3\Gamma_8 v_6 \;\; \overline{u}_5\Gamma_9 u_2 \quad , \end{array}$$

Clearly, we have left out an enormous amount of detail here, and the Γ's can be anything. Note that we have written the three diagrams in such a way that the conjugate spinors $\overline{u}_1$, $\overline{v}_3$ and $\overline{u}_5$ are in the same order in each diagram : this is always possible. Now, we see that to go from diagram 1 to diagram 2, the positions of u_4 and v_6 must be interhanged, whereas one can

[42]The process $e^-e^-e^+ \;\to\; e^-e^-e^+$ is an example.

go from diagram 1 to diagram 3 by, say, interchanging first u_2 and u_4, and then u_2 and v_6. Therefore, diagram 1 and 3 have no relative minus sign, and diagram 2 has a minus sign with respect to 1 and 3. In actual practice, the determination of the relative signs can be made even easier ; simply decide on *some preferred ordering* of all your u's, v's, $\overline{u}$'s and $\overline{v}$'s , and compare the ordering in your given diagram with your preferred one. Note that, since spinor sandwiches always contain two spinors, spinor sandwiches may be interchanged at will without destroying this simple rule.

Before finishing this section we want to make an important observation. The loop and interchange minus signs as we have discussed them depend on the structure of the diagrams, and *not* on the *type* of the Dirac particles ; even if a neutrino and a top quark were interchanged, the minus sign would crop up[43]. The minus signs depend only on the fact that they are Dirac particles, that is, spin-1/2 fermions. No notion of 'identical particles' is relevant here. **E 38**

E 39

7.4.4 The Pauli principle

Let us consider a possible experiment in which we attempt to produce two Dirac particles of the same type (two electrons, say), with exactly the same momentum and spin. Any such process is, in principle, described by Feynman diagrams. We can say immediately that the number of diagrams must be *even*, since for every diagram there must be a corresponding one in which the two electons are interchanged. Now, if the momenta and the spins of the two electrons are precisely the same, they will be described by identical conjugate spinors, and in fact the two diagrams of the pair will have exactly the same value — apart from the relative minus sign ! The total amplitude is therefore identically zero. We conclude that *it is not possible two* **produce** *two Dirac particles in exactly the same state*. By considering *incoming* electrons, we can also conclude that *it is not possible to* **observe** *two Dirac particles if they are in exactly the same state*, since the observation process is also describable (presumaby !) by Feynman diagrams. This is the Pauli exclusion principle[44].

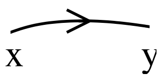
[43]Of course, the interactions in the theory may be such that no such interchange is *possible* : but this is beside the point.

[44]Note that I do not comment on the possibility that electrons in identical states might simply *exist* : they would not be observable by any process describable by Feynman diagrams. Their only influence could arise through some non-diagrammatic process, involving possibly gravity since that appears not to be amenable to diagrammatics. Of course, classi-

7.5 The Dirac equation

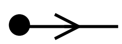
7.5.1 The classical limit

So far we have not mentioned the Dirac equation, nor have we had need for it. As an illustration, we shall show how it can be obtained. To this end, we need to provide a few Feynman rules in position, rather than in momentum space. The Dirac propagator, oriented from spacetime point x to spacetime point y, is

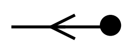


$$\leftrightarrow \quad \frac{i\hbar}{(2\pi)^4}\int d^4\; e^{-ik\cdot(y-x)}\frac{\not{k}+m}{k^2-m^2} \quad , \tag{7.113}$$

where we have dropped the $i\epsilon$ for simplicity. The Dirac particles are created by a *spinorial* source $J(x)$, and absorbed by a *conjugate-spinorial* source $\bar{J}(x)$, with the rules

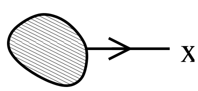


$$\leftrightarrow \quad -\frac{i}{\hbar}J(x) \quad ,$$



$$\leftrightarrow \quad -\frac{i}{\hbar}\bar{J}(x) \quad . \tag{7.114}$$

If we forget about any other couplings, the Dirac field is free, and its SDe is *exactly* its own classical limit. Now, consider the following form of it :



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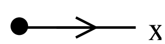


$$. \tag{7.115}$$

With the field function of the Dirac field denoted by $\psi(x)$, this SDe reads

$$\psi(x) = \frac{1}{(2\pi)^4}\int d^4y\; d^4k\; e^{-ik\cdot(x-y)}\frac{\not{k}+m}{k^2-m^2}J(y) \quad , \tag{7.116}$$

where matrix multiplication is implied as usual. We can now study the object

$$\begin{aligned}\Big(i\not{\partial}-m\Big)\,\psi(x) &= \\ &= \frac{1}{(2\pi)^4}\int d^4y\; d^4k\; e^{-ik\cdot(x-y)}(\not{k}-m)\frac{\not{k}+m}{k^2-m^2}J(y)\end{aligned}$$

---

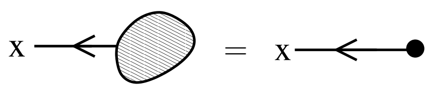
cal quantum mechanics finds that the combined wave function for identical-state electrons vanishes identically, but again quantum and gravity do not see completely eye to eye.

$$
\begin{aligned}
&= \frac{1}{(2\pi)^4} \int \; d^4y \; d^4k \; e^{-ik\cdot(x-y)} J(y) \\
&= \int d^4y \; \delta^4(x-y) \; J(y) = J(x) \quad , \qquad (7.117)
\end{aligned}
$$

which is the classical Dirac equation :

$$
\Big(i\partial\!\!\!/ - m \Big) \, \psi(x) = J(x) \quad . \qquad (7.118)
$$

We can also consider the ‘Dirac-conjugate’ SDe :



$$, \qquad (7.119) $$

which is written as

$$
\bar{\psi}(x) = \frac{1}{(2\pi)^4} \int d^4y \; d^4k \; \bar{J}(y) \frac{k\!\!\!/ + m}{k^2 - m^2} \; e^{-ik\cdot(y-x)} \quad . \qquad (7.120)
$$

By the same simple manipulation as above, we can then show that the conjugate Dirac equation reads

$$
\bar{\psi}(x) \Big(- i \overleftarrow{\partial\!\!\!/} - m \Big) = \bar{J}(x) \quad , \qquad (7.121)
$$

where the leftward arrow indicates that the derivative must be taken towards the *left*[45].

7.5.2 The free Dirac action

We can cast the above in the form of the – possibly more familiar – Lagrangian treatment. The action for the free Dirac field including sources is then given by

$$
S[\psi, \bar{\psi}, J, \bar{J}] = \int d^4x \; \mathcal{L}(x) \quad , \qquad (7.122)
$$

where the Dirac Lagrangian is given by

$$
\mathcal{L}(x) = \bar{\psi}(x) \left(i\partial\!\!\!/ - m \right) \psi(x) - \bar{J}(x)\psi(x) - \bar{\psi}(x)J(x) \quad . \qquad (7.123)
$$

[45]A word of caution is in order here. The operator $i\partial\!\!\!/$ is self-conjugate and does not change under Hermitian conjugation. The minus sign in front of it comes from the fact that the *direction* of the derivative is now also reversed.

This Lagrangian does not contain a derivative of $\bar{\psi}$: the Euler-Lagrange equation is therefore simply

$$\frac{\delta S}{\delta\bar{\psi}(x)} = \int d^4y\; \frac{\delta\mathcal{L}(y)}{\delta\bar{\psi}(x)} = 0 \quad , \tag{7.124}$$

which is seen to be exactly Eq.(7.118). By partial integration we can see that the same action can also be obtained from the Lagrangian

$$\hat{\mathcal{L}}(x) = \bar{\psi}(x)\left(-i\overleftarrow{\not\partial} - m\right)\psi(x) - \bar{J}(x)\psi(x) - \bar{\psi}(x)J(x) \quad , \tag{7.125}$$

which is now independent of any derivative of ψ. The Euler-Lagrange equation for ψ,

$$\frac{\delta S}{\delta\psi(x)} = \int d^4y\; \frac{\delta\hat{\mathcal{L}}(y)}{\delta\psi(x)} = 0 \quad , \tag{7.126}$$

gives us precisely Eq.(7.121). Finally, it is easily seen that the dimensionality of the field ψ is given by

$$\mathbf{dim}\Big[\psi\Big] = \mathbf{dim}\left[\frac{\hbar^{1/2}}{L^{3/2}}\right] \quad . \tag{7.127}$$

7.6 The standard form for spinors

7.6.1 Definition of the standard form for massless particles

In the special case where the momentum is massless, a very handy form for the spinors may be chosen, which we shall call the *standard form*. Let p^μ be the momentum of the spinor, so that $p^2 = 0$. We now choose two *basis vectors* k_0^μ and k_1^μ, which satisfy

$$k_0\cdot k_0 = k_0\cdot k_1 = 0 \quad , \quad k_1\cdot k_1 = -1 \quad . \tag{7.128}$$

Furthermore we require that $k_0\cdot p \neq 0$ for *any* massless momentum p^μ encountered in the problem at hand ; this is usually not difficult to arrange. Since k^0 is massless, it may serve to define the *basis spinor*

$$u_0 \equiv u_-(k_0) \quad \Rightarrow \quad u_0\overline{u}_0 = \omega_-\not{k}_0 \quad . \tag{7.129}$$

The reversal of this object gives us

$$(u_0\overline{u}_0)^R = (\omega_- \not{k})^R = \omega_+ \not{k}_0 = u_+(k_0)\overline{u}_+(k_0) = \not{k}_1\ u_0\overline{u}_0\ \not{k}_1 \quad . \tag{7.130}$$

Using the basis spinor, we now define *all* other massless spinors by

$$u_+(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_0 \quad , \quad u_-(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}\not{k}_1 u_0 \quad . \tag{7.131}$$

We can immediately check that $u_\pm(p)\overline{u}_\pm(p) = \omega_\pm\not{p}$, so that these spinorial objects are indeed admissible choices ; in fact, the standard form is nothing more than a (very useful) phase convention of all occurring spinors. This choice is at the basis of the so-called *spinor techniques* : the above definition will be applied to good effect in what follows.

7.6.2 Some useful identities

At this point we prove a few results that often turn out to be useful. In the first place, from the property $\mathrm{Tr}\,(\Gamma) = \mathrm{Tr}\left(\Gamma^R\right)$, we can see that

$$\begin{aligned}
\overline{u}_+(p_1)\ \gamma^\mu\ u_+(p_2) &= K\ \overline{u}_0\ \not{p}_1\ \gamma^\mu\ \not{p}_2 u_0 \\
&= K\ \mathrm{Tr}\,(u_0\overline{u}_0\ \not{p}_1\ \gamma^\mu\ \not{p}_2) \\
&= K\ \mathrm{Tr}\left(\not{p}_2\ \gamma^\mu\ \not{p}_1 (u_0\overline{u}_0)^R\right) \\
&= K\ \mathrm{Tr}\,(\not{p}_2\ \gamma^\mu\ \not{p}_1\ \not{k}_1\ u_0\overline{u}_0\ \not{k}_1) \\
&= K\ \overline{u}_0\ \not{k}_1\ \not{p}_2\ \gamma^\mu\ \not{p}_1\ \not{k}_1\ u_0 \quad ,
\end{aligned} \tag{7.132}$$

with $K = (4p_1\cdot k_0\ p_2\cdot k_0)^{-1/2}$, which leads to the useful *spinor reversal* :

$$\overline{u}_+(p_1)\gamma^\mu u_+(p_2) = \overline{u}_-(p_2)\gamma^\mu u_-(p_1) \quad . \tag{7.133}$$

In fact this can easily be generalized to

$$\overline{u}_{\lambda_1}\,\Gamma\, u_{\lambda_2}(q) = \lambda_1\lambda_2\ \overline{u}_{-\lambda_2}(q)\,\Gamma^R\, u_{-\lambda_1}(p) \tag{7.134}$$

In the second place, the standard form for the spinors allows us to relate $+$ and $-$ helicities, for instance, for massless p and q, and with $K^{-2} = 4(p\cdot k_0)(q\cdot k_0)$:

$$\begin{aligned}
&\gamma_\alpha\ u_\pm(p)\ \overline{u}_\pm(q)\ \gamma^\alpha = K\ \ \gamma_\alpha\ \not{p}\ \omega_\mp\ \not{k}_0\ \not{q}\ \gamma^\alpha \\
&\quad = \ -2K\ \ \not{q}\ \omega_\pm\ \not{k}_0\ \not{p}\ \ = \ -2\ u_\mp(q)\ \overline{u}_\mp(p)
\end{aligned} \tag{7.135}$$

Since the standard form of spinors is just a phase convention, a relation like Eq.(7.135) holds in other conventions as well ; only the factor -2 may pick up a complex phase that is elegantly absent here. In the last place, the Chisholm identity of Eq.(7.58) can be applied to simple spinor sandwiches so as to yield

$$\left(\overline{u}_\pm(p_1)\gamma^\mu u_\pm(p_2)\right)\ \gamma_\mu = 2\left\{u_\pm(p_2)\overline{u}_\pm(p_1) + u_\mp(p_1)\overline{u}_\mp(p_2)\right\}\ . \tag{7.136}$$

7.6.3 Spinor products

We may compute an explicit expression for the product of two spinors for massless momenta : we shall define

$$s_\pm(p,q) \equiv \overline{u}_\pm(p)\ u_\mp(q)\ . \tag{7.137}$$

For standard spinors, this can be evaluated using the Casimir trick

$$\begin{aligned} s_+(p,q) &= (4(p\cdot k_0)(q\cdot k_0))^{-1/2}\ \overline{u}_0\ \not{p}\ \not{q}\ \not{k}_1\ u_0 \\ &= (4(p\cdot k_0)(q\cdot k_0))^{-1/2}\ \mathrm{Tr}\left(\omega_- \not{k}_0 \not{p} \not{q} \not{k}_1\right) \\ &= \frac{1}{\sqrt{(p\cdot k_0)(q\cdot k_0)}}\Big((p\cdot k_0)(q\cdot k_1) - (p\cdot k_1)(q\cdot k_0) \\ &\qquad - \ i\epsilon_{\mu\nu\alpha\beta}{k_0}^\mu {k_1}^\nu p^\alpha q^\beta\Big)\ . \end{aligned} \tag{7.138}$$

This is antisymmetric in $p\ \leftrightarrow\ q$, and moreover

$$s_-(p,q) = -\ s_+(p,q)^*\ . \tag{7.139}$$

In addition, it is easily seen that

$$s_+(p,q)s_-(q,p) = |s_+(p,q)|^2 = \overline{u}_+(p)\ \not{q}\ u_+(p) = 2(p\cdot q)\ . \tag{7.140}$$

Spinor products are therefore somewhat like 'square roots' of vector products.

Finally, we may consider an explicit choice for the vectors $k_{0,1}^\mu$:

$${k_0}^\mu = (1,1,0,0)\quad ,\quad {k_1}^\mu = (0,0,1,0)\quad : \tag{7.141}$$

this gives the explicit form for the spinor product

$$s_+(p,q) = \left(p^2 + ip^3\right)\sqrt{\frac{q^0-q^1}{p^0-p^1}}\ -\ \left(q^2+iq^3\right)\sqrt{\frac{p^0-p^1}{q^0-q^1}}\ , \tag{7.142}$$

which is very useful for actual numerical applications. Note that this choice presupposes that none of the light-like vectors in the problem is oriented exactly along the x-axis. Since the 'special' direction in many problems is traditionally chosen to be the z-axis, this is usually safe. **E 40**

7.6.4 The Schouten identity

There exists a useful identity for massless-momentum spinors in the standard representation. For massless $p_{1,2,3,4}$, there is the truism

$$\overline{u}_+(p_1)\not{p}_2\not{p}_3 u_-(p_4) + \overline{u}_+(p_1)\not{p}_3\not{p}_2 u_-(p_4) - 2(p_2\cdot p_3)\;\overline{u}_+(p_1)u_-(p_4) = 0 \;\; . \quad (7.143)$$

Writing this out in terms of spinor products, we have

$$\begin{aligned} & s_+(p_1,p_2)s_-(p_2,p_3)s_+(p_3,p_4) + s_+(p_1,p_3)s_-(p_3,p_2)s_+(p_2,p_4) \\ & -s_+(p_2,p_3)s_-(p_3,p_2)s_+(p_1,p_4) \;\; = \;\; 0 \;\; . \end{aligned} \quad (7.144)$$

Using the antisymmetry property of s, and dividing out the factor $s_-(p_2,p_3)$, we obtain the so-called *Schouten identity* :

$$s_+(p_1,p_2)s_+(p_3,p_4) + s_+(p_1,p_3)s_+(p_4,p_2) + s_+(p_1,p_4)s_+(p_2,p_3) = 0 \;\; . \quad (7.145)$$

Note the cyclicity in $p_{2,3,4}$. Obviously, the identity holds for s_- as well.

7.6.5 Summary of relations for the standard form

Here we briefly summarize the relations and tricks that can be used in calculations with helicity states for massless fermions in the standard formulation. All momenta are massless here, and the symbols λ take on the values + or –. It is allowable to only use the u spinors and not the v antispinors, and in
E 41 fact that is what is done in most calculations.

E 42

- Dirac equations :

$$\not{p}\, u_\lambda(p) = 0 \quad , \quad \omega_\lambda\, u_\lambda(p) = u_\lambda(p) \quad , \quad \omega_{-\lambda}\, u_\lambda(p) = 0$$

- Projection operator :

$$u_\lambda(p)\, \overline{u}_\lambda(p) = \omega_\lambda\, \not{p}$$

- Spinor products :

$$\begin{aligned}
&\overline{u}_\lambda(p)\, u_\lambda(q) = 0 \\
&s_\lambda(p,q) = \overline{u}_\lambda(p)\, u_{-\lambda}(q) = -s_\lambda(q,p) = -s_{-\lambda}(p,q)^* \\
&|s_\lambda(p,q)|^2 = s_\lambda(p,q)\, s_{-\lambda}(q,p) = 2(p\cdot q)
\end{aligned}$$

- Elimination of repeated indices :

$$\gamma_\alpha\, u_\lambda(p)\, \overline{u}_\lambda(q)\, \gamma^\alpha = -2\, u_{-\lambda}(q)\, \overline{u}_{-\lambda}(p)$$

- Chisholm identity :

$$\left[\overline{u}_\lambda(p)\, \gamma_\alpha\, u_\lambda(q)\right]\ \gamma^\alpha = 2\ \left[u_\lambda(q)\, \overline{u}_\lambda(p) + u_{-\lambda}(p)\, \overline{u}_{-\lambda}(q)\right]$$

- Reversal :

$$\overline{u}_{\lambda_1}(p)\, \Gamma\, u_{\lambda_2}(q) = \lambda_1\lambda_2\, \overline{u}_{-\lambda_2}(q)\, \Gamma^R\, u_{-\lambda_1}(p)$$

- Schouten identity :

$$s_\lambda(q,p_1)\, s_\lambda(p_2,p_3) + s_\lambda(q,p_2)\, s_\lambda(p_3,p_1) + s_\lambda(q,p_3)\, s_\lambda(p_1,p_2) = 0$$

E 43

7.6.6 The standard form for massive particles

The standard form for Dirac spinors given in Eq.(7.131) can be simply expanded to the case of massive particles. Let p^μ be the momentum of such a particle, and let m be its mass. We then define

$$
\begin{aligned}
u_\pm(p) &= \frac{1}{\sqrt{2p\cdot k_0}}(\not{p}+m)u_\mp(k_0) \quad , \\
v_\pm(p) &= \frac{1}{\sqrt{2p\cdot k_0}}(\not{p}-m)u_\mp(k_0) \quad .
\end{aligned}
\tag{7.146}
$$

From Eqns.(7.69, 7.70) we can find out the spin vector for these two cases : writing $u_\pm(p) = u(p, \pm s_0)$ we obtain

$$
\begin{aligned}
{s_0}^\mu &= -\frac{1}{2m} u_+(p)\,\gamma^5\gamma^\mu\, u_+(p) \\
&= -\frac{1}{4m\,p\cdot k_0}\mathrm{Tr}\left(\omega_-\not{k}_0(\not{p}+m)\gamma^5\gamma^\mu(\not{p}+m)\right) \\
&= \frac{1}{m}p^\mu \;-\; \frac{m}{(pk_0)}{k_0}^\mu \quad ,
\end{aligned}
\tag{7.147}
$$

which is indeed the only vector built from p and k_0 that can have the right properties ${s_0}^2 = -1$ and $(ps_0) = 0$. Note that for small(ish) m and generally positioned k_0, $\vec{s}_0$ points in the general direction of $\vec{p}$. Therefore we call $u_+(p)$ a *right-handed spinor*, and $u_-(p)$ a *left-handed spinor*. In addition, from the fact that, for the antispinor $v_\pm(p)$,

$$
\frac{1}{2m}\overline{v}_+(p)\,\gamma^5\gamma^\mu\, v_+(p) = -{s_0}^\mu \quad , \tag{7.148}
$$

we see that $v_+(p)$ is a *left-handed antispinor* and $v_-(p)$ is a *right-handed antispinor*.

The standard spinors suffice to build up other spinors as well. To see this, consider a general superposition of $u_+(p)$ and $u_-(p)$:

$$
\xi = \alpha\, u_+(p) + \beta\, u_-(p) \quad , \quad |\alpha|^2 + |\beta|^2 = 1 \quad . \tag{7.149}
$$

Without loss of generality we may take

$$
\alpha = \sin\left(\frac{\theta}{2}\right)\, e^{-i\varphi} \quad , \quad \beta = \cos\left(\frac{\theta}{2}\right) \quad . \tag{7.150}
$$

The spin vector hidden inside the general spinor ξ is seen to be

$$\begin{aligned} -\frac{1}{2m}\bar{\xi}\gamma^5\gamma^\mu\xi &= \cos(\theta){s_0}^\mu + \sin(\theta)\cos(\varphi){s_{//}}^\mu + \sin(\theta)\sin(\varphi){s_\perp}^\mu \quad , \\ {s_{//}}^\mu &= {k_1}^\mu - \frac{(pk_1)}{(pk_0)}{k_0}^\mu \quad , \\ {s_\perp}^\mu &= \frac{1}{(pk_0)}\epsilon^\mu{}_{\nu\rho\sigma}{k_1}^\nu {k_0}^\rho p^\sigma \quad . \end{aligned} \tag{7.151}$$

Since

$$p \cdot s_0 = p \cdot s_{//} = p \cdot s_\perp = s_0 \cdot s_{//} = s_0 \cdot s_\perp = s_{//} \cdot s_\perp = 0 \tag{7.152}$$

and

$${s_{//}}^2 = {s_\perp}^2 = -1 \quad , \tag{7.153}$$

we see that every allowed spin vector is, in fact, accessible by taking a superposition of two standard forms : the vectors p^μ/m, ${s_0}^\mu$, ${s_{//}}^\mu$ and ${s_\perp}^\mu$ form an orthonormal basis.

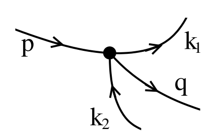
7.7 Muon decay in the Fermi model

7.7.1 The amplitude

An example of an actually occurring process involving *only* Dirac particles is provided by muon decay in the Fermi model. The process is[46]

$$\mu^-(p) \quad \to \quad e^-(q) \;\; \nu_\mu(k_1) \;\; \overline{\nu}_e(k_2)$$

and is pictured by the single Feynman diagram



Here, a muon at rest undergoes a three-particle decay into an electron, a muon neutrino and an electron antineutrino. We shall assume the neutrinos

[46]In this section, the vector k_1 is a momentum, and has nothing to do with the auxiliary vector of section 7.6.

to be massless. The Fermi amplitude introduced to describe the phenomenology of this process contains only a single pointlike vertex where four fermions meet with a coupling constant called $G_F/\sqrt{2}$, and is given by

$$\mathcal{M} = i\frac{G_F\,\hbar}{\sqrt{2}}\;\overline{u}(q)\,(1+\gamma^5)\gamma_\alpha\,v(k_2)\;\overline{u}(k_1)\,(1+\gamma^5)\gamma^\alpha\,u(p)\;\;. \tag{7.154}$$

The decision to 'hook up' the muon and the muon neutrino is in principle arbitrary[47], but as we have seen in section 7.2.6 we may easily interchange the muon neutrino and the electron, and end up with the matrix element in the 'charge retention form' :

$$\mathcal{M} = -i\frac{G_F\,\hbar}{\sqrt{2}}\;\overline{u}(q)\,(1+\gamma^5)\gamma_\alpha\,u(p)\;\overline{u}(k_1)\,(1+\gamma^5)\gamma^\alpha\,v(k_2)\;\;.$$

The amplitude (7.154) implies that the neutrinos must have negative helicity[48] : we can write

$$\mathcal{M} = i\frac{4G_F\,\hbar}{\sqrt{2}}\;\overline{u}(q)\,\gamma_\alpha\,v_-(k_2)\;\overline{u}_-(k_1)\,\gamma^\alpha\,u(p)\;\;. \tag{7.155}$$

We can now apply the result (7.135) to arrive at the very compact form

$$\mathcal{M} = -i\frac{8G_F\,\hbar}{\sqrt{2}}\;\overline{u}(q)\,u_+(k_1)\;\overline{v}_+(k_2)\,u(p)\;\;. \tag{7.156}$$

The transition rate can now easily computed with a few simple traces :

$$\begin{aligned}\left\langle|\mathcal{M}|^2\right\rangle &= \frac{1}{2}\sum_{\text{spins of }\mu,e}|\mathcal{M}|^2\\ &= 16\,{G_F}^2\,\hbar^2\,\mathrm{Tr}\left((\not{q}+m_e)\omega_+\not{k}_1\right)\;\;\mathrm{Tr}\left((\not{q}+m_\mu)\omega_+\not{k}_2\right)\\ &= 64\,G_F^2\,\hbar^2\,(q\cdot k_1)\,(p\cdot k_2)\;\;.\end{aligned} \tag{7.157}$$

It is practical to evaluate this in the muon rest frame. We shall write $E_{1,2}$ for ${k_{1,2}}^0$ in this frame. Then $(p\cdot k_2)$ is equal to $m_\mu E_2$, and by momentum conservation we find

$$(q\cdot k_1) = \frac{1}{2}\left((q+k_1)^2-{m_e}^2\right) = \frac{1}{2}\left((P-k_2)^2-{m_e}^2\right) = m_\mu(K-E_2)\;\;, \tag{7.158}$$

[47]Unless lepton flavour number is invoked.

[48]In the standard form of spinors, the helicity for antispinors is reversed. The antineutrino therefore actually has *positive* handedness.

where

$$K = \frac{{m_\mu}^2 - {m_e}^2}{2m_\mu} \quad . \tag{7.159}$$

The transition rate then takes the form

$$\left\langle |\mathcal{M}|^2 \right\rangle = 64\ {G_F}^2\ \hbar^2\ {m_\mu}^2\ E_2(K - E_2) \quad , \tag{7.160}$$

and for the partial decay width we find

$$d\Gamma(\mu \to e\nu_\mu\overline{\nu}_e) = 32\ {G_F}^2\ \hbar^2\ m_\mu E_2(K - E_2)\ dV(p; q, k_1, k_2) \quad . \tag{7.161}$$

7.7.2 Three-body phase space

The phase space for the muon decay process reads

$$\begin{aligned} dV(p; q, k_1, k_2) \quad &= \quad \frac{1}{(2\pi)^5}\ d^4q\ d^4k_1\ d^4k_2\ \delta^4(p - q - k_1 - k_2) \\ &\qquad \delta(q^2 - {m_e}^2)\ \delta({k_1}^2)\ \delta({k_2}^2) \quad . \end{aligned} \tag{7.162}$$

Since the rate depends only on E_2, we shall implicitly integrate over all other phase space variables. By cancelling the q integration against the Dirac delta for momentum conservation, we arrive at

$$\begin{aligned} dV(p; q, k_1, k_2) \quad &= \quad \frac{1}{(2\pi)^5}\ \frac{E_1 E_2}{4}\ dE_1\ dE_2\ d\Omega_1\ d\Omega_2 \\ &\qquad \delta((p - k_1 - k_2)^2 - {m_e}^2) \quad . \end{aligned} \tag{7.163}$$

The Dirac delta function can be written as

$$\delta\left({m_\mu}^2 - {m_e}^2 - 2m_\mu E_1 - 2m_\mu E_2 + 2E_1E_2 - 2E_1E_2\cos\theta\right) \quad ,$$

where θ is the angle between the neutrino momenta. Hence we can integrate trivially over the other polar and the two azimuthal angles (leading to a factor $8\pi^2$), and the integral over θ is resolved by the delta function. The result is

$$dV(p; q, k_1, k_2) = \frac{\pi^2}{(2\pi)^5}\ dE_1\ dE_2 \quad . \tag{7.164}$$

In terms of these variables, the phase space is perfectly flat[49]. Since $|\cos\theta|$ cannot exceed unity, we also have the restrictions

$$\begin{aligned} {m_\mu}^2 - {m_e}^2 - 2m_\mu E_1 - 2m_\mu E_2 &\leq 0 \;, \\ {m_\mu}^2 - {m_e}^2 - 2m_\mu E_1 - 2m_\mu E_2 + 4E_1E_2 &\geq 0 \;, \end{aligned} \tag{7.165}$$

which we can work into bounds on E_1 :

$$K - E_2 \leq E_1 \leq \hat{K}(E_2) \equiv \frac{{m_\mu}^2 - {m_e}^2 - 2m_\mu E_2}{2(m_\mu - 2E_2)} \;, \tag{7.166}$$

while E_2 is seen to run from 0 to K.

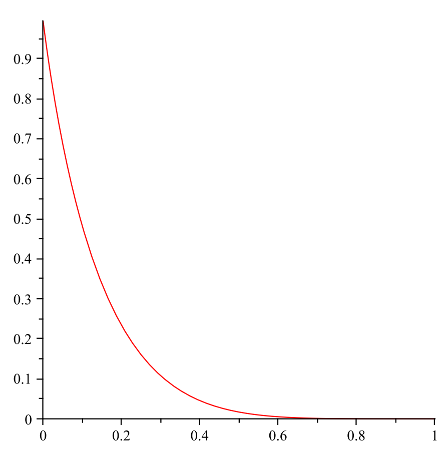
7.7.3 The muon decay width

After the simple integration over E_1, we have the muon partial decay width

$$\frac{d}{dE_2}\Gamma(\mu \to e\nu_\mu\overline{\nu}_e) = \pi^2\; {G_F}^2\; \hbar^2\; m_\mu\; E_2(K-E_2)(\hat{K}(E_2)+E_2-K) \;. \tag{7.167}$$

The remaining integral over E_2 can now be performed, and the final result is

$$\begin{aligned} \Gamma(\mu \to e\nu_\mu\overline{\nu}_e) &= \frac{{G_F}^2\;\hbar^2\;{m_\mu}^5}{192\;\pi^3}\;F({m_e}^2/{m_\mu}^2) \;, \\ F(x) &= 1 - 8x + 8x^3 - x^4 - 12x^2\log(x) \;. \end{aligned} \tag{7.168}$$



This displays the function $F(x)$. It is strictly decreasing since with increasing m_e/m_μ the available phase space shrinks. For the realistic values of m_e and m_μ $F(x)$ is smaller than 1 by about 2×10^{-4}. The effects of nonzero electron mass are therefore completely negligible, certainly if we realize that we have not included any loop diagrams, the contribution of which is very consider-

[49]This flatness does *not* depend on the masslessness of the neutrinos. For massive neutrinos the same phase space density s found, only the *boundaries* of the phase space become (horriby) complicated.

ably larger than this. Before finishing, it is instructive to inspect the muon width formula

$$\Gamma(\mu \to e\nu_\mu\overline{\nu}_e) = \frac{{G_F}^2\,\hbar^2\,{m_\mu}^5}{192\,\pi^3}$$

from the point of view of dimensional analysis. In the first place, the matrix element $\mathcal{M}$, being of $2 \to 2$ type, must be strictly dimensionless. Since every spinor carries half a power of momentum[50], the Fermi coupling constant G_F must carry dimension momentum^{-2}. Since decay widths carry the dimension of momentum, as do masses like m_μ, and the only mass scale in the problem is m_μ if we neglect the electron mass, the width is necessarily proportional to ${G_F}^2{m_\mu}^5$. The discussion at the end of section 6.5.4 shows that the factor
E 44 $1/\pi^3$ was also to be expected. It is a somewhat sobering thought that all the work of this section amounts to no more than computing the number 1/192 !

7.7.4 Observable distributions in muon decay

We can look more closely into the behavior of the electron in muon decay, in particular its angular and energy distribution : in fact, these are the only quantities of interest since the neutrinos can essentially never be detected. If the sample of decaying muons is unpolarized, no particular direction is favored in its rest frame, and the electrons may be expected to emerge isotropically. Let us therefore assume that the muons are fully polarized, with polarization vector $s^\mu = (0, \vec{s})$ in their rest frame. Since we are interested in distributions rather than the overall decay rate, we can afford to be sloppy with pre-factors in this computation. Additionally we shall assume $m_e = 0$. The matrix element is now given by

$$\begin{aligned} \mathcal{M} &\propto \overline{u}_-(q)\,\gamma_\mu\,v_-(k_2)\,\overline{u}_-(k_1)\,\gamma^\mu\,u(p,s) \\ &\propto \overline{u}_-(q)u_+(k_1)\,\overline{u}_+(k_2)u(p,s) \quad , \end{aligned} \tag{7.169}$$

where we have used Eq.(7.135). Squaring this, we obtain

$$\begin{aligned} |\mathcal{M}|^2 &\propto \mathrm{Tr}\,(\omega_-\not{q}\not{k}_1)\ \mathrm{Tr}\left(\omega_+\not{k}_2(1+\gamma^5\not{s})(\not{p}+m_\mu)\right) \\ &\propto (q\cdot k_1)\,\left((k_2\cdot p) - m_\mu(k_2\cdot s)\right) \\ &\propto k_2^0\,(m_\mu - 2k_2^0)\,(1+\cos(\theta)\cos(\theta_2)+\sin(\theta)\sin(\theta_2)\cos(\phi_2)) \quad , \end{aligned} \tag{7.170}$$

[50]Since the spin sum of $u\overline{u}$ contains $\not{p}$.

where θ is the angle between $\vec{q}$ and $\vec{s}$, and θ_2, ϕ_2 are the polar and azimuthal angles of $\vec{k}_2$ with respect to $\vec{q}$. As we have seen above, the phase space integration element is, up to overall factors,

$$dq^0\, dk_2^0\, d\cos(\theta)\, d\phi\, d\phi_2 \ \ ,$$

where ϕ is $\vec{q}$'s azimuthal angle around $\vec{s}$, while

$$\cos(\theta_2) = 1 - \frac{m_\mu(q^0 + k_2^0 - m_\mu/2)}{q^0\, k_2^0} \ \ . \tag{7.171}$$

Performing the integral over the (unobservable) angle ϕ_2, and disregarding the trivial ϕ dependence, we therefore have

$$d\Gamma \propto k_2^0\,(m_\mu - 2k_2^0)\left(1 + \cos(\theta)\left(1 - \frac{m_\mu(q^0 + k_2^0 - m_\mu/2)}{q^0\, k_2^0}\right)\right) dq^0\, dk_2^0\, d\cos(\theta) \ \ . \tag{7.172}$$

The integral over k_2^0 from $m_\mu/2 - q^0$ to $m_\mu/2$ is straightforward, leading to the following, properly normalised distribution :

$$\frac{1}{\Gamma}\frac{d^2\Gamma}{dy\, d\cos(\theta)} = y^2\left(3 - 2y + \cos(\theta)(1 - 2y)\right) \ \ , \ \ y = \frac{2q^0}{m_\mu} \ \ . \tag{7.173}$$

The overall angular distribution is

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\cos(\theta)} = \frac{1}{6}\left(3 - \cos(\theta)\right) \ \ , \tag{7.174}$$

and the overall distribution of the electron energy reads

$$\frac{1}{\Gamma}\frac{d\Gamma}{dy} = 2y^2(3 - 2y) \ \ . \tag{7.175}$$

Let us now consider the parity properties of the decay. Under the parity transform, Eq.(7.110) tells us that $\vec{q} \to -\vec{q}$ and $\vec{s} \to \vec{s}$, so that effectively $\cos(\theta) \to -\cos(\theta)$. We see that the muon decay amplitude, which is not symmetric in $\cos(\theta)$, displays *parity violation*. This is most prominent when the electron has maximal energy :

$$\left.\frac{1}{\Gamma}\frac{d^2\Gamma}{dy\, d\cos(\theta)}\right|_{y=1} = 1 - \cos(\theta) \ \ . \tag{7.176}$$

Qualitatively, we can understand this as follows. At $y = 1$, the two neutrinos recoil in parallel against the electron. Since ν_μ is left-handed and $\bar{\nu}_e$ is right-handed (because of the factors $1+\gamma^5$ in the coupling), their spins cancel, and the spin of the electron must therefore be in the direction of the muon spin by conservation of angular momentum. If the electron were to be emitted in the direction of the muon spin, it would therefore be right-handed, which is again forbidden by the coupling[51]. Hence the amplitude must vanish at $\theta = 0$.

7.8 Excercises for chapter 7

Excercise 27 Lorentz-contracted gamma matrices
Prove the results given in Eq.(7.14).

Excercise 28 Five spacetime dimensions
Let γ^4 be defined as $\gamma^4 = i\gamma^5$. Show that this γ^4 has just the right properties so that the relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\, g^{\mu\nu} \quad , \quad \overline{\gamma^\mu} = \gamma^\mu$$

are valid in a five-dimensional spacetime ($\mu, \nu = 0, 1, 2, 3, 4$) with metric $\gamma_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1, -1)$.

Excercise 29 Cyclic sigmas
Prove, by explicit calculation, that

$$\gamma^5\gamma^0\gamma^j = \sigma^{kn} \quad , \quad \gamma^5\gamma^j\gamma^k = \sigma^{n0}$$

where (j, k, n) is a *cyclic* permutation of $(1, 2, 3)$.

Excercise 30 Another relation between σ's
Use trace identities and Fierzing to prove that

$$\gamma^5\,\sigma^{\alpha\beta} = -\frac{i}{2}\,\epsilon^{\alpha\beta\mu\nu}\,\sigma_{\mu\nu}$$

Excercise 31 Projections playing around
Prove the results given in Eq.(7.67).

[51] Remember that we took $m_e = 0$.

Excercise 32 Some spinor products
Prove the results in Eq.(7.68).

Excercise 33 Finding mass, spin and momentum
Prove Eq.(7.69) and Eq.(7.70)

Excercise 34 Dirac spinors: positive energy, on the mass shell
Let $u = u(p, s)$. Using the results from the previous exercises about finding mass, momentum and spin from given spinors, prove that $p^0 > 0$, $p \cdot p = m^2$, $p \cdot s = 0$ and $s \cdot s = -1$.

Excercise 35 Building a Dirac spinor
Let ξ be an arbitrary spinorial object (*i.e.* a four-component column). Show that

$$(\not{p} + m)(1 + \gamma^5 \not{s})\xi$$

is proportional to $u(p, s)$.

Excercise 36 Spin basis for given momentum
Consider the two spinors $u(p, s)$ and $u(p, -s)$. Show that any other spinor with momentum p can be written as

$$u(p, s') = a_+ \, u(p, s) + a_- \, u(p, -s)$$

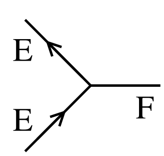
for some coefficients $a_\pm$, and write $a_\pm$ as spinor products.

Excercise 37 Rotating over 60 degrees and so on

$$p^\mu = x^\mu \quad , \quad q^\mu = \cos(\theta)\, x^\mu + \sin(\theta)\, y^\mu$$

Compute $\Sigma(p \to q)$ for $\theta = \pi/3$, and from there, by taking powers, for $\theta = 2\pi/3$ and finally for $\theta = 2\pi$.

Excercise 38 Dirac FEE model
We reconsider the FEE model,but this time the E are Dirac particles. This means that we have not only E particles but also $\bar{E}$ particles. There is one vertex :



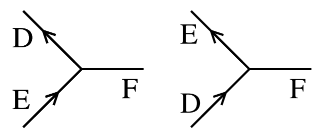
$$\leftrightarrow \quad \frac{i}{\hbar}\lambda$$

where λ is a dimensionless coupling constant. As before, F has mass M and E has mass m.

1. There are now 3 diagrammatic SDe's for this model. Write them out.
2. Assume $M > 2m$. Compute the decay width $\Gamma(F \to EE)$ at the tree level.
3. Assume $0 < M < 2m$.
   (a) Write the tree diagrams for the process $E(p_1)\,\bar{E}(p_2) \;\to\; E(q_1)\,\bar{E}(q_2)$ by F exchange. We have indicated the momenta of the E particles.
   (b) Compute $\langle|\mathcal{M}|^2\rangle$ for this process, where we *sum* over the final-state spins and *average* over the initial-state spins of the E's.
   (c) Compute the total cross section $\sigma(E\bar{E} \to E\bar{E})$, as a function of the total invariant mass-squared $s \equiv (p_1 + p_2)^2$. This is best done in the centre-of-mass frame.
4. In the above process, assume now $M = m = 0$ (and still $\Gamma = 0$). Determine $\langle|\mathcal{M}|^2\rangle$, using momentum conservation to simplify the result as much as possible.
5. Assume $M = 0$ and $m > 0$.
   (a) Write down the 2 diagrams for $E(p_1)\,F(k_1) \;\to\; E(p_2)\,F(k_2)$ at the tree level.
   (b) Work out $\langle|\mathcal{M}|^2\rangle$ for this process.
   (c) Compute the total cross section as a function of $s = (p_1 + k_1)^2$. This is best done in the centre-of-mass frame.
   (d) Compute the total cross section in the limit $s \to m^2$.
6. Make no assumption on m or M.
   (a) Write down the 24 diagrams for $EF \to EF$ at one loop.
   (b) Assume that there are also FFF and $FFFF$ vertices.
      i. Write the 3 diagrams for $EF \to EF$ at the tree level.
      ii. Write the 13 diagrams for $F \to EE$ at the one-loop level.
      iii. Determine the *number* of Feynman diagrams at the tree level for the following processes:
         A. $E\bar{E} \;\to\; E\bar{E}F$

B. $E\bar{E} \to E\bar{E}FF$
C. $E\bar{E} \to EE\bar{E}\bar{E}$
D. $E\bar{E} \to EEE\bar{E}\bar{E}\bar{E}$

Excercise 39 Dirac FED model
We consider a variation on the FEE model: there are now 2 types of Dirac particles, E and D, with masses m_E and m_D, respectively. The F particle is massless. There are now 2 vertices :



where both vertices have Feynman rule $i\lambda/\hbar$ as before. We shall assume $m_E > m_D$.

1. Compute the decay width $\Gamma(E \to FD)$ at the tree level.

2. Compute the cross section $\sigma(FF \to E\bar{E})$ at the tree level, as a function of the total invariant mass-squared s.

Excercise 40 Spinor products and vector products
Using the form of Eq.(7.142) for $s_+(p,q)$, prove that

$$|s_+(p,q)|^2 = 2(p \cdot q)$$

Excercise 41 Long Dirac strings
Let p_j^μ, $(j = 1,2,3,\ldots)$ be massless momenta. Write the expression

$$A = \overline{u}_+(p_1)\not{p}_2\not{p}_3\not{p}_4\not{p}_5 u_-(p_6)$$

in terms of spinor products, and compute $|A|^2$.

Excercise 42 Why is it zero ?
for massless momenta (as in the previous exercise), let

$$A_1 = \overline{u}(p_1)\not{p}_2\not{p}_3 u(p_7) \quad , \quad A_2 = \overline{u}(p_1)\not{p}_4\not{p}_5\not{p}_6 u(p_7)$$

where you can chooce the helicities of $p_{1,7}$ yourself. Show that in all cases

$$A_1 {A_2}^* = 0$$

Excercise 43 Towards a real theory prediction
We consider the process

$$e^+(p_1)\,e^-(p_2) \quad\to\quad e^+(q_1)\,e^-(q_2)$$

where the electrons/positrons are massless spin-1/2 particles. In Quantum Electrodynamics (QED), the amplitude for this process is given by

$$\begin{aligned}
&\mathcal{M}(\lambda_1,\lambda_2,\rho_1,\rho_2) = \\
&\quad i\hbar {Q_{\mathrm{e}}}^2\left(\frac{1}{(p_1+p_2)^2}\,\overline{u}_{\lambda_1}(p_1)\gamma^\mu u_{\lambda_2}(p_2)\;\overline{u}_{\rho_2}(q_2)\gamma_\mu u_{\rho_1}(q_1)\right. \\
&\quad \left.-\frac{1}{(p_1-q_1)^2}\,\overline{u}_{\lambda_1}(p_1)\gamma^\mu u_{\rho_1}(q_1)\;\overline{u}_{\rho_2}(q_2)\gamma_\mu u_{\lambda_2}(p_2)\right)
\end{aligned}$$

where (as usual for massless fermions) we have disregarded the distinction between spinors and antispinors. The quantity Q_{e} is a fixed coupling constant. The helicities $\lambda_{1,2}$ and $\rho_{1,2}$ are explicitly indicated. There are, in total, 16 helicity combinations.

1. Using spinor techniques, compute $\mathcal{M}$ for the sixteen helicity cases.

2. Assume that we are in the centre-of-mass frame, where $\vec{p}_1 + \vec{p}_2 = 0$. Let θ be the angle between $\vec{p}_1$ and $\vec{q}_1$, and $E = {p_1}^0$. Give the simplest form you can find for $\langle|\mathcal{M}|^2\rangle$.

3. Show that $\langle|\mathcal{M}|^2\rangle$ is really divergent for $\theta = 0$, and that this has nothing to do with neglecting the electron mass.

Excercise 44 Muon decay revisited
We reconsider muon decay, again with vanishing neutrino and electron mass. We shall investigate the result of leaving out the factors $(1+\gamma^5)$ in the amplitude, so that we take it to have the form

$$\mathcal{M} = \overline{u}(k_1)\,\gamma^\alpha\,u(p)\;\;\overline{u}(q)\,\gamma_\alpha\,v(k_2)$$

and the massless fermions can take on either helicity. The muon can, of course, also be in two spin states, to be summed over.

1. Compute $\langle|\mathcal{M}|^2\rangle$ for all the helicity configurations, summed over the muon spins. Note : there are 2 different forms only.

2. Use the results n three-body kinematics of section 7.7.2 to write the above expressions in terms of q^0, ${k_2}^0$, and m_μ.

3. After the angular integrations, as explained in section 7.7.2 there remains the energy integral

$$\int_0^{m_\mu/2} dq^0 \int_{m_\mu/2-q^0}^{m_\mu/2} d{k_2}^0$$

   The electron energy q^0 is observable, but the antineutrino energy ${k_2}^0$ is not. Compute the two integrals over ${k_2}^0$ to get the two electron energy spectra.

4. Show that the two spectra give the same result when integrated over q^0.

5. Investigate the two normalized spectra,

$$\frac{1}{\Gamma}\,\frac{d\Gamma}{d\,q^0}$$

   to show how one can experimentally infer the presence of the $1+\gamma^5$.

Chapter 8

Vectors particles

8.1 Massive vector particles

8.1.1 The propagator

In the last chapter we have studied the consequences of embellishing the scalar propagator by endowing it with a numerator linear in the momentum. The next obvious generalization is to let $\mathcal{T}(p)$ depend on *two* powers of the momentum. That is, we assume it to be of the form

$$\mathcal{T}(p) \to \mathcal{T}(p)^{\mu\nu} = Ag^{\mu\nu} + Bp^\mu p^\nu \quad , \quad B \neq 0 \quad ,$$

for some A and B that may depend on p^2. The numerator now carries two Lorentz indices, one of each to be contracted with a corresponding index in the vertices between which the propagator runs. The discussion of the last chapter leads us to require that for momenta on the mass shell $\mathcal{T}(p)$ must be proportional to a projection operator :

$$\mathcal{T}(p)^{\mu\alpha}\mathcal{T}(p)_\alpha{}^\nu = k\mathcal{T}(p)^{\mu\nu} \quad \text{if} \quad p^2 = m^2 \quad , \tag{8.1}$$

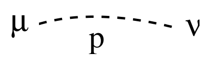
for some $k \neq 0$, in other words

$$A^2 = kA \quad , \quad B^2m^2 + 2AB = kB \quad . \tag{8.2}$$

We might choose the solution $A = 0$, but then the resulting form $\mathcal{T}(p)^{\mu\nu} \sim p^\mu p^\nu$ would be immediately absorbable into the vertices at either side, and a scalar propagator would result again. It follows that A must equal $-m^2B$, and therefore we shall use

$$\mathcal{T}(p)^{\mu\nu} = -g^{\mu\nu} + \frac{1}{m^2}p^\mu p^\nu \quad , \tag{8.3}$$

as before also (and mostly) using this form for off-shell momenta. The first Feynman rule for these particles, that we call *vector particles* since they carry a Lorentz index, is therefore established :



$$\leftrightarrow \quad i\hbar\,\frac{-g^{\mu\nu} + p^\mu p^\nu/m^2}{p^2 - m^2 + i\epsilon}$$ internal lines

Feynman rules, version 8.1 (8.4)

Note that this propagator is *even* in p and therefore has no orientation[1].

8.1.2 The Feynman rules for external vector particles

From the form of $\mathcal{T}(p)$ we must be able to derive the form of the external-line factors. Indeed, let us assume p^μ to be in its rest frame. There, we have

$$\mathcal{T}(p)^{\mu\nu} = -g^{\mu\nu} + g^{0\mu}g^{0\nu} = \mathrm{diag}(0,1,1,1) \quad , \tag{8.5}$$

that is, the unit tensor in the spatial sector of Minkowski space. We see that we can write

$$\mathcal{T}(p)^{\mu\nu} = -\left(x^\mu x^\nu + y^\mu y^\nu + z^\mu z^\nu\right) \quad , \tag{8.6}$$

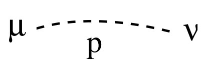
which means that, for the objects $\mathcal{U}, \mathcal{W}$ three mutually orthogonal choices can be made, for instance $\mathcal{U}^{(1)} = x$, $\mathcal{U}^{(2)} = y$, and $\mathcal{U}^{(3)} = z$. Of course, complex linear combinations of these are also possible : in general, we can say that there can be found three *polarization vectors* ϵ^μ_λ, with $\lambda = -1, 0, 1$, such that

$$(\epsilon_\lambda)^\mu \overline{(\epsilon_{\lambda'})}_\mu = -\delta_{\lambda,\lambda'} \quad , \quad \mathcal{T}(p)^{\mu\nu} = \sum_{\lambda=-1}^{1} (\epsilon_\lambda)^\mu \overline{(\epsilon_\lambda)}^\nu \quad . \tag{8.7}$$

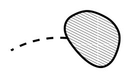
E 45 We can now go once more through the truncation argument of chapter 6,

[1] That is, its spacetime part is unoriented. There may of course be other properties such as charge that do impose a distinction between production and decay of the particle.

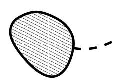
with the obvious result that the polarization vectors are to be assigned to the external lines, and we immediately arrive at the full set of Feynman rules for massive vector particles :



$$\leftrightarrow \quad i\hbar\,\frac{-g^{\mu\nu}+p^\mu p^\nu/m^2}{p^2-m^2+i\epsilon}$$ internal lines



$$\leftrightarrow \quad \sqrt{\hbar}\;\epsilon_\lambda{}^\mu$$ incoming lines



$$\leftrightarrow \quad \sqrt{\hbar}\;\overline{\epsilon_\lambda}{}^\mu$$ outgoing lines

Feynman rules, version 8.2 (8.8)

Owing to the lack of orientation, the rules for the external lines are quite simple, and fortunately no Dirac indices appear, nor do any curious and cumbersome minus signs.

8.1.3 The spin of vector particles

To ascertain the spin of vector particles[2], we need to establish the form of the Lorentz transformation in the space of the polarization vectors, *i.e.* Minkowski space. We can do this conveniently using the transform in Clifford space, as follows. Let us denote by $\Lambda(p;q)^\mu{}_\nu$ the representation of the minimal Lorentz transformation between p^μ and q^μ in Minkowski space : that is, if an arbitrary vector a^μ is transformed into b^μ, we have

$$\Lambda(p;q)^\mu{}_\nu\; a^\nu = b^\mu \quad . \tag{8.9}$$

Since $\not{a}$ and $\not{b}$ encode exactly the same information as do a^μ and b^μ, consistency requires that

$$\not{b} = \Lambda(p;q)^\mu{}_\nu\; a^\nu\; \gamma_\mu = \Sigma\not{a}\overline{\Sigma} = \Sigma\; a^\nu\; \gamma_\nu\; \overline{\Sigma} \quad , \tag{8.10}$$

[2]The fact that there are three polarization vectors of course *suggests* that the spin is 1.

with Σ as defined in section 7.3.5 ; since this must hold for arbitrary a, we have the relation

$$\Lambda(p;q)^{\mu}{}_{\nu}\,\gamma_{\mu} = \Sigma\,\gamma_{\nu}\,\overline{\Sigma}\quad, \tag{8.11}$$

By multiplying with γ^{α} on both sides and taking the trace, we immediately find the form of $\Lambda(p;q)$ in Minkowski space :

$$\begin{aligned}\Lambda(p;q)^{\alpha}{}_{\nu} = \frac{1}{4}\mathrm{Tr}\left(\Lambda(p;q)^{\mu}{}_{\nu}\,\gamma_{\mu}\,\gamma^{\alpha}\right) &= \frac{1}{4}\mathrm{Tr}\left(\Sigma\,\gamma_{\nu}\,\overline{\Sigma}\,\gamma^{\alpha}\right)\\ = \quad \frac{p^2}{4(p+q)^2}\mathrm{Tr}&\left(\left(1+\frac{\not{q}\not{p}}{p^2}\right)\gamma_{\nu}\left(1+\frac{\not{p}\not{q}}{p^2}\right)\gamma^{\alpha}\right)\\ = \quad \delta^{\alpha}{}_{\nu} - \frac{2}{(p+q)^2}(p+q)^{\alpha}(p+q)_{\nu} &+ \frac{2}{p^2}q^{\alpha}p_{\nu}\quad .\end{aligned} \tag{8.12}$$

E 46 Let us now specialize to the case of infinitesimal rotations, as in section 7.3.5: again, we take $p^{\mu} = x^{\mu}$ and $q^{\mu} = x^{\mu} + \theta y^{\mu}$ (θ infinitesimal), and then find to first order in θ :

$$\begin{aligned}\Lambda(p;q)^{\mu}{}_{\nu} &\approx \delta^{\mu}{}_{\nu} + \frac{1}{2}(2x+\theta y)^{\mu}(2x+\theta y)_{\nu} - 2(x+\theta y)^{\mu}x_{\nu}\\ &\approx \delta^{\mu}{}_{\nu} - \theta\left(x^{\mu}y_{\nu} - y^{\mu}x_{\nu}\right)\quad ,\end{aligned} \tag{8.13}$$

so that the generators of the rotation group must in this case have the form

$$\begin{gathered}(T_x)^{\mu}{}_{\nu} = \beta(y^{\mu}z_{\nu} - z^{\mu}y_{\nu})\quad,\quad (T_y)^{\mu}{}_{\nu} = \beta(z^{\mu}x_{\nu} - x^{\mu}z_{\nu})\quad,\\ (T_z)^{\mu}{}_{\nu} = \beta(x^{\mu}y_{\nu} - y^{\mu}x_{\nu})\quad ,\end{gathered} \tag{8.14}$$

with the constant β again to be determined from the commutation algebra :

$$\begin{aligned}[T_x,T_y]^{\mu}{}_{\nu} &= (T_x)^{\mu}{}_{\alpha}(T_y)^{\alpha}{}_{\nu} - (T_y)^{\mu}{}_{\alpha}(T_x)^{\alpha}{}_{\nu}\\ &= \beta^2\left(x^{\mu}y_{\nu} - y^{\mu}x_{\nu}\right) = \beta(T_z)^{\mu}{}_{\nu}\quad .\end{aligned} \tag{8.15}$$

We conclude that $\beta = i\hbar$ in the Minkowski space. We find

$$({T_x}^2)^{\mu}{}_{\nu} = -\hbar^2\left(y^{\mu}y_{\nu} + z^{\mu}z_{\nu}\right)\quad , \tag{8.16}$$

etcetera, so that the total-spin operator takes the form

$$(\vec{L}^2)^{\mu}{}_{\nu} = -2\hbar^2\ \left(x^{\mu}x_{\nu} + y^{\mu}y_{\nu} + z^{\mu}z_{\nu}\right) = 2\hbar^2\left(-\delta^{\mu}{}_{\nu} + \frac{1}{m^2}p^{\mu}p_{\nu}\right)\quad . \tag{8.17}$$

we conclude that the spin is indeed unity. The total spin operator contains, as it must, the projection of all vectors on the spatial subspace. In words: to be a good polarization vector, ϵ^μ must satisfy the *Lorenz condition*[3] :

$$\epsilon \cdot p = 0 \ . \tag{8.18}$$

Any part of a polarization vector that is parallel to p^μ does, of course, *not* transform under rotations in the space orthogonal to p^μ (in our case, the spatial part of Minkowski space since p^μ is at rest). That part, therefore, corresponds to a *scalar* degree of freedom. Returning to $\mathcal{T}(p)$ we may interpret the form

$$\mathcal{T}(p)^{\mu\nu} = -g^{\mu\nu} + \frac{1}{m^2}p^\mu p^\nu \tag{8.19}$$

as a propagator in which *a priori* four degrees of freedom propagate (the $g^{\mu\nu}$ part), and where the scalar part (the $p^\mu p^\nu$ term) is carefully excised. The $p^\mu p^\nu$ term is sometimes loosely called the 'longitudinal part' of the propagator, but this is *wrong* ; we should do better by calling it the 'scalar part'.

8.1.4 Full rotations in vector space

In analogy with the rotations over 90 degrees that we studied in section 7.3.6, we may cast a quick look at the behaviour of states under the transformation (**??**) when applied to a 90-degree rotation in the $x-y$ plane. The minimal Lorentz transformation then reads

$$\Lambda(\pi/2)^\mu{}_\nu = \delta^\mu{}_\nu + x^\mu x_\nu + y^\mu y_\nu + x^\mu y_\nu - y^\mu x_\nu \ . \tag{8.20}$$

Taking powers, we obtain

$$\begin{aligned} \Lambda(\pi)^\mu{}_\nu &= \delta^\mu{}_\nu + 2x^\mu x_\nu + 2y^\mu y_\nu \ , \\ \Lambda(2\pi)^\mu{}_\nu &= \delta^\mu{}_\nu \ . \end{aligned} \tag{8.21}$$

In contrast to the Dirac case, it now only needs a rotation over 2π to restore any state of a vector particle to its original form ; a conclusion which was already reached in section 7.3.6. There are, of course, polarization vectors that are not affected by the rotation at all, namely those that point in the z direction : the point is that a rotation over 2π restores *any* polarization vector.

[3]Note the spelling ! This does not refer to the famous Dutchman Hendrik Antoon Lorentz (1853-1928) of transformation fame, but to the Dane Ludvig Valentin Lorenz (1829-1891), quite another person. A relation between the density and the refractive index of a medium goes by the funky name of the *Lorentz-Lorenz* equation.

8.1.5 Polarization vectors for helicity states

As usual, the helicity of a state refers to its spin as measured along the direction of its motion. For definitiveness, let us assume that our massive vector particle moves along the z direction. If we boost carefully (and minimally !) back to the rest frame, $\vec{p}$ of course vanishes, but we shall remember that to go back to the original situation we must boost along the z direction. The operator for the helicity is therefore T_z in this case. Good polarization vectors for helicity 1,0 and -1 are then

$$\epsilon_+{}^\mu = \frac{1}{\sqrt{2}}(x^\mu + iy^\mu) \quad , \quad \epsilon_0{}^\mu = z^\mu \quad , \quad \epsilon_-{}^\mu = -\frac{1}{\sqrt{2}}(x^\mu - iy^\mu) \quad , \tag{8.22}$$

which is easily checked by verifying that

$$(T_z)^\mu{}_\nu \epsilon_+{}^\nu = \hbar\epsilon_+{}^\mu \quad , \quad (T_z)^\mu{}_\nu \epsilon_0{}^\nu = 0 \quad , \quad (T_z)^\mu{}_\nu \epsilon_-{}^\nu = -\hbar\epsilon_-{}^\mu \quad . \tag{8.23}$$

The vectors $\epsilon_{\pm 1}$ are said to describe *transverse polarization*, and the vector ϵ_0 is called *longitudinal*. If we now perform the boost back to the original system in which p^μ is moving along the z direction, the transverse polarizations remain unaffected, while the longitudinal one takes the form[4]

$$\epsilon_0{}^\mu \to \left(\frac{|\vec{p}|}{mp^0}\right) p^\mu + \left(\frac{m}{p^0}\right) z^\mu \quad . \tag{8.24}$$

Very fast-moving particles, for which $m \ll p^0 \approx |\vec{p}|$, have longitudinal polarization vector

$$\epsilon_0{}^\mu \to \frac{1}{m} p^\mu + \mathcal{O}\left(\frac{m}{p^0}\right) \quad . \tag{8.25}$$

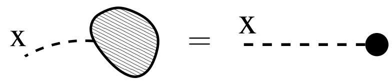
8.1.6 The Proca equation

Massive vector particles have their own 'classical' equation, which we shall now uncover. The coupling of a massive vector particle to a source is given by the following Feynman rule for position space :

$$\mu\text{- - - - -}\bullet \quad \leftrightarrow \quad -\frac{i}{\hbar} J^\mu(x) \tag{8.26}$$

[4] In a somewhat simpler notation, if $p^\mu = (p^0, \vec{p})$, with $p = |\vec{p}|$ and $\vec{e} = \vec{p}/p$, then the longitudinal polarization vector reads $\epsilon_0{}^\mu = (p, p^0\vec{e})/m$.

The SDe for a free vector particle's field function V^μ is then again very simple :



, (8.27)

or, more explicitly,

$$V^\mu(x) = \frac{1}{(2\pi)^4} \int d^4y\, d^4k\, \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2} \left(-g^{\mu\nu} + \frac{1}{m^2}k^\mu k^\nu\right) J_\nu(y) \quad . \tag{8.28}$$

We can then form the following derivative operator acting on V^μ :

$$\begin{aligned} &\partial^\alpha\partial_\alpha\, V^\mu(x) - \partial^\mu\partial_\alpha\, V^\alpha(x) + m^2 V^\mu(x) = \\ &= \frac{1}{(2\pi)^4} \int d^4y\, d^4k\, \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2}\, W^{\mu\nu}\, J_\nu(y) \quad , \end{aligned} \tag{8.29}$$

where $W^{\mu\nu}$ can be evaluated as

$$\begin{aligned} W^{\mu\nu} &= (-k^2 + m^2)\left(-g^{\mu\nu} + \frac{1}{m^2}k^\mu k^\nu\right) + k^\mu k_\alpha\left(-g^{\alpha\nu} + \frac{1}{m^2}k^\alpha k^\nu\right) \\ &= (k^2 - m^2) g^{\mu\nu} \quad . \end{aligned} \tag{8.30}$$

The remaining integrals over y and k now lead immediately to the so-called *Proca* equation for V^μ :

$$\partial\cdot\partial\, V^\mu - \partial^\mu\, \partial\cdot V + m^2 V^\mu = J \quad . \tag{8.31}$$

This is the 'Maxwell equation' for massive vector fields. It is instructive to examine this equation in empty space, that is, for $J = 0$. Multipying it by ∂_μ, we find that the first two terms cancel, and we are left the Lorenz condition $\partial \cdot V = 0$: all physical polarizations must be orthogonal to the momentum, as we had already found. Reinserting this condition in Eq.(8.31), we are left with the Klein-Gordon equation $(\partial\cdot\partial + m^2)V^\mu = 0$, which essentially requires the particles to be on the mass shell. Note that this nicely compact way of enforcing the Lorenz condition only works for $m \neq 0$: for *massless* vector particles, it must be put in by hand.

We can also write down the Lagrangian corresponding to the Proca equation, that is, that Lagrangian that has the Proca equation as its Euler-Lagrange equation. It reads

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu V_\nu)(\partial^\mu V^\nu) - \frac{1}{2}(\partial_\mu V_\nu)(\partial^\nu V^\mu) + \frac{1}{2}m^2 V^\mu V_\mu \\ &= \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 V^\mu V_\mu \quad , \end{aligned} \tag{8.32}$$

where the *field strength* tensor is defined as

$$F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu \quad . \tag{8.33}$$

8.2 The spin-statistics theorem

8.2.1 Spinorial form of vector polarizations

Although there is no special need for it, we can define the polarization vectors for a massive vector particle using Dirac spinors. Let the momentum of the vector particle be q^μ and its mass m. We can find two *massless* momenta p_1^μ and p_2^μ whose spatial parts are parallel (or antiparallel) to $\vec{q}$ and that sum to q^μ :

$$q^\mu = p_1^\mu + p_2^\mu \quad , \quad {p_{1,2}}^2 = 0 \quad , \quad 2\,(p_1 \cdot p_2) = m^2 \quad . \tag{8.34}$$

The helicity states can now be constructed by standard-form spinors as follows :

$$\begin{aligned}
{\epsilon_+}^\mu &= \frac{1}{m\sqrt{2}}\,\overline{u}_+(p_1)\gamma^\mu u_+(p_2) \quad , \\
{\epsilon_0}^\mu &= \frac{1}{2m}\left(\overline{u}_+(p_1)\gamma^\mu u_+(p_1) - \overline{u}_+(p_2)\gamma^\mu u_+(p_2)\right) \quad , \\
{\epsilon_-}^\mu &= \frac{1}{m\sqrt{2}}\,\overline{u}_-(p_1)\gamma^\mu u_-(p_2) \quad .
\end{aligned} \tag{8.35}$$

In fact, the longitudinal polarization ${\epsilon_0}^\mu$ can (by the Casimir trick, as usual) be seen to be nothing else than

$${\epsilon_0}^\mu = \frac{1}{m}(p_1 - p_2)^\mu \quad . \tag{8.36}$$

This polarization, then, is properly normalized and orthogonal to ${\epsilon_\pm}^\mu$. Furthermore, we have

$$\epsilon_+ \cdot \overline{\epsilon}_- = \frac{1}{2m^2}\;\overline{u}_+(p_1)\gamma^\mu u_+(p_2)\;\overline{u}_-(p_2)\gamma_\mu u_-(p_1) \quad . \tag{8.37}$$

By virtue of the standard choice of the spinors, we can see that

$$\begin{aligned}
\gamma^\mu u_+(p_2)\,\overline{u}_-(p_2)\gamma_\mu &\propto \gamma^\mu\,\not{p}_2\,\not{k}_0\,\not{k}_1\,\not{p}_2\,\gamma_\mu \\
&= -\not{p}_2\,\gamma^\mu\,\not{k}_0\,\not{k}_1\,\not{p}_2\,\gamma_\mu \\
&= 2\not{p}_2\,\not{p}_2\,\not{k}_1\,\not{k}_0 = 0 \quad ,
\end{aligned} \tag{8.38}$$

where we have used twice that ${p_2}^2 = 0$. The vectors are therefore all orthogonal to each other. To check the normalization of ϵ_+, we write

$$\begin{aligned}\epsilon_+ \cdot \overline{\epsilon}_+ &= \frac{1}{2m^2}\,\overline{u}_+(p_1)\gamma^\mu u_+(p_2)\;\overline{u}_+(p_2)\gamma_\mu u(p_1)\\ &= \frac{1}{2m^2}\,\overline{u}(p_1)\gamma^\mu \not{p}_2\gamma_\mu u_+(p_1)\\ &= -\frac{1}{m^2}\,\overline{u}_+(p_1)\not{p}_2 u_+(p_1) = -\frac{2(p_1\cdot p_2)}{m^2} = -1 \ . \end{aligned} \tag{8.39}$$

It remains to ascertain that these states are, indeed, pure helicity states. To this end, let us assume that $\vec{p}_1$ and $\vec{p}_2$ are aligned with the z axis. The helicity operator is then $(T_z)^\mu{}_\nu = i\hbar(x^\mu y_\nu - y^\mu x_\nu)$, so that ϵ_0 trivially has helicity zero. We have

$$(T_z)^\mu{}_\nu {\epsilon_+}^\nu = \frac{1}{m\sqrt{2}}\Big(x^\mu\,\overline{u}_+(p_1)\not{y}u_+(p_2) - (x\ \leftrightarrow\ y)\Big) \ . \tag{8.40}$$

Again employing the properties of the standard form, we can show that this is orthogonal to ϵ_- :

$$\begin{aligned}((T_z)^\mu{}_\nu {\epsilon_+}^\nu)\,\overline{\epsilon}_- &= \frac{1}{2m^2}\Big(\overline{u}_+(p_1)\not{y}u_+(p_2)\;\overline{u}_-(p_2)\not{x}u_-(p_1) - (x\ \leftrightarrow\ y)\Big)\\ &= \frac{1}{2m^2}\Big(\overline{u}_+(p_1)\not{y}u_+(p_2)\;\overline{u}_+(p_1)\not{x}u_+(p_2) - (x\ \leftrightarrow\ y)\Big) = 0 \ . \end{aligned} \tag{8.41}$$

Finally, we can examine

$$((T_z)^\mu{}_\nu {\epsilon_+}^\nu)\,\overline{\epsilon}_+ = \frac{1}{2m^2}\Big(\overline{u}_+(p_1)\not{y}u_+(p_2)\;\overline{u}_+(p_2)\not{x}u_+(p_1) - (x\ \leftrightarrow\ y)\Big) \ . \tag{8.42}$$

The first term in brackets can be evaluated by trace techniques :

$$\overline{u}_+(p_1)\not{y}u_+(p_2)\;\overline{u}_+(p_2)\not{x}u_+(p_1) = \mathrm{Tr}\,(\omega_+\ \not{p}_1\ \not{y}\ \not{p}_2\ \not{x}) = 2iA \ , \tag{8.43}$$

so that

$$((T_z)^\mu{}_\nu {\epsilon_+}^\nu)\,\overline{\epsilon}_+ = -\frac{2\hbar}{m^2}A \ , \tag{8.44}$$

where

$$A = \epsilon_{\mu\nu\alpha\beta}\ p_1^\mu\ y^\nu\ p_1^\alpha\ x^\beta \ , \tag{8.45}$$

which is real ; moreover,

$$A^2 = (p_1\cdot p_2)^2 = m^4/4 \ . \tag{8.46}$$

We conclude that

$$((T_z)^\mu{}_\nu \epsilon_+{}^\nu)\,\bar{\epsilon}_+ = -\hbar\,\mathrm{sign}(A) \ . \tag{8.47}$$

The chosen form do therefore indeed represent correct helicity states[5].

Before finishing this sector, we point out that also the (trivial) external-line Feynman factor for scalar particles can be written in terms of spinors. For a massive *scalar* with momentum q^μ, the same choice of $p_{1,2}^\mu$ is of course possible. We simply note that

$$|\overline{u}_+(p_1)u_-(p_2)|^2 = \mathrm{Tr}\,(\omega_+\, \not{p}_1\, \not{p}_2) = 2(p_1\cdot p_2) \ , \tag{8.48}$$

so that we can always find a complex phase $e^{i\varphi}$ such that the external-line factor $\sqrt{h}$ can be cast in a form containing two spinors :

$$\sqrt{\hbar} \ \ \to \ \ \sqrt{\hbar}\ \frac{e^{i\varphi}}{\sqrt{2\,p_1\cdot p_2}}\ \overline{u}_+(p_1)u_-(p_2) \ . \tag{8.49}$$

It should not come as a surprise that an external integer-spin particle can conventiently be represented by a spinor-antispinor pair. After all, this is precisely the way in which particles like the W and Z are most often seen in experiment : namely, through their decay into a fermion-antifermion pair.

8.2.2 Proof of the spin-statistics theorem

The treatment of the previous section may appear somewhat academic, but it has an interesting consequence. Integer-spin particles (scalars and vectors) can be represented in their external lines with an *even* number of spinors, that is an even number of Dirac particles. Particles with half-integer spin are represented by an *odd* number of Dirac particles. This persists : spin-3/2 particles can be formulated using 3 spinors, spin-2 particles by 4 spinors, and so on. This implies that the interchange of two external *half-integer*-spin particles involves the interchange of an *odd* number of Dirac particles, and will therefore lead to a minus sign. The interchange of two external *integer*-spin particles involves the interchange of an *even* number of Dirac particles, and hence no minus sign. These particles, therefore, obey opposite statistics : **integer-spin particles are bosons, half-integer spin particles are fermions**[6].

---

[5] We have not established that ϵ_+ is ϵ_{+1}; it is actually ϵ_{-1} if A is negative. This is easily remedied if necessary, by interchanging p_1 and p_2.

[6] Traditionally, the spin-statistics theorem, like the CPT theorem, is considered to be very deep and difficult. Make up your mind.

8.3 Massless vector particles

8.3.1 Polarizations of massless vector particles

Let us reconsider the helicity states of Eq.(8.22). These are defined in the rest frame of the particle, with the understanding that we have to boost back to the frame in which the particle moves, in our case along the z axis. Under this boost the longitudinal polarization takes the form of Eq.(8.24). Let us now imagine that the particle approaches masslessness, that is, we let m/p^0 decrease towards zero. The boost necessary to reach the original frame then becomes enormous, and the longitudinal polarization will go to infinity when the particle becomes massless. The only way to avoid matrix elements becoming arbitrarily large, and hence violating unitarity sooner or later, is to arrange the *interactions* of the theory in such a way that the effect of longitudinal polarization are suppressed by a factor of order $\mathcal{O}\left(m/p^0\right)$: we shall use this extensively later on. In the strictly massless case, the longitudinal polarization vector must decouple completely, and we arrive at the result that **for massless particles, only the two states of maximal helicity are physical**[7].

8.3.2 Current conservation from the polarization

A photon is a vector particle ; as far as we know it is massless. Its polarization vectors must therefore be transverse. For a photon moving in the z direction, any possible polarization vector must be a superposition of $(x+iy)^\mu/\sqrt{2}$ and $(x-iy)^\mu/\sqrt{2}$. If k^μ is the photon momentum, and ϵ^μ its polarization, we must therefore have not only $k \cdot \epsilon$ but also

$$\epsilon^0 = 0 \quad , \quad \vec{k} \cdot \vec{\epsilon} = 0 \ . \tag{8.50}$$

However, a problem immediately arises: for the above equations are not invariant under Lorentz boosts. If we boost k^μ and ϵ^μ to a generically other frame, they no longer hold. Let us assume that we are in such a frame ; there we have the Lorentz-invariant conditions

$$(k^0)^2 = |\vec{k}|^2 \quad , \quad (\epsilon^0)^2 - |\vec{\epsilon}|^2 = -1 \quad , \quad k^0\epsilon^0 = \vec{k} \cdot \vec{\epsilon} \ . \tag{8.51}$$

[7]This can also be proven for particles of higher spin, see Appendix 13.12.

We can decompose $\vec{\epsilon}$ into a parallel and a perpendicular part :

$$\vec{\epsilon} = \vec{\epsilon}_{\parallel} + \vec{\epsilon}_{\perp} \quad , \quad \vec{\epsilon}_{\parallel} \; // \; \vec{k} \quad , \quad \vec{\epsilon}_{\perp} \cdot \vec{k} = 0 \ \ . \tag{8.52}$$

Inserting this into the last equation of Eq.(8.51), we find immediately that $\epsilon_0 = |\vec{\epsilon}_{\parallel}|$, and the second equation then gives $|\vec{\epsilon}_{\perp}| = 1$. We see that, *whatever the value of* ϵ^{μ}, we can always write

$$\epsilon^{\mu} = {\epsilon_{\perp}}^{\mu} \quad + \quad \frac{\epsilon^0}{k^0} k^{\mu} \ \ , \tag{8.53}$$

where ${\epsilon_{\perp}}^{\mu}$ *does* satisfy Eq.(8.50). We can therefore have a consistent and unitary theory of massless vector particles, provided that the k^{μ} term decouples from the physics. Now, any matrix element involving an external massless vector particle with momentum k^{μ} and polarization vector ϵ^{μ} will be of the form

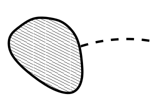
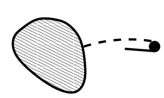
$$\mathcal{M} = \mathcal{J}(k)^{\mu} \, \epsilon_{\mu} \ \ , \tag{8.54}$$

where $\mathcal{J}^{\mu}(k)$ stands for the rest of the amplitude. Note that $\mathcal{J}^{\mu}$ does not carry any information about ϵ_{μ}, but it *does* know what k^{μ} is, by momentum conservation. Our requirement then is that the interactions of the theory be such that

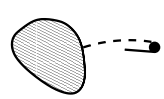
$$\mathcal{J}^{\mu}(k) \, k_{\mu} = 0 \ \ . \tag{8.55}$$

That is, if we replace the polarization vector by the momentum, the amplitude must vanish.

Diagrammatically, we may indicate the replacing of polarization by momentum by attaching a 'handlebar' to the external line, so that we may write

 $= \mathcal{M}$,  $= \mathcal{M}\rfloor_{\epsilon \to k}$. (8.56)

We shall use the convention that the momentum under the handlebar is counted *outgoing*. The requirement for strictly massless external vector particles then becomes

 $= 0$. (8.57)

What, finally, is the physical content of the requirement ? This is simply answered if we let our massless vector particle be a photon. The object $\mathcal{J}^{\mu}$

is then seen as a source of photons, that is, an electromagnetic *current*[8]. If we now briefly return from a momentum-language formulation to a position-language one, we see that the Fourier transform of the requirement (8.55) is written as

$$\partial_\mu \, \mathcal{J}(x)^\mu = 0 \quad . \tag{8.58}$$

We see that our requirement is nothing but *current conservation* in the case of electromagnetism ! The fact that electric charge is conserved ensures that longitudinally polarized photons are safely absent from our experience[9].

8.3.3 Current conservation from the propagator

A message similar to that of the previous section can be gotten from the propagator. After all, the massive-vector propagator

$$i\hbar \, \frac{-g^{\mu\nu} + k^\mu k^\nu/m^2}{k^2 - m^2}$$

clearly becomes horribly singular at $m = 0$. The solution, as before, is to require that in our theory the $k^\mu k^\nu$ term should drop out. There is a catch, however: whereas external vector particles must be on the mass shell, the momentum of *internal* lines is off the mass shell. We therefore arrive at the sharper requirement that Eq.(8.57) must hold *even if the particle is off-shell*.

8.3.4 Handlebar condition for massive vector particles

Let us examine the situation where a vector particle does have a mass, but the mass m is very small compared to the vector particle's energy E or its momentum. Clearly, it would be unacceptable[10] if the limit $m \to 0$ would be singular while the case $m = 0$ is not[11]. We shall therefore require that, for

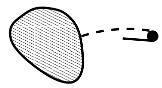
[8]One may for instance have the source $\mathcal{J}$ represent a charge whose momentum changes, thereby emitting radiation.

[9]Whether they *exist* is another question ; at any rate we cannot *produce* them, not *observe* them.

[10]Or at least embarassing — after all, we do not know for certain if the mass of the photon is strictly zero or just a measly 10^{-137} kilograms. The most trustworthy current limit is $m_\gamma c^2 < 10^{-18}$eV, or $m_\gamma < 2 \times 10^{-54}$ kg.

[11]Note that we do not even insist that $m \to 0$ gives the *same* result as $m = 0$, only that the limit is nonsingular.

massive vector particles partaking in a process at high energy, the handlebar condition (8.57) holds in a milder form :

 $$= \mathcal{O}\,(m) \quad . \tag{8.59}$$

The meaning of this condition is the following. The *longitudinal* polarization vector of a massive vector boson has energy behaviour different from its two *transverse* ones : it grows at high energy $E \gg m$ with an extra power of E. If for transverse polarization the amplitude is well-behaved at high energy it may not be so for longitudinal polarization. The requirement implied by the handlebar condition is that the extra power E inserted into the expression because of longitudinal polarization is softened, by cancellations over at least one order of magnitude in terms of E/m. We shall presently see that this condtion is sufficiently severe to determine, to a large extent, the possible couplings of a theory containing such particles.

8.3.5 Helicity states for massless vectors

The spinor-based helicity states for massive vector particles of section 8.2.1 are apparently not well suited to the massless case. Note, however, that we may generalize the method of Eq.(8.34) as follows :

$$q^\mu = p_1 + \alpha\, p_2 \quad , \quad {p_{1,2}}^2 = 0 \quad , \quad m^2 = 2\alpha\,(p_1 \cdot p_2) \quad . \tag{8.60}$$

Using the fact that the spinors of massless particles are homogeneous of degree 1/2 in the argument :

$$u_\pm(\alpha p_2) = \sqrt{\alpha}\; u_\pm(p_2) \quad , \tag{8.61}$$

we see that (for instance) the polarization vector ϵ_+ can be written, in analogy to Eq.(8.35), as

$${\epsilon_+}^\mu = \frac{1}{2\sqrt{p_1 \cdot p_2}} u_+(p_1)\; \gamma^\mu\; u_+(p_2) \quad . \tag{8.62}$$

Since α does not occur in the polarization vector, we may consider the limit $\alpha \to 0$. In that case, $q = p_1$ is massless, and the only condition on the massless vector p_2 is that $(p_1 \cdot p_2)$ must not vanish. By a judicious choice of overall complex phase, this leads us to propose, for a massless vector particle with momentum k^μ, states of definite helicity as follows, where the spinors are again in the standard form :

$$\epsilon_\lambda{}^\mu = \frac{\lambda}{s_{-\lambda}(k,r)\,\sqrt{2}}\,\overline{u}_\lambda(k)\gamma^\mu u_\lambda(r) \quad , \quad \lambda = \pm \;\; . \tag{8.63}$$

Here, the vector r^μ is an arbitrarily chosen massless vector not parallel to k^μ ; it is called the *gauge vector*. We can ascertain that

$$\epsilon_+ \cdot \overline{\epsilon}_- = \frac{1}{4k\cdot r}\overline{u}_+(k)\gamma^\mu u_+(r)\;\overline{u}_-(r)\gamma_\mu u_-(k) = 0 \;\; , \tag{8.64}$$

in the same manner we employed in Eq.(8.38). Furthermore,

$$\epsilon_+ \cdot \overline{\epsilon}_+ = \frac{1}{4k\cdot r}\overline{u}_+(k)\gamma^\mu u_+(r)\;\overline{u}_+(r)\gamma_\mu u_+(k) = \frac{-1}{2k\cdot r}\overline{u}_+(k)\not{r}u_+(k) = -1 \;\; . \tag{8.65}$$

These, then, are acceptable helicity states.

A few useful properties of these polarization vectors are

$$\omega_\lambda \not{\epsilon}_\lambda = \frac{\lambda\sqrt{2}}{s_{-\lambda}(k,r)}\;u_\lambda(r)\overline{u}_\lambda(k) \quad , \quad \omega_{-\lambda} \not{\epsilon}_\lambda = \frac{\lambda\sqrt{2}}{s_{-\lambda}(k,r)}\;u_{-\lambda}(r)\overline{u}_{-\lambda}(k) \tag{8.66}$$

and

$$\not{k}\not{\epsilon}_\lambda = \lambda\,\sqrt{2}\;u_{-\lambda}(k)\;\overline{u}_\lambda(k) \;\; , \tag{8.67}$$

and this object is *explicitly* gauge-invariant.

8.3.6 The massless propagator : the axial gauge

We can perform the sum over the physical polarization states of a massless vector from the helicity states :

$$\begin{aligned} \sum_{\lambda=\pm} \epsilon_\lambda{}^\mu\overline{\epsilon_\lambda}{}^\nu &= \sum_{\lambda=\pm}\frac{1}{4k\cdot r}\overline{u}_\lambda(k)\gamma^\mu u_\lambda(r)\;\overline{u}_\lambda(r)\gamma^\nu u_\lambda(k) \\ &= \sum_{\lambda=\pm}\frac{1}{4k\cdot r}\overline{u}_\lambda(k)\gamma^\mu\not{r}\gamma^\nu u_\lambda(k) \\ &= \frac{1}{4k\cdot r}\;\mathrm{Tr}\left(\not{k}\;\gamma^\mu\;\not{r}\;\gamma^\nu\right) \\ &= -g^{\mu\nu} + \frac{1}{k\cdot r}\left(k^\mu r^\nu + r^\mu k^\nu\right) \;\; . \end{aligned} \tag{8.68}$$

The form of the massless vector propagator *in which only physical degrees of freedom propagate* is therefore given by the following Feynman rule :

$$\leftrightarrow \; i\hbar\, \frac{-g^{\mu\nu} + (k^\mu r^\nu + r^\mu k^\nu)/(k\cdot r)}{k^2 + i\epsilon} \quad \text{massless internal lines}$$

$$\boxed{\text{Feynman rules, version 8.3}} \tag{8.69}$$

Note the appearance of the arbitrary vector r. This way of writing the propagator is called the *axial gauge*. The propagator is constructed to be orthogonal to r^μ whatever the value of k. The vector r acts as an 'axis' with respect to which the field is always orthogonal, hence the name. The fact that the vector r is arbitrary is of course bothersome, in the same way that the arbitrariness of the representation chosen for the Dirac matrices in the case of Dirac particles is bothersome. We solve it in the same way, by insisting that we ought to be able to remove r from the final expressions for matrix elements. This can of course not be by virtue of any property of r itself, but must come from the handlebar condition, since every term containing r also contains k. Two things are worthy of remark here. In the first place, the propagator is homogeneous of degree zero in r, so any result cannot depend on the *length* of r anyway. In the second place, in contrast to the propagator proposed before, with $p^\mu p^\nu/m^2$, the propagator in the axial gauge does not diverge. We are therefore freed from the requirement that the handlebar condition must also hold off-shell.

8.3.7 Gauge vector shift

Let us consider helicity states for massless vector particles as defined in sect.8.3.5. We shall denote these by ${\epsilon_\lambda}^\mu(k,r)$. If we change the gauge vector r from one value into another, another perfectly acceptable helicity state is obtained. What is the relation between these states ? To answer this we simply compute the difference between the states with different gauge vector :

$$\begin{aligned}
&{\epsilon_\lambda}^\mu(k,r_1) - {\epsilon_\lambda}^\mu(k,r_2) = \\
&\qquad \frac{\lambda}{\sqrt{2}}\left(\frac{\overline{u}_\lambda(k)\gamma^\mu u_\lambda(r_1)}{s_{-\lambda}(k,r_1)} - \frac{\overline{u}_\lambda(k)\gamma^\mu u_\lambda(r_2)}{s_{-\lambda}(k,r_2)}\right) \\
&\quad = \; -\frac{\lambda}{\sqrt{2}}\left(\frac{\overline{u}_{-\lambda}(r_1)\gamma^\mu u_{-\lambda}(k)}{s_{-\lambda}(r_1,k)} + \frac{\overline{u}_\lambda(k)\gamma^\mu u_\lambda(r_2)}{s_{-\lambda}(k,r_2)}\right)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\lambda}{\sqrt{2}}\frac{\Big(\overline{u}_{-\lambda}(r_1)\gamma^\mu u_{-\lambda}(k)\overline{u}_{-\lambda}(k)u_\lambda(r_2)+\overline{u}_{-\lambda}(r_1)u_\lambda(k)\overline{u}_\lambda(k)\gamma^\mu u_\lambda(r_2)\Big)}{s_{-\lambda}(r_1,k)s_{-\lambda}(k,r_2)}\\
&= \frac{\lambda}{\sqrt{2}}\frac{\overline{u}_{-\lambda}(r_1)(\gamma^\mu \not{k}+\not{k}\gamma^\mu)u_\lambda(r_2)}{s_{-\lambda}(k,r_1)\ s_{-\lambda}(k,r_2)}\\
&= \lambda\sqrt{2}\frac{s_{-\lambda}(r_1,r_2)}{s_{-\lambda}(k,r_1)\ s_{-\lambda}(k,r_2)}\ k^\mu \quad . \qquad (8.70)
\end{aligned}$$

we see that the two states differ only by the vector particle's momentum. In any current-conserving set of diagrams we may therfore choose the gauge vector at will ; there is no risk of picking up a phase difference if two different gauge vectors are used for two different current-conserving sets of diagrams.

As an illustration of how the gauge vector can disappear from a current-conserving object, let us consider

$$\epsilon_\lambda\cdot\left(\frac{p}{2k\cdot p}-\frac{q}{2k\cdot q}\right)\quad ,$$

with p and q two massless momenta. The form of section 8.3.5 turns this into

$$\begin{aligned}
&\frac{\lambda}{\sqrt{2}\ s_{-\lambda}(k,r)}\left(\frac{s_\lambda(k,p)s_{-\lambda}(p,r)}{2k\cdot p}-\frac{s_\lambda(k,q)s_{-\lambda}(q,r)}{2k\cdot q}\right)\\
&\qquad = \frac{\lambda}{\sqrt{2}\ s_{-\lambda}(k,r)}\left(\frac{s_{-\lambda}(p,r)}{s_{-\lambda}(p,k)}-\frac{s_{-\lambda}(q,r)}{s_{-\lambda}(q,k)}\right)\\
&\qquad = \frac{\lambda}{\sqrt{2}}\ \frac{s_{-\lambda}(p,r)s_{-\lambda}(q,k)-s_{-\lambda}(q,r)s_{-\lambda}(p,k)}{s_{-\lambda}(k,r)s_{-\lambda}(p,k)s_{-\lambda}(q,k)} \qquad (8.71)
\end{aligned}$$

Now, the Schouten identity tells us that

$$s_{-\lambda}(p,r)s_{-\lambda}(q,k)+s_{-\lambda}(p,k)s_{-\lambda}(r,q)=-s_{-\lambda}(p,q)s_{-\lambda}(k,r) \qquad (8.72)$$

so that the gauge vector indeed drops out, and

$$\epsilon_\lambda\cdot\left(\frac{p}{2k\cdot p}-\frac{q}{2k\cdot q}\right)=-\frac{\lambda}{\sqrt{2}}\ \frac{s_{-\lambda}(p,q)}{s_{-\lambda}(k,p)s_{-\lambda}(k,q)}\quad . \qquad (8.73)$$

One can easily check that the same form is obtained without using the Schouten identity if we choose either $r=p$ or $r=q$. **E 47**

8.4 Exercises for chapter 8

Excercise 45 Polarization averages
If, for vector particles, we take not a spin sum but a spin *average*, we find

$$\frac{1}{3}\sum_{j=1}^{3} {\epsilon_j}^{\mu}\bar{\epsilon}_j^{\nu} = \frac{1}{3}\left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right)$$

for an on-shell vector particle of momentum p and mass m. We can also consider it *classically*, in the rest frame where $\vec{p} = 0$. In that case there is a polarization vector $\epsilon^\mu = (0, \vec{\epsilon})$ (we only consider *real* polarizations). Show that, if we also *average* the polarization vector over all possible orientations, we find

$$\frac{1}{4\pi}\int d\Omega\ \epsilon^\mu\,\epsilon^\nu = \frac{1}{3}\left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right)$$

as well.

Excercise 46 A character-building calculation
Let $b^\mu = \Lambda(p;q)^\mu{}_\nu\, a^\nu$ for some arbitrary vector a. Use the explicit form (8.12) of the minimal Lorentz transform to prove by explicit calculation that $b \cdot b = a \cdot a$.

Excercise 47 A check
Do this.

Chapter 9

Quantum Electrodynamics

9.1 Introduction

In this chapter we shall start to work our way to realistic theories about the actual elementary particles encountered in nature[1]. All elementary particles seen so far have nonzero spin, apart from the newly-discovered Higgs boson. We shall defer the discussion of charged spin-1 particles to a later chapter ; at this point we shall only discuss how to set up a consistent theory of spin-1/2 particles (charged leptons and/or quarks) and photons. This is the theory of *quantum electro-dynamics*, or QED.

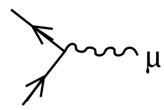
9.2 Setting up QED

9.2.1 The QED vertex

Since the propagators of spin-1/2 particles and of the massless spin-1 photon have already been fixed, the only ingredient which we still have to determine is the coupling between them ; and on this coupling rests the burden of ensuring the current-conservation requirement as embodied in Eq.(8.57). The vertex coupling Dirac particles must have one upper, and one lower Dirac index : and since the photon is involved, it must also carry a Lorentz index.

[1]It may of course be possible that the elementary particles discussed in this text are not truly elementary and that a yet deeper level of substructure will be discovered. In that case, please insert in whatever follows the addendum (as of March 26, 2017).

The simplest, and – as we shall see – indeed the correct form of the vertex is that of a Dirac matrix. We therefore propose the following Feynman rule :



$$\leftrightarrow \quad i\frac{Q}{\hbar}\gamma^{\mu}$$

QED vertex

Feynman rules, version 9.1 (9.1)

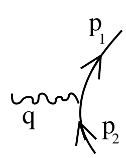
Here Q is the strength of the fermion-photon coupling : the *charge* of the fermion[2]. By dimensional analysis, we see that is has dimension

$$\mathbf{dim}\Big[Q\Big] = \mathbf{dim}\Big[\hbar^{-1/2}\Big] \quad . \tag{9.2}$$

The Dirac delta function imposing momentum conservation is implied. As is conventional, we shall employ wavy lines to indicate photons. As stressed in the previous chapter, this choice of vertex can only been argued to be reasonable if the photon current is conserved ; this we shall show in what follows.

9.2.2 Handlebars : a first look

Let us now start to investigate the requirements of current conservation for our theory. One of the simplest possible processes is the decay of a photon into a fermion-antifermion pair, shown below :

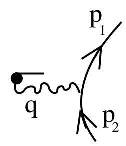


Of course the photon has to be off-shell here, but that is no problem since also off-shell photons must obey current conservation. The part of the amplitude depicted is given by

$$\mathcal{M} = -Q\,\overline{u}(p_1)\gamma^{\mu}v(p_2) \quad , \tag{9.3}$$

[2]Or, rather, it is *related to* the charge. The precise form of this relation must, of course, be established by investigating the coupling in a well-defined physical situation.

where the index μ of the photon is coupled to a corresponding index somewhere else in the larger Feynman diagram. Let us now attach the handlebar, so that we get



With the convention, to which we shall try to adhere, that the momentum assigned in the handlebar must be counted *outgoing* from the vertex, so in this case should read $-q$, the handlebarred $\mathcal{M}$ becomes

$$\mathcal{M}\rfloor = Q\,\overline{u}(p_1)\,\not{q}\,v(p_2) \quad . \tag{9.4}$$

Note that we indicate the handlebar algebraically by the symbol $\rfloor$. Now we apply momentum conservation that tells us that $q = p_1 + p_2$:

$$\mathcal{M}\rfloor = Q\,\overline{u}(p_1)\left(\not{p}_1 + \not{p}_2\right)v(p_2) \quad . \tag{9.5}$$

To the expression in the middle we add zero in a clever way :

$$\mathcal{M}\rfloor = Q\,\overline{u}(p_1)\left(\not{p}_1 - m + \not{p}_2 + m\right)v(p_2) \quad , \tag{9.6}$$

where m is the mass of the fermion. Now, we know that the spinors $\overline{u}$ and v satisfy the Dirac equations

$$(\not{p}_1 - m)u(p_1) = 0 \quad \text{and} \quad (\not{p}_2 + m)v(p_2) = 0 \tag{9.7}$$

for on-shell momenta, so that half of the expression 9.6 'cancels to the left' and the other half 'cancels to the right', and we end up with

$$\mathcal{M}\rfloor = 0 \quad . \tag{9.8}$$

We shall see that this is the general mechanism by which unitarity and current conservation are ensured.

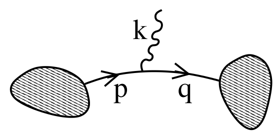
The above is of course only the simplest example of current conservation in QED, and in the following we shall in fact study *all* conceivable QED process at once, but already we can learn a few useful things. In the first place, a possible alternative coupling, with $\gamma^5\gamma^\mu$ instead of γ^μ, is ruled out since we cannot obtain two Dirac equations :

$$\gamma^5\,\not{q} = -\not{p}_1\gamma^5 + \gamma^5\not{p}_2 = -(\not{p}_1 \pm m)\gamma^5 + \gamma^5(\not{p}_2 \pm m) \quad , \tag{9.9}$$

so that either the cancellation to the left would be spoiled, or that to the right. In the second place, it is necessary that both fermions have precisely the same mass. Since all known different fermion types have different masses, this means that the QED interaction must conserve fermion type, or 'flavour'. Electromagnetic muon decay, $\mu \to e\gamma$, is therefore forbidden, not by conservation of the electric charge (which is indeed the same for muons
E 49 and electrons) but by conservation of the whole electromagnetic current [3].

9.2.3 Handlebar diagrammatics

The argument for current conservation in the previous section went through because both fermions were on their mass shell. Since fermions in internal lines in Feynman diagrams are *not* on the mass shell, we have to extend our approach to off-shell fermions. Consider an arbitrary diagram in which a fermion of mass m propagates and couples to a photon, as depicted below.



The fermion momenta p and q are indicated and for the photon momentum k we have $k^\mu = (p-q)^\mu$. The momenta p and q *may* be on-shell (in which case the corresponding blob is left out), but any of them may be off-shell, and hence leads into a further piece of Feynman diagram. In that case the blobs stand for the other vertices, where the fermion is created and absorbed[4]. The part of the diagram between the blobs is of course given by

$$\left(i\hbar\frac{\not{q}+m}{q^2-m^2}\right)\ \left(i\frac{Q\gamma^\mu}{\hbar}\right)\left(i\hbar\frac{\not{p}+m}{p^2-m^2}\right)\quad ,$$

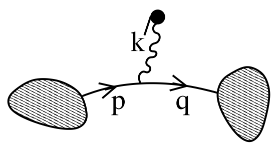
where μ is the index belonging to the photon line ; in an actual process, μ may be coupled to the photon's polarization vector if the photon is external, or to the photon's propagator if the photon happens to be an internal line. In case p, say, is on-shell we have to write

$$\left(i\hbar\frac{\not{q}+m}{q^2-m^2}\right)\ \left(i\frac{Q\gamma^\mu}{\hbar}\right)\left(u(p)\,\sqrt{\hbar}\right)\quad .$$

[3]Fortunately, the decay $\mu \to e\gamma$ has never been observed, and the branching ratio is smaller than about 10^{-11}.

[4]Actually, the p and q lines are attached to a semi-connected graph rather than two separate connected ones, but here the distinction is irrelevant.

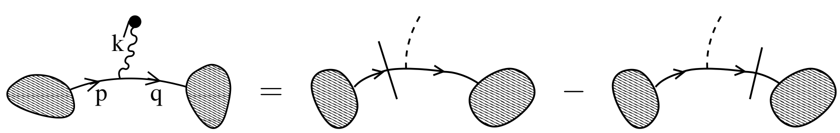
Let us now put the handlebar on the photon leg :



Algebraically, we must multiply the above expression by k_μ, and then

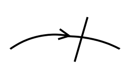
$$
\begin{aligned}
&\left(\frac{iQ}{\hbar}\right)(i\hbar)^2\,\left(\frac{\not{q}+m}{q^2-m^2}\,\gamma^\mu\,\frac{\not{p}+m}{p^2-m^2}\right)\,k_\mu = \\
&= \left(\frac{iQ}{\hbar}\right)(i\hbar)^2\,\frac{\not{q}+m}{q^2-m^2}\,(\not{p}-\not{q})\,\frac{\not{p}+m}{p^2-m^2} \\
&= \left(\frac{iQ}{\hbar}\right)(i\hbar)^2\,\frac{\not{q}+m}{q^2-m^2}\,\Big((\not{p}-m)-(\not{q}-m)\Big)\,\frac{\not{p}+m}{p^2-m^2} \\
&= \left(\frac{iQ}{\hbar}\right)(i\hbar)^2\,\left(\frac{\not{q}+m}{q^2-m^2}\,-\,\frac{\not{p}+m}{p^2-m^2}\right)\quad .
\end{aligned}
\tag{9.10}
$$

We see that under the handlebar the double propagator splits up into two single ones. Note that, for this to be possible, it is again essential that the mass of the fermion does not change at the vertex. We may write this operation diagrammatically as

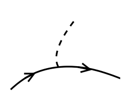


, (9.11)

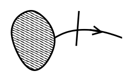
where we have introduced two new diagrammatic ingredients: a slashed fermion line, with a trivial Feynman rule :


$$\leftrightarrow \quad i\hbar \ , \tag{9.12}$$

and a new vertex, also carrying a trivial rule :

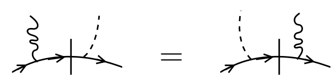

$$\leftrightarrow \quad i\frac{Q}{\hbar} \ . \tag{9.13}$$

The handlebarred photon line is replaced by a dotted line which evaluates trivially to unity, but we do not want to leave it out of the diagram since the dashed propagator still carries an amount of momentum, so that without it momentum conservation would not hold at the new vertex. Like the handlebar this rule is not intended to represent some physical interaction, but serves only as a computational device. For *external* Dirac lines we find even simpler rules, since the external spinors satisfy the Dirac equation :



$= 0$ for on-shell external lines , (9.14)

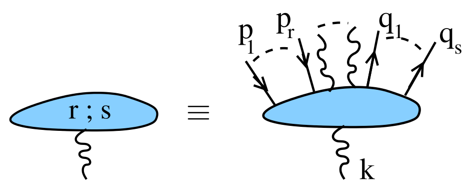
where the external line may belong to the initial or final state, and the arrow orientation may be also reversed. An important result follows immediately from the triviality of our new Feynman-rule tools :



. (9.15)

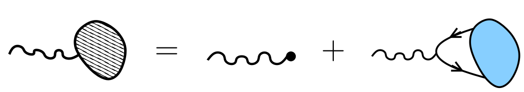
9.2.4 Current conservation : the Ward-Takahashi identity

We shall now prove that the Feynman rule (9.2.1) is a good one, in the sense that a handlebar on any photon gives a zero result, both for on-shell (external) and off-shell (internal) photon lines. This is quite a tall order, since we have to consider a literal infinity of possible processes. We shall base the proof on - what else ? - the SDe's of the theory. Throughout this section we shall use semi-connected diagrams only. Let us consider a general Green's function that contains r fermion lines flowing in, and s fermion lines flowing out, together with any number (≥ 1) of photon lines, one of which we single out :

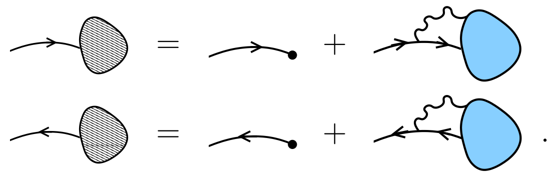


(9.16)

The fermions' momenta are reckoned along their respective arrows, and the photon momentum k is counted outgoing. Note that, since we are considering here a Green's function and not an amplitude, any of the external lines may be off-shell. It is this object that we shall submit to a handlebar operation. The SDe's of QED are, in our notation :

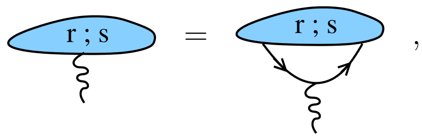


,



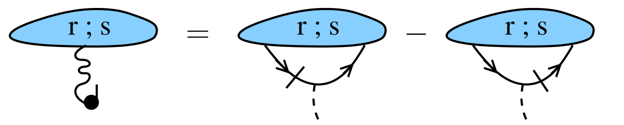
(9.17)

We therefore have



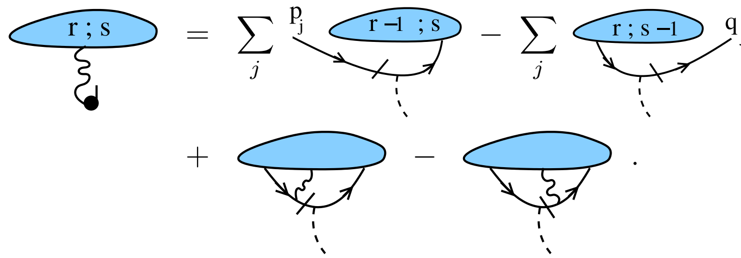
(9.18)

and the handlebar operation gives us



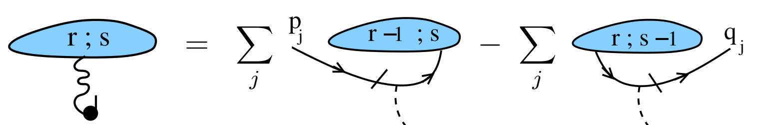
(9.19)

Each term on the right-hand side can be subjected to its own SDe, to give



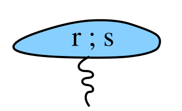
(9.20)

By virtue of Eq.(9.15) the last two terms cancel precisely, and we are left with the *Ward-Takahashi identity* :



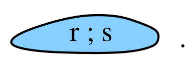
(9.21)

We can conveniently express this in a more analytic form. Let us denote our Green's function by

$$G^{\mu}(p_1, \ldots, p_r; q_1, \ldots, q_s; k) \equiv$$ 

, (9.22)

where the explicit Lorentz index is that of the photon line, and the same Green's function, only with the special photon removed, by

$$G_0(p_1, \ldots, p_r; q_1, \ldots, q_s) \equiv$$ 

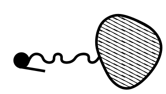
. (9.23)

Taking into account the flow of the momenta and the fact that the two trivial Feynman rules (9.12) and (9.13) together yield a factor of $(iQ/\hbar)(1\hbar) = -Q$, we can write the diagrammatic Ward-Takahashi identity (9.21) as follows[5] :

$$\begin{aligned} &G^\mu(p_1,\ldots,p_r;q_1,\ldots,q_s;k)\;k_\mu = \\ &\qquad Q\sum_{j=1}^{s} G_0(p_1,\ldots,p_r;q_1,\ldots,q_j+k,\ldots,q_s) \\ &\qquad -Q\sum_{j=1}^{r} G_0(p_1,\ldots,p_j-k,\ldots,p_r;q_1,\ldots,q_s)\quad . \end{aligned} \tag{9.24}$$

It is this result that proves that, indeed, the choice of the vertex (9.2.1) leads to an acceptable theory.

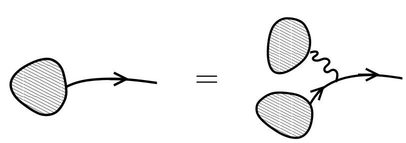
So far we have considered the general case, with no constraint on the external momenta. If we now specialize to amplitudes, in which all external momenta except perhaps for k^μ, must be on their mass shell, the rule (9.14) applies, and we find the even more attractive *Ward identity* :



$$= \; 0 \quad . \tag{9.25}$$

9.2.5 The charged Dirac equation

We still have to determine the precise relation between the coupling constant Q in the Feynman rule, and the classical electric charge q of the particle. We shall do this by establishing a relation with classical electrodynamics. The classical (*i.e.* non-loop) SDe for ψ in the presence of a photon field A is given by



$$, \tag{9.26}$$

[5] In many, or even most, cases of interest fermion number is conserved, which means that in every (fundamental or effective) vertex the number of incoming and outgoing fermions is the same ; in that case we have $r = s$. But since we have nowhere used this, the Ward-Takahashi identity may be expected to hold also for processes in which fermion number is not conserved, for example in supersymmetry where Majorana fermions occur. Note, however, that Majorana fermions are necessarily neutral and themselves do not couple to photons.

in other words

$$\psi(x) \;=\; \int d^4y\, \frac{1}{(2\pi)^4} \int d^4k\, e^{-ik\cdot(x-y)} \\ i\hbar\, \frac{\not{k}+m}{k^2-m^2+i\epsilon} \left(i\frac{Q}{\hbar}\right) \gamma_\mu \psi(y) A^\mu(y) \;\;, \tag{9.27}$$

whence

$$\left(i\not{\partial} - m + Q\not{A}(x)\right)\psi(x) = 0 \;\;, \tag{9.28}$$

which is the Dirac equation in the presence of an electromagnetic field. Let us work this expression towards classical physics. In the first place, the derivative is, by the standard assignment rules for quantum mechanics, related to the momentum operator :

$$p^\mu = i\hbar\, \partial^\mu \;\;, \tag{9.29}$$

and the mass m to the mechanical mass M by (as we have seen)

$$m = \frac{Mc}{\hbar} \;\;. \tag{9.30}$$

The Dirac equation can therefore be written as

$$\Big((p^\mu + \hbar Q A(x)^\mu)\gamma_\mu - Mc\Big)\psi(x) = 0 \;\;, \tag{9.31}$$

which is to be compared with the standard expression for the *electromagnetic* momentum if a charged particle in classical electrodynamics:

$$p_{\mathrm{em}}{}^\mu = p^\mu - \frac{q}{c} A^\mu \;\;. \tag{9.32}$$

where q is the classical charge of the particle and A_c the classical electromagnetic field. In the Gaussian system of units, the charges have dimensionality $\mathbf{dim}[q^2] = \mathrm{kg\ m^3/sec^2}$ and the Coulomb field strength E therefore obeys $\mathbf{dim}[E] = \mathbf{dim}[q]\,/\mathrm{m}^2$. Since this is the gradient of the classical e.m. vector potential A_c we have $\mathbf{dim}[A_c] = \mathbf{dim}[q]\,/\mathrm{m}$, and because the photon field A has dimensionality $\mathbf{dim}[A^2] = \mathrm{kg/sec}$, it follows the correct relation between the photon field and the classical e.m. field must read

$$A_c{}^2 = c\, A^2 \;\;. \tag{9.33}$$

From this it follows that the coupling Q and the charge q are related by

$$Q = -q/(\hbar\sqrt{c}) \;\;, \tag{9.34}$$

which implies the correct dimensionality $\mathbf{dim}[Q] = \mathbf{dim}\Big[1/\sqrt{\hbar}\Big]$; moreover, we find immediately that, for particles with unit electric charge,

$$Q^2 = \frac{4\pi}{\hbar}\alpha \ \ , \tag{9.35}$$

where α stands for the electromagnetic fine structure constant :

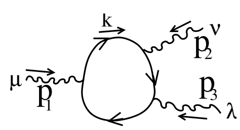
$$\alpha \ \approx \ 1 \ / \ 137.036 \ \ . \tag{9.36}$$

Since in QED every next loop order contains two extra powers of Q and one (effective) power of $\hbar$, the loop expansion is in QED equivalent to an expansion in powers of α.

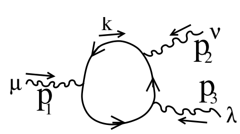
A final remark is in order. The above analysis is purely dimensional *i.e.* it does not decide between $Q = q/\hbar\sqrt{c}$ and $Q = 2q/\hbar\sqrt{c}$. Later on, the discussion of Thomson scattering will assure us that we have made the right choice. Of course, any extra numerical factor in Q can be compensated for by a rescaling of A.

9.2.6 Furry's theorem

An interesting observation concerns closed fermion loops in QED. Let us consider a fermion loop that is attached by three QED vertices to the rest of a Feynman diagram:

$D_- \equiv$ 

Here, we have indicated the Lorentz indices on the photon lines, and the momenta across the photon lines are considered incoming into the loop. In addition to this diagram, there is also a similar diagram in which the orientation of the loop is reversed :

$D_+ \equiv$ 

Note that these graphs cannot be twisted into one another. For loops with only one or two vertices they *can* be so twisted, and then do not count as separate diagrams ; for three or more vertices, there are two distinct ones.

Without pretending to evaluate the whole loop, let us concentrate on the Dirac structure of their numerators. The first diagram contains the trace[6]

$$D_- \;\to\; \mathrm{Tr}\left((\not{k}+m)\,\gamma^\mu\,(\not{k}-\not{p}_1+m)\,\gamma^\lambda\,(\not{k}+\not{p}_2+m)\,\gamma^\nu\right)\;\equiv T_- \quad , \qquad (9.37)$$

whereas the corresponding trace for the other diagram reads

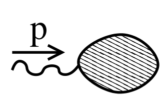
$$D_+ \;\to\; \mathrm{Tr}\left((-\not{k}+m)\,\gamma^\nu\,(-\not{k}-\not{p}_2+m)\,\gamma^\lambda\,(-\not{k}+\not{p}_1+m)\,\gamma^\mu\right)\;\equiv T_+ \quad . \qquad (9.38)$$

Note that the rest of the loops, and in particular the propagator denominators, are identical for both graphs. By using the reversibility inside traces of Clifford algebra elements, we can write

$$\begin{aligned} T_+ &= -\,\mathrm{Tr}\left((\not{k}-m)\,\gamma^\nu\,(\not{k}+\not{p}_2-m)\,\gamma^\lambda\,(\not{k}-\not{p}_1-m)\,\gamma^\mu\right)\\ &= -\,\mathrm{Tr}\left((\not{k}-m)\,\gamma^\mu\,(\not{k}-\not{p}_1-m)\,\gamma^\lambda\,(\not{k}+\not{p}_2-m)\,\gamma^\nu\right)\\ &= -\,T_- \quad , \end{aligned} \qquad (9.39)$$

since no terms with an odd power of m survives the trace. We see that the two loops cancel each other precisely ! This can obviously be extended to loops with more vertices, and we find *Furry's theorem* : **fermion loops with an odd number of vector vertices[7] and opposite orientation cancel each other ; with an even number of vector vertices, they are identical[8]**. Furry's theorem does not hold if one or more of the vertices are of axial-vector type, and so it is not generally valid for the weak interactions. For QCD, in which the quark-gluon couplings have the Dirac-matrix form as in QED, Furry's theorem holds in a more restricted form : the *spacetime part* of the two quark loops with even(odd) number of vertices are equal(opposite), but the additional colour structures of the diagrams are different. This implies, for instance, that the two quark loops with three gluon vertices do not cancel completely. We shall come back to that case later on.

Let us consider a diagram (or set of diagrams) N with only one single external line which is a photon :

$N \;\equiv$  , (9.40)

[6]By the rules of Dirac particles, closed loops automatically evaluate to traces.

[7]That is, vertices consisting of a single Dirac matrix, such as in QED.

[8]Furry's theorem is usually proved by invoking the charge-conjugation matrix, discussed in section 13.10.2. However, this is not strictly necessary as we see.

where we have indicated the momentum of the photon. Such objects go under the name of *tadpoles*[9]. By Lorentz covariance we see that, whatever goes on inside the blob, N must always be of the form

$$N^\mu = p^\mu \; f(p^2) \;\; . \tag{9.41}$$

But ... by momentum conservation, $p^\mu = 0$ since no momentum is coming out on the other side of the blob. Therefore $N = 0$; photon tadpoles vanish identically, even if Furry's theorem does not apply in this case. It is easy to see that the same must hold for the tadpoles of any other vector particle. For scalar particles it does *not* apply, since $N = f(0)$ has no particular reason to vanish. Also, for particles that are described not by vectors but by symmetric tensors, the tadpole $N^{\mu\nu} = g^{\mu\nu} f(0)$ does not necessarily vanish[10].

9.3 Some QED processes

9.3.1 A classic calculation : muon pair production

We are now in a position to compute, for the first time, a realistic cross secttion. We shall follow the classic steps that lead to our final result.

Description of the process with momentum assignments

The simplest calculation is that of the cross section for muon pair production in e^+e^- collisions:

$$e^-(p_1) \;\; e^+(p_2) \quad \to \quad \mu^-(q_1) \;\; \mu^+(q_2) \;\; .$$

since it is described, at tree level, by only a single diagram[11].

---

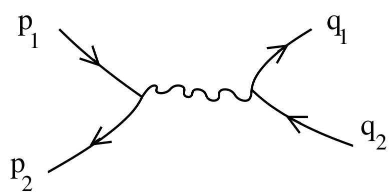
[9]Of course, 'spermatozoon' would be more appropriate...

[10]This is not the graviton, since that is described by a symmetric *traceless* tensor ; it is in fact another representaion of a scalar particle.

[11]It might be thought that processes involving only electrons would be simpler, but as we shall see these contain always at least two diagrams.

Drawing and writing out the diagram(s)

The single lowest-order Feynman diagram is given by



Both the electron and muon are Dirac particles. We shall denote the electron charge by Q_e, and the muon charge by Q_μ, and their masses by m_e and m_μ, respectively. The total invariant mass squared is conventionally denoted by s, and of course momentum is conserved :

$$p_1{}^\alpha + p_2{}^\alpha = q_1{}^\alpha + q_2{}^\alpha \quad , \quad s = (p_1 + p_2)^2 = (q_1 + q_2)^2 \ . \tag{9.42}$$

The amplitude corresponding to the Feynman diagram is

$$\mathcal{M} = i\frac{\hbar Q_e Q_\mu}{s} \quad \overline{v}(p_2)\, \gamma^\alpha\, u(p_1) \quad \overline{u}(q_1)\, \gamma_\alpha\, v(q_2) \ \ , \tag{9.43}$$

and is strictly dimensionless: $\mathbf{dim}[\mathcal{M}] = \mathbf{dim}[1]$, as it ought to be for a $2 \to 2$ process at tree order.

Squaring and averaging, eliminating Dirac stuff

In a typical muon pair production process, we shall accept muons with any polarization in the final state ; also, usually the beams of incoming electrons and positrons are unpolarized. The amplitude, squared and averaged over the incoming electron and positron spins can be evaluated using the Casimir trick :

$$\begin{aligned}
\left\langle |\mathcal{M}|^2 \right\rangle &= \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \\
&= \frac{\hbar^2 Q_e{}^2 Q_\mu{}^2}{4s^2} \quad \sum_{\text{spins}} \overline{v}(p_2)\gamma^\alpha u(p_1)\, \overline{u}(p_1)\gamma^\beta v(p_2) \\
&\qquad \times \sum_{\text{spins}} \overline{u}(q_1)\gamma_\alpha v(q_2)\, \overline{v}(q_2)\gamma_\beta u(q_1)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\hbar^2 {Q_e}^2 {Q_\mu}^2}{4s^2} \ \mathrm{Tr}\Big((\not{p}_2 - m_e)\gamma^\alpha(\not{p}_1 + m_e)\gamma^\beta\Big) \\
&\qquad\qquad\qquad \mathrm{Tr}\Big((\not{q}_1 + m_\mu)\gamma_\alpha(\not{q}_2 - m_\mu)\gamma_\beta\Big) \\
&= \frac{4\hbar^2 {Q_e}^2 {Q_\mu}^2}{s^2} \ \Big({p_2}^\alpha {p_1}^\beta + {p_1}^\alpha {p_2}^\beta - (p_1 \cdot p_2) g^{\alpha\beta} - {m_e}^2 g^{\alpha\beta}\Big) \\
&\qquad\qquad\qquad \Big(q_{1\alpha} q_{2\beta} + q_{2\alpha} q_{2\beta} - (q_1 \cdot q_2) g_{\alpha\beta} - {m_\mu}^2 g_{\alpha\beta}\Big) \\
&= \frac{4\hbar^2 {Q_e}^2 {Q_\mu}^2}{s^2} \\
&\quad \Big(2(p_1 \cdot q_1)(p_2 \cdot q_2) + 2(p_1 \cdot q_2)(p_2 \cdot q_1) \\
&\quad - s(p_1 \cdot p_2) - s(q_1 \cdot q_2) + s^2\Big) \qquad (9.44)
\end{aligned}
$$

Choosing a Lorentz frame, working out dot products

We shall work in the centre-of-mass frame of the colliding electron-positron pairs. In that frame, we have

$$
{p_{1,2}}^0 = {q_{1,2}}^0 = E \quad , \quad |\vec{p}_{1,2}| = p \quad , \quad |\vec{q}_{1,2}| = q \quad , \qquad (9.45)
$$

where

$$
s = 4E^2 \quad , \quad p^2 = E^2 - {m_e}^2 \quad , \quad q^2 = E^2 - {m_\mu}^2 \quad . \qquad (9.46)
$$

The various vector products are therefore given by

$$
\begin{aligned}
&(p_1 \cdot p_2) = s/2 - {m_e}^2 \quad , \quad (q_1 \cdot q_2) = s/2 - {m_\mu}^2 \quad , \\
&(p_1 \cdot q_1) = (p_2 \cdot q_2) = s/4 - pq \ \cos(\theta) \\
&(p_1 \cdot q_2) = (p_2 \cdot q_1) = s/4 + pq \ \cos(\theta) \quad , \qquad (9.47)
\end{aligned}
$$

where θ is the *polar* scattering angle, that is, the angle between $\vec{p}_1$ and $\vec{q}_1$. We also use the fact that Q_μ and Q_e are the negative of the unit charge, so that $Q_\mu Q_e = 4\pi\alpha/\hbar$.

The final transition rate

We now arrive at

$$
\left\langle |\mathcal{M}|^2 \right\rangle = \frac{16\pi^2\alpha^2}{s^2} \ \Big(s^2(1 + \cos(\theta)^2) + 4s({m_e}^2 + {m_\mu}^2)\sin(\theta)^2
$$

$$+\,16{m_e}^2{m_\mu}^2\cos(\theta)^2\Big) \tag{9.48}$$

The differential cross section

Using what we have already learned about the flux factor and the two-body phase space, we can write the differential cross section as

$$\begin{aligned} d\sigma &= \frac{1}{64\pi^2 s}\left[\frac{s-4{m_\mu}^2}{s-4{m_e}^2}\right]^{1/2}\ \left\langle|\mathcal{M}|^2\right\rangle\ d\Omega \\ &= \frac{\alpha^2}{4s}\left[\frac{s-4{m_\mu}^2}{s-4{m_e}^2}\right]^{1/2}\left((1+\cos(\theta)^2)+4\frac{{m_e}^2+{m_\mu}^2}{s}\sin(\theta)^2\right. \\ &\qquad\qquad \left.+\,16\frac{{m_e}^2{m_\mu}^2}{s^2}\cos(\theta)^2\right)\ d\Omega\ \ . \end{aligned} \tag{9.49}$$

This cross section therefore only depends on s and the polar scattering angle: there is, for unpolarized incoming beams, no azimuthal direction singled out and there is therefore no azimuthal angle dependence[12].

The total cross section

The *total* cross section is obtained by simple angular integration, and reads

$$\sigma = \frac{4\pi\ \alpha^2}{3\ s}\left(1+2\frac{{m_e}^2}{s}\right)\ \left(1+2\frac{{m_\mu}^2}{s}\right)\ \left[\frac{s-4{m_\mu}^2}{s-4{m_e}^2}\right]^{1/2}\quad . \tag{9.50}$$

The cross section is only nonzero above the muon pair-production threshold, $s > 4{m_\mu}^2$. Since the muon mass m_μ is much larger than the electron mass m_e, we may accurately approximate by putting $m_e \approx 0$:

$$\sigma \approx \frac{4\pi\ \alpha^2}{3\ s}\ \left(1+2\frac{{m_\mu}^2}{s}\right)\ \left(1-4\frac{{m_\mu}^2}{s}\right)^{1/2}\quad . \tag{9.51}$$

For large s, furthermore, we have

$$\sigma \approx\ \frac{4\pi\ \alpha^2}{3\ s}\ \left(1-6\frac{{m_\mu}^4}{s^2}+\cdots\right)\ \ . \tag{9.52}$$

By accidental cancellation of the leading ${m_\mu}^2/s$ terms, the large-s limit is reached quite rapidly. Note that the cross section does not depend on $\hbar$.

---

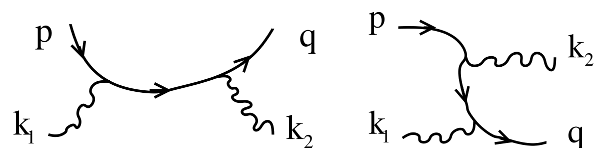
[12] This could be different, *e.g.* in the case of transversely polarized beams.

9.3.2 Compton and Thomson scattering

We next consider the Compton scattering process, an elastic collision between a photon and an elecron :

$$e^-(p) \;\; \gamma(k_1) \quad \to \quad e^-(q)\; \gamma(k_2)$$

Now, there are two Feynman diagrams,



The amplitude is given by

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_1 + \mathcal{M}_2 \quad , \\
\mathcal{M}_1 &= -i\hbar {Q_e}^2 \, \frac{\mathcal{A}_1}{2(p \cdot k_1)} \quad , \\
\mathcal{M}_2 &= -i\hbar {Q_e}^2 \, \frac{\mathcal{A}_2}{-2(q \cdot k_1)} \quad , \\
\mathcal{A}_1 &= \overline{u}(q) \, \not\epsilon_2 \, (\not p + \not k_1 + m) \, \not\epsilon_1 \, u(p) \quad , \\
\mathcal{A}_2 &= \overline{u}(q) \, \not\epsilon_1 \, (\not q - \not k_1 + m) \, \not\epsilon_2 \, u(p) \quad ,
\end{aligned} \tag{9.53}$$

where $\epsilon_{1,2}$ are the polarization vectors of the respective photons. Taking into account the averaging factor 1/4, we find[13] (with m for m_e)

$$\begin{aligned}
\left\langle |\mathcal{A}_1|^2 \right\rangle &= \frac{1}{4}\mathrm{Tr}\Big((\not q + m)\, \gamma^\alpha \, (\not p + \not k_1 + m) \, \gamma^\beta \, (\not p + m) \, \gamma_\beta \, (\not p + \not k_1 + m) \, \gamma_\alpha \Big) \\
&= 16m^4 - 8(pq)m^2 + 8(pk_1)(qk_1) + 16(pk_1)m^2 - 8(qk_1)m^2 \quad , \\
\left\langle |\mathcal{A}_2|^2 \right\rangle &= \frac{1}{4}\mathrm{Tr}\Big((\not q + m)\, \gamma^\beta \, (\not q - \not k_1 + m) \, \gamma^\alpha \, (\not p + m) \, \gamma_\alpha \, (\not q - \not k_1 + m) \, \gamma_\beta \Big) \\
&= 16m^4 - 8(pq)m^2 + 8(pk_1)(qk_1) + 8(pk_1)m^2 - 16(qk_1)m^2 \quad , \\
\langle \mathcal{A}_1 \mathcal{A}_2^* \rangle &= \langle \mathcal{A}_2 \mathcal{A}_1^* \rangle \\
&= \frac{1}{4}\mathrm{Tr}\Big((\not q + m)\, \gamma^\alpha \, (\not p + \not k_1 + m) \, \gamma^\beta \, (\not p + m) \, \gamma_\alpha \, (\not q - \not k_1 + m) \, \gamma_\beta \Big) \\
&= 8(pq)(pk_1) - 8(pq)(qk_1) + 16(pq)m^2 - 8(pq)^2 \\
&\quad -4(pk_1)m^2 + 4(qk_1)m^2 \quad .
\end{aligned} \tag{9.54}$$

[13] Both the incoming electron and the incoming photon have 2 degrees of freedom, hence (1/2)(1/2)=1/4.

We can most easily evaluate this in the photon-electron centre-of-mass frame[14]. In this frame, we have

$$p^0 = q^0 = \frac{s+m^2}{2\sqrt{s}} \quad , \quad |\vec{p}| = |\vec{q}| = |\vec{k}_1| = |\vec{k}_2| = \frac{K}{2\sqrt{s}} \quad , \tag{9.55}$$

where $K = 2(pk_1) = s - m^2$: and the angle between $\vec{q}$ and $\vec{k}_1$ is denoted by θ. Putting everyhting together, we find

$$\begin{aligned} \left\langle |\mathcal{M}|^2 \right\rangle &= 16\pi^2\,\alpha^2 \left(\frac{8m^2}{K} + \frac{8m^4}{K^2} + \frac{2m^4}{(qk_1)^2} - \frac{4m^2}{(qk_1)} \right. \\ &\qquad \left. - \frac{8m^4}{(qk_1)K} + \frac{K}{(qk_1)} + \frac{4(qk_1)}{K} \right) \quad . \end{aligned} \tag{9.56}$$

The phase space integration element is given by

$$dV(p+k_1; q, k_2) = \frac{1}{(2\pi)^2}\,\frac{1}{8}\,\frac{K}{s}\,d\Omega \quad , \tag{9.57}$$

where Ω is the solid angle of the emitted electron. The flux factor is

$$\frac{1}{2\lambda(s,m^2,0)^{1/2}} = \frac{1}{2K} \quad . \tag{9.58}$$

The only nontrivial quantity in the computation is

$$(qk_1) = {k_1}^0\left(q^0 \;-\; |\vec{q}|\cos\theta\right) = \frac{K}{4s}\left((s+m^2) - K\cos\theta\right) \quad , \tag{9.59}$$

and we can find the *angular averages*

$$\begin{aligned} \frac{1}{4\pi}\int d\Omega\;(qk_1) &= \frac{K(s+m^2)}{4s} \quad , \\ \frac{1}{4\pi}\int d\Omega\;\frac{1}{(qk_1)} &= \frac{2s}{K^2}\log\left(1+\frac{K}{m^2}\right) \quad , \\ \frac{1}{4\pi}\int d\Omega\;\frac{1}{(qk_1)^2} &= \frac{4s}{m^2K^2} \quad . \end{aligned} \tag{9.60}$$

[14] In the actual experiment, the photon will of course be impingeing on the *stationary* electron ; but since the cross section is invariant we may choose any frame we want.

We therefore have for the transition rate, now *also* averaged over the scattering angle :

$$\begin{aligned}\left\langle\left\langle|\mathcal{M}|^2\right\rangle\right\rangle &= 16\pi^2\,\alpha^2\left\{1+\frac{m^2}{s}+16\frac{m^2 s}{K^2}\right.\\ &\left.+\left(-8\frac{m^2 s}{K^2}-16\frac{m^4 s}{K^3}+2\frac{s}{K}\right)\log\left(1+\frac{K}{m^2}\right)\right\} \qquad (9.61)\end{aligned}$$

The total cross section

$$\sigma = \frac{1}{16\pi s}\left\langle\left\langle|\mathcal{M}|^2\right\rangle\right\rangle \quad . \qquad (9.62)$$

It is interesting[15] to note that the 'static' limit $K \to 0$ is well-defined :

$$\lim_{K\to 0}\sigma = \frac{8\pi\,\alpha^2}{3\,m^2} \quad . \qquad (9.63)$$

This is called the *Thomson* cross section. It may serve as the 'measurement' prediction by which the electric charge of the electron is defined.

Note that, just as in the case of muon pair production, the cross section does not depend on $\hbar$. This means that this cross section had better coincide with the prediction from *classical* electromagnetism — as, indeed, it does. Returning to the arguments that led us to make the identifications

$$A_{cl}{}^2 = c\,A^2 \quad , \quad Q = \frac{q}{\hbar\sqrt{c}} \quad , \qquad (9.64)$$

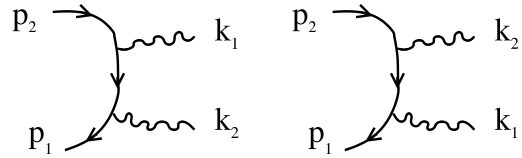
we see that we have made the correct choice.

9.3.3 Electron-positron annihilation

The process

$$e^+(p_1)\;e^-(p_2) \quad \to \quad \gamma(k_1)\;\gamma(k_2)$$

is related by crossing to Compton scattering, and is described at the tree level by the two Feynman diagrams



.

[15] And comforting.

We shall study it in the context of the way it is actually observed at high-energy e^+e^- colliders, that is, in the centre-of-mass frame with the photons emerging an nonnegligible angles with respect to the electron and positron beams. In that case, no invariant vector products are small, and we may neglect the electron mass. We then have an example of a process in which spinor techniques can be usefully employed. The amplitude is given by

$$\begin{aligned}\mathcal{M} &= i\hbar {Q_e}^2\left(\frac{\mathcal{A}_1}{2(p_2k_1)}+\frac{\mathcal{A}_2}{2(p_2k_2)}\right) \;, \\ \mathcal{A}_1(\lambda_e,\lambda_1,\lambda_2) &= \overline{u}_{\lambda_e}(p_1)\;\not{\epsilon}_{\lambda_2}(k_2)\;(\not{p}_2-\not{k}_1)\;\not{\epsilon}_{\lambda_1}(k_1)\;u_{\lambda_e}(p_2)\;\;, \\ \mathcal{A}_2(\lambda_e,\lambda_1,\lambda_2) &= \overline{u}_{\lambda_e}(p_1)\;\not{\epsilon}_{\lambda_1}(k_1)\;(\not{p}_2-\not{k}_2)\;\not{\epsilon}_{\lambda_2}(k_2)\;u_{\lambda_e}(p_2)\;\;. \end{aligned} \tag{9.65}$$

Since $m_e = 0$ we may as well employ the symbol u for both the positron and the electron. Also, the helicity of the electron fixes that of the positron, and both are indicated by λ_e. The helicities of the two photons are denoted by $\lambda_{1,2}$. We shall use the following spinorial representation of the polarization vectors given in Eq.(8.63), without bothering overmuch about the complex phase of the polarization vector[16] :

$$\epsilon(k_j)_\lambda{}^\mu = \frac{1}{2\sqrt{(k_jr_j)}}\;\overline{u}_\lambda(k_j)\;\gamma^\mu\;u_\lambda(r_j)\;\;, \tag{9.66}$$

with $r_j{}^\alpha$ the gauge vector as discussed before. It is important to note that the choice of r_j can be made for different photons, and for different helicity configurations, independently[17]. We shall usefully employ also Eq.(8.66) :

$$\omega_\lambda\;\not{\epsilon}(k)_\lambda = \frac{u_\lambda(r)\overline{u}_\lambda(k)}{\sqrt{(kr)}}\quad,\quad \omega_{-\lambda}\;\not{\epsilon}(k)_\lambda = \frac{u_{-\lambda}(k)\overline{u}_{-\lambda}(r)}{\sqrt{(kr)}}\quad. \tag{9.67}$$

Let us first take the case where the two photon polarizations are equal. With $N = 1/\sqrt{(k_1r_1)(k_2r_2)}$, we have

$$\mathcal{A}_1(+,+,+) \;=\; N\;\overline{u}_+(p_1)u_-(k_2)\overline{u}_-(r_2)(\not{p}_2-\not{k}_1)u_-(k_1)\overline{u}_-(r_1)u_+(p_2)\;\;,$$

[16]Because the process is described by only one single current-conserving object. For more complicated processes we *do* have to ensure the correct complex phase ; this is however greatly helped by the observation of section 8.3.7, that the complex phase of the polarization is independent of the choice of gauge vector.

[17]But of course we have better choose the *same* r for all diagrams in the amplitude, or at least in each of its current-conserving subsets.

$$\begin{aligned}
\mathcal{A}_2(+,+,+) &= N\,\overline{u}_+(p_1)u_-(k_1)\overline{u}_-(r_1)(\not{p}_2-\not{k}_2)u_-(k_2)\overline{u}_-(r_2)u_+(p_2)\ ,\\
\mathcal{A}_1(+,-,-) &= N\,\overline{u}_+(p_1)u_-(r_2)\overline{u}_-(k_2)(\not{p}_2-\not{k}_1)u_-(r_1)\overline{u}_-(k_1)u_+(p_2)\ ,\\
\mathcal{A}_2(+,-,-) &= N\,\overline{u}_+(p_1)u_-(r_1)\overline{u}_-(k_1)(\not{p}_2-\not{k}_2)u_-(r_2)\overline{u}_-(k_2)u_+(p_2)\ .
\end{aligned} \tag{9.68}$$

If, now, we choose $r_1 = r_2 = p_2$ for the $(+,+,+)$ configuration and $r_1 = r_2 = p_1$ for the $(+,-,-)$ configuration, the amplitude is seen to vanish identically in either case[18] ! We also see that the same must happen for electron-positron annihilation into *any* number of photons : if they all have the same helicity, the amplitude vanishes. Next, we have the $(+,+,-)$ configuration :

$$\begin{aligned}
\mathcal{A}_1(+,+,-) &= N\,\overline{u}_+(p_1)u_-(r_2)\overline{u}_-(k_2)(\not{p}_2-\not{k}_1)u_-(k_1)\overline{u}_-(r_1)u_+(p_2)\ ,\\
\mathcal{A}_2(+,+,-) &= N\,\overline{u}_+(p_1)u_-(k_1)\overline{u}_-(r_1)(\not{p}_2-\not{k}_2)u_-(r_2)\overline{u}_-(k_2)u_+(p_2)\ .
\end{aligned} \tag{9.69}$$

We can now choose, say, $r_1 = p_2$ and $r_2 = p_1$. Then $\mathcal{A}_1$ is again zero, and

$$\begin{aligned}
\mathcal{A}_2(+,+,-) &= N\,\overline{u}_+(p_1)u_-(k_1)\overline{u}_-(p_2)(\not{p}_2-\not{k}_2)u_-(p_1)\overline{u}_-(k_2)u_+(p_2)\\
&= -s_+(p_1,k_1)s_-(p_2,k_2)^2 s_+(k_2,p_1)/\sqrt{(k_1p_2)(k_2p_1)}\ ,
\end{aligned} \tag{9.70}$$

so that up to an irrelevant overall phase we have

$$\mathcal{M}(+,+,-) = 8\pi\ \alpha\ \left[\frac{(p_1k_1)}{(p_2k_1)}\right]^{1/2}\quad . \tag{9.71}$$

By symmetry, the configuration $(+,-,+)$ is obtained by replacing k_1 by k_2. The configurations with $\lambda_e = -$ follow from complex conjugation. The final result is, therefore,

$$\left\langle |\mathcal{M}|^2 \right\rangle = 32\pi^2\ \alpha^2\ \left(\frac{(p_1k_1)}{(p_2k_1)} + \frac{(p_1k_2)}{(p_2k_2)}\right)\quad . \tag{9.72}$$

The computation of the cross section is left as an excercise. We have discussed this process, rather, to show how spinor techniques may be usefully employed to compute amplitudes for massless-particle processes in a fast and efficient manner ; moreover, we can gain results (such as the vanishing of the amplitude when the photons helicities are equal) that are not so easily obtained by more traditional approaches[19].

---

[18] This is of course independent of our using the standard-spinor techniques ; these just make it *simpler* to see the vanishing.

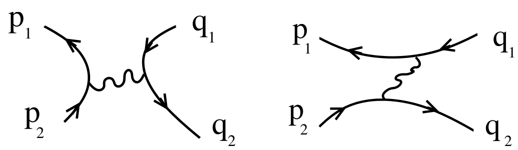
[19] A word of caution is in order here. The Minkowski products (p_ik_j) can become small if the photons are emitted collinearly. In that case these products are of order m^2 rather

9.3.4 Bhabha scattering

Our final $2 \to 2$ QED process is that of *Bhabha scattering*:

$$e^+(p_1)\; e^-(p_2) \quad \to \quad e^+(q_1)\; e^-(q_2) \;\; ,$$

described by the two following Feynman graphs:



We shall use, in addition to s, the following conventional invariants :

$$t = (p_1 - q_1)^2 = (p_2 - q_2)^2 \quad , \quad u = (p_1 - q_2)^2 = (p_2 - q_1)^2 \;\; . \tag{9.73}$$

For $m_e = 0$ we have $s + t + u = 0$ by momentum conservation. As before, we shall work in the high-energy limit so that m_e is neglected. The helicity-dependent amplitude is

$$\begin{aligned}
\mathcal{M}(\lambda_1,\lambda_2,\rho_1,\rho_2) &= i\hbar {Q_e}^2\; A(\lambda_1,\lambda_2,\rho_1,\rho_2) \;\; , \\
\mathcal{A}(\lambda_1,\lambda_2,\rho_1,\rho_2) &= \frac{1}{s}\; \overline{u}_{\lambda_1}(p_1)\; \gamma^\mu\; u_{\lambda_2}(p_2)\;\; \overline{u}_{\rho_2}(q_2)\; \gamma_\mu\; u_{\rho_1}(q_1) \\
&- \frac{1}{t}\; \overline{u}_{\lambda_1}(p_1)\; \gamma^\mu\; u_{\rho_1}(q_1)\;\; \overline{u}_{\rho_2}(q_2)\; \gamma_\mu\; u_{\lambda_2}(p_2) \; .
\end{aligned} \tag{9.74}$$

Note the relative minus sign between the two diagrams ! By the Chisholm identity, we can now evaluate the various helicity configurations :

$$\begin{aligned}
\mathcal{A}(+,+,+,+) &= \frac{2}{s}\; s_+(p_1,q_2)s_-(q_1,p_2) - \frac{2}{t}\; s_+(p_1,q_2)s_-(p_2,q_1) \\
&\sim\; 2\,u\left(\frac{1}{s}+\frac{1}{t}\right) \quad \sim \quad 2\,\frac{u^2}{st} \;\; , \\
\mathcal{A}(+,+,-,-) &= \frac{2}{s}\; s_+(p_1,q_1)s_-(q_2,p_2) \quad \sim \quad 2\,\frac{t}{s} \;\; , \\
\mathcal{A}(+,-,+,-) &= -\frac{2}{t}\; s_+(p_1,p_2)s_-(q_2,q_1) \quad \sim \quad 2\,\frac{s}{t} \;\; ,
\end{aligned} \tag{9.75}$$

---

than of order s. It is therefore not adviseable to blindly put $m = 0$ in any process in which photons are emitted, since then we might miss terms looking like $m^2/(p_ik_j)^2$. As can be seen from the matrix element for Compton scattering, in this case the double-pole term is actually suppressed by m^4 rather than by m^2, and therefore at high energies we do not have to worry about double poles *for this process*. For other Bremsstrahlung processes such as $e^+e^- \;\to\; \mu^+\mu^-\gamma$, the double poles *are* important : see section 9.3.5.

where the symbol $\sim$ denotes our throwing away unimportant complex phases. The other helicity configurations with $\lambda_1 = +$ give zero, and those with $\lambda_1 = -$ follow again trivially by conjugation. We find

$$\left\langle |\mathcal{M}|^2 \right\rangle = 2\hbar^2 {Q_e}^4 \; \frac{s^4+t^4+u^4}{s^2t^2} = 16\pi^2\;\alpha^2\;\left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2 \quad , \tag{9.76}$$

where θ is the angle between $\vec{p}_1$ and $\vec{q}_1$ in the centre-of-mass frame in which most e^+e^- scattering experiments are performed. Note that, in this case, the singularity is not due to our neglecting the electron mass ; indeed, for nonzero mass we have

$$\begin{aligned} t &= (p_1-q_1)^2 = 2m^2 - 2({p_1}^0)^2 + 2|\vec{p}_1|^2\cos\theta \\ &= -2|\vec{p}_1|^2(1-\cos\theta) \;\; . \end{aligned} \tag{9.77}$$

To this order in perturbation theory, the total cross section for Bhabha scattering is therefore indeed divergent[20].

E 50

E 51

9.3.5 Bremsstrahlung in Mœller scattering

The nonradiative process

Mœller scattering is the mutual scattering of two electrons :

$$e^-(p_1)\;e^-(p_2) \quad\to\quad e^-(q_1)\;e^-(q_2)$$

and is just a crossed version of Bhabha scattering. The relevant expression is therefore, for negligible electron mass,

$$\left\langle |\mathcal{M}|^2 \right\rangle = 2\hbar^2 {Q_e}^4 \; \frac{s^4+t^4+u^4}{s^2u^2} \tag{9.78}$$

The radiative process

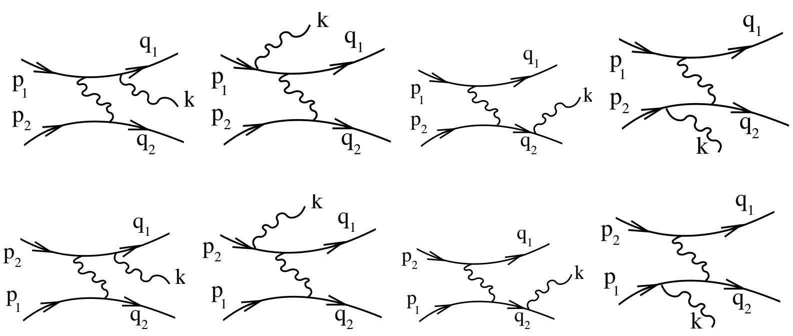
We shall now consider the so-called Bremsstrahlung[21] process :

$$e^-(p_1)\;e^-(p_2) \quad\to\quad e^-(q_1)\;e^-(q_2)\;\gamma(k)$$

[20] The importance of the Fermi minus sign is very visible here. If inadvertently we would forget it, the cross section would be overestimated by as much as 50 % for $\cos\theta = -2+\sqrt{5}$, *i.e.* a scattering angle $\theta = 76.345$ degrees.

[21] The term originated in studies of the motion of charged particles through a medium ; they may lose energy by emitting photons, and slow down, or 'brake', or – in the language of early-twentieth-century physics, which was German rather than American English – 'Bremsen'.

At the tree level, it is described by the eight Feynman diagrams



which we may conveniently put in four groups of two diagrams each :

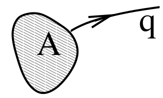
$$
\begin{aligned}
\mathcal{M} &= \sum_{i=1}^{4} \mathcal{M}_i \;\;, \\
\mathcal{M}_1 &= -i(Q_e\sqrt{\hbar})^3\,\overline{u}(q_1)\left[\not{\epsilon}\frac{\not{q}_1+\not{k}+m_e}{2q_1\cdot k}\gamma^\alpha - \gamma^\alpha\frac{\not{p}_1-\not{k}+m_e}{2p_1\cdot k}\not{\epsilon}\right]u(p_1) \\
&\quad\times\frac{1}{(p_2-q_2)^2}\,\overline{u}(q_2)\gamma_\alpha u(p_2)\;\;, \\
\mathcal{M}_2 &= \mathcal{M}_1\rfloor_{p_1\leftrightarrow p_2\;,\;q_1\leftrightarrow q_2}\;\;, \\
\mathcal{M}_3 &= -\,\mathcal{M}_1\rfloor_{p_1\leftrightarrow p_2}\;\;,\quad \mathcal{M}_4 = -\,\mathcal{M}_2\rfloor_{p_1\leftrightarrow p_2}\;\;.
\end{aligned}
\tag{9.79}
$$

Note the Fermi minus sign between $\mathcal{M}_{1,2}$ and $\mathcal{M}_{3,4}$. The four pairs of diagrams are separately current-conserving, *i.e.*

$$
M_i\rfloor_{\epsilon\to k} = 0\;\;,\;\; i=1,2,3,4\;\;. \tag{9.80}
$$

The soft-photon approximation

Since the emitted photon is a massless particle, its energy can be arbitrarily low. A useful result can be obtained if we take this limit, that is, the photon energy is taken to be negligible with respect to the other particle energies. Consider an arbitrary process in which a fermion with momentum q and mass m is produced during a scattering :

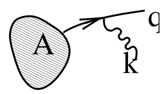


(9.81)

this amplitude can be written as

$$\mathcal{M}_0 \equiv \overline{u}(q)\ A(q)\ \ , \tag{9.82}$$

where A denotes the rest of the diagram(s). The corresponding radiative process will (amongst others) contain diagrams in which the photon is emitted by this particular fermion :



(9.83)

which evaluates to

$$\mathcal{M}_s \equiv -(Q_e\sqrt{\hbar})\ \overline{u}(q)\not\epsilon\frac{\not q + \not k + m}{2q\cdot k}\ A(q+k)\ \ . \tag{9.84}$$

Notice that the denominator $q\cdot k$ goes to zero as the photon energy vanishes, and hence the diagram diverges in the soft-photon limit. In the soft-photon approximation (and assuming that the object A does not depend on q in too drastic a manner[22]) we have

$$\mathcal{M}_s\ \approx\ -(Q_e\sqrt{\hbar})\ \overline{u}(q)\not\epsilon\frac{\not q + m}{2q\cdot k}\ A(q)\ \ . \tag{9.85}$$

Anticommuting $\not\epsilon$ and $\not q$, and using the property of the Dirac spinor, which tells us that $\overline{u}(q)\not q = m\overline{u}(q)$, we then find

$$\mathcal{M}_s\ \approx\ -(Q_e\sqrt{\hbar})\frac{q\cdot\epsilon}{q\cdot k}\overline{u}(q)A(q)\ \ , \tag{9.86}$$

that is, the diagrams factorizes into the nonradiative result and an 'infrared factor'[23]. We can repeat this procedure for those diagrams in which the photon is emitted by the other external particles. There are, of course, also (possibly) diagrams in which the photon is emitted from *internal* lines ; but, as can easily be checked, such diagrams do not diverge as $k^0 \to 0$. In the soft-photon approximation, they do therefore not contribute. For radiative Mœller scattering, we therefore have the nicely factorized form

$$\mathcal{M} = -(Q_e\sqrt{\hbar})\left(\frac{q_1\cdot\epsilon}{q_1\cdot k}+\frac{q_2\cdot\epsilon}{q_2\cdot k}-\frac{p_1\cdot\epsilon}{p_1\cdot k}-\frac{p_1\cdot\epsilon}{p_1\cdot k}\right)\mathcal{M}_0\ \ , \tag{9.87}$$

[22]This assumption fails, for instance, close to a resonance. However, since every resonance has a finite width, the soft-photon approximation is formally correct for *infinitesimal* photon energies.

[23]Since infrared light has low energy compared to visible light.

where $\mathcal{M}_0$ is the amplitude for the nonradiative process ; and, using the polarization sum rule $\Sigma\epsilon^\mu\bar{\epsilon}^\nu = -g^{\mu\nu}$, we find

$$\begin{aligned}\left\langle|\mathcal{M}|^2\right\rangle &= -2{Q_e}^6\hbar^3\,\frac{s^4+t^4+u^4}{t^2u^2}\,\left(V_{\mathrm{IR}}\cdot V_{\mathrm{IR}}\right)\ ,\\ {V_{\mathrm{IR}}}^\mu &= \frac{{p_1}^\mu}{k\cdot p_1}+\frac{{p_2}^\mu}{k\cdot p_2}-\frac{{q_1}^\mu}{k\cdot q_1}-\frac{{q_2}^\mu}{k\cdot q_2}\ .\end{aligned} \tag{9.88}$$

As has already been intimated, the double poles are indeed suppressed by a factor ${m_e}^2$.

Hard Bremsstrahlung: massless case

Next, we consider 'hard Bremsstrahlung' (*i.e.* any photon emission which is not soft) in the limit of vanishing electron mass. It is then most useful to assign definite helicities to the electrons, so that the scattering process is

$$e^-(p_1,\mu_1)\;e^-(p_2,\mu_2)\quad\to\quad e^-(q_1,\nu_1)\;e^-(q_2,\nu_2)\;\gamma(k,\lambda)$$

with $\mu_{1,2},\nu_{1,2},\lambda=\pm$. The amplitude is then a function of the helicities, and we write $\mathcal{M}(\mu_1,\mu_2;\nu_1,\nu_2;\lambda)$. We first consider $\mathcal{M}_1(+,+;+,+;+)$. Using Eq.(8.66) this can be written as

$$\begin{aligned}\mathcal{M}_1(+,+;+,+;+) &= i\frac{(Q_e\sqrt{\hbar})^3\sqrt{2}}{2(p_2\cdot q_2)s_-(k,r)}\\ &\times\overline{u}_+(q_1)\left[u_-(k)\overline{u}_-(r)\frac{\not{q}_1+\not{k}}{2k\cdot q_1}\gamma^\alpha-\gamma^\alpha\frac{\not{p}_1-\not{k}}{2k\cdot p_1}u_-(k)u_-(r)\right]u_+(p_1)\\ &\times\overline{u}_+(q_2)\gamma_\alpha u_+(p_2)\ ,\end{aligned} \tag{9.89}$$

and since $\mathcal{M}_1$ is current-conserving by itself we may choose r at will ; in this case $r=p_1$ appears to be optimal since it kills the second term. Applying standard (hopefully, by now) spinor techniques we arrive at

$$\begin{aligned}&\mathcal{M}_1(+,+;+,+;+) =\\ &\quad i(Q_e\sqrt{\hbar})^3\sqrt{8}\,\frac{s_+(q_1,k)\overline{u}_-(p_1)(\not{q}_1+\not{k})u_-(q_2)s_-(p_2,p_1)}{(2p_2\cdot q_2)(2k\cdot q_1)s_-(k,p_1)}\ .\end{aligned} \tag{9.90}$$

We may employ momentum conservation and masslessness for a further manipulation :

$$\overline{u}_-(p_1)(\not{q}_1+\not{k})u_-(q_2)\quad=\quad\overline{u}_-(p_1)(\not{q}_1+\not{k}+\not{q}_2)u_-(q_2)$$

$$
\begin{aligned}
&= \overline{u}_-(p_1)(\not{p}_1 + \not{p}_2)u_-(q_2) \\
&= \overline{u}_-(p_1)\ \not{p}_2\ u_-(q_2) \\
&= s_-(p_1,p_2)s_+(p_2,q_2)\ \ , \qquad (9.91)
\end{aligned}
$$

so that

$$
\mathcal{M}(+,+;+,+;+) = i(Q_e\sqrt{\hbar})^3\sqrt{8}\ \frac{s_-(p_1,p_2)^2}{s_-(p_2,q_2)s_-(k,p_1)s_-(k,q_1)}\ \ . \qquad (9.92)
$$

Note the fact that in this expression no s_+'s occur, but only s_-'s. This is a quite general feature of such processes. Finally, we can make use of the identity of Eq.(8.73) to arrive at the form

$$
\mathcal{M}_1(+,+;+,+;+) = -2i(Q_e\sqrt{\hbar})^3\ \frac{s_-(p_1,p_2)^2}{s_-(p_1,q_1)s_-(p_2,q_2)}\ \left(\frac{\epsilon_+\cdot p_1}{k\cdot p_1} - \frac{\epsilon_+\cdot q_1}{k\cdot q_1}\right)\ \ . \qquad (9.93)
$$

The infrared factor also appears in this case ! Performing the appropriate subtitutions we can write the complete amplitude as

$$
\begin{aligned}
\mathcal{M}(+,+;+,+;+) &= -2i(Q_e\sqrt{\hbar})^3\ s_-(p_1,p_2)^2\ \ (V_{\mathrm{IR}}\cdot\epsilon_+) \\
&\times\left(\frac{1}{s_-(p_1,q_1)s_-(p_2,q_2)} - \frac{1}{s_-(p_1,q_2)s_-(p_2,q_1)}\right)\ \ . \qquad (9.94)
\end{aligned}
$$

The minus sign in the last term is the Fermi sign ; it helps us to simplify our expression even further using the Schouten identity, and the final form for the amplitude is

$$
\mathcal{M}(+,+;+,+;+) = 2i(Q_e\sqrt{\hbar})^3\ \frac{s_-(p_1,p_2)^3s_-(q_1,q_2)\ \ (V_{\mathrm{IR}}\cdot\epsilon_+)}{s_-(p_1,q_1)s_-(p_2,q_2)s_-(p_1,q_2)s_-(p_2,q_1)}\ \ . \qquad (9.95)
$$

For the other helicity configurations, the above treatment can be repeated straightforwardly. We simply list the final results :

$$
\begin{aligned}
&\mathcal{M}(\mu_1,\mu_2;\nu_1,\nu_2;\lambda) = \\
&\quad 2i(Q_e\sqrt{\hbar})^3\ \frac{(V_{\mathrm{IR}}\cdot\epsilon_\lambda)\quad K(\mu_1,\mu_2;\nu_1,\nu_2;\lambda)}{s_{-\lambda}(p_1,q_1)s_{-\lambda}(p_2,q_2)s_{-\lambda}(p_1,q_2)s_{-\lambda}(p_2,q_1)}\ \ , \\
&\quad K(+,+;+,+;+) = +s_-(p_1,p_2)^3\ s_-(q_1,q_2)\ \ , \\
&\quad K(+,+;+,+;-) = +s_+(q_1,q_2)^3\ s_+(p_1,p_2)\ \ ,
\end{aligned}
$$

$$\begin{aligned}
K(+,-;+,-;+) &= -s_-(p_1,q_2)^3\ s_-(p_2,q_1) \quad , \\
K(+,-;+,-;-) &= -s_+(p_2,q_1)^3\ s_+(p_1,q_2) \quad , \\
K(+,-;-,+;+) &= +s_-(p_1,q_1)^3\ s_-(p_2,q_2) \quad , \\
K(+,-;-,+;-) &= +s_+(p_2,q_2)^3\ s_+(p_1,q_1) \quad , \\
K(-,-;-,-;+) &= +s_-(q_1,q_2)^3\ s_-(p_1,p_2) \quad , \\
K(-,-;-,-;-) &= +s_+(p_1,p_2)^3\ s_+(q_1,q_2) \quad , \\
K(-,+;-,+;+) &= -s_-(p_2,q_1)^3\ s_-(p_1,q_2) \quad , \\
K(-,+;-,+;-) &= -s_+(p_1,q_2)^3\ s_+(p_2,q_1) \quad , \\
K(-,+;+,-;+) &= +s_-(p_2,q_2)^3\ s_-(p_1,q_1) \quad , \\
K(-,+;+,-;-) &= +s_+(p_1,q_1)^3\ s_+(p_2,q_2) \quad .
\end{aligned} \tag{9.96}$$

No other helicity configurations contribute. The spin-averaged matrix element squared therefore has the following form in the strictly massless case :

$$\begin{aligned}
\left\langle |\mathcal{M}|^2 \right\rangle_{m_e=0} &= -2{Q_e}^6\ \hbar^3\ (V_{\mathrm{IR}} \cdot V_{\mathrm{IR}}) \\
&\quad \times \frac{ss'(s^2+s'^2) + uu'(u^2+u'^2) + tt'(t^2+t'^2)}{uu'tt'} \quad ,
\end{aligned} \tag{9.97}$$

with $s = (p_1+p_2)^2$, $s' = (q_1+q_2)^2$, $t = (p_1-q_1)^2$, $t' = (p_2-q_2)^2$, $u = (p_1-q_2)^2$, and $u' = (p_2-q_1)^2$. The final result is surprisingly simple. It consists of the 'soft-photon' factor ${V_{\mathrm{IR}}}^2$ (evaluated for non-soft photon momenta), multiplying a 'symmetrized' form of the nonradiative cross section.

Double-pole terms at high energy

We have already mentioned that putting $m_e = 0$ strictly may be too strict since there are invariant products of momenta that may become equally small. To see how this works, let us again inspect the radiation emitted from a produced fermion, as given in figure 9.83, that can be written as

$$\mathcal{M}_c \equiv -(Q_e\sqrt{\hbar})\ \overline{u}(q)\not{\epsilon}\frac{\not{q}+\not{k}+m}{2q\cdot k}\ A(q+k) \tag{9.98}$$

where, as before, A stands for the rest of the diagram(s). We shall not assume the soft-photon limit, however. Let us assume that the photon is emitted as small angle θ with respect to the fermion momentum. We then find, assuming the fermion energy to be large compared to its mass m :

$$\begin{aligned}(k\cdot q) &= k^0\left(q^0-|\vec{q}|\cos\theta\right)\\ &\approx k^0\left((q^0-|\vec{q}|)+|\vec{q}|\theta^2/2\right)\approx\frac{1}{2}k^0q^0\left(\theta^2+\left(\frac{m_e}{q^0}\right)^2\right)\quad ,\end{aligned}\tag{9.99}$$

where we have used the fact that $q^0-|\vec{q}|=m_e{}^2/(q^0+|\vec{q}|)\approx m_e{}^2/(2q^0)$. we conclude that as soon as θ is of order m_e/q^0 or smaller[24], the product $(k\cdot q)$ becomes of order m_e^2 ; and this means that in that case the 'single pole' $(k\cdot q)^{-1}$ and the 'double pole' $m_e{}^2(k\cdot q)^{-2}$ are of the same order[25]. The squared matrix element (summed over fermion and photon spins) contains of course

$$\begin{aligned}\left\langle|\mathcal{M}_c|^2\right\rangle &= -\frac{Q_e{}^2\hbar}{4(k\cdot q)^2}\\ &\quad\times\overline{A}(q+k)(\not{q}+\not{k}+m)\gamma^\alpha(\not{q}+m)\gamma_\alpha(\not{q}+\not{k}+m)A(q+k)\end{aligned}\tag{9.100}$$

Using standard Dirac algebra we can write

$$\begin{aligned}&(\not{q}+\not{k}+m)\gamma^\alpha(\not{q}+m)\gamma_\alpha(\not{q}+\not{k}+m)\\ &\quad= 4m^2(\not{q}+\not{k}+m)+4(k\cdot q)(m-\not{k})\quad .\end{aligned}\tag{9.101}$$

The second term in this expression enters into the 'massless' result since it will give rise only to single-pole terms, whereas the first term tells us that the double-pole term coming from this $\mathcal{M}_c$ must read

$$\left\langle|\mathcal{M}_c|^2\right\rangle=-Q_e{}^2\hbar\;\frac{m^2}{(k\cdot q)^2}\;\overline{A}(q+k)(\not{q}+\not{k})A(q+k)\quad ,\tag{9.102}$$

where we have again discarded terms of order m. The nonradiative transition rate was given by $\overline{(A)}\not{q}A(q)$, and in this expression we have now substituted $q+k$ for q. We can, by momentum conservation, always express the invariants s, t and u in Eq.(9.78) into a form that does not contain q, and this then gives us the double-pole terms : keeping all four collinear situations in sight, we can write the transition rate including the double-pole terms as

$$\left\langle|\mathcal{M}|^2\right\rangle=Q_e{}^6\;\hbar^2\;\Bigg($$

[24]This is sometimes called the 'ultra-collinear case'.

[25]For this reason we use the subscript c which stands for 'collinear'.

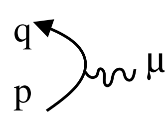
$$
\begin{aligned}
&\frac{ss'(s^2+s'^2)+tt'(t^2+t'^2)+uu'(u^2+u'^2)}{tt'uu'}\\
&\times\left[-\frac{2(p_1\cdot p_2)}{(k\cdot p_1)(k\cdot p_2)}-\frac{2(q_1\cdot q_2)}{(k\cdot q_1)(k\cdot q_2)}+\frac{2(p_1\cdot q_1)}{(k\cdot p_1)(k\cdot q_1)}\right.\\
&\left.+\frac{2(p_2\cdot q_2)}{(k\cdot p_2)(k\cdot q_2)}-\frac{2(p_1\cdot q_2)}{(k\cdot p_1)(k\cdot q_2)}-\frac{2(p_2\cdot q_1)}{(k\cdot p_2)(k\cdot q_1)}\right]\\
&-\frac{{m_e}^2}{(k\cdot p_1)^2}\,\frac{s'^4+t^4+u'^4}{t'^2u'^2}-\frac{{m_e}^2}{(k\cdot p_2)^2}\,\frac{s'^4+t^4+u^4}{t^2u^2}\\
&\left.-\frac{{m_e}^2}{(k\cdot q_1)^2}\,\frac{s^4+t'^4+u^2}{t'^2u^2}-\frac{{m_e}^2}{(k\cdot q_2)^2}\,\frac{s^4+t^4+u'^2}{t^2u'^2}\right)
\end{aligned}
\tag{9.103}
$$

This is the final expression for unpolarized Mœller scattering ; it is accurate in the limit of small m_e even for collinear[26] photon emission.

9.4 Scalar electrodynamics

9.4.1 The vertices

We can also consider the possibility of interactions between photons and charged *scalar* particles[27]. The simplest vertex is then given by

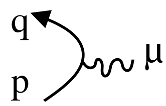


where the charge flow is indicated by the arrow. The photon index is μ. The momenta p and q are counted along the arrow. Note that the *propagator* of scalar particles may be unoriented, but the vertices do not have to, in particular if there is a quantum number, such as charge, that distinguishes between particle and antiparticle. In the absence of Dirac indices, the only quantities in this vertex that carry a Lorentz index are the momenta p and q (and of course the photon's own momentum, but that is fixed by p and q).

[26]Or ultra-collinear.

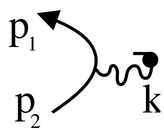
[27]*Elementary* charged scalar particles have to date not been observed, although they are predicted in extensions of the standard model. We include them here since they will provide indications on how to treat charged *vector* particles.

We therefore propose a Feynman rule of the form



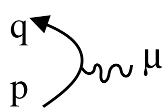
$$\leftrightarrow \quad i\frac{Q}{\hbar}\left(c_1 p^\mu + c_2 q^\mu\right) \;\;,$$

with constants $c_{1,2}$ to be determined. This is simple, since we can study the annihilation of the charged scalar-antiscalar pair into an off-shell photon : under the handlebar operation, the amplitude becomes



$$\begin{aligned} &= iQ\sqrt{\hbar}\;\left(c_2 {p_2}^\mu - c_1 {p_1}^\mu\right)\; k_\mu \\ &= iQ\sqrt{\hbar}\;\left(c_2 {p_2}^\mu - c_1 {p_1}^\mu\right)\left({p_1}_\mu + {p_2}_\mu\right) \\ &= \frac{iQ\sqrt{\hbar}}{2}(c_2 - c_1)s \;\;. \end{aligned} \tag{9.104}$$

We see that $c_1 = c_2$ is required, and therefore the first Feynman rule for scalar electrodynamics (sQED) reads

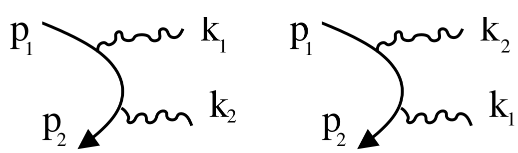


$$\leftrightarrow \; i\frac{Q}{\hbar}(p+q)^\mu$$

sQED vertex

sQED Feynman rules, version 9.1 (9.105)

Let us now consider the more complicated process of annihilation into two on-shell photons. With the above vertex two diagrams are involved :



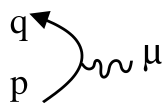
The amplitude is then given, with m indicating the scalar's mass, by

$$\begin{aligned} \mathcal{M} &= -i\hbar Q^2\;\frac{(p_1 + (p_1 - k))\cdot\epsilon_1\;\;((p_1 - k) + (-p_2))\cdot\epsilon_2}{(p_1 - k_1)^2 - m^2} + (k_1 \;\leftrightarrow\; k_2) \\ &= -2i\hbar Q^2\left(\frac{(p_1\cdot\epsilon_1)(p_2\cdot\epsilon_2)}{(p_1\cdot k_1)} + \frac{(p_1\cdot\epsilon_2)(p_2\cdot\epsilon_1)}{(p_2\cdot k_1)}\right) \end{aligned} \tag{9.106}$$

The test of current conservation now fails, since

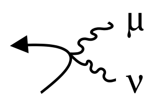
$$\mathcal{M}\rfloor_{\epsilon_1 \to k_1} = -2i\hbar Q^2((p_2 \cdot \epsilon_2) + (p_1 \cdot \epsilon_2)) = -2i\hbar Q^2\ (k_1 \cdot \epsilon_2)\ \ . \tag{9.107}$$

The solution is to introduce a four-point vertex into the Feynman rules[28] :



$$\leftrightarrow\ i\frac{Q}{\hbar}(p+q)^\mu$$

sQED 3-vertex

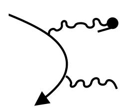
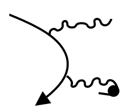
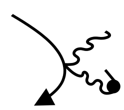


$$\leftrightarrow\ 2i\frac{Q^2}{\hbar}\ g^{\mu\nu}$$

sQED 4-vertex

$\boxed{\text{sQED Feynman rules, version 9.2}}$ (9.108)

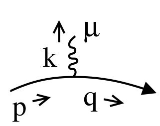
Now we find immediately the desired current conservation :

 $+$  $+$  $=\ 0\ \ .$ (9.109)

It might be supposed that annihiliation into three photons would necessitate a five-point vertex, and so on. Fortunately, the above two vertices are sufficient to guarantee current conservation in all sQED processes, as we shall now show using some more handlebar diagrammatics.

9.4.2 Proof of current conservation in sQED

Consider a charged scalar propagator somewhere in a Feynman diagram, and assume a photon attached to it :



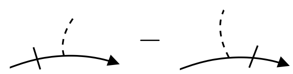
$$= \ \frac{i\hbar}{p^2-m^2}\ \left(i\frac{Q}{\hbar}(p+q)^\mu\right)\ \frac{i\hbar}{q^2-m^2}\ \ .$$

As in our proof of regular QED, none of these lines is necessarily on-shell. . Momentum conservation again fixes the photon momentum to be $k = p - q$.

[28]For reasons lost in the mists of time, such a vertex is called a *sea-gull* vertex, although to me it does not look very gully nor even particularly birdy.

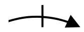
In analogy to regular QED we can now invent some handlebar diagrammatics as follows :

$$
\begin{aligned}
&= -iQ\hbar\,\frac{(p-q)\cdot(p+q)}{(p^2-m^2)(q^2-m^2)} \\
&= \frac{i\hbar}{q^2-m^2}\left(i\frac{Q}{\hbar}\right)(i\hbar) + (i\hbar)\left(i\frac{Q}{\hbar}\right)\frac{i\hbar}{p^2-m^2} \\
&=
\end{aligned}
$$

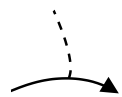


$$, \tag{9.110}$$

with the trivial auxiliary rules



$$= i\hbar \quad ,$$

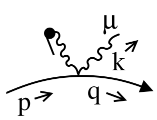


$$= i\frac{Q}{\hbar} \quad . \tag{9.111}$$

These rules are very similar to those we adopted in regular QED : however, in general we have

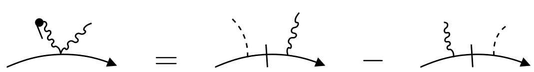
$$\tag{9.112}$$

since the scalar-scalar-photon vertex still depends on the various momenta. We now turn to the second vertex, with two photon lines. Not denoting the two scalar propagators, we have



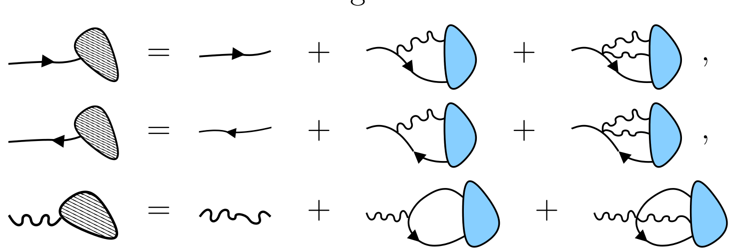
$$
\begin{aligned}
&= 2i\frac{Q^2}{\hbar}\,(p-q-k)s^\mu \\
&= \left(i\frac{Q}{\hbar}\right)(i\hbar)\left(i\frac{Q}{\hbar}(2q+k)^\mu\right) - \left(i\frac{Q}{\hbar}(2p-k)^\mu\right)(i\hbar)\left(i\frac{Q}{\hbar}\right) \quad ,
\end{aligned}
\tag{9.113}
$$

in other words,



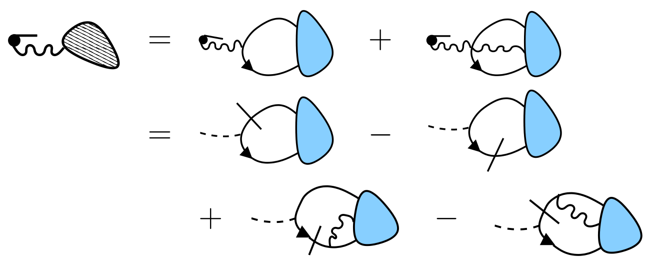
$$. \tag{9.114}$$

The proof of current conservation again relies on the SDe's for this model :



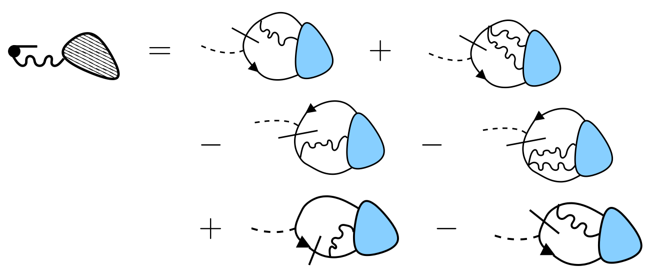
$$\tag{9.115}$$

where again we have used semi-connected graphs. The handlebar operation is now seen to lead to



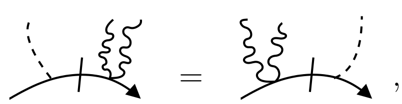
(9.116)

If we now iterate the SDe cleverly for the first two of these four diagrams, we obtain



$$= \;\; 0 \;\; , \tag{9.117}$$

since we *do* have



(9.118)

owing to the simple, momentum-independent structure of the seagull vertex. Comparing the lines of the proof for sQED with that of regular QED, the general proof strategy becomes clear : if in a diagram a slashed propagator occurs as one of the indicated lines of a (semi-)connected graph, we must iterate de SDe for that line, and then we can collect the various canceling contributions.

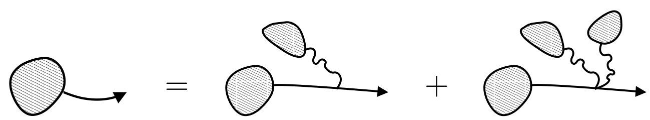
9.5 Electrons in external fields : $g = 2$

9.5.1 The charged Klein-Gordon equation

A point of particular interest is the way in which electrons react to external fields. Here 'external' is used for fields that are not of the fluctuating quantum type, but rather applied 'from outside', under our experimental control ; this is the sense in which it was used in section 9.2.5. In the absence of an explicit source, the Dirac equation with external field A reads (cf. Eq.(9.28)) :

$$\Big(\gamma_\mu(i\,\partial^\mu - eA^\mu(x)) - m\Big)\,\psi(x) = 0 \tag{9.119}$$

in the position representation, where e denotes the coupling. Let us first see how a *scalar* electron would behave. For a charged scalar in an external field we can write down a classical (tree-level) SDe



(9.120)

or, by explicit use of the Fourier transforms of the fields :

$$\begin{aligned}
\phi(x) \;\; &= \;\; \int d^4y\,\frac{1}{(2\pi)^4}\int\; d^4k\; e^{-ik\cdot(x-y)}\frac{i\hbar}{k^2-m^2+i\epsilon} \\
&\quad \left(\left(-i\frac{e}{\hbar}\right)\frac{1}{(2\pi)^8}\int d^4p d^4q e^{-ip\cdot y - iq\cdot y}(2p+q)_\mu A^\mu(q)\;\phi(p)\right. \\
&\quad \left. +\frac{1}{2}\left(2i\frac{e^2}{\hbar}\right)A_\mu(y)A^\mu(y)\phi(y)\right)\quad .
\end{aligned} \tag{9.121}$$

Note the occurence of the symmetry factor $1/2$ in the last line. Applying the differential operator $-\partial^2 - m^2$, and using the expressions for derivatives in Fourier representation we arrive at

$$\begin{aligned}
&\Big(-\partial^2 - m^2\Big)\,\phi(x) = \\
&\qquad 2ieA^\mu(x)\partial_\mu\phi(x) + ie\,\Big(\partial_\mu A^\mu(x)\Big)\,\phi(x) - e^2A^\mu(x)A_\mu(x)\phi(x)
\end{aligned} \tag{9.122}$$

or

$$\left(\Big(i\partial - eA(x)\Big)^2 - m^2\right)\phi(x) = 0\;\; . \tag{9.123}$$

This is the Klein-Gordon equation for charged scalar fields. We see that the same 'minimal substitution rule' $p^\mu \;\;\to\;\; p^\mu - eA^\mu$ as in the Dirac case is employed to account for the presence of the e.m. field ; and we see that the charge coupling constant e is defined in the same way for both scalar and Dirac particles.

9.5.2 The relativistic Pauli equation

We can cast the Dirac equation for spin-1/2 electrons in a form closely resembling that of the Klein-Gordon equation, by multiplying Eq.(9.119) :

$$\begin{aligned}
0 &= \Big(\gamma_\mu \big(i\partial^\mu - eA^\mu(x)\big) + m\Big)\Big(\gamma_\nu \big(i\partial^\nu - eA^\nu(x)\big) - m\Big)\psi(x) \\
&= \Big(- \partial^2 + e^2 A(x)^2 - m^2\Big)\psi(x) \\
&\quad -ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu(x)\psi(x) - ie\gamma^\nu\gamma^\mu A_\nu(x)\partial_\mu\psi(x) \;\; . \qquad (9.124)
\end{aligned}$$

The second line can be rewritten as

$$-ie\Big(\big(g^{\mu\nu} - i\sigma^{\mu\nu}\big)\,\partial_\mu A_\nu(x)\Big)\,\psi(x) - 2ieA^\mu(x)\,\partial_\mu\psi(x) \qquad (9.125)$$

We find, for spin-1/2 electrons, the *relativistic Pauli equation*, the Klein-Gordon equation with an extra spin term, the so-called *Stern-Gerlach term* added on :

$$\Big(\big(i\partial - eA(x)\big)^2 - m^2 - \big(e\sigma^{\mu\nu}\partial_\mu A_\nu(x)\big)\Big)\,\phi(x) = 0 \;\; . \qquad (9.126)$$

9.5.3 A constant magnetic field

The spin of particles manifests itself most clearly in a magnetic field. Since the notions of electric and magnetic fields are of dubious Lorentz covarience, we adopt the following. Let t^μ be a constant vector with $t^2 = 1$, and let B^μ be a constant vector with $(B\cdot t) = 0$. We now choose the external A field as follows :

$$A^\mu(x) = \frac{1}{2}\,\varepsilon^\mu(t,B,x) \;\; . \qquad (9.127)$$

In the 'rest frame' where $\vec{t} = 0$ this reduces to $A^0 = 0$, $\vec{A} = \vec{B}\times\vec{x}/2$, which corresponds to a constant magnetic field of strength $\vec{B}$.

In the Klein-Gordon equation, the term linear in A is given by[29]

$$\begin{aligned}
-2ieA(x)\cdot\partial\psi(x) &= -ie\,\varepsilon(\partial,t,B,x)\,\psi(x) = e\,\varepsilon(t,B,x,i\partial)\,\psi(x) \\
&= -e\,B_\mu\,\varepsilon^\mu(t,x,i\partial)\,\psi(x) \;\; . \qquad (9.128)
\end{aligned}$$

[29] The term proportional to $(\partial\cdot A)$ vanishes because of the Lorenz condition.

We can introduce the angular momentum operator

$$L^\mu = \varepsilon^\mu(t, x, i\partial) \tag{9.129}$$

since, for $\vec{t} = 0$, this reduces to $\vec{L} = i\vec{x} \times \vec{\partial} = \vec{x} \times \vec{p}$. In the rest frame we therefore have the interaction term

$$-2ieA(x) \cdot \partial\psi(x) = e\,\vec{B} \cdot \vec{L}\,\psi(x) \tag{9.130}$$

which represents the coupling between the magnetic field and the angular momentum of the moving charge. For the Stern-Gerlach term, we have

$$\begin{aligned} -e\,\sigma^{\mu\nu}\,\partial_\mu A_\nu(x) &= \frac{e}{2}\,\sigma^{\mu\nu}\,\varepsilon_{\mu\nu\alpha\beta}t^\alpha B^\beta \\ &= ie\gamma^5\sigma_{\alpha\beta}\,t^\alpha B^\beta = e\,\gamma^5 \not{B}\not{t}\ . \end{aligned} \tag{9.131}$$

We now make the approximation that the particle is moving *nonrelativistically* slow in the rest frame $\vec{t} = 0$. In that case we may write, in momentum language,

$$\not{t}\,u(p,s) \to \frac{1}{m}\not{p}\,u(p,s) = u(p,s)\ , \tag{9.132}$$

so that the Stern-Gerlach interaction term in this limit reads

$$-e\,\sigma^{\mu\nu}\,\partial_\mu A_\nu(x)\,\psi(x) = eB_\mu\,\gamma^5\gamma^\mu\psi(x)\ . \tag{9.133}$$

Now, as we have seen in Eq.(7.69), the operator $-\gamma^5\gamma^\mu$ describes the *spin vector* s^μ, which is normalized to $s^2 = -1$. However, as we have seen the actual *spin* S is one-half this amount since the electron has total spin one half. We therefore write

$$-e\,\sigma^{\mu\nu}\,\partial_\mu A_\nu(x)\,\psi(x) = 2e\,\vec{B} \cdot \vec{S}\,\psi(x)\ , \tag{9.134}$$

where $\vec{S} = -\gamma^5\vec{\gamma}/2$ is the operator for the spin of a nonrelativistic electron.

We see that the coupling of a (nonrelativistic) electron to an external magnetic field is proportional to

$$(\vec{L} + g_e\vec{S}) \cdot \vec{B} \quad , \quad g_e = 2\ ,$$

where the factor g_e comes somewhat as a surprise : it is called the *gyromagnetic ratio* of the particle[30]. The prediction $g_e = 2$ was one of the first and most welcome results from Dirac's description of the electron.

[30] *Composite* spin-1/2 particles can have different values for g : the proton and neutron have about 5.586 and -3.826, repsectively.

9.5.4 The Gordon decomposition

There is a way to pinpoint the gyromagnetic behaviour of an electron in a more precise and useful manner. Consider a charged Dirac particle that scatters by emitting (or absorbing) a single photon. The corresponding current reads

$$J^\mu = \frac{ie}{\hbar}\,\overline{u}(q)\;\gamma^\mu\;u(p)\quad , \tag{9.135}$$

where p is the incoming, and q the outgoing momentum. By the properties of the Dirac spinors we can write this as

$$J^\mu = \frac{ie}{2m\hbar}\,\overline{u}(q)\;\left(\not{q}\gamma^\mu + \gamma^\mu\not{p}\right)\;u(p)\quad . \tag{9.136}$$

Since

$$\not{q}\gamma^\mu = q^\mu - iq_\alpha\,\sigma^{\alpha\mu}\quad,\quad \gamma^\mu\not{p} = p^\mu + ip_\alpha\,\sigma^{\alpha\mu}\quad , \tag{9.137}$$

the current takes the form

$$J^\mu = \frac{ie}{2m\hbar}\,\overline{u}(q)\;\left((p+q)^\mu + i(p-q)_\alpha\,\sigma^{\alpha\mu}\right)\;u(p)\quad . \tag{9.138}$$

This is called the *Gordon decomposition* : the vertex is split up into a piece that we recognize as the sQED vertex, which is called the *convection term*, and a tensorial part, called the *spin term*. Both terms vanish individually under the handlebar operation. It is the spin term which must be responsible for the result $g_e = 2$ for the electron. The above implies that, by calculating loop corrections to the electron-photon vertex, we can isolate the σ part of the loop-corrected vertex and infer the loop corrections to g_e.

9.6 Selected topics in QED

9.6.1 Three-photon production

A fine example of a quite nontrivial computation using standard-spinor techniques is provided by the process of three-photon annihiliation in e^+e^- collisions :

$$e^+(p_1)\;e^-(p_2)\quad\to\quad\gamma(k_1)\;\gamma(k_2)\;\gamma(k_3)\quad . \tag{9.139}$$

We shall calculate this in the massless case, $m_e = 0$. At the tree level there are 6 Feynman diagrams, so that the amplitude reads

$$\begin{aligned}\mathcal{M}(\lambda;\lambda_1,\lambda_2,\lambda_3) &= i\hbar^{3/2}e^3 \\ &\times\ \overline{u}_\lambda(p_1)\not\epsilon_{\lambda_3}(k_3)\frac{(\not k_3 - \not p_1)}{2(k_3p_1)}\not\epsilon_{\lambda_2}(k_2)\frac{(\not p_2 - \not k_1)}{2(k_1p_2)}\not\epsilon_{\lambda_1}(k_1)u_\lambda(p_2) \\ &+\ (\text{perm})\ \ . \end{aligned} \tag{9.140}$$

Here we have explicitly indicated the various helicities, and 'perm' stands for the other 5 permutations of the photons. For the photon polarizations we take the standard form

$$\epsilon_{\lambda_j}(k_j)^\mu = \frac{\lambda_j}{\sqrt{2}}\,\frac{\overline{u}_{\lambda_j}(k_j)\,\gamma^\mu\,u_{\lambda_j}(r_j)}{s_{-\lambda_j}(k_j,r_j)}\ \ , \tag{9.141}$$

where the massless gauge vector r_j can be chosen independently for each photon and each helicity. The frist helicity amplitude that we consider is $\mathcal{M}(+;+++)$. As we shall see, it pays to choose $r_{1,2,3} = p_2$ since then we have, at the right-hand end of Eq.(9.140),

$$\not\epsilon_+(k_1)u_+(p_2) \propto u_-(k_1)\overline{u}_-(p_2)u_+(p_2) \tag{9.142}$$

which vanishes, not because of helicity but because ${p_2}^2 = 0$. All the other permuations vanish likewise so that we immediately see that

$$\mathcal{M}(+;+++) = \mathcal{M}(-;---) = 0\ \ . \tag{9.143}$$

Similarly, by taking $r_{1,2,3} = p_1$ we can see that

$$\mathcal{M}(+;---) = \mathcal{M}(-;+++) = 0 \tag{9.144}$$

as well. We see that the only helicity amplitude that we have to work hard on is, say, $\mathcal{M}(+;-++)$. Negelecting, for now, the overall factor $i(2\hbar e^2)^{3/2}$ we can write

$$\begin{aligned}&\mathcal{M}(+;-++) = \\ &\quad \overline{u}_+(p_1)\frac{u_-(k_3)\overline{u}_-(p_2)}{s_-(k_3,p_2)}\frac{(\not k_3 - \not p_1)}{2(k_3p_1)}\frac{u_-(k_2)\overline{u}_-(p_2)}{s_-(k_2,p_2)}\frac{(\not p_2 - \not k_1)}{2(k_1p_2)}\frac{u_-(p_2)\overline{u}_-(k_1)}{s_+(k_1,p_2)}u_+(p_2) \\ &\quad +\ (\text{perm})\ \ . \end{aligned} \tag{9.145}$$

Actually, only two diagrams contribute here, namely the one written down
52 and the one where k_2 and k_3 are interchanged. With

$$\frac{\overline{u}_+(p_1)u_-(k_3)}{2(k_3p_1)} = \frac{1}{s_-(k_3,p_1)} \tag{9.146}$$

and

$$\frac{\overline{u}_-(p_2)(\not{p}_2 - \not{k}_1)u_-(p_2)}{2(k_1p_2)} = -1 \tag{9.147}$$

we can rewrite

$$\mathcal{M}(+;-++) = -\frac{\overline{u}_-(p_2)(\not{k}_3 - \not{p}_1)u_-(k_2)\ s_-(k_1,p_2)}{s_-(k_3,p_1)\,s_-(k_3,p_2)\,s_-(k_2,p_2)\,s_+(k_1,p_2)} \ + \ (k_2 \leftrightarrow k_3) \ . \tag{9.148}$$

Using masslessness and momentum conservation, we moreover have

$$\begin{aligned} \overline{u}_-(p_2)(\not{k}_3 - \not{p}_1)u_-(k_2) &= \overline{u}_-(p_2)(\not{k}_2 + \not{k}_3 - \not{p}_1 - \not{p}_2)u_-(k_2) \\ &= -\overline{u}_-(p_2)\not{k}_1 u_-(k_2) \ , \end{aligned} \tag{9.149}$$

Making the denominator symmetric in k_2 and k_3 gives us

$$\begin{aligned} &\mathcal{M}(+;-++) \\ &\quad = \frac{\overline{u}_-(p_2)\not{k}_1 u_-(k_2)\ s_-(k_2,p_1)\ s_-(k_1,p_2)}{s_-(k_3,p_1)\,s_-(k_3,p_2)\,s_-(k_2,p_1)\,s_-(k_2,p_2)\,s_+(k_1,p_2)} \\ &\quad\quad + (k_2 \leftrightarrow k_3) \ . \end{aligned} \tag{9.150}$$

After yet another manipulation :

$$\begin{aligned} &\overline{u}_-(p_2)\not{k}_1 u_-(k_2)\ s_-(k_2,p_1) + (k_2 \leftrightarrow k_3) \\ &\quad = \overline{u}_-(p_2)\not{k}_1\not{k}_2 u_+(p_1) + (k_2 \leftrightarrow k_3) \\ &\quad = \overline{u}_-(p_2)\not{k}_1(\not{k}_2 + \not{k}_3)u_+(p_1) \\ &\quad = \overline{u}_-(p_2)\not{k}_1\not{p}_2 u_+(p_1) = s_-(p_2,k_1)s_+(k_1,p_2)s_-(p_2,p_1) \ , \end{aligned} \tag{9.151}$$

we arrive at

$$\mathcal{M}(+;-++) = \frac{s_-(k_1,p_2)^2\ s_-(p_1,p_2)}{s_-(k_3,p_1)s_-(k_3,p_2)s_-(k_2,p_1)s_-(k_2,p_2)} \ ; \tag{9.152}$$

putting it more symmetrically, and reinserting the overall factor,

$$\mathcal{M}(+;-++) = i(2\hbar e^2)^{3/2}\ s_-(p_1,p_2)\ \frac{s_-(k_1,p_2)^3 s_-(k_1,p_1)}{\prod\limits_{j=1}^{3} s_-(k_j,p_1)s_-(k_j,p_2)} \ . \tag{9.153}$$

The final answer is, therefore,

$$\left\langle |\mathcal{M}|^2 \right\rangle = 2\hbar^2 e^4\ (p_1p_2)\ \frac{\sum\limits_{j=1}^{3}(k_jp_1)(k_jp_2)\left[(k_jp_1)^2+(k_jp_2)^2\right]}{\prod\limits_{j=1}^{3}(k_jp_1)(k_jp_2)}\ \ . \tag{9.154}$$

Looking at this result, we notice two interesting things. In the first place, it is *very simple*, something you would not have guessed right off, and certainly would have had to work on very hard using the classical Casimir-trick approach. In the second place, Eq.(9.153) contains *only* s_- and not s_+. This is a general feature : conceding that $\mathcal{M}(+;+++)=0$ is the simplest possible amplitude, the 'next-simplest', in our case $\mathcal{M}(+;-++)$, is both simple and holomorphic in the spinor products[31]. Such amplitudes are called *maximal helicity violating* (MHV) and are an object of research in their own right, occurring in many theories with massless particles such as QCD with massless quarks and gluons, and even in gravity.

9.6.2 The Thomson limit : *scalar vs spinor*

It is interesting to note that the Thomson cross section of Eq.(9.63) is actually a *classical* object, derivable within classical, nonrelativistic electrodynamics[32]. It should therefore not depend on the fact that the electron is a Dirac particle : the amplitudes must be the same for spin-0 and spin-1/2 electrons in the low-energy limit. We shall now investigate this in some detail, at the tree level.

We will start by defining the so-called *static limit*, the situation where the kinetic energies are negligible with respect to the rest energies of massive particles. We shall, for our process

$$e(p)\,\gamma(k_1)\quad\to\quad e(q)\,\gamma(k_2)\ \ , \tag{9.155}$$

define the momenta as follows, in the centre-of-mass frame:

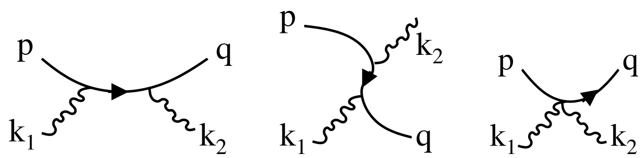
$$\begin{aligned} q^\mu &= (E,0,k\sin\theta,k\cos\theta)\ \ ,\ \ {k_2}^\mu = (k,0,-k\sin\theta,-k\cos\theta)\ \ ,\\ p^\mu &= (E,0,0,k)\ \ ,\ \ {k_1}^\mu = (k,0,0,-k)\ \ , \end{aligned} \tag{9.156}$$

[31] In the sense that s_+ is the *complex conjugate* of s_- up to a sign.

[32] After all, Thomson didn't know either about quantum physics at the time, nor about relativity.

where $E^2 = k^2 + m^2$ for electron mass m, the polar scattering angle of the photons is θ, and we have arbitrarily fixed the irrelevant azimuthal scattering angle. The static limit (SL) is defined by $k \to 0$ and $E \to m$.

We start with the scalar case. The scattering is described by the three diagrams



(9.157)

in which the charge flow is indicated. The amplitude reads

$$\begin{aligned}\mathcal{M}^{\lambda_1\lambda_2} &= -ie^2\hbar\left(\frac{2\left(p\cdot\epsilon_1(\lambda_1)\right)\left(q\cdot\epsilon_2(\lambda_2)\right)}{(p\cdot k_1)} - \right.\\ &\left.\frac{2\left(q\cdot\epsilon_1(\lambda_1)\right)\left(p\cdot\epsilon_2(\lambda_2)\right)}{(q\cdot k_1)} - 2\left(\epsilon_1(\lambda_1)\cdot\epsilon_2(\lambda_2)\right)\right)\ . \end{aligned} \tag{9.158}$$

Here e is the electron charge. The photon helicities are denoted by $\lambda_{1,2}$, and we shall use the representation

$$\epsilon_1(\lambda)^\alpha = \frac{\overline{u}_\lambda(k_1)\gamma^\alpha u_\lambda(k_2)}{\sqrt{4(k_1\cdot k_2)}}\ \ ,\ \ \epsilon_2(\lambda)^\alpha = \frac{\overline{u}_\lambda(k_2)\gamma^\alpha u_\lambda(k_1)}{\sqrt{4(k_1\cdot k_2)}}\ \ , \tag{9.159}$$

so that $(\epsilon_j\cdot p) = (\epsilon_j\cdot q)$ $(j = 1,2)$ and $\epsilon_2(\lambda) = \epsilon_1(\lambda)^* = \epsilon_1(\lambda)$. The first amplitude we consider is

$$\mathcal{M}^{++} = -ie^2\hbar\left(2|p\cdot\epsilon_1(+)|^2\left(\frac{1}{(pk_1)} - \frac{1}{(qk_1)}\right) + 2\right)\ \ . \tag{9.160}$$

Using

$$|p\cdot\epsilon_1(+)|^2 = \frac{\mathrm{Tr}\left(\omega_+\not{k}_1\not{p}\not{k}_2\not{p}\right)}{4(k_1k_2)} = \frac{(pk_1)(qk_1)}{(k_1k_2)} - \frac{m^2}{2} \tag{9.161}$$

and

$$\frac{1}{(pk_1)} - \frac{1}{(qk_1)} = -\frac{(k_1k_2)}{(pk_1)(qk_1)} \tag{9.162}$$

and we find the exact result

$$\mathcal{M}^{++} = -ie^2\hbar\,\frac{m^2(k_1k_2)}{(pk_1)(qk_1)} = -ie^2\hbar\frac{(E-k)(1-\cos\theta)}{E+k\cos\theta}\ \ . \tag{9.163}$$

In the SL this becomes

$$\mathcal{M}^{++}_{\text{SL}} = -ie^2\hbar(1-\cos\theta) \ . \tag{9.164}$$

Since the amplitude must be dimensionless in energy units and m is the only scale in the SL, it is not surprising that m drops out altogether. The second amplitude is given by

$$\begin{aligned}
\mathcal{M}^{+-} &= -2ie^2\hbar\left(\frac{1}{(pk_1)} - \frac{1}{(qk_1)}\right)\Big(p\cdot\epsilon_1(+)\Big)^2 \\
&= 2ie^2\hbar\,\frac{(k_1k_2)}{(pk_1)(qk_1)}\,e^{i\phi}\,|p\cdot\epsilon_1(+)|^2 \\
&= ie^2\hbar\,e^{i\phi}\,\frac{2(pk_1)(qk_1) - m^2(k_1k_2)}{(pk_1)(qk_1)} \\
&= ie^2\hbar\,e^{i\phi}\,\frac{(E+k)(1+\cos\theta)}{E+k\cos\theta} \ .
\end{aligned} \tag{9.165}$$

Here we have introduced the phase factor

$$e^{i\phi} = \frac{p\cdot\epsilon_1(+)}{p\cdot\epsilon_1(+)^*} = \frac{\overline{u}_+(k_1)\,\not{p}\,u_+(k_2)}{\overline{u}_-(k_1)\,\not{p}\,u_-(k_2)} \ . \tag{9.166}$$

In the SL we arrive at

$$\mathcal{M}^{+-}_{\text{SL}} = -ie^2\hbar\,e^{i\phi}\,(1+\cos\theta) \ . \tag{9.167}$$

We now turn to the spin-1/2 case. In section 9.3.2 we already computed the cross section using the Casimir trick, but now we want the individual spin amplitudes. The amplitude is now given by two, rather than three, diagrams and reads

$$\begin{aligned}
\mathcal{M} &= -ie^2\hbar\,\overline{u}(q)\left(\frac{\not{\epsilon}_2(\lambda_2)\,(\not{p}+\not{k}_1+m)\,\not{\epsilon}_1(\lambda_1)}{2(pk_1)}\right. \\
&\qquad\qquad \left. - \frac{\not{\epsilon}_1(\lambda_2)\,(\not{q}-\not{k}_1+m)\,\not{\epsilon}_1(\lambda_2)}{2(qk_1)}\right)u(p) \ .
\end{aligned} \tag{9.168}$$

We do not bother to indicate the spins of the electrons explicitly, although we shall have to discuss them later. As before, we start with

$$\mathcal{M}^{++} = \frac{-ie^2\hbar}{(k_1k_2)}\left(\frac{A_1^{++}}{2(pk_1)} - \frac{A_2^{++}}{2(qk_1)}\right) \ ,$$

$$\begin{aligned}
A_1^{++} &= \overline{u}(q)\left(u_+(k_1)\overline{u}_+(k_2) + u_-(k_2)\overline{u}_-(k_1)\right)(\not{p} + \not{k}_1 + m) \times \\
&\quad \left(u_+(k_2)\overline{u}_+(k_1) + u_-(k_1)\overline{u}_-(k_2)\right)u(p) \;\;, \\
A_2^{++} &= \overline{u}(q)\left(u_+(k_2)\overline{u}_+(k_1) + u_-(k_1)\overline{u}_-(k_2)\right)(\not{q} - \not{k}_1 + m) \times \\
&\quad \left(u_+(k_1)\overline{u}_+(k_2) + u_-(k_2)\overline{u}_-(k_1)\right)u(p) \;\;. \qquad (9.169)
\end{aligned}$$

Using the Dirac equation and some algebra, we can rewrite

$$\begin{aligned}
A_1^{++} &= \overline{u}(q)\left(2(qk_2)\omega_+\not{k}_1 + 2(pk_1)\omega_-\not{k}_2 - m\omega_+\not{k}_1\not{k}_2 - m\omega_-\not{k}_2\not{k}_1\right)u(p) \\
&= \overline{u}(q)\left(2(pk_1)\not{k}_2 - 2m(k_1k_2)\omega_+\right)u(p) \;\;, \\
A_2^{++} &= \overline{u}(q)\left(2(qk_1)\omega_+\not{k}_2 + 2(pk_2)\omega_-\not{k}_1 - m\omega_-\not{k}_1\not{k}_2 - m\omega_+\not{k}_2\not{k}_1\right)u(p) \\
&= \overline{u}(q)\left(2(qk_1)\not{k}_1 - 2m(k_1k_2)\omega_+\right)u(p) \;\;. \qquad (9.170)
\end{aligned}$$

Putting everything back together :

$$\mathcal{M}^{++} = -ie^2\,\overline{u}(q)\left(\frac{\not{k}_2 - \not{k}_1}{(k_1k_2)} - \frac{m}{(pk_1)}\omega_+ + \frac{m}{(qk_1)}\omega_+\right)u(p) \;\;. \qquad (9.171)$$

Since

$$u(q)\left(\not{k}_2 - \not{k}_1\right)u(p) = \overline{u}(q)\left(\not{p} - \not{q}\right)u(p) = 0 \;\;, \qquad (9.172)$$

we arrive at the exact result

$$\mathcal{M}^{++} = -ie^2\hbar\frac{m(k_1k_2)}{2(pk_1)(qk_1)}\,\overline{u}(q)\left(1 + \gamma^5\right)u(p) \;\;. \qquad (9.173)$$

In the SL, we may in this expression approximate q by p. It is now time to consider the spin s of the electron, for which we have the following relations :

$$\begin{aligned}
&\overline{u}(p,s)\,u(p,s) = 2m \;\;,\;\; \overline{u}(p,-s)\,u(p,s) = 0 \;\;,\;\; \overline{u}(p,\pm s)\,\gamma^5\,u(p,s) = 0 \;\;, \\
&\overline{u}(p,s)\,\gamma^\beta\,u(p,s) = 2p^\beta \;\;,\;\; \overline{u}(p,-s)\,\gamma^\beta\,u(p,s) = 0 \;\;. \qquad (9.174)
\end{aligned}$$

We conclude that $\mathcal{M}^{++}_{\text{SL}}$ takes again the form (9.164), and that the electron spin is not influenced in the SL. The second amplitude is given by

$$\begin{aligned}
\mathcal{M}^{+-} &= \frac{-ie^2\hbar}{(k_1k_2)}\left(\frac{A_1^{+-}}{2(pk_1)} - \frac{A_2^{+-}}{2(qk_1)}\right) \;\;, \\
A_1^{+-} &= \overline{u}(q)\left(u_-(k_1)\overline{u}_-(k_2) + u_+(k_2)\overline{u}_+(k_1)\right)(\not{p} + \not{k}_1 + m) \times \\
&\quad \left(u_+(k_2)\overline{u}_+(k_1) + u_-(k_1)\overline{u}_-(k_2)\right)u(p) \;\;, \\
A_2^{+-} &= \overline{u}(q)\left(u_+(k_2)\overline{u}_+(k_1) + u_-(k_1)\overline{u}_-(k_2)\right)(\not{q} - \not{k}_1 + m) \times \\
&\quad \left(u_-(k_1)\overline{u}_-(k_2) + u_+(k_2)\overline{u}_+(k_1)\right)u(p) \;\;. \qquad (9.175)
\end{aligned}$$

Again using the various standard-form techniques we can establish that

$$\begin{aligned} A_1^{+-} &= \overline{u}(q)u_-(k_1)\overline{u}_-(k_2)\not{p}u_-(k_1)\overline{u}_-(k_2)u(p) \\ &\quad + \overline{u}(q)u_+(k_2)\overline{u}_+(k_1)\not{p}u_+(k_2)\overline{u}_+(k_1)u(p) \\ &= e^{i\phi}\,\overline{u}(q)\left(\omega_-\not{k}_1\not{p}\not{k}_2 + \omega_+\not{k}_2\not{p}\not{k}_1\right)u(p) \;\;, \end{aligned} \tag{9.176}$$

where $e^{i\phi}$ is the phase factor of Eq.(9.166). For A_2^{+-} we find the exact same result, and thus

$$\mathcal{M}^{+-} = \frac{ie^2\hbar\,e^{i\phi}}{2(pk_1)(qk_1)}\,\overline{u}(q)\left(\omega_-\not{k}_1\not{p}\not{k}_2 + \omega_+\not{k}_2\not{p}\not{k}_1\right)u(p) \;\;. \tag{9.177}$$

In this expression, the denominator is of order $\mathcal{O}\left(k^2\right)$ which is already compensated by the occurrence of $\not{k}_1$ and $\not{k}_2$ in the numerator. In the SL we can therefore again replace q by p since $q = p + \mathcal{O}\left(k\right)$, and use

$$\omega_-\not{k}_1\not{p}\not{k}_2 + \omega_+\not{k}_2\not{p}\not{k}_1 = (pk_1)\not{k}_2 + (pk_2)\not{k}_1 - (k_1k_2)\not{p} - i\gamma^\alpha\epsilon_\alpha(k_1,k_2,p) \tag{9.178}$$

which yields $4(pk_1)(pk_2) - 2m^2(k_1k_2)$ when sandwiched between $\overline{u}(p,s)$ and $u(p,s)$, but zero when sandwiched between $\overline{u}(p,-s)$ and $u(p,s)$. We see that also $\mathcal{M}_{\mathrm{SL}}^{+-}$ is the same as in the scalar case, while the spin again remains unaffected by the Thomson scattering.

We have thus established that Thomson scattering will not distinguish between scalar or Dirac electrons. It is interesting to note, however, that the amplitudes depend on θ even in the SL, while of course in that precise limit (zero photon energy) the value of θ becomes undetermined ! We see that Thomson scattering (and consequently the determination of the electron charge by this process) is only meaningful if there is *some* momentum transfer, no matter how small[33].

9.6.3 The Landau-Yang theorem

The photon polarisation revisited

As stated above, any good amplitude for processes in which a photon is absorbed or produced must vanish under the handlebar operation. That

[33]This situation is, of course, the same in classical physics : to determine the elasticity modulus of a spring you will have to stretch or compress it, no matter by how small an amount.

means that, *provided the amplitude is acceptable*, we may add to any photon polarisation a piece of photon momentum. Let us consider a process with several photons present, with momenta ${q_i}^\mu$ and polarisation vectors ${\epsilon_i}^\mu$. We have, obviously, $(q_i \cdot q_i) = (q_i \cdot \epsilon_i) = 0$ and $(\epsilon_i \cdot \epsilon_i) = -1$. From the above, we see that, if we wish, we may employ instead of ϵ_i the more complicated object

$$ {\eta_i}^\mu = {\epsilon_i}^\mu - \frac{(p \cdot \epsilon_i)}{(p \cdot q_i)} {q_i}^\mu \quad , \tag{9.179} $$

where p is *any* vector not proportional to q_i. This has the properties

$$ (\eta_i \cdot q_i) = (\eta_i \cdot p) = 0 \quad , \quad {\eta_i}^2 = -1 \quad . \tag{9.180} $$

In numerous applications, η is actually more profitable to use than ϵ. But we should note that, in any amplitude described by more than one Feynman diagram, the shift from ϵ to η simply means that parts of some Feynman diagrams are 'transferred' to other diagrams : the total result must, of course, be the same. The most important difference between ϵ and η is in the handlebar, since η then vanishes :

$$ \eta_i \rfloor_{\epsilon_i \to q_i} = 0 \quad ; \tag{9.181} $$

therefore, *any* expression written in terms of η's vanishes automatically under the handlebar. On the one hand this is, of course, nice ; on the other hand, it deprives us of a powerful check on the correctness of our diagrams, since almost any mistake made in writing them out will show up as a failure under the handlebar.

The Landau-Yang result

Although this may seem to fall somewhat outside the province of QED, we can consider the decay of a spin-1 particle into photons. But even within QED this can be envisaged, since we may have a bound state of electron and positron (*positronium*) that may, of course, have some angular momentum. Such a positronium state can, unless we look *really* closely, be considered a single particle. In its ground state, positronium comes in two varieties : *para-positronium* in which the electron and positron's spin are antiparallel and hence form total spin zero, and *ortho-positronium* in which the spins are parallel, leading to a total spin of one.

Without knowing anything much about the bound-state structure of positronium, let us consider the amplitude for its decay into a pair of photons. Let us denote by P^μ the positronium momentum (in its rest frame), and by $q_{1,2}$ and $\epsilon_{1,2}$ the photon momenta and polarizations. We shall define $q^\mu = (q_1^\mu - q_2^\mu)/2$. In addition, the positronium being a spin-1 particle, we need *its* polarisation vector ϵ_0. Any amplitude for the decay must necessarily be linear in ϵ_0, ϵ_1 and ϵ_2 ; and to have current conservation we can, rather, take $\eta_{1,2}$ instead of $\epsilon_{1,2}$, where here $\eta_i = \epsilon_i - (P \cdot \epsilon_i)/(P \cdot q_i)q_i$. Since also $(P \cdot \epsilon_0) = 0$, the three polarisations (as well as the vector q) have no timelike component. Noting that, in this case, $(q \cdot \eta_{1,2}) = 0$ as well, we see that to build an amplitude $\mathcal{M}$ we actually have but a very few structures that we can use[34] :

$$\mathcal{M} = A_1\,(q\cdot\epsilon_0)(\eta_1\cdot\eta_2) + A_2\,\varepsilon(P,\epsilon_0,\eta_1,\eta_2) + A_3\,\varepsilon(P,q,\eta_1,\eta_2)(q\cdot\epsilon_0)\quad . \tag{9.182}$$

The coefficients A are of course undetermined, but they can only depend on P^2, q^2 and $(P\cdot q)$. This last product is zero, and $P^2 = -4q^2 = M^2$ where M is the positronium mass, so the A's are effectively just constants. We now come to the main observation : under *interchange* of the two photons we have $\eta_1 \leftrightarrow \eta_2$ and $q \to -q$. It is immediately seen that *all* possible terms in $\mathcal{M}$ are *antisymmetric* under this operation, and hence cannot occur if we are to have Bose statistics. It is obvious that this results holds to all orders of perturbation theory, nor is restricted to the case of positronium. We conclude that *a spin-1 particle cannot decay into two photons*, which is the Landau-Yang theorem. And so it is : para-positronium has a lifetime of
E 53 1.25×10^{-10} seconds, while ortho-positronium, having to perform the much more cumbersome decay into *three* photons, lives for as long as 1.39×10^{-7} seconds.

In the literature and most textbooks, the Landau-Yang theorem, especially when applied to positronium, appears to be based on fairly complicated reasonings having to do with the charge-conjugation properties of the various states. In our more simple-minded approach, we see that it is simply a consequence of the relative paucity of building blocks available when you start to imagine what a decay amplitude could look like. Indeed, as soon as you envisage three-photon decay, a host of terms can be written down that re-

[34]You might be tempted to write down a term like $\varepsilon(q,\epsilon_0,\eta_1,\eta_2)$ but since all these vectors have vanishing zeroth component, this Levi-Civita product is simply zero.

spect Bose symmetry, so that it is easily understood why three-photon decay is not forbidden[35].

9.7 Exercises for Chapter 9

Excercise 48 Compton Current Conservation
Consider the process

$$e^-(p_1)\,\gamma(k_1,\epsilon_1) \quad \to \quad e^-(p_2)\,\gamma(k_2,\epsilon_2)$$

where the momenta and polarizations are indicated, and write the two diagrams that describe it at tree level. Then, substitute $\epsilon_1 \to k_1$ and show that the amplitude, so treated, vanishes.

Excercise 49 Rare or impossible ? The process $\mu \to e\,\gamma$
The process

$$\mu^-(p) \to e^-(q)\,\gamma(k)$$

is not allowed in standard QED since it violates current conservation. Nevertheless, it *could* be possible with a different interaction vertex coming from some 'new physics'. The amplitude would then read

$$\mathcal{M} = i\sqrt{\hbar}g\,\overline{u}(q)\,\left(\cos\theta + \sin\theta\gamma^5\right)\,\not{k}\,\not{\epsilon}\,u(p)$$

where g is the coupling in the new physics, and the angle θ simply parametrizes the relative weight of the two alternative couplings.

1. Show that this amplitude is current-conserving by applying the handlebar.

2. Show that the coupling g must have the following dimensionality :

$$\mathbf{dim}[g] = \frac{L}{\sqrt{\hbar}}$$

   We shall therefore write

$$g = \frac{e}{\Lambda}$$

   with e the QED coupling constant and Λ the scale of the new physics.

[35]In the words of Feynman, 'everything that is not explicitly forbidden is allowed'.

3. Compute $\langle|\mathcal{M}|^2\rangle$ using the Casimir trick and the trace identities.

4. Compute the total decay width $\Gamma(\mu \to e\gamma)$.

5. Current limits on this process are expressed as

$$\frac{\Gamma(\mu \to e\gamma)}{\Gamma(\mu \to \text{all})} \leq B \quad , \quad B \approx 10^{-11} \quad .$$

We shall assume that the decay $\mu \to e\nu_\mu\bar{\nu}_e$ is by far the dominant one. Show that we can relate this to B as follows :

$$\Lambda \geq \frac{K}{\sqrt{B}} \quad .$$

Compute K, and find the current lower limit on Λ.

Excercise 50 Bhabha scattering got wrong
In section 9.3.4 it is mentioned in a footnote how important the correct application of the Fermi minus sign can be. Investigate this by re-computing the cross section for Bhabha scattering using the wrong sign, and finding the ratio between the two expressions ; then find the maximum ratio and the scattering angle at which this is reached.

Excercise 51 Bhabha scattering the hard way
Compute $\langle|\mathcal{M}|^2\rangle$ for Bhabha scattering at tree level, this time keeping m_e nonzero. To do this, use the Casimir (trace) method.

Excercise 52 Multi-photon production
Consider the process $e^+e^- \;\to\; n\,\gamma$, the n-photon analogue of the process 9.139. Show the following :

1. There are, at the tree level, $n!$ diagrams.

2. The amplitude vanishes if all photons have the same helicity.

3. If all photon helicities *except one* are equal, only $(n-1)!$ diagrams contribute if we choose the gauge vectors cleverly.

Excercise 53 The Landau-Yang theorem in action

1. Consider the proces

$$e^+(p_1, s_1)\, e^-(p_2, s_2) \quad \to \quad \gamma(k_1, \epsilon_1)\, \gamma(k_2, \epsilon_2)$$

   where the polarizations and spins are indicated along with the momenta. Write out the amplitude at the tree level, keeping the electron mass m nonzero.

2. Choose the gauge vectors such that not only $(k_1\epsilon_1) = (k_2\epsilon_2) = 0$ but also $(k_1\epsilon_2) = (k_2\epsilon_1) = 0$. Show that in that case ${\epsilon_{1,2}}^0 = 0$ in the centre-of-mass frame.

3. Take the static limit, in which p_1 and p_2 become equal. Show that in this limit the amplitude is proportional to

$$\epsilon^\mu(\epsilon_1, k_1 - k_2, \epsilon_2)\, \overline{v}(p_1, s_1)\, \gamma^5\, \gamma_\mu\, u(p_1, s_2)$$

4. Show that the amplitude vanishes if $s_1 = s_2$, but does *not* vanish if $s_1 = -s_2$. Hint: show that $\epsilon^\mu(\epsilon_1, k_1 - k_2, \epsilon_2)\ \propto\ {p_1}^\mu$.

Chapter 10

Quantum Chromodynamics

10.1 Introduction: coloured quarks and gluons

In chapter 9 we have studied the behaviour of electrically charged particles and the electromagnetic field embodied by photons. Notwithstanding the fact that particles can have different charges, all these charges are of the same *type* in the sense that they can be added. For instance, atoms are electrically neutral when studied from the 'outside', since the positive charge of the nucleus is cancelled out by the negative charge of the electron cloud. It is interesting to see what happens if we enlarge our view to the possibility of 'different types of charge', that cannot be meaningfully added in a simple way. In that case, a bound state of particles with a different charge type might not look 'neutral' when seen from the outside : the charges of the constituents would show through. To avoid confusion with the electric charge we shall let the 'new charges' go by the name of *colours*, and the dynamical theory of their interactions is called Quantum Chromodynamics, or QCD.

We shall start our investigation with coloured fermions, called *quarks*[1]. The *number* of colours is denoted by N, where of course $N \geq 2$. The quarks are described by Dirac spinors for given momentum and spin, and also by a *colour label* which we shall denote by $a, b, c, \ldots$. All these labels (or indices)

[1]Historically, the notion of quark predates that of colour, and the colouring of quarks was invented to explain the possibility of the existence of curious particles such as the Δ^{++} or the Ω^-. In this chapter, we are less interested in describing the world of hadrons than in constructing an internally consistent theory, hence the unhistorical line of reasoning.

run from 1 to N. A conjugate fermion ($\overline{u}$ or $\overline{v}$) will carry a lower, a regular fermion (u or v) an upper index.

In addition we expect vector particles to be present, that carry the colour force. These we call *gluons*. In analogy to QED, we shall assume the gluons to be massless, but since we have different colour types there must also be different gluon types. The gluon type will be denoted by $j, k, l, m, \ldots$, and it is up to us to determine[2] how many gluon types occur for given N.

We now postulate a few properties that we want our world of colour to possess:

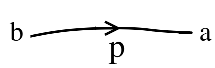
1. Colour is conserved in interactions, just like electric charge. This must hold for every type of colour charge separately.

2. All colours are equal and none are 'more equal than others', which means that particles that only differ by their colours propagate through spacetime in the same way.

10.2 Quarks and gluons : first Feynman rules

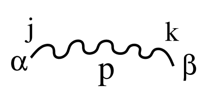
10.2.1 The propagators

Under the assumption of colour conservation, the quark propagator is defined by



$$\leftrightarrow \quad i\hbar \, \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \, \delta^a{}_b \tag{10.1}$$

where the incoming and outgoing quark colours are denoted by a and b. The gluon propagator requires some more care. We shall assume that it is described by the axial propagator of chapter 8. That is, we take an *axis* n^μ, and require the propagator to be orthogonal to this. Anticipating that also gluons may carry colour information, we have



$$\leftrightarrow \quad \Pi^{\alpha\beta}(p)\, \delta^{jk} \tag{10.2}$$

[2] The usual approch is simply to postulate a local $SU(N)$ gauge symmetry, from which the number of gluons immediately follows ; but our (or rather my) interest is to see how we can arrive at that result from simpler, or rather physical, requirements.

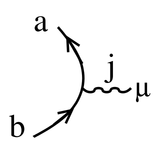
where j and k denote the gluon colours, and

$$\begin{aligned}\Pi^{\alpha\beta}(p) &= i\hbar\,\frac{-1}{p^2+i\epsilon}\left(g^{\alpha}{}_{\lambda}-\frac{p^{\alpha}n_{\lambda}}{(p\cdot n)}\right)\left(g^{\lambda\beta}-\frac{n^{\lambda}p^{\beta}}{(p\cdot n)}\right)\\ &= \frac{i\hbar}{p^2+i\epsilon}\left(-g^{\alpha\beta}+\frac{p^{\alpha}n^{\beta}+n^{\alpha}p^{\beta}}{(p\cdot n)}-\frac{n^2\,p^{\alpha}p^{\beta}}{(p\cdot n)^2}\right)\quad . \end{aligned}\qquad (10.3)$$

The first line of Eq.(10.3) shows that, indeed, $\Pi^{\alpha\beta}(p)\,n_{\alpha}=\Pi^{\alpha\beta}(p)\,n_{\beta}=0$. Note that we have extended the definition of the axial-gauge propagator to nonzero values of n^2. However, since $\Pi^{\alpha}{}_{\alpha}(p)\propto -2+n^2p^2/(p\cdot n)^2$, the number of degrees of freedom is still equal to 2 on the mass shell, where $p^2=0$. The choice of this propagator has ultimately, of course, to be justified at least by the handlebar investigation.

10.2.2 The quark-gluon vertex

We start by defining the quark-gluon vertex, as a close analogue of the QED fermion-photon vertex :



$$\leftrightarrow\ \frac{i}{\hbar}g\,\gamma^{\mu}\,(T^{j})^{a}{}_{b}\quad , \qquad (10.4)$$

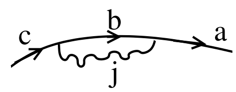
where we have explicitly indicated the quark and gluon colour types. Here, g is the coupling constant, and $(T^{j})^{a}{}_{b}$ is recognized as an element of an $N\times N$ matrix, the properties of which we still need to derive. Allowing for complex matrices, we see that the number of different gluon colours j cannot exceed $2N^2$. It is clear that an overall factor in the matrices T can always be absorbed in a redefinition of g, and we shall use this to normalize the T matrices. The above vertex, inspired by the example of QED, must ultimately be justified by investigating the handlebar condition[3]. Because of the presence of colour we may expect that this will be more complicated than in the simple electrodynamics case.

10.2.3 A closer look at the T matrices

We require the structure of the colour part of the interactions to take care of colour conservation and colour equality in the presence of interactions.

[3]And, of course, by comparison to experiment !

Consider the following diagram :



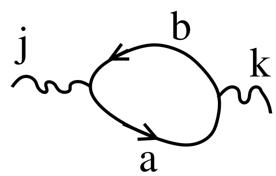
The colour part of this diagram reads

$$\sum_{j,b} (T^j)^a{}_b (T^j)^b{}_c$$

and colour conservation/equality hence demands that

$$\sum_j \left(T^{j\,2}\right)^a{}_b = k\,\delta^a{}_b \tag{10.5}$$

for some constant k. Similarly, the diagram



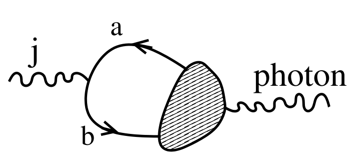
contains the colour factor

$$\sum_{a,b} (T^j)^a{}_b (T^k)^b{}_a = \mathrm{Tr}\left(T^j T^k\right) \quad ,$$

and using the normalization freedom we may take

$$\mathrm{Tr}\left(T^j T^k\right) = \frac{1}{2}\,\delta^{jk} \quad . \tag{10.6}$$

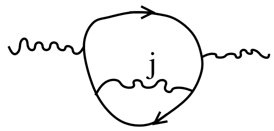
Since colour must be conserved, a gluon cannot lose its colour charge and therefore gluons and photons cannot mix : in all diagrams of the form



we must have $b = a$ since the colour is conserved and the photon is colourless ; therefore the T matrices must be traceless :

$$\mathrm{Tr}\left(T^j\right) = 0 \quad \text{for all gluon colours } j \quad . \tag{10.7}$$

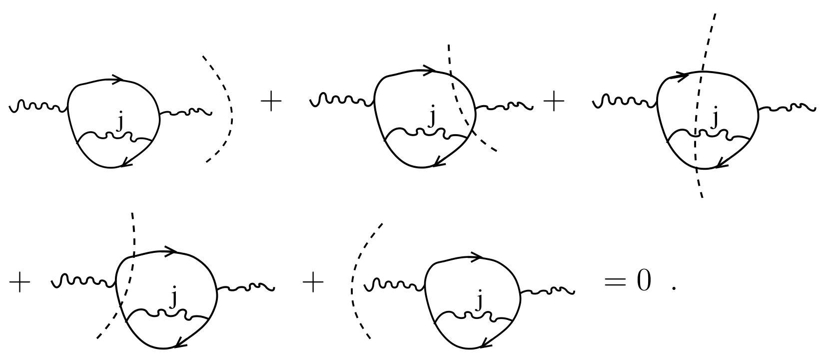
Finally, we consider the following two-loop self-energy diagram of the photon :



Here, the fermions are quarks and the internal line labelled j is a gluon of colour type j : of course, we have to sum over all j values. If we compare this diagram to the corresponding QED one, we see that apart from the overall charges (g^2 instead of Q^2) the only difference is the colour factor, in this case

$$\sum_j \mathrm{Tr}\left(T^j\, T^j\right)$$

Now, if our theory is to be unitary, it must obey the Cutkosky rules, and therefore we demand that



$$= 0 \quad . \tag{10.8}$$

For the QED diagram, this indeeds holds. In the coloured case, however, the colour structures of the diagram cut in the various ways are no longer the same : the three lines in Eq.(10.8) are proportional to, respectively,

$$\sum_j \mathrm{Tr}\left(T^j T^j\right) \quad , \quad \sum_j \mathrm{Tr}\left(T^j {T^j}^\dagger\right) \quad , \text{ and } \quad \sum_j \mathrm{Tr}\left({T^j}^\dagger {T^j}^\dagger\right) \quad .$$

Unitarity can therefore only be safe if these three different traces are, in fact, equal to one another. We may therefore write

$$\sum_j \mathrm{Tr}\left(A^j A^j\right) = 0 \quad , \quad A_j = i\left(T^j - {T^j}^\dagger\right) \quad . \tag{10.9}$$

The matrices A^j are obviously Hermitean, so that Eq.(10.9) can also be written as

$$\sum_j \mathrm{Tr}\left(A^j {A^j}^\dagger\right) = \sum_j \sum_{a,b=1}^{N} \left|(A^j)^a{}_b\right|^2 = 0 \quad , \tag{10.10}$$

hence all A^j are actually identically zero, and the matrices T^j must be Hermitean. The number of different gluon colours type is therefore $N^2 - 1$, and the constant k of Eq.(10.5) is equal to $(N^2 - 1)/2N$.

10.2.4 The Fierz identity for T matrices

We have now zoomed in quite efficiently on the matrices T^j. On the other hand, just as in the case of Dirac particles we would prefer if predictions for cross sections and the like dit not depend on the particular choice of the matrices[4]. We can, in fact, derive a relation between the T's that holds independently of any representation : it goes under the name of the Fierz identity[5]. Any $N \times N$ matrix M can be written[6] as

$$M = a_0 1 + \sum_j a_j T^j \quad . \tag{10.11}$$

By taking traces we can determine the coefficients :

$$\mathrm{Tr}\left(M\right) = a_0 N \quad , \quad \mathrm{Tr}\left(M T^k\right) = a_k/2 \quad . \tag{10.12}$$

Therefore we have

$$M = 2\sum_j \mathrm{Tr}\left(T^j M\right) T^j + \mathrm{Tr}\left(M\right)/N \quad , \tag{10.13}$$

or, in terms of the matrix components,

$$M^d{}_c\, \delta^c{}_b\, \delta^a{}_d = 2\sum_j M^d{}_c\, (T^j)^c{}_d (T^j)^a{}_b + \frac{1}{N} M^d{}_c\, \delta^c{}_d\, \delta^a{}_b \quad , \tag{10.14}$$

[4]In the Dirac case this was indispensable since any dependence would destroy Lorentz invariance. In the present case one might argue that the T^j could, in principle, just be *measured*. Nevertheless having a representation-independent theory just feels so much more comfortable.

[5]Same Fierz.

[6]You might be tempted to think that this holds only for Hermitean matrices. But since iT^j is *anti*Hermitean we can accomodate *any* M provided the a's can be complex.

whence the following, representation-independent identity :

$$(T^j)^a{}_b (T^j)^c{}_d = \frac{1}{2}\left(\delta^a{}_d\,\delta^c{}_b - \frac{1}{N}\delta^a{}_b\,\delta^c{}_d\right) \quad . \tag{10.15}$$

Since[7] the colour of quarks and gluons cannot be observed, any cross section will involve a summation over all colours, and therefore every cross section is expressed as (a product of) traces of strings of T matrices, in which every matrix T^k occurs exactly twice, and the index k is summed over. The Fierz identity comes in useful here, since we can write (with summation implied)

$$\begin{aligned}
\mathrm{Tr}\left(T^j A\right)\,\mathrm{Tr}\left(T^j B\right) &= \frac{1}{2}\left(\mathrm{Tr}\,(AB) - \frac{1}{N}\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(B)\right) \quad , \\
\mathrm{Tr}\left(T^j A\, T^j B\right) &= \frac{1}{2}\left(\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(B) - \frac{1}{N}\mathrm{Tr}\,(AB)\right) \quad .
\end{aligned} \tag{10.16}$$

With these trace identities we can simplify and compute any set of colour traces without recourse to any explicit representation, especially if we recall that $\mathrm{Tr}\,(1) = N$, $\mathrm{Tr}\,(T^j) = 0$ and $T^jT^j = (N^2-1)/2N$ times unity. For instance,

$$\begin{aligned}
&\mathrm{Tr}\left(T^jT^kT^l\right)\mathrm{Tr}\left(T^jT^kT^l\right) \\
&= \frac{1}{2}\left(\mathrm{Tr}\left(T^kT^lT^kT^l\right) - \frac{1}{N}\mathrm{Tr}\left(T^kT^l\right)\mathrm{Tr}\left(T^kT^l\right)\right) \\
&= \frac{1}{4}\left(\mathrm{Tr}\left(T^l\right)\mathrm{Tr}\left(T^l\right) - \frac{2}{N}\mathrm{Tr}\left(T^lT^l\right) + \frac{1}{N^2}\mathrm{Tr}\left(T^l\right)\mathrm{Tr}\left(T^l\right)\right) \\
&= -\frac{1}{2N}\mathrm{Tr}\left(T^lT^l\right) = -\frac{N^2-1}{4N} \quad ,
\end{aligned} \tag{10.17}$$

and

$$\begin{aligned}
&\mathrm{Tr}\left(T^jT^kT^l\right)\mathrm{Tr}\left(T^jT^lT^k\right) \\
&= \frac{1}{2}\left(\mathrm{Tr}\left(T^kT^lT^lT^k\right) - \frac{1}{N}\mathrm{Tr}\left(T^kT^l\right)\mathrm{Tr}\left(T^lT^k\right)\right) \\
&= \frac{1}{4}\left(\mathrm{Tr}\left(T^kT^k\right)\mathrm{Tr}\,(1) - \frac{2}{N}\mathrm{Tr}\left(T^kT^k\right) + \frac{1}{N^2}\mathrm{Tr}\left(T^k\right)\mathrm{Tr}\left(T^k\right)\right) \\
&= \frac{N^2-2}{4N}\mathrm{Tr}\left(T^kT^k\right) = \frac{(N^2-1)(N^2-2)}{8N} \quad .
\end{aligned} \tag{10.18}$$

[7] Empirically.

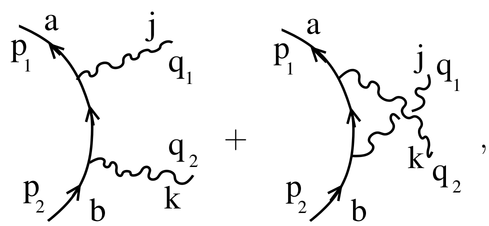
10.3 The three-gluon interaction

10.3.1 The need for three-gluon vertices

It is now time to investigate our theory using handlebars. In the first place, in the process $g \to q\bar{q}$ the current is conserved in the same way as in QED, since there is only a single Feynman diagram and the colour structure is therefore irrelevant to any cancellation. The situation becomes more delicate in the case of more complicated interactions, so let us consider the process

$$\bar{q}(p_1, a)\ q(p_2, b) \quad \to \quad g(q_1, \epsilon_1, j)\ g(q_2, \epsilon_2, k)$$

where we have explicitly indicated the momenta, polarizations, and colours. We have *at least* the following two diagrams[8] :

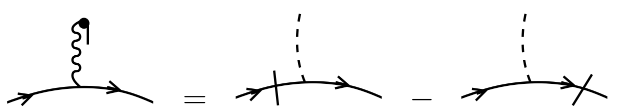
$\mathcal{M} =$ 

, (10.19)

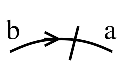
where we have indicate the colours explicitly. Explicitly, they read

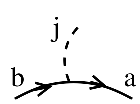
$$\begin{aligned}
\mathcal{M}_1 &= -i\hbar g^2\ \overline{v}(p_1)\not{\epsilon}_1 \frac{\not{q}_1 - \not{p}_1 + m}{-2(q_1 \cdot p_1)} \not{\epsilon}_2 u(p_2)\ (T^j T^k)^a{}_b\ , \\
\mathcal{M}_2 &= -i\hbar g^2\ \overline{v}(p_1)\not{\epsilon}_2 \frac{\not{p}_2 - \not{q}_1 + m}{-2(q_1 \cdot p_2)} \not{\epsilon}_1 u(p_2)\ (T^k T^j)^a{}_b\ .
\end{aligned} \tag{10.20}$$

Let us now put the handlebar on gluon 1, so that we replace ϵ_1^μ by q_1^μ. By the same reasoning as in chapter 9, we arrive at the handlebar rule


, (10.21)

with the auxiliary Feynman rules



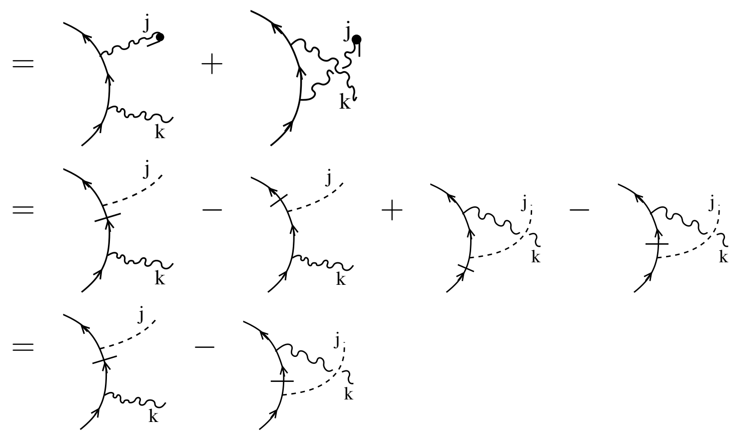
$\leftrightarrow \quad i\hbar\,\delta^a{}_b$, 

$\leftrightarrow \quad i\dfrac{g}{\hbar}(T^j)^a{}_b$. (10.22)

[8]In the second diagram the two gluon lines form an overpass, without a vertex.

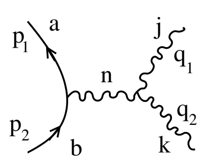
As before, slashed propagators vanish on external lines. Applying this to the two Feynman diagrams of Eq.(10.19) yields[9]

$$\mathcal{M}_{1+2}\big\rfloor_{\epsilon_1 \to q_1} =$$



$$= -i\hbar g^2\, \overline{v}(p_1)\not{q}_2 u(p_2)\, [T^j, T^k]^a{}_b \;\;, \qquad (10.23)$$

where the square brackets denote, of course, the commutator of the matrices T^j and T^k. Because of the colour structure we have a non-vanishing result, and current conservation is in trouble ! The remedy must be to introduce a *third* diagram, with a nontrivial ggg vertex :

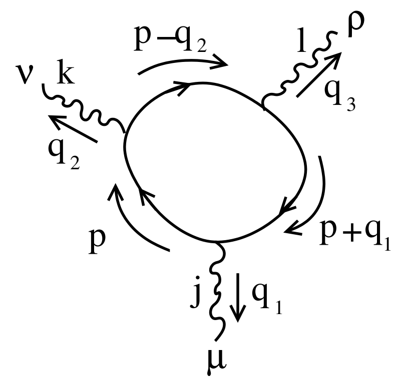


It is now our job to determine the form of the new three-gluon vertex. We shall do this by investigating loop diagrams.

[9]In calculations such as this one it is often sufficient to simply label the gluons by their colour, thus simplifying the typography.

10.3.2 Furry's failure

Consider the Feynman diagram depicted on the right, in which three gluons are effectively coupled by a quark loop. We have explicitly indicated the momentum flows. Note especially that the gluon momenta are all counted flowing *out* of the vertex, so that we have



$$q_1 + q_2 + q_3 = 0 \ . \tag{10.24}$$

Apart from overall coupling constants and the like, the loop diagram is given by

$$T = \int d^4p \, \frac{\mathrm{Tr}\,((\not p + m)\gamma^\mu(\not p + \not q_1 + m)\gamma^\rho(\not p - \not q_2 + m)\gamma^\nu)}{(p^2 - m^2)((p+q_1)^2 - m^2)((p-q_2)^2 - m^2)}\, \mathrm{Tr}\left(T^jT^lT^k\right) \ . \tag{10.25}$$

There is also a loop diagram in which the quark runs counterclockwise instead of clockwise. In our discussion of Furry's theorem in sect. 9.2.6, we have seen that the space-time part of the second diagram is exactly opposite to the one of the first, so that in QED these two diagrams cancel. In QCD, however, they do not since the second diagram contains the colour matrices in the opposite order, that is to say it contains $\mathrm{Tr}\left(T^jT^kT^l\right)$ instead of $\mathrm{Tr}\left(T^jT^lT^k\right)$. The sum of the two diagrams must, if we take into account the Lorentz-covariant nature of the loop integral, and the fact that out of q_1, q_2 and q_3 only two momenta are independent, be of the form

$$T = Y(q_1,\mu;q_2,\nu;q_3,\rho)\,\mathrm{Tr}\left(T^j[T^k,T^l]\right) \ , \tag{10.26}$$

with

$$\begin{aligned} Y(q_1,\mu;q_2,\nu;q_3,\rho) &= \{(a_1q_1 + a_2q_2)^\rho g^{\mu\nu} + (a_3q_2 + a_4q_3)^\mu g^{\nu\rho} \\ &\quad + (a_5q_3 + a_6q_1)^\nu g^{\rho\mu}\} \ , \end{aligned} \tag{10.27}$$

for some numbers $a_1, \ldots, a_6$. For large p, each of the three propagators goes as $1/p$, and the loop integral is therefore divergent. We see that indeed there *has* to be a three-gluon coupling in the action, otherwise the theory would not be renormalizable ; and the *form* of the three-gluon vertex must be that of Eq.(10.27).

Without evaluating the loop integral completely, we can glean all the information we need. Consider the following transformation on T :

$$q_1 \leftrightarrow -q_2 \quad , \quad q_3 \to -q_3 \quad , \quad \mu \leftrightarrow \nu \quad . \tag{10.28}$$

This transformation leaves the momentum conservation law (10.24) intact, and also preserves the value of T (by the reversal property (7.29) of Dirac traces). The same holds, of course, for the transformations

$$\begin{aligned} q_1 \leftrightarrow -q_3 \quad &, \quad q_2 \to -q_2 \quad , \quad \mu \leftrightarrow \rho \quad , \\ q_2 \leftrightarrow -q_3 \quad &, \quad q_1 \to -q_1 \quad , \quad \nu \leftrightarrow \rho \quad . \end{aligned} \tag{10.29}$$

The function Y must therefore satisfy

$$\begin{aligned} & Y(q_1,\mu;q_2,\nu;q_3,\rho) = Y(-q_2,\nu;-q_1,\mu;-q_3,\rho) = \\ & = Y(-q_3,\rho;-q_2,\nu;-q_1,\mu) = Y(-q_1,\mu;-q_3,\rho;-q_2,\nu) \;\; ; \end{aligned} \tag{10.30}$$

and by inspection we then find that $c_1 = c_3 = c_5 = -c_2 = -c_4 = -c_6$. We shall therefore from now on use the definition

$$\begin{aligned} & Y(q_1,\mu;q_2,\nu;q_3,\rho) \\ & \quad \equiv \;\; (q_1-q_2)^\rho g^{\mu\nu} + (q_2-q_3)^\mu g^{\nu\rho} + (q_3-q_1)^\nu g^{\rho\mu} \quad . \end{aligned} \tag{10.31}$$

Note that this form is antisymmetric in the interchange of any two gluons, and therefore invariant under a *cyclic* permutation.

A final remark is in order here. If *one* of the couplings were not of vector type (with γ^μ) but of axial-vector type (with $\gamma^5\gamma^\mu$), then the integral would change sign under the above transformations. In that case the function Y would read

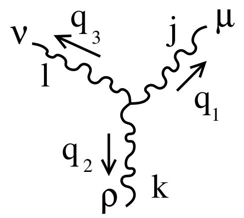
$$\begin{aligned} & (q_1+q_2)^\rho g^{\mu\nu} + (q_2+q_3)^\mu g^{\nu\rho} + (q_3+q_1)^\nu g^{\rho\mu} \\ & = -q_3^\rho g^{\mu\nu} - q_1^\mu g^{\nu\rho} - q_2^\nu g^{\rho\mu} \end{aligned}$$

and hence be completely transverse to any external polarisation vector[10].

[10]This effect forbids, for example, the decay of a Z^0 boson into two photons or two gluons.

10.3.3 The ggg vertex and its handlebar

On the basis of the previous section, we see that the only reasonable form of the three-gluon vertex Feynman rule is



$$\leftrightarrow \quad \frac{i}{\hbar} g_3\, Y(q_1,\mu;q_2,\nu;q_3,\rho)\, h^{jkl} \tag{10.32}$$

Note that the gluon momenta are counted *outgoing* from the vertex. The value of g_3 must be determined, as well as the colour factor h^{jkl}. The Y function's total antisymmetry strongly suggests that we take the h symbols antisymmetric as well, and later on we shall show that this is indeed the case.

The following object will turn out to be useful :

$$\Delta(q)^{\alpha\beta} \equiv q^\alpha q^\beta - q^2 g^{\alpha\beta} \quad , \tag{10.33}$$

for which

$$\Delta(q)^{\alpha\beta} = \Delta(q)^{\beta\alpha} \quad , \quad \Delta(q)^{\alpha\beta}\, q_\beta = 0 \quad . \tag{10.34}$$

Also,

$$\Delta(q)^{\alpha\beta}\, \epsilon_\beta = q^\alpha (q\cdot\epsilon) - q^2 \epsilon^\alpha = 0 \tag{10.35}$$

if ϵ is the polarisation vector of an on-shell gluon with momentum q.

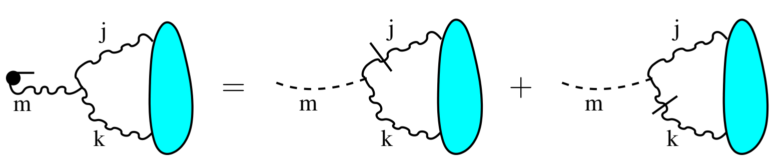
We now come to an important result. Let us consider the vertex of Eq.(10.32), and let us put a handlebar on gluon q_3. We find, using momentum conservation in the form $q_3 = -q_1 - q_2$,

$$\begin{aligned} Y(q_1,\mu;q_2,\nu;q_3,q_3) &= (q_1 - q_2\cdot q_3) g^{\mu\nu} + (q_2-q_3)^\mu {q_3}^\nu + (q_3-q_1)^\nu {q_3}^\mu \\ &= \ (q_2 - q_1\cdot q_2 + q_1) g^{\mu\nu} - {q_2}^\mu (q_1+q_2)^\nu + {q_1}^\nu (q_1+q_2)^\mu \\ &= \ \Delta(q_1)^{\mu\nu} - \Delta(q_2)^{\mu\nu} \quad . \end{aligned} \tag{10.36}$$

Some algebra tells us, moreover, that *in the axial gauge*

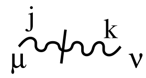
$$\Pi^{\mu\alpha}(q_1)\, \Delta_{\alpha\beta}(q_1)\, \Pi^{\beta\nu}(q_2) = (i\hbar g^{\mu\alpha})\,(g_{\alpha\beta})\, \Pi^{\beta\nu}(q_2) \quad , \tag{10.37}$$

so that we find the handlebar rule



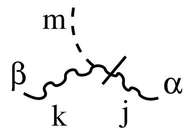
$$, \tag{10.38}$$

with the auxiliary Feynman rules



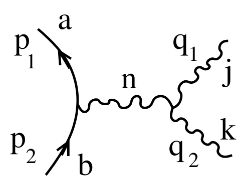
$$\leftrightarrow \quad i\hbar\, g^{\mu\nu}\, \delta^{jk} \tag{10.39}$$

and



$$\leftrightarrow \quad i\frac{g_3}{\hbar}\, g^{\alpha\beta}\, h^{jkm} \quad . \tag{10.40}$$

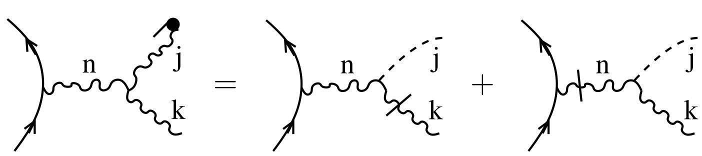
It is now time to return to the $q\bar{q} \to gg$ process. The new available Feynman diagram, given by



reads

$$\mathcal{M}_3 = -gg_3\overline{v}(p_1)\gamma_\mu u(p_2)\Pi^{\mu\nu}(p_1+p_2)Y(q_1,\epsilon_1;q_2,\epsilon_2;-q_1-q_2,\nu)\, h^{jkn}(T^n)^a{}_b \quad . \tag{10.41}$$

with summation over the colour n implied. Putting the handlebar on gluon 1 as before, we get



(10.42)

so that

$$\mathcal{M}_3\rfloor = -i\hbar g g_3\, \overline{v}(p_1)\not{\epsilon}_2 u(p_2)\, h^{nkj}(T^n)^a{}_b \quad . \tag{10.43}$$

The total handlebarred amplitude now reads

$$\mathcal{M}_{1+2+3}\rfloor = i\hbar g\, \overline{v}(p_1)\not{\epsilon}_2 u(p_2)\left(g_3 h^{jkn}T^n - g[T^j,T^k]\right)^a{}_b \tag{10.44}$$

The colour current will therefore be conserved if we choose

$$g_3 = g \tag{10.45}$$

and

$$[T^j, T^k] = h^{jkn}\, T^n \quad . \tag{10.46}$$

Note that since the matrices T are Hermitean, the constants h must be purely imaginary[11]. Moreover, we can compute them, using Eq.(10.6), as

$$h^{jkl} = 2\,\mathrm{Tr}\left(T^jT^kT^l - T^lT^kT^j\right) \quad , \tag{10.47}$$

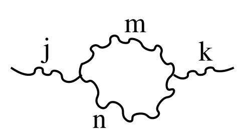
from which we see that the h symbols must be totally antisymmetric. Since they are related to commutators, we can use the Jacobi identity to find relations between them[12] :

$$\begin{aligned} 0 &= [[T^j,T^k],T^l] + [[T^k,T^l],T^j] + [[T^l,T^j],T^k] \\ &= h^{jk}{}_n[T^n,T^l] + h^{kl}{}_n[T^n,T^j] + h^{lj}{}_n[T^n,T^k] \\ &= h^{jk}{}_n h^{nl}{}_m T^m + h^{kl}{}_n h^{nj}{}_m T^m + h^{lj}{}_n h^{nk}{}_m T^m \quad , \end{aligned} \tag{10.48}$$

which after a few interchanges of indices leads to

$$h^{jk}{}_n h^{lm}{}_n + h^{jl}{}_n h^{mk}{}_n + h^{jm}{}_n h^{kl}{}_n = 0 \quad . \tag{10.49}$$

More information comes from colour conservation/equality in the diagram



from which we find the requirement that

$$\sum_{m,n} h^{mnj}\, h_{mnk} = C\,\delta^j{}_k \quad , \tag{10.50}$$

with some constant C. Eq.(10.50) is the gluonic equivalent of the property (10.5) of the T matrices. It does *not* follow from the Jacobi identity. But

[11]It is customary to write $[T^j,T^k] = i\,f^{jk}{}_nT^n$. The f's are then called the structure constants, and the set of T matrices are then the generators of the Lie algebra of the group $SU(N)$. The i is then combined with the overall i of the vertex to give a Feynman rule without any i. This is of course a matter of taste.

[12]Here and in the following, raising or lowering colour labels has no physical meaning ; I do it only for typographical reasons.

since we have already *defined* the h symbols by Eq.(10.47), it is not an extra condition bur rather has to be proven. To this end, we use Eq.(10.47) :

$$\begin{aligned} h^{mnj}h^{mnk} = & \\ = \; & 8\left(\mathrm{Tr}\left(T^mT^nT^j\right)\mathrm{Tr}\left(T^mT^nT^k\right) - \mathrm{Tr}\left(T^mT^nT^j\right)\mathrm{Tr}\left(T^mT^kT^n\right)\right) \end{aligned} \tag{10.51}$$

and the reduction formulæ(10.16) then give us, for instance,

$$\begin{aligned} \mathrm{Tr}\left(T^mT^nT^j\right)\mathrm{Tr}\left(T^mT^nT^k\right) &= -\frac{1}{4N}\,\delta^{jk} \quad , \\ \mathrm{Tr}\left(T^mT^nT^j\right)\mathrm{Tr}\left(T^mT^kT^n\right) &= \left(\frac{N}{8} - \frac{1}{4N}\right)\delta^{jk} \quad . \end{aligned} \tag{10.52}$$

Thus we arrive at the desired property :

$$h^{mnj}h_{mnk} = -N\,\delta^j{}_k \quad . \tag{10.53}$$

10.3.4 On coupling quantisation

In the previous chapter we have discussed QED, characterized by the fact that only fermion-fermion-photon couplings occur. The coupling constant Q_f for a given fermion is not constrained in any way, and there is no *a priori* reason why different fermion species ought to have the same charge[13]. In QCD the situation is different. In deriving the ggg vertex we have started with the process $q\bar{q} \to gg$, without specifying the quark type, only saying that its coupling to the gluon has coupling constant ('colour charge') g. We then find that the three-gluon vertex has *the same* coupling constant g. But *that* implies that all quark types must have the same coupling constant to gluons, since two different quark types must lead to the same value of the Yang-Mills coupling. We see that the presence of bosonic self-interactions enforces a uniformity on the quark colour charges that was not there before. In the next chapter we shall see how the presence of a $WW\gamma$ vertex forces the electron and muon charges to be, indeed, identical.

[13]Of course, leptons and quarks have different charge ; but I rather refer to the ratio of muon and electrron charge, which could be $\sqrt{\pi}$ for all that QED cares.

10.4 The four-gluon interaction

10.4.1 Colourful manipulations

Before proceeding it is useful to prepare some groundwork. Let us define, first, the 'four-colour' object

$$[jklm] = h^{jk}{}_a\, h^{lm}{}_a = h^{jk}{}_a h_a{}^{lm} \quad . \tag{10.54}$$

This has the symmetries[14]

$$[jklm] = -[kjlm] = -[jkml] = [lmjk] \quad ; \tag{10.55}$$

and the Jacobi identity reads

$$[jklm] + [jlmk] + [jmkl] = 0 \quad . \tag{10.56}$$

In addition, we have the 'five-colour' object

$$[jklmn] = [jkla]\, h^{amn} = h^{jk}{}_a\, h_a{}^l{}_b\, h_b{}^{mn} \quad . \tag{10.57}$$

We can easily verify the symmetries

$$[jklmn] = -[kjlmn] = [jklnm] = -[mnljk] \tag{10.58}$$

and the two Jacobi identities

$$[jklmn] + [jkmnl] + [jknlm] = [jklmn] + [kljmn] + [ljkmn] = 0 \quad . \tag{10.59}$$

With such an arsenal of identities quite a few results can be derived: for instance, by using each of the three symmetries and the two Jacobi identities once, we can prove that

$$\begin{aligned} &[jklmn] - [jklmn] + [lmjkn] - [lmjkn] = 0 \quad , \\ &[jlkmn] + [kmjln] - [jlmkn] - [kmljn] = 0 \quad . \end{aligned} \tag{10.60}$$

[14]These are precisely those of the Riemann-Christoffel tensor, familiar from the theory of general relativity.

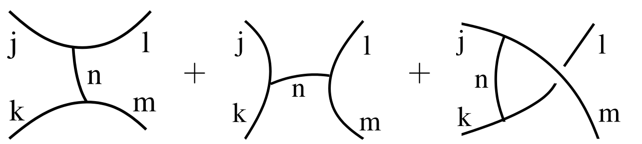
10.4.2 A purely gluonic process

We have seen that gluons, in contrast to photons, exhibit self-interactions. We can therefore consider the process

$$g(q_1, \epsilon_1, j)\ g(q_2, \epsilon_2, k) \quad \to \quad g(q_3, \epsilon_3, l)\ g(q_4, \epsilon_4, m)$$

which so far is given (at the tree level) by three Feynman diagrams[15] :

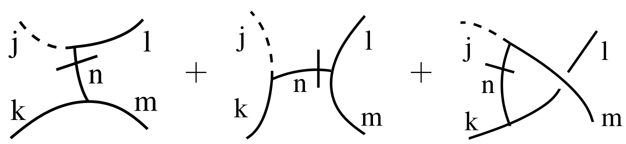
$\mathcal{M} =$



(10.61)

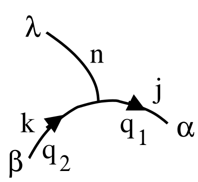
As before, we put a handlebar on gluon 1 :

$\mathcal{M}\rfloor_{\epsilon_1 \to q_1} =$



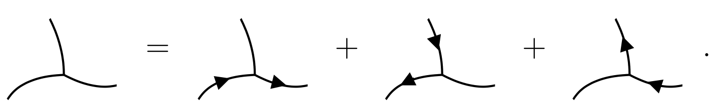
(10.62)

and this combination does not obviously vanish. The three-gluon vertex is somewhat cumbersome, but we can streamline our calculations a bit by introducing a 'partial' three-gluon vertex :



$$\leftrightarrow \quad i\frac{g}{\hbar}\,(q_1 + q_2)^\lambda\, g^{\alpha\beta}\, h^{jkn} \quad . \tag{10.63}$$

Here, the momenta are counted *in the direction of the arrows.* Note that reversing the arrows leaves the vertex unchanged owing to the antisymmetry of the h symbol. There is therefore no ambiguity. We may write



(10.64)

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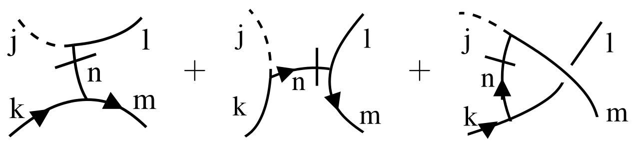
[15]Since in this section we only consider gluons anyway, I shall here denote them by smooth rather than wriggly lines ; this makes them somewhat easier to read, and certainly easier to draw.

This makes it easier to single out particular terms in Eq.(10.62). For instance, the terms proportional to $(\epsilon_2 \cdot \epsilon_4)$ are given by the diagrams



that, apart from an overall factor $-i\hbar g^2(\epsilon_2 \cdot \epsilon_4)$, evaluate to

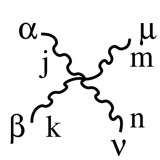
$$\begin{aligned}
&(\epsilon_3 \cdot q_2 + q_4)\, h^{mkn}\, h^{nlj} + (\epsilon_3 \cdot q_1 + q_2 + q_4)\, h^{mnl}\, h^{nkj} + (\epsilon_3 \cdot 2q_2 - q_3)\, h^{nkl}\, h^{nmj} \\
&\quad = \quad (\epsilon_3 \cdot q_2 + q_4)\, [mklj] - (\epsilon_3 \cdot q_1 + q_2 + q_4)\, [mlkj] + (\epsilon_3 \cdot 2q_2 - q_3)\, [klmj] \\
&\quad = \quad (\epsilon_3 \cdot q_2 + q_4)\, \Big(-[mljk] - [mjkl]\Big) \\
&\qquad\quad +(\epsilon_3 \cdot q_1 + q_2 + q_4)\, [mljk] - (\epsilon_3 \cdot 2q_2 - q_3)\, [mjlk] \\
&\quad = \quad [mljk]\, (\epsilon_3 \cdot q_1 + q_2 + q_4 - q_2 - q_4) + [mjlk]\, (\epsilon_3 \cdot q_2 + q_4 - 2q_2 + q_3) \\
&\quad = \quad (q_1 \cdot \epsilon_3)\, \Big([mjlk] + [mljk]\Big) \quad . \qquad (10.65)
\end{aligned}$$

I have displayed this computation in detail in order to emphasize that we only use momentum conservation, and *not* for instance the property $(\epsilon_3 \cdot q_3) = 0$. In addition, in the third line the Jacobi identity comes into play. The other terms are of course treated in the same way, so that we find

$$\begin{aligned}
\mathcal{M}\big\rfloor_{\epsilon_1 \to q_1} \quad = \quad -i\hbar g^2 \Big\{ & (q_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4)\, \Big([jlmk] + [jmlk]\Big) \\
& + (q_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4)\, \Big([jmkl] + [jkml]\Big) \\
& + (q_1 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3)\, \Big([jklm] + [jlkm]\Big) \Big\} \quad . \qquad (10.66)
\end{aligned}$$

10.4.3 The *gggg* vertex and its handlebar

The handlebar requirement can now be satisfied by introducing a four-gluon interaction vertex as follows:

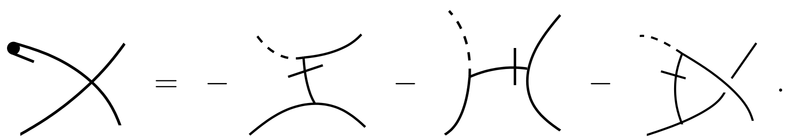


$$\leftrightarrow \quad i\frac{g^2}{\hbar}\, X(\alpha, j; \beta, k; \mu, m; \nu, n) \qquad (10.67)$$

with

$$X(\alpha, j; \beta, k; \mu, m; \nu, n) = g^{\alpha\beta}\, g^{\mu\nu} \Big([jmnk] + [jnmk]\Big) + g^{\alpha\mu}\, g^{\beta\nu} \Big([jknm] + [jnkm]\Big) + g^{\alpha\nu}\, g^{\mu\beta} \Big([jkmn] + [jmkn]\Big) \quad . \tag{10.68}$$

In addition, we have found the handlebar rule



(10.69)

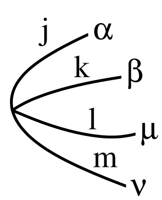
10.5 Current conservation in QCD

10.5.1 More vertices ?

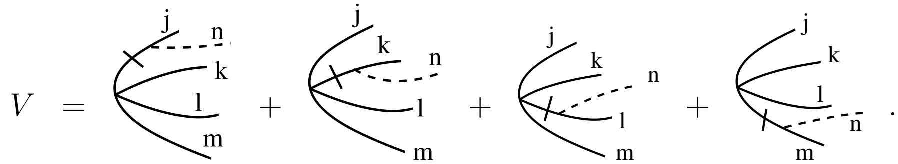
After having introduced the three-gluon vertex, we have seen that we also need a four-gluon vertex in order to save current conservation in $qq \to qq$. What if we now consider$gg \to ggg$? Wil we also need a *five*-gluon vertex ? And then what about $gg \to gggg$? Fortunately, as we shall see, no such bad luck : with the three- and four-gluon vertices all amplitudes will vanish under the handlebar. We shall prove this in the same manner as for QED, only obviously the proof will be somewhat more involved. Before we turn to our reliable working horse, the SDe, we first need one more result.

10.5.2 Antkaz

Consider the four-gluon vertex in the following form :



Let us now attach a fifth gluon inall possible ways, and slash the intervening propagators. This leads to the following expression :



(10.70)

In V, no gluon momenta enter. Let us now concentrate on the term with $g^{\alpha\beta}\,g^{\mu\nu}$, say. The colour part of this term reads

$$
\begin{aligned}
V_{\alpha\beta;\mu\nu} &= [plmk]\,h^{pjn} + [pmlk]\,h^{pjn} + [jlmp]\,h^{pkn} + [jmlp]\,h^{pkn}\\
&+ [jpmk]\,h^{pln} + [jmpk]\,h^{pln} + [jlpk]\,h^{pmn} + [jplk]\,h^{pmn}\\
&= -[mkljn] - [lkmjn] + [jlmkn] + [jmlkn]\\
&\quad +[mkjln] - [jmkln] - [jlkmn] + [lkjmn] \qquad (10.71)
\end{aligned}
$$

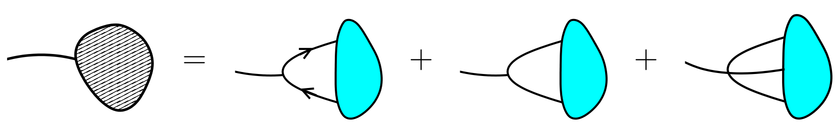
We now call upon the various symmetry properties and Jacobi identities :

$$
\begin{aligned}
V_{\alpha\beta;\mu\nu} &= +[mklnj] + [lkmnj] + [jlmkn] + [jmlkn]\\
&\quad +[mkjln] + [lkjmn] + [jlknm] + [jmknl]\\
&= -[mknjl] - [jlnmk] - [lknjm] - [jmnlk] = 0 \;\; . \qquad (10.72)
\end{aligned}
$$

So we find that V vanishes completely, and we shall employ that fact in what follows next.

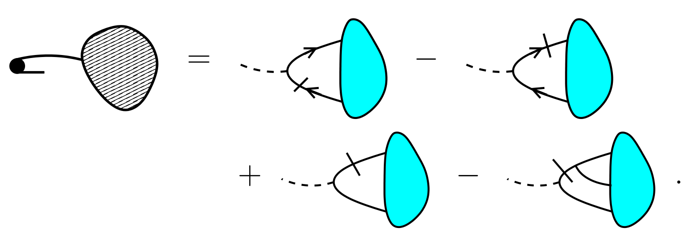
10.5.3 Proof of current conservation

Let us consider an amplitude with a gluon line sticking out, and see what the SDe has to tell us about it



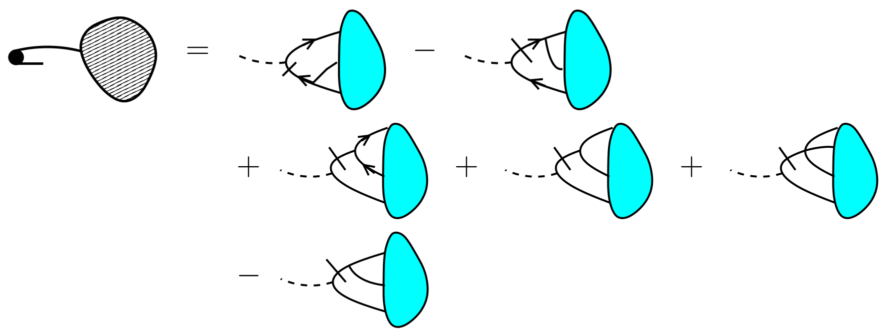
(10.73)

Let us now apply the handlebar rules we have developed in this chapter :



(10.74)

It is important to realize that the internal lines in the SDe take on all possible identities, and therefore the third diagram stands for both graphs in Eq.(10.38), and the fourth stands for all three diagrams in Eq.(10.69)[16]. We now iterate the SDe for the slashed propagators in the first three diagrams, and this gives us



$$= \; 0 \;, \tag{10.75}$$

since the first diagram on the second line cancels against those of the first line, and the last diagram on the second line vanishes all by itself, as we have seen in the previous section. This establishes the proof of current conservation in QCD.

It is important that we realize that the above proof relies on our use of the axial gauge for the gluon propagator. Indeed, that choice is what makes the identity (10.38) possible.

[16]Are you worried about possible double-counting here ? Don't worry, be happy : the symmetry factors are there for precisely *that* reason.

Chapter 11

Electroweak theory

In this chapter we shall introduce the electroweak interactions of the Minimal Standard Model. We will *not* use the gauge principle to do this, but rather build up the theory by introducing new particles and/or vertices as the need arises. This is more or less the exact opposite of the usual exposition, but is (hopefully) rather closer to physics than to mathematics.

11.1 Muon decay

11.1.1 The Fermi coupling constant

Let us return to the Fermi model of muon decay as discussed in chapter 7. There, the (phenomenological) amplitude for this decay was proposed to be of the form of Eq.(7.154). The resulting width was

$$\Gamma_\mu \equiv \Gamma(\mu^- \to e^-\ \overline{\nu}_e \nu_\mu) = \frac{{G_F}^2\ \hbar^2 {m_\mu}^5}{192\pi^3} \quad . \tag{11.1}$$

The measured values of the mechanical mass M_μ and the lifetime τ_μ of the muon are

$$M_\mu \ \approx\ 1.88353\ 10^{-28}\ \mathrm{kg} \quad , \quad \tau_\mu \ \approx\ 2.19703\ 10^{-6}\ \mathrm{sec} \ ; \tag{11.2}$$

the muon mass may be more familiar under its appellation of $M_\mu c^2 \approx 0.106$ GeV. From these we can construct the more useful quantities

$$m_\mu = \frac{M_\mu c}{\hbar} \ \approx\ 5.35446\ 10^{14}\ \mathrm{m}^{-1} \quad , \quad \Gamma_\mu = \frac{1}{c\tau_\mu} \ \approx\ 1.51825\ 10^{-3}\ \mathrm{m}^{-1} \quad . \tag{11.3}$$

From Eq.(11.1) we then find

$$G_F\,\hbar \;\approx\; 4.53167\;10^{-37}\;\mathrm{m}^2\quad , \tag{11.4}$$

or

$$\frac{G_F}{\hbar c^2} \;\approx\; 1.16383\;10^{-5}\;\mathrm{GeV}^{-2}\quad . \tag{11.5}$$

We can therefore derive the 'energy scale' of the interaction responsible for muon decay[1] :

$$\Lambda_{\mathrm{W}} = \sqrt{\frac{\hbar c^2}{G_F}} \approx 292.5\;\mathrm{GeV}\quad . \tag{11.6}$$

11.1.2 Failure of the Fermi model in $\mu^-\;\overline{\nu}_\mu \to e^-\;\overline{\nu}_e$

If the phenomenologically motivated Fermi interaction is to have any claim on global validity, it must also describe the process

$$\mu^-(p_1)\,\overline{\nu}_\mu(p_2) \quad\to\quad e^-(q_1)\,\overline{\nu}_e(q_2)\quad , \tag{11.7}$$

which amounts to the previous process, only with the outgoing muon neutrino moved to an incoming anti-muon neutrino. No matter that we cannot, at present, build $\mu\overline{\nu}_\mu$ colliders ; the very, very, *very* early universe *did* provide such processes, and their description must be correct. By the rules of the Fermi model, the amplitude is given by

$$\begin{aligned}
\mathcal{M} &= i\frac{G_F\,\hbar}{\sqrt{2}}\;\overline{v}(p_2)(1+\gamma^5)\gamma^\mu u(p_1)\;\;\overline{u}(q_1)(1+\gamma^5)\gamma_\mu v(q_2)\\
&= i\frac{4}{\sqrt{2}}\,G_F\,\hbar\;\;\overline{v}_-(p_2)\gamma^\mu u_-(p_1)\;\;\overline{u}_-(q_1)\gamma_\mu v_-(q_2)\\
&= i\frac{8}{\sqrt{2}}\,G_F\,\hbar\;s_-(p_2,q_1)\;s_+(q_2,p_1)
\end{aligned} \tag{11.8}$$

Here, we have neglected both the muon and the electron mass since the scattering takes place at high energy, and we have applied the Chisholm identity in order to remove the contracted Lorentz index. Disregarding overall complex phases and using momentum conservation, we then find

$$\mathcal{M} \approx i\frac{16\;G_F\,\hbar}{\sqrt{2}}\,(p_1\cdot q_2)\quad . \tag{11.9}$$

[1]What precisely constitutes the scale is of course to some extent a matter of taste. If we include a factor $\sqrt{2}$ in G_F the scale is reduced by a factor $(\sqrt{2})^{1/2}$ to 246 GeV, which is the more commonly used number.

Neutrinos[2] have only *one* helicity state, and therefore the averaged matrix element square is given, in the centre-of-mass system, by

$$\left\langle|\mathcal{M}|^2\right\rangle = 64\; {G_F}^2\; \hbar^2\; (p_1 \cdot q_2)^2 = 4\; {G_F}^2\; \hbar^2\; s^2\; (1 + \cos\theta)^2 \;\; , \tag{11.10}$$

where θ is the angle between the muon and electron momenta. By taking also the angular average we obtain

$$\left\langle\left\langle|\mathcal{M}|^2\right\rangle\right\rangle = \frac{16}{3}\; {G_F}^2\; \hbar^2\; s^2 \;\; . \tag{11.11}$$

The total cross section is therefore given by

$$\sigma(\mu^-\; \overline{\nu}_\mu \to e^-\; \overline{\nu}_e) = \frac{{G_F}^2\; \hbar^2}{3\pi}\; s \tag{11.12}$$

As we have seen before, only the factor $1/3$ cannot be established straightaway in this expression, but has to be computed from the Feynman diagrams.

The scattering cross section rises linearly with s, and will therefore violate the unitarity bound at sufficiently high energy. Since the the muon and its antineutrino couple with a Dirac matrix, we may conclude that they must be in a $J = 1$ state. The unitarity bound on this cross section is therefore

$$\sigma(\mu^-\; \overline{\nu}_\mu \to e^-\; \overline{\nu}_e) \leq \frac{1}{2}\; \frac{16\pi}{s}\; (2J + 1) = \frac{24\pi}{s} \;\; , \tag{11.13}$$

which leads to a fundamental failure of the Fermi model (at least, at the tree level) at a scattering energy of $\sqrt{s} \approx 1.5$ TeV.

11.2 The W particle

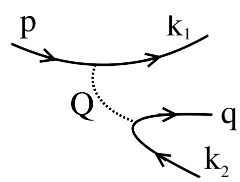
11.2.1 The IVB strategy

We are faced with the task of modifying the Fermi model in such a way that its success in the low-energy description of muon decay is preserved, while at high energies unitarity remains inviolate. One possible way out might be to simply make G_F depend on the energy scale of the process so that it

[2] We shall assume, in this section, that neutrinos are strictly massless.

decreases at high energies, making the $\mu^- \ \overline{\nu}_\mu \to e^- \ \overline{\nu}_e$ cross section well-behaved. We see that this would necessitate a modification that leads to a $1/s$ behaviour at high values of s. Such energy-dependent couplings, called *form factors*, are employed in for instance 'low-energy' hadronic physics ; in such cases, however, this approach is generally viewed as an admission of ignorance of, and an attempt to cope with, some underlying and simpler physics at a smaller distance scale[3].

The more elegant, and (as it turns out) the correct way to go is to make the Fermi model look more 'QED-like': instead of using a contact interaction between four fermions, we postulate the existence of a new particle, the so-called W boson. This couples to fermion-antifermion pairs in a way reminiscent of the photon. The four-fermion interaction then resolves into two $f\bar{f}W$ interactions, with the W boson mediating between the two vertices ; the corresponding Feynman diagram for the process $\mu^-(p) \ \ to \ \ e^-(q) \ \nu_\mu(k_1) \ \overline{\nu}_e(k_2)$ is therefore given by



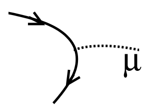
At the time this model was first seriously discussed, it went under the name of *Intermediate Vector-Boson* (IVB) hypothesis. We take the W to couple to the fermion pairs $e\nu_e$ and $\mu\nu_\mu$, so that (as we shall check!) the W must be electrically charged, and assume that the coupling is in both cases of equal strength[4] (for now). We therefore postulate the following Feynman rules :

---

[3]This is particularly evident in some modifications of QED where the 'dimensionless' coupling Q is replaced by an s-dependent form $Q(s/\Lambda^2)$ which equals Q at low s but deviates from it at high s. With the commissioning of each higher-energy accelerator, such deviations are always looked for (and have, so far, not been found). Note that in this case the quantity Λ for which search limits are obtained establishes an energy scale (or $1/\Lambda$ establishes a length scale) at which 'new physics' sets in. In the present case, G_F, being dimensionful, sets such a scale by itself.

[4]At this point, these are of course just assumptions. Since 1983, when the W boson was first freely produced, they have been tested with great accuracy. The alternative scenario of the 'charge-retention' form in which an electrically neutral W couples to $e\mu$ and $\nu_e\nu_\mu$ is for instance completely ruled out by the fact that the decay $W \to e^+\mu^-$ is never seen. The equality of the couplings is verified by the fact that the branching ratios for $W \to e\overline{\nu}_e$ and $W \to \mu\overline{\nu}_\mu$ are the same up to computable mass effects.

$$\leftrightarrow \quad i\hbar\, \frac{-g^{\mu\nu} + k^\mu k^\nu / m_{\rm W}{}^2}{k^2 - m_{\rm W}{}^2 + i\epsilon} \qquad \text{internal } W \text{ lines}$$



$$\leftrightarrow \quad \frac{i}{\hbar} g_{\rm W} \, \Big((1+\gamma^5) \Big) \gamma^\mu \qquad f f' W \text{ vertices}$$

EW Feynman rules, part 11.1 (11.14)

The W propagator is the standard one for a vector particle. Note that the occurrence of the $(1+\gamma^5)$ in the vertex is suggested by the form of the Fermi interaction ; and, that the two fermions meeting in the vertex must be of different type. The values of $m_{\rm W}$ and $g_{\rm W}$ are to be determined. Another attractive property of this model is that here the coupling constant, $g_{\rm W}$, has the same dimensionality as the QED one, and does not formally contain a length scale.

With the above Feynman rules, the muon decay amplitude can now be written as

$$\begin{aligned} \mathcal{M} \;\; &= \;\; \frac{i\hbar g_{\rm W}{}^2}{Q^2 - m_{\rm W}{}^2} \Bigg[\overline{u}(k_1)(1+\gamma^5)\gamma^\alpha u(p) \;\; \overline{u}(q)(1+\gamma^5)\gamma_\alpha v(k_2) \\ & \quad - \frac{1}{m_{\rm W}{}^2}\, \overline{u}(k_1)(1+\gamma^5)\not{Q} u(p) \;\; \overline{u}(q)(1+\gamma^5)\not{Q} v(k_2) \Bigg] \quad , \end{aligned} \qquad (11.15)$$

where the momentum of the internal W is given by

$$Q^\mu = (p - k_1)^\mu = (q + k_2)^\mu \quad . \qquad (11.16)$$

The last term in Eq.(11.15) appears to deviate significantly from the spinorial structure of the first term, which coincides with the Fermi model. Hoewever, notice that

$$\begin{aligned} \overline{u}(k_1)(1+\gamma^5)\not{Q} u(p) \;\; &= \;\; \overline{u}(k_1)(1+\gamma^5)(\not{p} - \not{k}_1) u(p) \\ &= \;\; \overline{u}(k_1) \Big(-\not{k}_1 (1-\gamma^5) + (1+\gamma^5)\not{p} \Big)\, u(p) \\ &= \;\; m_\mu \, \overline{u}(k_1)(1+\gamma^5) u(p) \end{aligned} \qquad (11.17)$$

upon application of the Dirac equation to the external spinors ; and since, in the same way,

$$\overline{u}(q)(1+\gamma^5)\not{Q} v(k_2) = m_e \, \overline{u}(q)(1-\gamma^5) v(k_2) \quad , \qquad (11.18)$$

the second term in Eq.(11.15) is actually suppressed by a factor $(m_e m_\mu)/m_{\rm W}{}^2$, which is small if $m_{\rm W}$ is sufficiently large[5]. Neglecting this term, we see that the Fermi-model amplitude is recovered with the single replacement of the coupling constant $G_F/\sqrt{2}$ by $g_{\rm W}{}^2/(Q^2 - m_{\rm W}{}^2)$. Now, the *maximum* value that Q^2 can take in this process is $m_\mu{}^2$, which is attained in the improbable case that the muon neutrino emerges with zero momentum from the decay. If, therefore, we assume that $m_{\rm W}$ is large compared to m_μ, we see that the successes of the Fermi model in describing muon decay will be completely reproduced provided[6]

$$\frac{g_{\rm W}{}^2}{m_{\rm W}{}^2} = \frac{G_F}{\sqrt{2}} \quad , \tag{11.19}$$

which we may also write in purely dimensionless terms as

$$\left(\frac{g_{\rm W}}{c\sqrt{\hbar}}\right) = \frac{1}{2^{1/4}}\,\left(\frac{m_{\rm W}c^2}{\Lambda_{\rm W}}\right) \quad . \tag{11.20}$$

11.2.2 The cross section for $\mu^-\overline{\nu}_\mu \to e^-\overline{\nu}_e$ revisited

We can now study the modification that the IVB hypothesis makes in the cross section for the process $\mu^-\overline{\nu}_\mu \to e^-\overline{\nu}_e$, where the Fermi model fails. In this case the total invariant mass is (assumed to be) much larger than the W mass, so that the modified prediction can immediately be seen to be

$$\sigma(\mu^-\ \overline{\nu}_\mu \to e^-\ \overline{\nu}_e) = \frac{2\hbar^2 g_{\rm W}{}^4}{3\pi}\frac{s}{(s - m_{\rm W}{}^2)^2} = \frac{\hbar^2 G_F{}^2 s}{3\pi}\left(\frac{m_{\rm W}{}^2}{s - m_{\rm W}{}^2}\right)^2 \quad , \tag{11.21}$$

and this cross section *does* decrease as $1/s$ for large s.

Of course, the unitarity limit (11.13) still has to be observed, which puts an upper limit[7] on the useful values of $m_{\rm W}$:

$$m_{\rm W}c^2 \quad \leq \quad (72\pi^2)^{1/4}\,\Lambda_{\rm W} \quad \approx \quad 1.5\ \text{TeV} \quad . \tag{11.22}$$

However, from Eq.(11.20) we see that for such large values the dimensionless coupling constant is so large that the tree-level approximation for the cross section is questionable.

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[5] In fact, for the actual values of the masses the suppression factor is about 10^{-7}.

[6] We disregard the overall sign difference between the two forms as $Q^2/m_{\rm W}{}^2 \to 0$.

[7] This value is close to the value of $\sqrt{s}$ at which unitairy breaks down in the unmodified Fermi model, see Eq.(11.13). This is not a coincidence. Whatever we do to the electroweak interactions, 1.5 TeV appears to be the energy régime where things get tricky.

One may wonder what happens at $s = m_{\rm W}{}^2$. There, the cross section would seem to diverge ! We must realize, however, that at that energy we are, in fact, producing an *on-shell* W that decays into a fermion-antifermion pair : that is to say, the W is an *unstable* particle, and has a decay width. We ought, therefore, to include the decay width into the propagator, so that in the neighbourhood of the *resonance* at $s \approx m_{\rm W}{}^2$ the cross section reads

$$\sigma(\mu^- \ \overline{\nu}_\mu \to e^- \ \overline{\nu}_e) = \frac{2\hbar^2 g_{\rm W}{}^4}{3\pi} \frac{s}{(s - m_{\rm W}{}^2)^2 + m_{\rm W}{}^2 \Gamma_W{}^2} \quad . \tag{11.23}$$

This is well below the unitarity limit. The IVB hypothesis therefore indeed **E 54**
cures the unitarity problem in this process. **E 55**

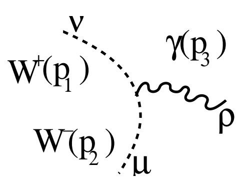
Because of these successes, we shall adopt the notion of an existing W particle of spin 1 (and hence obeying the lines laid out in chapter 8), coupling to pairs of fermions separated by one unit of charge[8].

11.2.3 The $WW\gamma$ vertex

Minimal coupling

Since the W particle couples to fermion pairs of unequal charge, it must itself also be charged[9], which means that it must couple to the photon in (at least !) a $WW\gamma$ vertex. It is our aim now to find the form of such a vertex.

Both W's and photons are characterised by the fact that, in addition to their momentum, they carry also a polarization vector, *i.e.* a Lorentz index: the $WW\gamma$ vertex must therefore carry no fewer than 3 Lorentz indices. As a first attempt, we can simply view the W particles as a kind of funny scalars, and adopt the sQED vertex dressed up with a metric tensor to take care of the W indices. That is, the Feynman rule for the vertex is taken to be



$$\leftrightarrow \quad \frac{i}{\hbar} Q_{\rm W} \ (p_1 - p_2)^\rho \ \gamma^{\mu\nu} \tag{11.24}$$

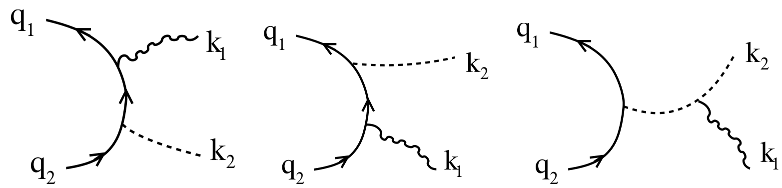
where the coupling constant (the W charge) is to be determined, and the particles are considered to be *outgoing* from the vertex. To this end, let us

[8] Note that this automatically rules out couplings between a W, a lepton, and a quark.
[9] At pain of charge *non*conservation, *i.e.* at pain of pain.

examine the process

$$\overline{D}(q_1)\,U(q_2) \quad\to\quad \gamma(k_1,\epsilon)\,W^+(k_2,\epsilon_+) \ \ .$$

ϵ and ϵ_W denote the polarization vectors of the photon and the W, respectively, and we have indicated the particle momenta. Here, and in the following, we shall denote by U and D two fermions of which the U has an electric charge one unit higher than the D: for instance, $U=\nu_e$ and $D=e$, or $U=u$ and $D=d$. Their respective charges are $Q_{\rm U}$ and $Q_{\rm D}$. At the tree level, we then have three Feynman diagrams :



The three diagrams correspond to the three partial matrix elements

$$\begin{aligned}
\mathcal{M}_1 &= -i\hbar g_{\rm W}Q_{\rm D}\,\overline{v}(q_1)\not{\epsilon}\frac{\not{k}_1-\not{q}_1+m_{\rm D}}{(k_1-q_1)^2-{m_{\rm D}}^2}\left((1+\gamma^5)\right)\not{\epsilon}_W u(q_2)\ \ ,\\
\mathcal{M}_2 &= -i\hbar g_{\rm W}Q_{\rm U}\,\overline{v}(q_1)\left((1+\gamma^5)\right)\not{\epsilon}_W\frac{\not{q}_2-\not{k}_1+m_{\rm U}}{(q_2-k_1)^2-{m_{\rm U}}^2}\not{\epsilon}u(q_2)\ \ ,\\
\mathcal{M}_3 &= +i\hbar g_{\rm W}Q_{\rm W}\,\overline{v}(q_1)\left((1+\gamma^5)\right)\gamma_\alpha u(q_2)\\
&\qquad \frac{g^{\alpha\beta}-P^\alpha P^\beta/{m_{\rm W}}^2}{s-{m_{\rm W}}^2}\epsilon_{W\beta}\left((2k_2+k_1)\cdot\epsilon\right)\ \ ,
\end{aligned}\tag{11.25}$$

where $s=P^2$, $P=q_1+q_2=k_1+k_2$.

Since this process involves a produced photon, the handlebar identity must hold : if we replace ϵ^μ by ${k_1}^\mu$ the amplitude must vanish. We shall investigate this is some detail. In the first place, we perform some simple Dirac algebra to note that

$$\begin{aligned}
\overline{v}(q_1)\not{\epsilon}(\not{k}_1-\not{q}_1+m_{\rm D})\Big|_{\epsilon\to k_1} &= \overline{v}(q_1)\not{k}_1(\not{k}_1-\not{q}_1+m_{\rm D})\\
&= \overline{v}(q_1)\left({k_1}^2-2(q_1\cdot k_1)+(\not{q}_1+m_{\rm D})\not{k}_1\right)\\
&= \left((k_1-q_1)^2-{m_{\rm D}}^2\right)\,\overline{v}(q_1)\ \ ,
\end{aligned}\tag{11.26}$$

where in the second line we have used anticommutation between $\not{k}_1$ and $\not{q}_1$, and in the third line the Dirac equation for $\overline{v}(q_1)$. This kind of operation will occuur very frequently in what follows. We see that

$$\mathcal{M}_1\rfloor_{\epsilon\to k_1} = -i\hbar g_{\rm W} Q_{\rm D}\; \overline{v}(q_1)(1+\gamma^5)\not\epsilon_W u(q_2) \;\;, \tag{11.27}$$

and similarly (see excercise **??**)

$$\mathcal{M}_2\rfloor_{\epsilon\to k_1} = +i\hbar g_{\rm W} Q_{\rm U}\; \overline{v}(q_1)(1+\gamma^5)\not\epsilon_W u(q_2) \;\;. \tag{11.28}$$

For the third diagram we find

$$\begin{aligned}\mathcal{M}_3\rfloor_{\epsilon\to k_1} \;\; = \;\; & +i\hbar g_{\rm W} Q_{\rm W}\; \overline{v}(q_1)\left((1+\gamma^5)\right)\not\epsilon_W u(q_2) \\ & -i\hbar g_{\rm W} Q_{\rm W}\; (k_1\cdot\epsilon_W) \\ & \times\, \overline{v}(q_1)\left(m_{\rm U}\left((1+\gamma^5)\right) - m_{\rm D}\left(1-\gamma^5\right)\right)u(q_2) \;\;. \end{aligned} \tag{11.29}$$

If we were allowed to consider only the first of the two terms of the result (11.29), we could obtain the desired cancellation :

$$\left.\sum_{j=1}^{3}\mathcal{M}_j\right\rfloor_{\epsilon\to k_1} = 0 \quad\Rightarrow\quad Q_{\rm W} = Q_{\rm D} - Q_{\rm U} \quad : \tag{11.30}$$

but the second term in Eq.(11.29) spoils this idea by having a quite different algebraic structure ; no tuning of coupling constants is going to ensure that a $WW\gamma$ vertex of the form (11.24) can do the job.

Yang-Mills coupling

Treating the $WW\gamma$ vertex as a prettified sQED vertex does not work. It means that the photon-W interactions cannot be obtained by the minimal-substitution rule. This should not come as a surprize since the vertex (11.24) is only designed for graceful behaviour towards longitudinal photons, not towards longitudinal W's. We therefore propose to replace Eq.(11.24) by a vertex of the form

$$i\frac{Q_{\rm W}}{\hbar}\left((a_1p_1+a_2p_2)^\rho g^{\mu\nu} + (a_3p_2+a_4p_3)^\mu g^{\nu\rho} + (a_5p_3+a_6p_1)^\nu g^{\rho\mu}\right) \;\;. \tag{11.31}$$

Note that because of momentum conservation each of the three terms need contain only two of the momenta; the constants $a_{1,\ldots,6}$ are to be determined. This we shall do by considering several situations.

First, we consider the process of decay of a photon in a W^+W^- pair :

$$\gamma^*(q) \quad \to \quad W^+(k_+,\epsilon_+)\,W^-(k_-,\epsilon_-) \quad .$$

Kinematically this is only possible if the photon is quite off-shell, and therefore we do not give it a polarization vector but leave its Lorentz index μ free. The matrix element is given by

$$\begin{aligned}
\mathcal{M} &= i\hbar^{1/2}Q_{\mathrm{W}}\,\mathcal{A}^\mu \quad , \\
\mathcal{A}^\mu &= (a_1k_+ + a_2k_-)^\mu(\epsilon_+\cdot\epsilon_-) \\
&\quad +((a_3k_- - a_4q)\cdot\epsilon_+){\epsilon_-}^\mu + ((-a_5q + a_6k_+)\cdot\epsilon_-){\epsilon_+}^\mu \\
&= (a_1k_+ + a_2k_-)^\mu(\epsilon_+\cdot\epsilon_-) \\
&\quad +(a_3-a_4)(q\cdot\epsilon_+){\epsilon_-}^\mu + +(a_6-a_5)(q\cdot\epsilon_-){\epsilon_+}^\mu \quad ,
\end{aligned} \tag{11.32}$$

where in the last line we have used $q = k_+ + k_-$ and $(k_\pm\cdot\epsilon_\pm) = 0$. Since even for off-shell photons the current must be strictly conserved we require that

$$\mathcal{A}^\mu q_\mu = \frac{1}{2}q^2(a_1+a_2)(\epsilon_+\cdot\epsilon_-) + (a_3-a_4-a_5+a_6)(q\cdot\epsilon_+)(q\cdot\epsilon_-) = 0, \tag{11.33}$$

which leads to the following relations between the six constants :

$$a_1 + a_2 = 0 \quad , \quad a_3 - a_4 = a_5 - a_6 \quad . \tag{11.34}$$

In the second place, we return to the process $\overline{D}U \to \gamma W^+$ discussed in the previous section. The third Feynman diagram now reads differently :

$$\begin{aligned}
\mathcal{M}_3 &= +i\hbar g_{\mathrm{W}}Q_{\mathrm{W}}\,\overline{v}(q_1)\left((1+\gamma^5)\right)\gamma_\alpha u(q_2)\frac{1}{2(k_1\cdot k_2)}Z^\alpha \quad , \\
Z^\alpha &= \left(\delta^\alpha{}_\beta - P^\alpha P_\beta/{m_{\mathrm{W}}}^2\right) \\
&\quad \Big\{((a_1k_2 - a_2P)\cdot\epsilon){\epsilon_+}^\beta + ((-a_3P + a_4k_1)\cdot\epsilon_+)\epsilon^\beta \\
&\qquad +(a_5k_1 + a_6k_2)^\beta(\epsilon_+\cdot\epsilon)\Big\} \\
&= \left(\delta^\alpha{}_\beta - P^\alpha P_\beta/{m_{\mathrm{W}}}^2\right) \\
&\quad \Big\{(a_1-a_2)(k_2\cdot\epsilon){\epsilon_+}^\beta + (a_4-a_3)(k_1\cdot\epsilon_+)\epsilon^\beta \\
&\qquad +(a_5k_1 + a_6k_2)^\beta(\epsilon_+\cdot\epsilon)\Big\} \quad .
\end{aligned} \tag{11.35}$$

The replacement $\epsilon \to k_1$ now leads, after some simple algebra (and use of momentum conservation !) to the form

$$\begin{aligned} Z^\alpha\rfloor_{\epsilon\to k_1} &= (a_1 - a_2)(k_1 \cdot k_2){\epsilon_+}^\alpha + \mathcal{T}^\alpha \quad , \\ \mathcal{T}^\alpha &= (-a_3 + a_4 + a_5 - a_6){k_1}^\alpha \\ &\quad - \frac{(k_1 \cdot k_2)}{{m_\mathrm{W}}^2}(a_1 - a_2 - a_3 + a_4 + a_5 + a_6)P^\alpha \quad . \end{aligned} \tag{11.36}$$

Now a complete cancellation of all diagrams in this case is only possible if only the first term in $Z^\alpha\rfloor$ survives. Using the assignment[10] $Q_\mathrm{W} = Q_\mathrm{D} - Q_\mathrm{U}$, we then come to the following additional relations between the a's :

$$a_1 - a_2 = 2 \quad , \quad a_1 - a_2 = a_3 - a_4 - a_5 - a_6 = 0 \quad . \tag{11.37}$$

A third result is obtained by considering the process $\overline{U}D \to \gamma W^-$. Because of the symmetry between this amplitude and the previous one, we can establish (see excercise **??**) that also

$$a_1 - a_2 = a_5 - a_6 + a_3 + a_4 \quad . \tag{11.38}$$

For the last necessary piece of information we must turn to the handlebar operation for the produced W rather than the photon. We can rewrite the three Feynman diagrams as

$$\begin{aligned} \mathcal{M}_1 &= -i\frac{\hbar g_\mathrm{W} Q_\mathrm{D}}{(q_2 - k_2)^2 - {m_\mathrm{D}}^2}\,\overline{v}(q_1)\not\epsilon\,(\not q_2 - \not k_2 + m_\mathrm{D})\left((1+\gamma^5)\right)\not\epsilon_+ u(q_2) \quad , \\ \mathcal{M}_2 &= -i\frac{\hbar g_\mathrm{W} Q_\mathrm{U}}{(k_2 - q_1)^2 - {m_\mathrm{U}}^2}\,\overline{v}(q_1)\left((1+\gamma^5)\right)\not\epsilon_+\,(\not k_2 - \not q_1 + m_\mathrm{U})\not\epsilon u(q_2) \quad , \\ \mathcal{M}_3 &= +i\frac{\hbar g_\mathrm{W} Q_\mathrm{W}}{s - {m_\mathrm{W}}^2}\,\overline{v}(q_1)\left((1+\gamma^5)\right)\gamma_\alpha u(q_2)\; Z^\alpha, \end{aligned} \tag{11.39}$$

with Z^α as in Eq.(11.35). The handlebar operation on ϵ_+ now gives the slightly more complicated result

$$\begin{aligned} &\overline{v}(q_1)\not\epsilon\,(\not q_2 - \not k_2 + m_\mathrm{D})\left((1+\gamma^5)\right)\not\epsilon_+ u(q_2)\Big\rfloor_{\epsilon_+\to k_2} = \\ &= \; - \left((q_2-k_2)^2 - {m_\mathrm{D}}^2\right)\overline{v}(q_1)\left((1+\gamma^5)\right)\not\epsilon u(q_2) \\ &\qquad - \overline{v}(q_1)\left(m_\mathrm{U}\left((1+\gamma^5)\right) - m_\mathrm{D}\left(1-\gamma^5\right)\right)\not\epsilon\not k_2 u(q_2) \\ &\qquad + \left({m_\mathrm{U}}^2 - {m_\mathrm{D}}^2\right)\overline{v}(q_1)\left((1+\gamma^5)\right)\not\epsilon u(q_2) \quad . \end{aligned} \tag{11.40}$$

[10]Any common factor in the a's is always absorbed in the value of Q_W so this is no loss of generality.

Of these three lines, the second is suppressed with respect to the first one by a factor (mass/energy), and the third line even by (mass/energy)^2. In the high-energy limit, therefore, the second and third line will not contribute to any unwanted high-energy behaviour of the amplitude : we shall call such terms *safe* terms[11]. We can therefore write

$$\mathcal{M}_1\rfloor_{\epsilon_+\to k_2} = +i\hbar g_{\mathrm{W}} Q_{\mathrm{D}}\ \overline{v}(q_1)\left((1+\gamma^5)\right)\not{\epsilon}u(q_2) + \cdots \quad , \tag{11.41}$$

where the ellipsis denotes safe terms. For the second diagram, we find in a similar way (see excercise **??**) :

$$\mathcal{M}_2\rfloor_{\epsilon_+\to k_2} = -i\hbar g_{\mathrm{W}} Q_{\mathrm{U}}\ \overline{v}(q_1)\left((1+\gamma^5)\right)\not{\epsilon}u(q_2) + \cdots \quad , \tag{11.42}$$

For the third graph we find, after some algebra,

$$\begin{aligned} Z^\alpha\rfloor_{\epsilon_+\to k_2} \quad = \quad & (a_4 - a_3)(k_1\cdot k_2)\epsilon^\alpha - a_3 {m_{\mathrm{W}}}^2\epsilon^\alpha \\ & - \frac{(k_2\cdot\epsilon)(k_1\cdot k_2)}{{m_{\mathrm{W}}}^2}(a_1 - a_2 - a_3 + a_4 + a_5 + a_6)P^\alpha \\ & + (k_2\cdot\epsilon)(-a_1 + a_2 + a_5 - a_6){k_1}^\alpha \quad . \end{aligned} \tag{11.43}$$

Requiring $\mathcal{M}_3$ to cancel against $\mathcal{M}_1 + \mathcal{M}_2$ up to safe terms therefore leads to yet more relations between the a's :

$$a_3 - a_4 = 2 \quad , \quad a_1 - a_2 = a_5 - a_6 \quad . \tag{11.44}$$

Combining the requirements (11.34), (11.37), (11.38) and (11.44) we find the unique solution

$$a_1 = a_3 = a_5 = 1 \quad , \quad a_2 = a_4 = a_6 = -1 \quad . \tag{11.45}$$

This leads us to introduce the *Yang-Mills* form of the three-boson vertex :

$$\begin{aligned} & Y(p_1,\mu;p_2,\nu;p_3,\rho) \equiv \\ & \quad (p_1 - p_2)^\rho g^{\mu\nu} + (p_2 - p_3)^\mu g^{\nu\rho} + (p_3 - p_1)^\nu g^{\rho\mu} \quad . \end{aligned} \tag{11.46}$$

Note that this is antisymmetric in the interchange of any two of its pairs of arguments. It is therefore invariant under cyclic permutations of the argument pairs[12].

We have thus established the $WW\gamma$ vertex to be

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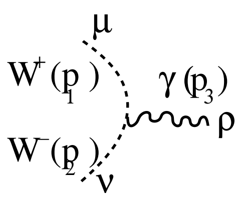
[11] Which is not to say that they are negligible ! The point here is that they do not contribute to any condition on the coupling constants.

[12] I have adopted the notation 'Y' for this vertex since it reminds us both of the name Yang(-Mills), and of the fact that in such a vertex *three* bosons meet.



$$\leftrightarrow \quad \frac{i}{\hbar} Q_\mathrm{W}\, Y(p_1,\mu;p_2,\nu;p_3,\rho)\ WW\gamma \text{ vertex}$$

All particles and momenta counted *outgoing*

$$\boxed{\text{EW Feynman rules, part 11.2}} \tag{11.47}$$

A very important identity for the Yang-Mills vertex is the following :

$$Y(p_1,p_1;p_2,\nu;p_3,\rho) = \left({p_2}^\nu {p_2}^\rho - {p_2}^2 g^{\nu\rho}\right) - \left({p_3}^\nu {p_3}^\rho - {p_3}^2 g^{\nu\rho}\right) \quad , \tag{11.48}$$

and its cyclic permutations. This identity, which follows directly from momentum conservation, is very important whenever we decide to put a handlebar on any of the three boson lines.

11.3 The Z particle

11.3.1 W pair production

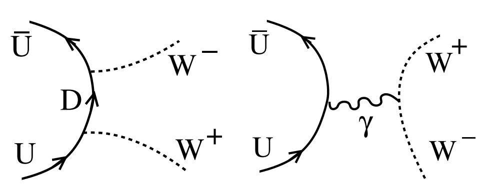
Unitarization from extra fermions

In the previous section we have investigated how the possible coupling between W's and photons are restricted by the requirements of the handlebar. We shall now pursue the same strategy for different processes. Since we shall be interested in the *high-energy* behaviour of amplitudes we shall allow ourselves to neglect particle masses wherever possible.

Let us consider the process

$$\overline{U}(p_1)\,U(p_2) \quad \to \quad W^+(q_+,\epsilon_+)\,W^-(q_-,\epsilon_-)$$

With the vertices available so far, we have the following two Feynman diagrams



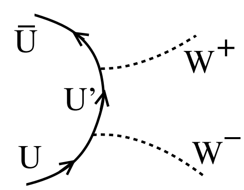
which contribute to the amplitude as follows :

$$\begin{aligned}
\mathcal{M}_1 &= -2i\frac{\hbar {g_{\mathrm{W}}}^2}{(p_2-q_+)^2}\,\overline{v}(p_1)\left((1+\gamma^5)\right)\not\epsilon_-\,(\not p_2-\not q_+)\not\epsilon_+ u(p_2)\ \ , \\
\mathcal{M}_2 &= i\frac{\hbar Q_{\mathrm{U}} Q_{\mathrm{W}}}{(q_++q_-)^2}\,\overline{v}(p_1)\gamma_\mu u(p_2)\ Y(q_+,\epsilon_+;q_-,\epsilon_-,-q_+-q_-,\mu)
\end{aligned} \tag{11.49}$$

Here we have neglected the masses as announced. The high-energy behaviour can be investigated by putting a handlebar on the W^+, say ; we then obtain

$$\begin{aligned}
\mathcal{M}_1\rfloor_{\epsilon_+\to q_+} &= 2i\hbar {g_{\mathrm{W}}}^2\,\overline{v}(p_1)(1+\gamma^5)\not\epsilon_- u(p_2)\ \ , \\
\mathcal{M}_2\rfloor_{\epsilon_+\to q_+} &= i\hbar Q_{\mathrm{U}} Q_{\mathrm{W}}\,\overline{v}(p_1)\not\epsilon_- u(p_2)\ \ ,
\end{aligned} \tag{11.50}$$

and we see that these two diagrams cannot possibly cancel one another. We must therefore introduce an additional ingredient in the model. A possible approach is the following. In the analogous process $\overline{U}U \to \gamma\gamma$ the handlebar requirement is satisfied because there are two diagrams, with the photons interchanged. We might do the same for the W by postulating the existence of another fermion type U', with charge one unit *higher* than Q_{U}, and the existence, in addition to the UDW vertex, of a $U'UW$ vertex with vector and axial-vector couplings. We then have a third diagram at hand :



with its own contribution

$$\begin{aligned}
\mathcal{M}_3 &= -i\frac{\hbar}{(p_1-q_+)^2}\ \overline{v}(p_1)\omega\not\epsilon_+\,(\not q_+-\not p_1)\,\omega\not\epsilon_- u(p_2)\ \ , \\
\omega &= g_1+g_2\gamma^5\ \ .
\end{aligned} \tag{11.51}$$

The mass of the U' is also neglected, and $g_{1,2}$ are to be determined. We have

$$\mathcal{M}_3\rfloor_{\epsilon_+\to q_+} = -i\hbar\,\overline{v}(p_1)\omega^2\not\epsilon_- u(p_2)\ \ , \tag{11.52}$$

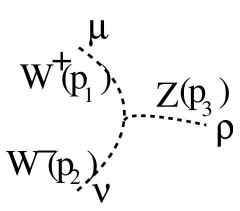
so that

$$\left.\sum_{j=1}^{3}\mathcal{M}_j\right\rfloor_{\epsilon_+\to q_+} = 0 \quad\Rightarrow\quad \left(g_1+g_2\gamma^5\right)^2 = 2{g_{\mathrm{W}}}^2(1+\gamma^5)+Q_{\mathrm{U}}Q_{\mathrm{W}}\ \ . \tag{11.53}$$

We see that it is *in principle* possible to attain good high-energy behaviour in the process $U\overline{U} \to W^+W^-$, at the cost of introducing new fermion types ; and the same is possible for $D\overline{D} \to W^+W^-$ (note, however, the problem raised in excercise **??**). But a very serious conundrum immediately arises. Having postulated the existence of the U', we of course also have to consider high-energy behaviour in the process $U'\overline{U}' \to W^+W^-$. It is easy to see that *that* can only be cured by postulating also a fermion U", of again one unit of charge higher ... An infinite tower of fermions with higher and higher charge becomes unavoidable. Not only is this extremely unattractive[13], but as the charges grow without bound perturbation theory is bound to break down since it is based on the assumption that the interactions are *not* large.

The Z boson to the rescue

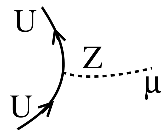
Since introducing additional Dirac particles does not seem a viable way to ensure good high-energy behaviour in $U\overline{U} \to W^+W^-$, we shall investigate the alternative of an additional *boson*. That is, we shall postulate the existence of a *neutral* spin-1 particle, coupling to W^+W^- pairs and to fermion-antifermion pairs. This particle, denoted by Z (or Z^0) is supposed to cure the high-energy behaviour in both $U\overline{U} \to W^+W^-$ and $D\overline{D} \to W^+W^-$ simultaneously[14]. For the WWZ vertex it stands to reason to employ the useful Yang-Mills form (11.46), with a coupling constant to be determined. Since the diagram with the Z must cancel against a combination of the purely vectorial photon diagram and the D-exchange diagram with its $(1+\gamma^5)$ structure, the Z must couple to the fermions with a mixture of vector and axial-vector terms. We therefore arrive at the following putative Feynman rules :



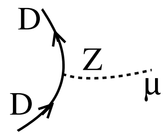
$$\to \quad \frac{i}{\hbar}\, g_{\mathrm{wwz}}\, Y(p_1,\mu;p_2,\nu;p_3,\rho) \quad ,$$

[13]Even leaving aside the fact that no higher-charge fermions have been found to date.

[14]This is the *simplest* scenario. Other possibilities could be explored, in which there is more than one type of Z, perhaps one type for the U fermions and one type for the D fermions. Experiment, however, has taught us that the simplest option appears, as usual, to be the one chosen by nature.

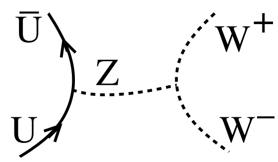


$$\rightarrow \quad \frac{i}{\hbar}\ \left(v_{\mathrm{U}} + a_{\mathrm{U}}\gamma^5\right)\gamma^\mu \quad ,$$



$$\rightarrow \quad \frac{i}{\hbar}\ \left(v_{\mathrm{D}} + a_{\mathrm{D}}\gamma^5\right)\gamma^\mu \quad ,$$

where as before in the Yang-Mills vertex every participant is counted in the *outgoing* manner. With these vertices a new Feynman diagram is available in the process $U\overline{U} \to W^+W^-$:



,

which evaluates to

$$\begin{aligned}\mathcal{M}_3 \quad &= \quad i\frac{\hbar\ g_{\mathrm{WWZ}}}{(q_+ + q_-)^2 - {m_Z}^2}\overline{v}(p_1)\left(v_{\mathrm{U}} + a_{\mathrm{U}}\gamma^5\right)\ \gamma_\mu u(p_2) \\ &\qquad Y(q_+, \epsilon_+; q_-, \epsilon_-; -q_+ - q_-, \mu) \quad . \end{aligned} \tag{11.54}$$

Note that nothing has been neglected in this expression ; the second term in the massive-boson propagator drops out when we multiply it into the Yang-Mills vertex. Since this diagram is so similar to $\mathcal{M}_2$ it is easy to perform the handlebar operation :

$$\mathcal{M}_3\rfloor_{\epsilon_+ \to q_+} \approx i\hbar\ g_{\mathrm{WWZ}}\ \overline{v}(p_1)\left(v_{\mathrm{U}} + a_{\mathrm{U}}\gamma^5\right)\not{\epsilon}_- u(p_2) \quad , \tag{11.55}$$

where we have assumed that $s = (q_+ + q_-)^2$ is also much larger than ${m_Z}^2$, and neglected safe terms. We now see that the high-energy behaviour is acceptable provided that the non-safe terms cancel under the relations

$$\begin{aligned} 0 \quad &= \quad v_{\mathrm{U}} g_{\mathrm{WWZ}} + 2{g_{\mathrm{W}}}^2 + Q_{\mathrm{U}} Q_{\mathrm{W}} \quad , \\ 0 \quad &= \quad a_{\mathrm{U}} g_{\mathrm{WWZ}} + 2{g_{\mathrm{W}}}^2 \quad . \end{aligned} \tag{11.56}$$

We can perform precisely the same procedure for the process $D\overline{D} \to W^+W^-$ and obtain (see excercise **??**)

$$\begin{aligned} 0 \quad &= \quad v_{\mathrm{D}} g_{\mathrm{WWZ}} - 2{g_{\mathrm{W}}}^2 + Q_{\mathrm{D}} Q_{\mathrm{W}} \quad , \\ 0 \quad &= \quad a_{\mathrm{D}} g_{\mathrm{WWZ}} - 2{g_{\mathrm{W}}}^2 \quad . \end{aligned} \tag{11.57}$$

A final piece of information is obtained if we realize that, the Z being a massive spin-1 particle, it must obey its own handlebar relations ; we can therefore investigate the process $U\overline{D} \to W^+Z$, which gives a single extra condition

$$0 = v_{\rm D} + a_{\rm D} - v_{\rm U} - a_{\rm U} - g_{\rm WWZ} \quad . \tag{11.58}$$

11.3.2 The weak mixing angle for couplings

We can handle (if not completely solve) the system of constraints as follows. Let us subtract Eqs.(11.56) from Eqs.(11.57). We then obtain

$$(v_{\rm D} + a_{\rm D} - v_{\rm U} - a_{\rm U})g_{\rm WWZ} + (Q_{\rm D} - Q_{\rm U})Q_{\rm W} = 8{g_{\rm W}}^2 \quad . \tag{11.59}$$

Using Eq.(11.58) and the definition of $Q_{\rm W}$, we find a relation between three couplings :

$${g_{\rm WWZ}}^2 + {Q_{\rm W}}^2 = 8{g_{\rm W}}^2 \quad . \tag{11.60}$$

There must, therefore, exist an angle θ_W such that

$$Q_{\rm W} = \sqrt{8}\; g_{\rm W}\; \sin\theta_W \quad , \quad g_{\rm WWZ} = \sqrt{8}\; g_{\rm W}\; \cos\theta_W \quad . \tag{11.61}$$

In the following we shall use the notation $s_{\rm W} = \sin\theta_W$ and $c_{\rm W} = \cos\theta_W$. This angle is called the *weak mixing angle*, and it parametrizes essentially all of the minimal model of electroweak interactions we are constructing here. In the first place, we know that the charge of the W must be equal to the charge of the electron (since neutrinos are neutral) and therefore we might prefer to write

$$g_{\rm W} = \frac{Q_{\rm W}}{\sqrt{8}\; s_{\rm W}} \tag{11.62}$$

which leads to a parametrization of the W mass itself[15] :

$$(\hbar\; c\; m_{\rm W})^2 = \frac{\pi\;\alpha}{\sqrt{2}\; 1.16\; 10^{-5}}\; \frac{1}{{s_{\rm W}}^2}\; {\rm GeV}^2 \quad , \tag{11.63}$$

or

$$\hbar\; c\; m_{\rm W} = \frac{37.3}{s_{\rm W}}\; {\rm GeV} \quad . \tag{11.64}$$

[15] To arrive at this experession we have used the definition (11.5) for G_F, and the result (9.35) of α.

As we see, the *assumption* of the existence of a single, neutral Z boson immediately implies that the W has a mass of at least 37.3 GeV. Notice that *no* prediction for the mass of the Z is obtained, however.

The other unknowns in our treatment can now be expressed in terms of θ_W. Adopting the usual convention of denoting by e the positive unit charge, we find by straightforward algebra

$$\begin{aligned} Q_{\mathrm{W}} &= -e \quad , \quad g_{\mathrm{WWZ}} = -e\frac{c_{\mathrm{W}}}{s_{\mathrm{W}}} \quad , \\ a_{\mathrm{U}} &= -a_{\mathrm{D}} \;=\; \frac{e}{4s_{\mathrm{W}}c_{\mathrm{W}}} \quad , \\ v_{\mathrm{U}} &= a_{\mathrm{U}}\left(1 - 4{s_{\mathrm{W}}}^2\frac{Q_{\mathrm{U}}}{e}\right) \quad , \\ v_{\mathrm{D}} &= a_{\mathrm{D}}\left(1 + 4{s_{\mathrm{W}}}^2\frac{Q_{\mathrm{D}}}{e}\right) \quad . \end{aligned} \tag{11.65}$$

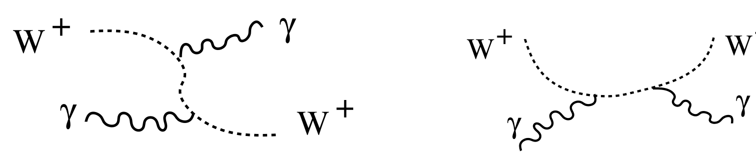
E 56 We note here that θ_W is defined at this stage as a relation between *coupling constants* ; later on we shall encounter it in another guise !

11.3.3 W, Z and γ four-point interactions

The $2 \to 2$ processes involving either four fermions or two fermions and two bosons have led us to postulate W and Z particles and their interactions with fermions, as well as their mutual three-point vertices. Since as excercise **??** shows we have pretty much quarried all possible information[16] about this sector, we now turn to the $2 \to 2$ processes involving four bosons. First we consider the process

$$W^+(p_1,\epsilon_1)\;\gamma(p_2,\epsilon_2) \quad\to\quad W^+(p_3,\epsilon_3)\;\gamma(p_4,\epsilon_4)$$

With the available vertices we have two Feynman diagrams for this process :



with the respective contributions

$$\mathcal{M}_1 \quad = \quad i\frac{\hbar {Q_{\mathrm{W}}}^2}{(p_2-p_3)^2 - {m_{\mathrm{W}}}^2}Y(p_3,\epsilon_3;p_2-p_3,\nu;-p_2,\epsilon_2)$$

[16] As long as the fermion masses are neglected, see later.

$$
\begin{aligned}
& \times\ \Big(g^{\mu\nu} + (p_1 - p_4)^\mu (p_2 - p_3)^\nu / {m_{\rm W}}^2\Big) \\
& \times\ Y(p_1 - p_4, \mu; -p_1, \epsilon_1; p_4, \epsilon_4) \\
= \ & -i\frac{\hbar {Q_{\rm W}}^2}{2(p_2 \cdot p_3)} \Big(- {m_{\rm W}}^2 (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot \epsilon_4) \\
& + Y(p_3, \epsilon_3; p_2 - p_3, \mu; -p_2, \epsilon_2) Y(p_1 - p_4, \mu; -p_1, \epsilon_1; p_4, \epsilon_4)\Big) \quad , \\
\mathcal{M}_2 \ = \ & i\frac{\hbar {Q_{\rm W}}^2}{(p_3 + p_4)^2 - {m_{\rm W}}^2} Y(p_3, \epsilon_3; -p_3 - p_4, \nu; p_4, \epsilon_4) \\
& \times\ \Big(g^{\mu\nu} + (p_1 + p_2)^\mu (-p_3 - p_4)^\nu / {m_{\rm W}}^2\Big) \\
& \times\ Y(p_1 + p_2, \mu; -p_1, \epsilon_1; -p_2, \epsilon_2) \\
= \ & i\frac{\hbar {Q_{\rm W}}^2}{2(p_3 \cdot p_4)} \Big(- {m_{\rm W}}^2 (\epsilon_3 \cdot \epsilon_4)(\epsilon_1 \cdot \epsilon_2) \\
& + Y(p_3, \epsilon_3; -p_3 - p_4, \mu; p_4, \epsilon_4) Y(p_1 + p_2, \mu; -p_1, \epsilon_1; -p_2, \epsilon_2)\Big) \quad ,
\end{aligned} \tag{11.66}
$$

where we have already used Eq.(11.48) in the internal W lines, as well as the fact that $(p_j \cdot \epsilon_j) = 0$, $j = 1, 2, 3, 4$. Let us now proceed to check current conservation for the outgoing photon. The following algebra applies to $\mathcal{M}_1$:

$$
\begin{aligned}
& Y(p_3, \epsilon_3; p_2 - p_3, \mu; -p_2, \epsilon_2) Y(p_1 - p_4, \mu; -p_1, \epsilon_1; p_4, \epsilon_4)\rfloor_{\epsilon_4 \to p_4} = \\
& = \ Y(p_3, \epsilon_3; p_2 - p_3, \mu; -p_2, \epsilon_2) \Big((p_4 \cdot \epsilon_1)(p_2 - p_3)^\mu + 2(p_2 \cdot p_3){\epsilon_1}^\mu \Big) \\
& = \ 2(p_2 \cdot p_3) Y(p_3, \epsilon_3; p_2 - p_3, \epsilon_1; -p_2, \epsilon_2) \\
& \quad\ + {m_{\rm W}}^2 (p_4 \cdot \epsilon_1)(\epsilon_2 \cdot \epsilon_3) \quad ,
\end{aligned} \tag{11.67}
$$

so that

$$
\mathcal{M}_1\rfloor_{\epsilon_4 \to p_4} = -i\hbar {Q_{\rm W}}^2\ Y(p_3, \epsilon_3; p_2 - p_3, \epsilon_1; -p_2, \epsilon_2) \tag{11.68}
$$

In the same manner we arrive at

$$
\mathcal{M}_2\rfloor_{\epsilon_4 \to p_4} = i\hbar {Q_{\rm W}}^2\ Y(p_1 + p_2, \epsilon_3; -p_1, \epsilon_1; -p_2, \epsilon_2) \tag{11.69}
$$

Adding these last two results we obtain

$$
\begin{aligned}
& \sum_{j=1}^{2} \mathcal{M}_j \Bigg\rfloor_{\epsilon_4 \to p_4} = \\
& = \ i\hbar {Q_{\rm W}}^2\ \big(2(\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot p_4) - (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot p_4) - (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot p_4)\big) \quad .
\end{aligned} \tag{11.70}
$$

We might also have chosen choose to put the handlebar on ϵ_2 instead ; the result would then have been

$$\left.\sum_{j=1}^{2}\mathcal{M}_j\right]_{\epsilon_2\to p_2} =$$
$$= \; i\hbar {Q_{\mathrm{W}}}^2 \;\left(2(\epsilon_1\cdot\epsilon_3)(p_2\cdot\epsilon_4)-(\epsilon_1\cdot p_2)(\epsilon_3\cdot\epsilon_4)-(p_2\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)\right) \;\; . \tag{11.71}$$

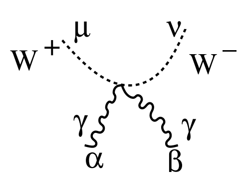
Going to the limit of large energies, we can also envisage putting a handlebar on ϵ_1 or ϵ_3. Neglecting safe terms leads to

$$\left.\sum_{j=1}^{2}\mathcal{M}_j\right]_{\epsilon_1\to p_1} =$$
$$= \; i\hbar {Q_{\mathrm{W}}}^2 \;\left(2(p_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)-(p_1\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4)-(\epsilon_2\cdot\epsilon_3)(p_1\cdot\epsilon_4)\right) \;\; , \tag{11.72}$$

and

$$\left.\sum_{j=1}^{2}\mathcal{M}_j\right]_{\epsilon_3\to p_3} =$$
$$= \; i\hbar {Q_{\mathrm{W}}}^2 \;\left(2(\epsilon_1\cdot p_3)(\epsilon_2\cdot\epsilon_4)-(\epsilon_1\cdot\epsilon_2)(p_3\cdot\epsilon_4)-(\epsilon_2\cdot p_3)(\epsilon_1\cdot\epsilon_4)\right) \;\; . \tag{11.73}$$

We can repair the high-energy behaviour of the amplitude, for all these cases at once, by introducing a four-boson vertex :



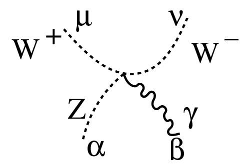
$$\leftrightarrow \quad -\frac{i}{\hbar}{Q_{\mathrm{W}}}^2\; X^{\mu\nu\alpha\beta}$$

where

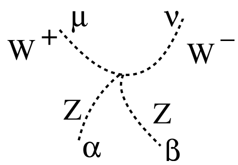
$$X^{\mu\nu\alpha\beta} = 2\, g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha} \;\; . \tag{11.74}$$

The *occurrence* of such a four-point vertex should not surprise us, with our experience of a similar vertex in sQED. Its precise algebraic structure can, of course, not be inferref from that example[17].

From the similarity between the $WW\gamma$ and WWZ vertices we can also immediately conclude that the analogous processes $WZ \to W\gamma$ and $WZ \to WZ$ will necessitate the existence of the following four-point vertices :



$$\leftrightarrow \quad -\frac{i}{\hbar} {Q_{\mathrm{W}}}^2 \frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}\ X^{\mu\nu\alpha\beta}$$

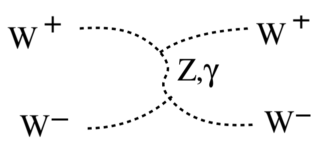


$$\leftrightarrow \quad -\frac{i}{\hbar} {Q_{\mathrm{W}}}^2 \frac{{c_{\mathrm{W}}}^2}{{s_{\mathrm{W}}}^2}\ X^{\mu\nu\alpha\beta}$$

fFinally, we consider the process

$$W^+(p_1,\epsilon_1); W^-(p_2,\epsilon_2) \quad \to \quad W^+(p_3,\epsilon_3)\ W^-(p_4,\epsilon_4) \quad ,$$

for which we have, so far, the four diagrams



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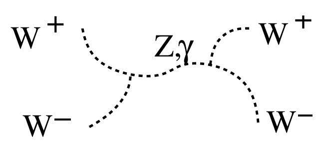


.

It will turn out to be useful to take the γ and Z exchanges together so that we have two contributions :

$$\begin{aligned}
\mathcal{M}_1 &= i\hbar {Q_{\mathrm{W}}}^2\, Y(p_3,\epsilon_3,-p_1,\epsilon_1,p_1-p_3,\mu) \\
&\quad \left(\frac{g^{\mu\nu}}{(p_1-p_3)^2} + \frac{{c_{\mathrm{W}}}^2}{{s_{\mathrm{W}}}^2}\frac{g^{\mu\nu}-(p_1-p_3)^\mu(p_1-p_3)^\nu/{m_Z}^2}{(p_1-p_3)^2-{m_Z}^2}\right) \\
&\quad Y(-p_2,\epsilon_2,p_4,\epsilon_4,p_2-p_4,\nu) \quad , \\
\mathcal{M}_2 &= i\hbar {Q_{\mathrm{W}}}^2\, Y(-p_2,\epsilon_2,-p_1,\epsilon_1,p_1+p_2,\mu) \\
&\quad \left(\frac{g^{\mu\nu}}{(p_1-p_3)^2} + \frac{{c_{\mathrm{W}}}^2}{{s_{\mathrm{W}}}^2}\frac{g^{\mu\nu}-(p_1+p_2)^\mu(p_1+p_2)^\nu/{m_Z}^2}{(p_1+p_2)^2-{m_Z}^2}\right) \\
&\quad Y(p_3,\epsilon_3,p_4,\epsilon_4,p_2-p_4,\nu) \quad .
\end{aligned} \tag{11.75}$$

[17]Except, perhaps, the idea that it contains only the metric tensor, and not any of the momenta.

Because the masses of the external particles are all equal, the second term in the Z propagator can be seen to drop out exactly. We can therefore afford to take the limit $s \gg {m_Z}^2$ without more ado, and combine the γ and Z propagators to arrive at the following high-energy form of the contributions :

$$\begin{aligned}
\mathcal{M}_1 &= i\frac{\hbar {Q_\mathrm{W}}^2}{{s_\mathrm{W}}^2}\frac{1}{(p_1-p_3)^2}\,Y(p_3,\epsilon_3,-p_1,\epsilon_1,p_1-p_3,\mu)\\
&\quad Y(-p_2,\epsilon_2,p_4,\epsilon_4,p_2-p_4,\mu)\ \ ,\\
\mathcal{M}_2 &= i\frac{\hbar {Q_\mathrm{W}}^2}{{s_\mathrm{W}}^2}\frac{1}{(p_1+p_2)^2}\,Y(-p_2,\epsilon_2,-p_1,\epsilon_1,p_1+p_2,\mu)\\
&\quad Y(p_3,\epsilon_3,p_4,\epsilon_4,p_2-p_4,\mu)\ \ . \qquad (11.76)
\end{aligned}$$

Let us now take the outgoing W^- longitudinal, $i.e$ apply the handlebar on ϵ_4, and drop safe terms :

$$\begin{aligned}
&Y(p_3,\epsilon_3,-p_1,\epsilon_1,p_1-p_3,\mu)Y(-p_2,\epsilon_2,p_4,\epsilon_4,p_2-p_4,\mu)\rfloor_{\epsilon_4\to p_4}\\
&\quad = Y(p_3,\epsilon_3,-p_1,\epsilon_1,p_1-p_3,\mu)\\
&\qquad \times\Big((p_1-p_3)^\mu((p_1-p_3)\cdot\epsilon_2)-((p_1-p_3)^2-{m_\mathrm{W}}^2){\epsilon_2}^\mu\Big)\\
&\quad \approx -(p_1-p_3)^2\,Y(p_3,\epsilon_3,-p_1,\epsilon_1,p_1-p_3,\epsilon_2) \qquad (11.77)
\end{aligned}$$

so that

$$\mathcal{M}_1\rfloor_{\epsilon_4\to p_4} = -i\frac{\hbar {Q_\mathrm{W}}^2}{{s_\mathrm{W}}^2}Y(p_3,\epsilon_3,-p_1,\epsilon_1,p_1-p_3,\epsilon_2)\ \ ; \qquad (11.78)$$

and the exactly analogous treatment gives

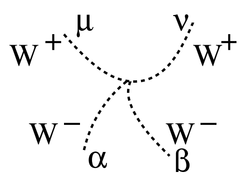
$$\mathcal{M}_2\rfloor_{\epsilon_4\to p_4} = -i\frac{\hbar {Q_\mathrm{W}}^2}{{s_\mathrm{W}}^2}Y(-p_2,\epsilon_2,-p_1,\epsilon_1,p_1+p2,\epsilon_3)\ \ . \qquad (11.79)$$

The total result of the handlebar operation is given by

$$\begin{aligned}
&\mathcal{M}_1+\mathcal{M}_2\rfloor_{\epsilon_4\to p_4} =\\
&\quad -i\frac{\hbar {Q_\mathrm{W}}^2}{{s_\mathrm{W}}^2}\left(2(p_4\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_3)-(p_4\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)-(p_4\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_2)\right)\ \ : \qquad (11.80)
\end{aligned}$$

we arrive at precisely the same algebraical structure as before, and we can immediately conclude that, in addition to the $WW\gamma\gamma$, $WWZ\gamma$ and $WWZZ$

couplings there must also be a $WWWW$ coupling :



$$\leftrightarrow \quad \frac{i}{\hbar}\frac{{Q_{\mathrm{W}}}^2}{{s_{\mathrm{W}}}^2}\; X^{\mu\nu\alpha\beta}$$

Note, however, a slight difference of this vertex as compared to the previous ones. There, the term that couples the two W Lorentz indices carries the factor 2 ; here, it is the term that couples the two W^+'s that is 'special'.

11.4 The Higgs sector

11.4.1 The Higgs hypothesis

Fully longitudinal scattering

Having pursued the consequences of unitarity in processes where a single external spin-1 particle is longitudinally polarized, we must of course also face the more taxing case in which, perhaps, *all* external spin-1 particles are longitudinally polarized : surely this is the most dangerous case from the point of view of unitarity. In doing so, we must however take into account the fact that the *notion* of longitudinal polarization is not strictly a Lorentz-invariant one since a generic Lorentz boost will mix longitudinal and transverse degrees of freedom. It therefore behooves us to specify in which particular Lorentz frame the particles are assumed to be longitudinally polarized. To this end we introduce a vector c^μ with

$$c \cdot c = 1 \;\; ;$$

the frame in which $\vec{c} = 0$ is defines the appropriate Lorentz frame. In these notes we shall take c^μ to be proportional to the total momentum involved in the scattering process, that is, the external vector particles are assumed to be purely longitudinal in the centre-of-mass frame of the scattering[18]. The

[18]That this is not a trivial point becomes clear when we realize that in 'WW scattering' at the LHC, say, the centre-of-mass frame of the scattering does not coincide with the laboratory frame, in which the *detector* is at rest, and in which the polarization analysis of the produced bosons is presumably performed.

longitudinal polarization of an on-shell vector particle with momentum p^μ and mass m is then given by

$$\epsilon_L{}^\mu = \frac{N_L}{m}\left(p^\mu - \frac{m^2}{c\cdot p}c^\mu\right) \quad , \quad N_L{}^{-2} = 1 - \frac{m^2}{(c\cdot p)^2} \quad , \tag{11.81}$$

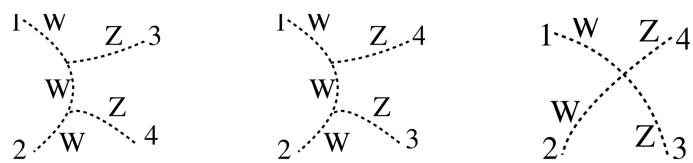
which expression is well-defined as long as $\vec{p} \neq 0$. We see that, as before, $\epsilon_L = p/m + \mathcal{O}\left(m/p^0\right)$. In the cases studied so far, the subleading terms in ϵ_L have only led to safe terms so that they could be neglected[19] ; now, this is no longer automatically the case.

$WW \to ZZ$

The first Gedanken process[20] is

$$W^+(p_1,\epsilon_1)\, W^-(p_2,\epsilon_2) \quad \to \quad Z^0(p_3,\epsilon_3)Z^0(p_4,\epsilon_4)$$

So far, we have the following three Feynman graphs available at the tree level :



,

and the following contributions :

$$\begin{aligned}
\mathcal{M}_j &= -i\hbar g_{\text{WWZ}}{}^2\frac{N_j}{\Delta_j} \quad , \quad j = 1,2 \quad , \\
N_1 &= Y(p_1 - p_3, \mu; -p_1, \epsilon_1; p_3, \epsilon_3) \\
&\quad \times \left(-g^{\mu\nu} + \frac{1}{m_{\text{W}}{}^2}(p_1-p_3)^\mu(p_1-p_3)^\nu\right) \\
&\quad \times Y(-p_2,\epsilon_2; p_2 - p_4, \nu; p_4, \epsilon_4) \quad , \\
\Delta_1 &= (p_1-p_3)^2 - m_{\text{W}}{}^2 = m_{\text{Z}}{}^2 - 2(p_1\cdot p_3) \quad , \\
N_2 &= Y(p_1 - p_4, \mu; -p_1, \epsilon_1; p_4, \epsilon_4)
\end{aligned}$$

[19]From the point of view of restoring unitarity, not that of actually getting the cross section right!

[20]As I write these notes, this is still a true Gedanken process. As usual, with improving technology and the commissioning of higher-energy machines, Gedanken processes are gradually turned into *actual* ones...

$$
\begin{aligned}
& \times \left(-g^{\mu\nu} + \frac{1}{{m_\mathrm{W}}^2}(p_1 - p_4)^\mu (p_1 - p_4)^\nu\right) \\
& \times Y(-p_2, \epsilon_2; p_2 - p_3, \nu; p_3, \epsilon_3) \ , \\
\Delta_2 \ &= \ (p_1 - p_4)^2 - {m_\mathrm{W}}^2 = {m_\mathrm{Z}}^2 - 2(p_1 \cdot p_4) \ , \\
M_3 \ &= \ -i\hbar {g_\mathrm{WWZ}}^2 N_3 \ , \\
N_3 \ &= \ X(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \ . \qquad (11.82)
\end{aligned}
$$

Owing to the work we have done so far, we may already anticipate some cancellations between the diagrams when we make all bosons longitudinal and the safe terms are therefore not the subleading ones, but rather the sub-subleading ones. We have to proceed carefully[21]. Denoting by the subscript L the 'fully longitudinal' case, it appears best to write the result as

$$
\begin{aligned}
\left.\sum_{j=1}^{3} \mathcal{M}_j \right|_L \ &= \ -i\hbar {g_\mathrm{WWZ}}^2 \ \frac{N_{123}}{\Delta_{12}} \ , \\
N_{123} \ &= \ N_1\Delta_2 + N_2\Delta_1 + \Delta_{12}N_3 = -4E^6 \ \frac{{m_\mathrm{Z}}^2}{{m_\mathrm{W}}^4} \ (\sin\theta)^2 + \cdots \ , \\
\Delta_{12} \ &= \ \Delta_1\Delta_2 = 4E^4 \ (\sin\theta)^2 + \cdots \ , \qquad (11.83)
\end{aligned}
$$

where $E = {p_1}^0 = {p_2}^0 = {p_3}^0 = {p_4}^0$ and $\theta = \angle(\vec{p_1}, \vec{p_3})$, all evaluated in the centre-of-mass frame. As before, the ellipses denote contributions that can only give rise to safe terms, and that therefore do not interest us here. Note that we have disregarded also the normalization factors N_L ; since the polarization vectors are overall factors in the scattering amplitude, the N_L can never play a rôle in any dynamical cancellation, and their subleading terms are therefore always safe. The non-safe contribution from our three Feynman graphs is therefore

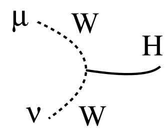
$$
\left.\sum_{j=1}^{3} \mathcal{M}_j \right|_L = i\hbar \ {g_\mathrm{WWZ}}^2 \ E^2 \ \frac{{m_\mathrm{Z}}^2}{{m_\mathrm{W}}^4} + \cdots \ , \qquad (11.84)
$$

and it violates unitarity at sufficiently large E. Note that each individual M_j will go as E^4 at high energy so, as already anticipated, some cancellation has already taken place, but not enough ; and since the vertices have already been fixed before, we have to introduce a new ingredient into the theory.

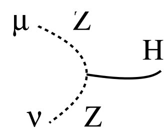
[21]This is most safely done using computer algebra, using *e.g.* `FORM`.

The Minimal Higgs approach

We shall assume that, in addition to the three graphs used so far, there is a fourth one available, mediated by a new particle type. We assume this to be a *neutral*, *scalar* particle, denoted by H, that couples to W^+W^- and ZZ as follows:

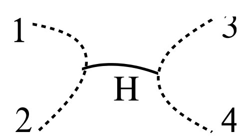


$$\leftrightarrow \quad \frac{i}{\hbar}\, g_{\mathrm{WWH}}\, g^{\mu\nu}$$



$$\leftrightarrow \quad \frac{i}{\hbar}\, g_{\mathrm{ZZH}}\, g^{\mu\nu}$$

A fourth Feynman diagram is now possible :



,

given by

$$\mathcal{M} = -i\hbar\, g_{\mathrm{WWH}} g_{\mathrm{ZZH}}\, (\epsilon_1 \cdot \epsilon_2)\, (\epsilon_3 \cdot \epsilon_4)\, \frac{1}{4E^2 - {m_{\mathrm{H}}}^2} \quad . \tag{11.85}$$

Its contribution to the fully longitudinal scattering reads

$$\mathcal{M}_4\rfloor_L = -i\hbar\, E^2\, g_{\mathrm{WWH}} g_{\mathrm{ZZH}}\, \frac{1}{{m_{\mathrm{W}}}^2\, {m_{\mathrm{Z}}}^2} \tag{11.86}$$

and good high-energy behaviour will be restored in the process $WW \to ZZ$ provided that

$$g_{\mathrm{WWH}}\; g_{\mathrm{ZZH}} = {g_{\mathrm{WWZ}}}^2\, \frac{{m_{\mathrm{Z}}}^4}{{m_{\mathrm{W}}}^2} \quad . \tag{11.87}$$

Before we proceed to the next Gedanken process, a few remarks are in order. In the first place, the choice for a *scalar* Higgs particle is almost unavoidable. It certainly cannot be a fermion ; if it were a vector particle, its propagator would contain unwanted higher powers of the energy E, the WWH and ZZH would presumably be of Yang-Mills type hence also E-dependent. The vertices given above are essentially the only ones possible for the interactions between two vectors and a scalar if we want them to be energy-independent.

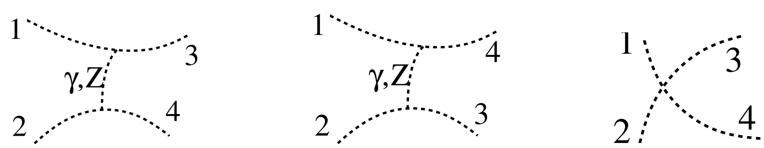
Note that g_{WWH} and g_{ZZH} may both be expected to contain a *mass*, that is, they are of dimension $L^{-1}/\sqrt{\hbar}$. The assumption that there is just *one* type of neutral scalar involved is, of course, based on nothing but a prejudice in favour of simplicity. Finally, at high energy all contributions from m_{H} end up in safe terms, and we do not expect to glean any information on the Higgs mass from our considerations.

$WW \to WW$ scattering

Another four-boson scattering process of interest is

$$W^+(p_1,\epsilon_1)W^+(p_2,\epsilon_2) \;\;\to\;\; W^+(p_3,\epsilon_3)W^+(p_4,\epsilon_4)$$

for which we have five purely vector-boson diagrams :



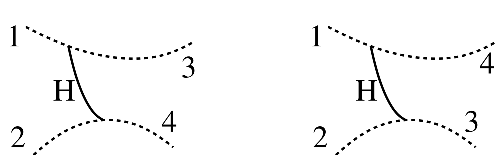
whose contributions can be conviently written as

$$\begin{aligned}
\mathcal{M}_1 &= -i\hbar\, Y(p_3,\epsilon_3;-p_1,\epsilon_1;p_1-p_3,\mu) \\
&\quad \times \left({Q_{\text{W}}}^2 \frac{-g_{\mu\nu}}{(p_1-p_3)^2} + {g_{\text{WWZ}}}^2 \frac{-g^{\mu\nu}+(p_1-p_3)^\mu (p_1-p_3)^\nu/{m_{\text{W}}}^2}{(p_1-p_3)^2-{m_{\text{Z}}}^2} \right) \\
&\quad \times Y(p_4,\epsilon_4;-p_2,\epsilon_2;p_2-p_4,\nu) \;\;, \\
\mathcal{M}_2 &= \mathcal{M}_1\rfloor_{p_3,\epsilon_3 \;\leftrightarrow\; p_4,\epsilon_4} \;\;, \\
\mathcal{M}_3 &= i\hbar\, \frac{{Q_{\text{W}}}^2}{{s_{\text{W}}}^2}\, X(\epsilon_3,\epsilon_4,\epsilon_1,\epsilon_2) \;\;.
\end{aligned} \tag{11.88}$$

By the same methods as used in the previous section we arrive at

$$\left.\sum_{j=1}^{3} \mathcal{M}_j \right\rfloor_L = i\frac{\hbar\, E^2\, {Q_{\text{W}}}^2}{{m_{\text{W}}}^4\, {s_{\text{W}}}^2}\; \left(-4{m_{\text{W}}}^2 + 3{m_{\text{Z}}}^2\, {c_{\text{W}}}^2\right) + \cdots \tag{11.89}$$

The Higgs hypothesis now provides for two additional diagrams :



with the contributions

$$\begin{aligned}\mathcal{M}_4 &= -i\hbar\, {g_{\mathrm{WWH}}}^2\, \frac{(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)}{(p_1-p_3)^2-{m_{\mathrm{H}}}^2}\ ,\\ \mathcal{M}_5 &= -i\hbar\, {g_{\mathrm{WWH}}}^2\, \frac{(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)}{(p_1-p_4)^2-{m_{\mathrm{H}}}^2}\ ,\end{aligned} \tag{11.90}$$

so that

$$\left.\sum_{j=4}^{5}\mathcal{M}_j\right|_L = i\hbar\, E^2\, \frac{{g_{\mathrm{WWH}}}^2}{{m_{\mathrm{W}}}^4} + \cdots \tag{11.91}$$

In this process, then, good high-energy behaviour is obtained under the condition

$${g_{\mathrm{WWH}}}^2 = \frac{{Q_{\mathrm{W}}}^2}{{s_{\mathrm{W}}}^2}\left(4{m_{\mathrm{W}}}^2 - 3{m_{\mathrm{Z}}}^2\,{c_{\mathrm{W}}}^2\right)\ . \tag{11.92}$$

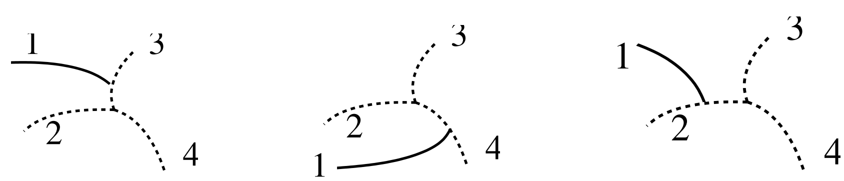
Again, no restrictions on m_{H} occur.

$HZ \to WW$ scattering

We have now run out of four-vector Gedanken processes. $ZZ \to ZZ$ scattering has no Yang-Mills contributions[22], and any four-vector process involving photons will have vanishing amplitudes under a handlebar on any photon. However, in the same spirit by which we boldly proposed the process $U\overline{D} \to WZ$ as soon as the Z was hypothesized, we can consider the process

$$H(p_1)Z^0(p_2,\epsilon_2) \ \ \to\ \ W^+(p_3,\epsilon_3)W^-(p_4,\epsilon^4)$$

Since only three out of four particles can become longitudinal here, the unitarity violations are not so bad, and the safe terms are of sub- rather than of sub-sub-leading type. We have three diagrams,



[22]Under the Higgs hypothesis $ZZ \to ZZ$ scattering is described by three diagrams containing Higgs exchange. Their sum, however, is safe by itself and hence does not lead to any constraints.

that contribute as

$$\begin{aligned}
\mathcal{M}_1 &= -i\hbar\; g_{\rm WWZ} g_{\rm WWH}\; Y(p_3,\epsilon_3;p_2-p_3,\mu;-p_2,\epsilon_2) \\
&\quad \times \frac{-g^{\mu\nu}+(p_2-p_3)^\mu(p_2-p_3)^\nu}{(p_2-p_3)^2-{m_{\rm W}}^2}\;(\epsilon_4)_\nu \;\;, \\
\mathcal{M}_2 &= -i\hbar\; g_{\rm WWZ} g_{\rm WWH}\; Y(p_2-p_4,\mu;p_4,\epsilon_4;-p_2,\epsilon_2) \\
&\quad \times \frac{-g^{\mu\nu}+(p_2-p_4)^\mu(p_2-p_4)^\nu}{(p_2-p_4)^2-{m_{\rm W}}^2}\;(\epsilon_3)_\nu \;\;, \\
\mathcal{M}_3 &= -i\hbar\; g_{\rm WWZ} g_{\rm ZZH}\; Y(p_3,\epsilon_3;p_4,\epsilon_4;-p_3-p_4,\mu) \\
&\quad \times \frac{-g^{\mu\nu}+(p_1+p_2)^\mu(p_1+p_2)^\nu}{(p_1+p_2)^2-{m_{\rm Z}}^2}\;(\epsilon_2)_\nu \;\;.
\end{aligned} \tag{11.93}$$

The kinematics of this process is a little different from that of the two pervious ones, since $m_{\rm H}$ and $m_{\rm Z}$ cannot be assumed to be equal. Still, at high energy we may apply massless kinematics since we only have to cancel the *leading* non-safe terms. Neglecting, therefore, $m_{\rm W}$, $m_{\rm Z}$ and $m_{\rm H}$ in the kinematics[23] we find

$$\left.\sum_{j=1}^{3}\mathcal{M}_j\right]_L = i\hbar\;E^2\;\cos\theta\;g_{\rm WWZ}\;\left(g_{\rm WWH}\,\frac{m_{\rm Z}}{{m_{\rm W}}^4} - g_{\rm ZZH}\frac{1}{m_{\rm Z}\;{m_{\rm W}}^2}\right) + \cdots \tag{11.94}$$

and find the final requirement

$$g_{\rm WWH}\;\frac{m_{\rm Z}}{{m_{\rm W}}^4} = g_{\rm ZZH}\frac{1}{m_{\rm Z}\;{m_{\rm W}}^2} \tag{11.95}$$

if good high-energy behaviour is to emerge.

11.4.2 Predictions from the Higgs hypothesis

The Higgs hypothesis has given us the three conditions of Eqs.(11.87), (11.92) and (11.95). If we consider $g_{\rm WWH}$ and $g_{\rm ZZH}$ as the two unknowns, this system is overconstrained, and we obtain additional information. The system of conditions can easily be solved and we find the two couplings

$$g_{\rm WWH} = \frac{Q_{\rm W}\;m_{\rm W}}{s_{\rm W}} \quad , \quad g_{\rm ZZH} = \frac{Q_{\rm W}\;m_{\rm Z}}{s_{\rm W}\;c_{\rm W}} \;\;, \tag{11.96}$$

[23]But not, of course, in the longitudinal polarizations!

and, in addition, the interesting relation

$$m_{\mathrm{W}} = m_{\mathrm{Z}}\, c_{\mathrm{W}} \quad . \tag{11.97}$$

E 57 It is apposite to dwell on this last result. The weak mixing angle θ_W was introduced to parametrize the system of *coupling constants*, as discussed in section 11.3.2 : we now see it come back here as a relation between *masses* instead ! From the treatment of the Electroweak Standard Model presented in these notes, it also becomes clear that the mixing angle as a description of coupling constants is, in a logical sense, prior to that as a description of masses. The assumption of a single Z^0 particle determines the couplings as described in section 11.3.2 : but it takes the supposition of a single, neutral Higgs particle to obtain Eq.(11.97). If the Higgs sector of the Standard Model turns out to be different, with more Higgs-like particles, say, the W and Z mass become uncorrelated ; but the couplings of W and Z with the fermions and each other remain unaffected. In the usual textbook derivation of the model this distinction tends to be obscured by the simultaneous obtention of all couplings at once after symmetry breaking.

As a final comment we remark that, if unitarity is restored by*whatever* Higgs-like phenomenon, the weak mixing angle must always obey the bound

$${c_{\mathrm{W}}}^2 < \frac{4}{3}\,\frac{{m_{\mathrm{W}}}^2}{{m_{\mathrm{Z}}}^2} \tag{11.98}$$

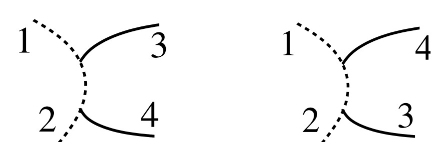
as can be seen from Eq.(11.92)[24].

11.4.3 W, Z and H four-point interactions

The class of bosonic four-particle scattering amplitudes is not yet completely exhausted. We can consider the process

$$Z^0(p_1,\epsilon_1)Z^0(p_2,\epsilon_2) \quad \to H(p_3)H(p_4)$$

given by two diagrams so far,



[24]For the actually observed values of W and Z mass this bound is itself somewhat larger than unity, and therefore not so significant; but it is nice to have it even so.

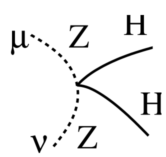
and the following amplitude :

$$
\begin{aligned}
\mathcal{M}_{1+2} \;=\; & -i\hbar\; g_{\mathrm{ZZH}}{}^2\;(\epsilon_1)_\mu\;(\epsilon_2)_\nu \\
& \left(\frac{-g^{\mu\nu} + (p_1-p_3)^\mu (p_1-p_3)^\nu / m_{\mathrm{Z}}{}^2}{(p_1-p_3)^2 - m_{\mathrm{Z}}{}^2} \right. \\
& \left. + \frac{-g^{\mu\nu} + (p_1-p_4)^\mu (p_1-p_4)^\nu / m_{\mathrm{Z}}{}^2}{(p_1-p_4)^2 - m_{\mathrm{Z}}{}^2} \right) \quad . \qquad (11.99)
\end{aligned}
$$

In the fully longitudinal case the non-safe terms are

$$
\mathcal{M}_{1+2}\rfloor_L = -i\hbar\; E^2\; \frac{g_{\mathrm{ZZH}}{}^2}{m_{\mathrm{Z}}{}^4} + \cdots \qquad (11.100)
$$

and the remedy ought to be straightforward by now. We introduce yet another vertex, involving two Z's and two H's :



$$
\leftrightarrow \quad \frac{i}{\hbar}\; g_{\mathrm{ZZHH}}\; g^{\mu\nu}
$$

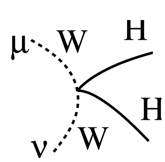
upon which we have a third diagram, whose nonsafe part is trivial :

$$
\mathcal{M}_3\rfloor_L = 2i\hbar\; E^2\; \frac{g_{\mathrm{ZZHH}}}{m_{\mathrm{Z}}{}^2} + \cdots \qquad (11.101)
$$

We see that the four-point coupling constant must be given by

$$
g_{\mathrm{ZZHH}} = \frac{g_{\mathrm{ZZH}}{}^2}{2m_{\mathrm{Z}}{}^2} = \frac{Q_{\mathrm{W}}{}^2}{2s_{\mathrm{W}}{}^2 c_{\mathrm{W}}{}^2} \quad . \qquad (11.102)
$$

As in the case of sQED and YM, this four-point coupling does not contain a length scale, in contrast to the ZZH coupling. For the case of $WW \to HH$ scattering, exactly the same treatment holds. It suffices to replace m_{Z} by m_{W} and g_{ZZH} by g_{WWH}. We find that also a $WWHH$ vertex is required :



$$
\leftrightarrow \quad \frac{i}{\hbar}\; g_{\mathrm{WWHH}}\; g^{\mu\nu} \quad ,
$$

with

$$
g_{\mathrm{WWHH}} = \frac{g_{\mathrm{WWH}}{}^2}{2m_{\mathrm{W}}{}^2} = \frac{Q_{\mathrm{W}}{}^2}{2s_{\mathrm{W}}{}^2} \quad . \qquad (11.103)
$$

11.4.4 Higgs-fermion couplings

Let us return to the process

$$\overline{U}(p_1)U(p_2) \quad \to \quad W^+(q_+,\epsilon_+)W^-(q_-,\epsilon_-)$$

which was used in section 11.3.1 to argue the existence of the Z boson. This time, however, we shall *not* neglect the fermion masses ; and we shall take *both* W's longitudinal. It can be seen that each individual diagram will go as E^2 when the energy E of the W's in their centre-of-mass frame becomes large. This means that, in the longitudinal polarization of Eq.(11.81), the second term will only contribute to the safe terms, and we may simply write $(\epsilon_\pm)_L = q_\pm/m_{\mathrm{W}}$, so that

$$Y(q_+,\epsilon_+;q_-,\epsilon_-;-q_+-q_-,\mu)\Big]_L \approx -\frac{s}{2{m_{\mathrm{W}}}^2}(q_+-q_-)^\mu + \cdots \qquad (11.104)$$

where once more the ellipsis denotes safe terms. In fact, the restriction to *non*safe terms in our treatment means that we may neglect the boson masses in the kinematics : every occurrence of boson masses from the kinematics is quadratic and hence gives safe terms. For the fermions this is not the case as we shall see.

Let us revisit the diagrams of our process. The first one now reads

$$\mathcal{M}1 = -i\hbar {g_{\mathrm{W}}}^2\; \overline{v}(p_1)(1+\gamma^5)\not{\epsilon}_-\frac{\not{q}_- - \not{p}_1 + m_{\mathrm{D}}}{(q_- - p_1)^2 - {m_{\mathrm{D}}}^2}(1+\gamma^5)\not{\epsilon}_+ u(p_2) \quad . \quad (11.105)$$

Note that the m_{D} in the numerator drops out by virtue of the $(1+\gamma^5)$'s. We can now perform some Diracology, using the Dirac equation and dropping safe contributions wherever opportune :

$$\begin{aligned}
\mathcal{M}_1\big]_L &= -i\frac{2\hbar {g_{\mathrm{W}}}^2}{{m_{\mathrm{W}}}^2((q_- - p_1)^2 - {m_{\mathrm{D}}}^2)}\;\overline{v}(p_1)\;\mathcal{A}\;u(p_2) \quad , \\
\mathcal{A} &= (1+\gamma^5)\;\not{q}_-\;(\not{q}_- - \not{p}_1)\;(1+\gamma^5)\;\not{q}_+ \\
&\to 2(1+\gamma^5)\;\not{q}_-\;(\not{q}_- - \not{p}_1)\;(\not{q}_+ - \not{p}_2 + m_{\mathrm{U}}) \\
&= 2(1+\gamma^5)\;\not{q}_-\;(\not{q}_- - \not{p}_1)\;(\not{p}_1 - \not{q}_- + m_{\mathrm{U}}) \\
&\to 2(1+\gamma^5)\;\Big(-(q_- - p_1)^2\not{q}_- + (\not{q}_- - \not{p}_1 - m_{\mathrm{U}})(\not{q}_- - \not{p}_1)m_{\mathrm{U}}\Big) \\
&\to 2(1+\gamma^5)\;(m_{\mathrm{U}} - \not{q}_-)\;(q_- - p_1)^2 \quad ; \qquad (11.106)
\end{aligned}$$

so that the fully longitudinal case gives for this diagram

$$\mathcal{M}_1\rfloor_L = 2i\hbar {g_{\mathrm{W}}}^2\, \overline{v}(p_1)(1+\gamma^5)(\not\epsilon_- - m_{\mathrm{U}})u(p_2) + \cdots \tag{11.107}$$

For the third diagram we can perform a similar analysis :

$$\begin{aligned}
\mathcal{M}_3\rfloor_L &= i\frac{\hbar\, g_{\mathrm{WWZ}}}{s - {m_{\mathrm{Z}}}^2}\frac{-s}{2{m_{\mathrm{W}}}^2}\,\overline{v}(p_1)\;\mathcal{B}\;u(p_2)\;\;, \\
\mathcal{B} &= (v_{\mathrm{U}} + a_{\mathrm{U}}\gamma^5)(\not\epsilon_+ - \not\epsilon_-) \\
&\to (v_{\mathrm{U}} + a_{\mathrm{U}}\gamma^5)(\not\epsilon_+ - \not\epsilon_- - \not p_2 + m_{\mathrm{U}} - \not p_1) - m_{\mathrm{U}}(v_{\mathrm{U}} - a_{\mathrm{U}}\gamma^5) \\
&= -2(v_{\mathrm{U}} + a_{\mathrm{U}}\gamma^5)\not\epsilon_- + 2m_{\mathrm{U}}a_{\mathrm{U}}\gamma^5\;\;;
\end{aligned} \tag{11.108}$$

and up to safe terms, we therefore have

$$\mathcal{M}_3\rfloor_L = i\frac{\hbar g_{\mathrm{WWZ}}}{{m_{\mathrm{W}}}^2}\,\overline{v}(p_1)\left((v_{\mathrm{U}} + a_{\mathrm{U}}\gamma^5)\not\epsilon_- - m_{\mathrm{U}}a_{\mathrm{U}}\gamma^5\right)\;u(p_2) + \cdots \tag{11.109}$$

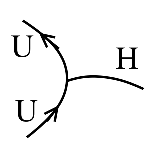
To obtain the contribution from the second diagram, we simply put $g_{\mathrm{WWZ}} \to Q_{\mathrm{W}}$, $v_{\mathrm{U}} \to Q_{\mathrm{U}}$, and $a_{\mathrm{U}} \to 0$ in the third diagram :

$$\mathcal{M}_2\rfloor_L = i\frac{\hbar Q_{\mathrm{W}}\;Q_{\mathrm{U}}}{{m_{\mathrm{W}}}^2}\,\overline{v}(p_1)\;\not\epsilon_-\;u(p_2) + \cdots \tag{11.110}$$

If we add the three diagrams, the contributions with $\overline{v}\not\epsilon_- u$ cancel precisely, as they should since that was what we imposed in section 11.3.1. We are left with terms proportional to m_{U} :

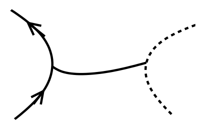
$$\begin{aligned}
\mathcal{M}_{1+2+3}\rfloor_L &= i\frac{\hbar m_{\mathrm{U}}}{{m_{\mathrm{W}}}^2}\,\overline{v}(p_1)\left(-2{g_{\mathrm{W}}}^2(1+\gamma^5) - g_{\mathrm{WWZ}}a_{\mathrm{U}}\gamma^5\right)u(p_2) + \cdots \\
&= -i\frac{\hbar}{{m_{\mathrm{W}}}^2}\;\frac{{Q_{\mathrm{W}}}^2 m_{\mathrm{U}}}{4{s_{\mathrm{W}}}^2}\,\overline{v}(p_1)u(p_2) + \cdots
\end{aligned} \tag{11.111}$$

so that an energy behaviour of E^1 at high energy is still uncompensated. The Higgs boson is usefully applied here as well. We simply assume the UUH vertex



$$\leftrightarrow \quad \frac{i}{\hbar}\, g_{\mathrm{UUH}}\; 1\;\;,$$

where we must realize that the Dirac unit matrix is involved[25]. For the process $\overline{U}U \to WW$ we then have a fourth available diagram :



which contributes to the amplitude the amount

$$\mathcal{M}_4 = -i\hbar\, g_{\mathrm{UUH}} g_{\mathrm{WWH}}\, \overline{v}(p_1)u(p_2)\, \frac{1}{s - {m_{\mathrm{H}}}^2}\, (\epsilon_+ . \epsilon_-) \quad . \tag{11.112}$$

In the fully longitudinal case we therefore have

$$\mathcal{M}_4\rfloor_{=} - i\frac{\hbar\, g_{\mathrm{UUH}} g_{\mathrm{WWH}}}{2{m_{\mathrm{W}}}^2}\overline{v}(p_1)u(p_2) + \cdots , \tag{11.113}$$

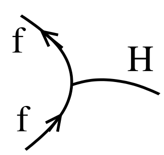
and the following requirement on g_{UUH} is obtained :

$$\frac{{Q_{\mathrm{W}}}^2 m_{\mathrm{U}}}{4{s_{\mathrm{W}}}^2} + \frac{\hbar\, g_{\mathrm{UUH}} g_{\mathrm{WWH}}}{2{m_{\mathrm{W}}}^2} = 0 \quad , \tag{11.114}$$

or

$$g_{\mathrm{UUH}} = -\frac{Q_{\mathrm{W}}}{2s_{\mathrm{W}}} \frac{m_{\mathrm{U}}}{m_{\mathrm{W}}} \quad . \tag{11.115}$$

This discussion can of course be applied to any fermion type[26], and we find
E 58 the general Feynman rule
E 59



$$\leftrightarrow \quad \frac{i}{\hbar}\, \frac{e}{2s_{\mathrm{W}}}\, \frac{m_f}{m_{\mathrm{W}}}\, 1$$

11.4.5 Higgs self-interactions

The triple H coupling

There remains the issue of possible self-interactions of the Higgs particle. To this end we examine not a $2 \to 2$ but a $2 \to 3$ process, namely

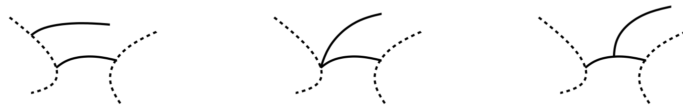
$$Z(p_1, \epsilon_1)\; Z(p_2, \epsilon_2) \;\; \to Z(p_3, \epsilon_3)\; Z(p_4, \epsilon_4)\; H(p_5) \quad .$$

---

[25] In fact, the observation that the nonsafe part in this process is proportional to $\overline{v}u$ is the strongest argument in favour of a *scalar* Higgs.

[26] Note that for D-type fermions, a_{D} has opposite sign ; but also the W^+ and W^- are interchanged in the first diagram.

At the tree level, this process is described by 21 Feynman diagrams provided we allow for three-point couplings between H's. These belong to one of the three following types :



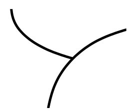
where as usual the dotted lines denotes Z's and the solid lines stand for H particles, and we have to take into account the appropriate permutations of the external Z particles. The amplitude is given by the three corresponding contributions :

$$
\begin{aligned}
\mathcal{M}_1 &= \mathcal{A}_1(1,2,3,4,5)+\mathcal{A}_1(2,1,3,4,5)+\mathcal{A}_1(3,4,1,2,5)\\
&+ \mathcal{A}_1(4,3,1,2,5)+\mathcal{A}_1(1,3,2,4,5)+\mathcal{A}_1(3,1,2,4,5)\\
&+ \mathcal{A}_1(2,4,1,3,5)+\mathcal{A}_1(4,2,1,3,5)+\mathcal{A}_1(1,4,3,2,5)\\
&+ \mathcal{A}_1(4,1,3,2,5)+\mathcal{A}_1(3,2,1,4,5)+\mathcal{A}_1(2,3,1,4,5)\ ,\\
\mathcal{A}_1(i_1,i_2,i_3,i_4,i_5) &= i\hbar^{3/2}\, {g_{\text{ZZH}}}^3\, {\epsilon_{i_1}}^{\mu}\, \Pi_{\mu\nu}(p_{i_1}+p_{i_3})\, {\epsilon_{i_2}}^{\nu}\, (\epsilon_{i_3}\cdot\epsilon_{i_4})\\
&\times\ \Delta_Z(p_{i_1}+p_{i_3})\Delta_H(p_{i_3}+p_{i_4})\ ,\\
\mathcal{M}_2 &= \mathcal{A}_2(1,2,3,4,5)+\mathcal{A}_2(3,4,1,2,5)+\mathcal{A}_2(1,3,2,4,5)\\
&+ \mathcal{A}_2(2,4,1,3,5)+\mathcal{A}_2(1,4,3,2,5)+\mathcal{A}_2(3,2,1,4,5)\ .\\
\mathcal{A}_2(i_1,i_2,i_3,i_4,i_5) &= -i\hbar^{3/2}\, g_{\text{ZZH}}\, g_{\text{ZZHH}}\, (\epsilon_{i_1}\cdot\epsilon_{i_2})(\epsilon_{i_3}\cdot\epsilon_{i_4})\\
&\times\ \Delta_H(p_{i_3}+p_{i_4})\ ,\\
\mathcal{M}_3 &= \mathcal{A}_3(1,2,3,4,5)+\mathcal{A}_3(1,3,2,4,5)+\mathcal{A}_3(1,4,2,3,5)\ ,\\
\mathcal{A}_3(i_1,i_2,i_3,i_4,i_5) &= i\hbar^{3/2}\, {g_{\text{ZZH}}}^2\, g_{\text{HHH}}\, (\epsilon_{i_1}\cdot\epsilon_{i_2})(\epsilon_{i_3}\cdot\epsilon_{i_4})\\
&\times\ \Delta_H(p_{i_3}+p_{i_4})\, \Delta_H(p_{i_3}+p_{i_4})\ ,\\
\Pi_{\mu\nu}(q) &= -g_{\mu\nu}+\frac{1}{{m_{\text{Z}}}^2}q_\mu q_\nu\ ,\\
\Delta_Z(q) &= \left(q^2-{m_{\text{Z}}}^2\right)^{-1}\ ,\ \ \Delta_H(q)=\left(q^2-{m_{\text{H}}}^2\right)^{-1}\ . \qquad (11.116)
\end{aligned}
$$

Here we have, for once, taken all momenta outgoing, which means that the momenta of the incoming Z's have negative zeroth component. For this 5-particle process the phase space is of course more complicated, and here we demonstrate a *numerical* method to investigate cancellations. This can be quite sensitive if done right[27]. Although naïvely each diagram $\mathcal{A}_1$ and $\mathcal{A}_2$

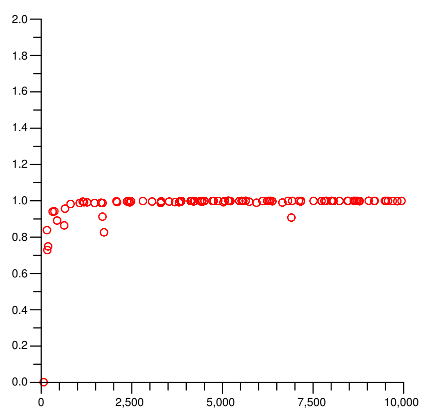
[27] A short description of how this is done follows. We first define an *energy scale* E. The 1,2, and 3-components of the momenta $\vec{p}_{3,4}$ are chosen as random values, uniformly

grow quadratically with the energy in the fully longitudinal case, both $\mathcal{M}_1$ and $\mathcal{M}_2$ actualy become energy-independent at sufficiently high energy E. But this is *not safe* : a $2 \to 3$ amplitude must go at most as E^{-1}, and therefore cancellations between $(\mathcal{M}_1 + \mathcal{M}_2)$ and $\mathcal{M}_3$ are still necessary. We find that the required HHH coupling is given by



$$\leftrightarrow \quad i\frac{g_{\text{HHH}}}{\hbar} \quad , \quad g_{\text{HHH}} = -\frac{3}{2}\,\frac{{Q_{\text{W}}}^2\;{m_{\text{H}}}^2}{m_{\text{W}}\;s_{\text{W}}}$$

if the necessary cancellations are to arise. In the figure below we have, somewhat arbitrarily, chosen $m_{\text{W}}c^2 = 80$ GeV, $m_{\text{Z}}c^2 = 90$ GeV, $m_{\text{H}}c^2 = 250$ GeV.



We plot $-\,\mathcal{M}_3\rfloor_L \,/\, \mathcal{M}_{1+2}\rfloor_L$ for various energy scales E. The sampling is performed as described in the footnote. The two contributions to the amplitude are seen to balance one another precisely, and the combined amplitude goes as E^{-2}, provided the right choice of g_{HHH} is made. Note that the amplitudes are heavily dependent on the various scattering angles: but their *ratio* is not.

A word of caution is in order on the interpretation of this picture. The high-energy limit is, strictly speaking, only obtained if *all* products of momenta grow large with respect to *all* masses involved. In a sampling over phase space it can always happen that some momentum products are comparable

---

distributed between $-E$ and E, and the corresponding momentum components of $\vec{p}_5$ are given by $\vec{p}_5 = -\vec{p}_3 - \vec{p}_4$. We then compute the energy components ${p_{3,4,5}}^0$ from the mass-shell condition. The energy components ${p_{1,2}}^0$ are then given by ${p_{1,2}}^0 = -(p_3+p_4+p_5)^0/2$, and their momenta are computed from *their* mass-shell condition. We take these to be along the z axis, say, and oppositely pointed. This is a crude but efficient way of obtaining momentum configurations satisfying all kinematical conditions, and the various polarization vectors are then easily obtained using Eq.(11.81). Repeating this procedure a number of times, we can map out the phase space for a given energy scale.

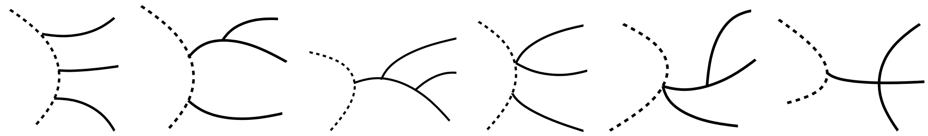
to squared masses ; these cases are responsible for the 'outlying' dots in the plot at large values of the energy scale.

The quartic H coupling

The last Gedanken process needed is

$$Z(p_1,\epsilon_1)\;Z(p_2,\epsilon_2)\quad\to\quad H(p_3)\;H(p_4)\;H(p_5)$$

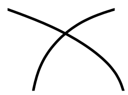
which is described by 25 Feynman diagrams in six types:



where we have already anticipated a quartic Higgs coupling in the last diagram. The contributions to the amplitude are

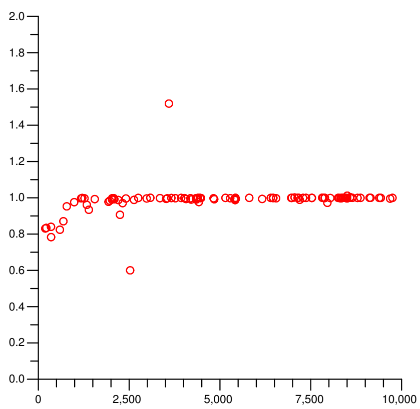
$$\begin{aligned}
\mathcal{M}_1 &= \mathcal{B}_1(1,2,3,4,5)+\mathcal{B}_1(1,2,4,5,3)+\mathcal{B}_1(1,2,5,3,4)\\
&+ \mathcal{B}_1(1,2,5,4,3)+\mathcal{B}_1(1,2,3,5,4)+\mathcal{B}_1(1,2,4,3,5)\;\;,\\
\mathcal{B}_1(1,2,i_3,i_4,i_5) &= i\hbar^{3/2}\;g_{\text{ZZH}}{}^3\;\epsilon_1{}^\mu\;\Pi_\mu{}^\lambda(p_1+p_{i_3})\;\Pi_{\lambda\nu}(p_2+p_{i_5})\;\epsilon_2{}^\nu\\
&\times\;\Delta_Z(p_1+p_{i_3})\;\Delta_Z(p_2+p_{i_5})\;\;,\\
\mathcal{M}_2 &= \mathcal{B}_2(1,2,3,4,5)+\mathcal{B}_2(1,2,4,5,3)+\mathcal{B}_2(1,2,5,3,4)\\
&+ \mathcal{B}_2(2,1,3,4,5)+\mathcal{B}_2(1,2,4,5,3)+\mathcal{B}_2(1,2,5,3,4)\;\;,\\
\mathcal{B}_2(i_1,i_2,i_3,i_4,i_5) &= i\hbar^{3/2}\;g_{\text{ZZH}}{}^2\;g_{\text{HHH}}\;\epsilon_{i_1}{}^\mu\;\Pi_{\mu\nu}(p_{i_2}+p_{i_5})\;\epsilon_{i_2}{}^\nu\\
&\times\;\Delta_Z(p_{i_2}+p_{i_5})\;\Delta_H(p_{i_3}+p_{i_4})\;\;,\\
\mathcal{M}_3 &= \mathcal{B}_3(1,2,3,4,5)+\mathcal{B}_3(1,2,4,5,3)+\mathcal{B}_3(1,2,5,3,4)\;\;,\\
\mathcal{B}_3(1,2,i_3,i_4,i_5) &= i\hbar^{3/2}\;g_{\text{ZZH}}\;g_{\text{HHH}}{}^2\;(\epsilon_1\cdot\epsilon_2)\;\Delta_H(p_1+p_2)\;\Delta_H(p_{i_4}+p_{i_5})\;\;,\\
\mathcal{M}_4 &= \mathcal{B}_4(1,2,3,4,5)+\mathcal{B}_4(1,2,4,5,3)+\mathcal{B}_4(1,2,5,3,4)\\
&+ \mathcal{B}_4(2,1,3,4,5)+\mathcal{B}_4(1,2,4,5,3)+\mathcal{B}_4(1,2,5,3,4)\;\;,\\
\mathcal{B}_4(i_1,i_2,i_3,i_4,i_5) &= -i\hbar^{3/2}\;g_{\text{ZZHH}}\;g_{\text{ZZH}}\;\epsilon_{i_1}{}^\mu\;\Pi_{\mu\nu}(p_{i_2}+p_{i_5})\;\epsilon_{i_2}{}^\nu\\
&\times\;\Delta_Z(p_{i_2}+p_{i_5})\;\;,\\
\mathcal{M}_5 &= \mathcal{B}_5(1,2,3,4,5)+\mathcal{B}_5(1,2,4,5,3)+\mathcal{B}_5(1,2,5,3,4)\;\;,\\
\mathcal{B}_5(1,2,i_3,i_4,i_5) &= -i\hbar^{3/2}\;g_{\text{ZZHH}}\;g_{\text{HHH}}\;(\epsilon_1\cdot\epsilon_2)\;\Delta_H(p_{i_3}+p_{i_4})\;\;,\\
\mathcal{M}_6 &= -i\hbar^{3/2}\;g_{\text{ZZH}}\;g_{\text{HHHH}}\;(\epsilon_1\cdot\epsilon_2)\;\Delta_H(p_1+p_2)\;\;. \qquad (11.117)
\end{aligned}$$

A treatment analogous to that of the previous paragraph leads to the following, final Feynman rule :



$$\leftrightarrow \quad \frac{i}{\hbar}\, g_{\mathrm{HHHH}} \quad , \quad g_{\mathrm{HHHH}} = -\frac{3}{4}\, \frac{{Q_{\mathrm{W}}}^2\, {m_{\mathrm{H}}}^2}{{m_{\mathrm{W}}}^2\, {s_{\mathrm{W}}}^2}$$

as indicated by the picture below.



We plot the ratio

$$-\left.\mathcal{M}_6\right\rfloor_L \Big/ \left.\mathcal{M}_{1+\cdot+5}\right\rfloor_L$$

obtained in the same manner as in the previous paragraph. Again, the choice of the factor 3/4 in g_{HHHH} is justified by the fact that the ratio geos to 1 with great accuracy as the scale increases.

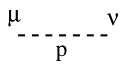
11.5 Private sector : the Abelian Higgs model

11.5.1 A sign of symmetry breaking

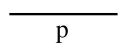
The Standard Model contains a small subset that has an interesting life of its own. It consists of the Z and H particles alone, with their four interactions ZZH, $ZZHH$, HHH and $HHHH$. It is easily seen that, at the tree level, any diagram having as external particles only Z's and H's, can only have these as internal lines as well. This makes the $Z-H$ sector an interestingly self-contained piece of the Standard Model. With the introduction of a single coupling constant

$$g \equiv \frac{Q_{\mathrm{W}}}{2\, s_{\mathrm{W}}\, m_{\mathrm{W}}} \quad \text{hence} \quad g^2 = G_F\sqrt{2} \;\; , \tag{11.118}$$

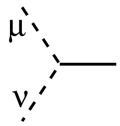
we can rewrite the Feynman rules for the Z, H vertices as follows :



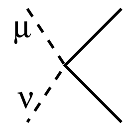
$$\leftrightarrow \quad \frac{i\hbar}{p^2 - {m_{\mathrm{Z}}}^2}\left(-g^{\mu\nu} + p^\mu p^\nu/{m_{\mathrm{Z}}}^2\right)$$



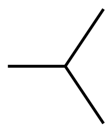
$$\leftrightarrow \quad \frac{i\hbar}{p^2 - {m_{\mathrm{H}}}^2}$$



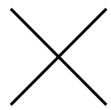
$$\leftrightarrow \quad 2\,\frac{i}{\hbar}\,g\,{m_{\mathrm{Z}}}^2\,g^{\mu\nu}$$



$$\leftrightarrow \quad 2\,\frac{i}{\hbar}\,g^2\,{m_{\mathrm{Z}}}^2\,g^{\mu\nu}$$



$$\leftrightarrow \quad -3\,\frac{i}{\hbar}\,g\,{m_{\mathrm{H}}}^2$$



$$\leftrightarrow \quad -3\,\frac{i}{\hbar}\,g^2\,{m_{\mathrm{H}}}^2$$

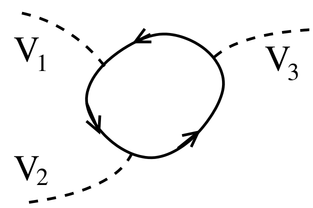
In this section, the Higgs will be denoted by a solid, the Z by a dashed line[28]. Things become interesting if we reconstruct the action corresponding to this set of Feynman rules (including the propagators). It reads

$$\begin{aligned} S &= \int d^4x\,\mathcal{L}\ , \\ \mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial^\mu H)(\partial_\mu H) \\ &\quad + \frac{1}{2}Q^2\,Z^\mu Z_\mu\,(v+H)^2 - \frac{1}{4}\lambda\left((v+H)^2 - v^2\right)^2 \quad , \end{aligned} \tag{11.119}$$

where $v = 1/g$, $Q = m_{\mathrm{Z}}g$, $\lambda = {m_{\mathrm{H}}}^2 g^2/2$, and $F_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$. We see that there is a 'primordial' combination $\phi = v + H$, in terms of which the action looks much simpler. On the other hand, if we use the ϕ field instead of the Higgs field, the ϕ^2 term will have a pre-factor $+{m_{\mathrm{H}}}^2/4$, the *wrong* sign to serve as a mass term ! The potential $(\phi^2 - v^2)^2$ does not have a minimum at $\phi = 0$, but rather a local maximum. Such a situation is considered inherently unstable, and the idea of *spontaneous symmetry breaking* is that the ground state of the world has not $\phi = 0$ but rather $\phi = v$. Therefore v is called the *vacuum expectation value*. The model we have here is, in fact, the resulting form of the simplest model that exhibits this idea of spontaneous symmetry breaking : it is called the *Abelian Higgs model*.

[28]There is a tendency among phenomenologists to always use the same line notation for the same particle. Understandable but not obligatory.

11.6 About axial anomalies



In sections 9.2.6 and 10.3.2, we have discussed triangle diagrams in which three vector particles couple to fermions running around a loop, as indicated on the left. It is always accompanied by a similar diagram in which the orientation of the fermion lines is reversed. In three or more than spacetime dimensions, such diagrams are individually ultraviolet divergent. A fermion propagator at large momentum p^μ goes as $1/p$, while the integration element in D dimensions goes as p^D, leading to a logarithmic divergence or worse if $d \geq 3$. In four dimensions, the diagrams are therefore linearly divergent, but this leads to trouble : there is no unambiguous intergation prescription. If all three fermion-fermion-vector couplings are of vector type, section 9.2.6 shows how the two diagrams have opposite sign, all other things being equal. Therefore, in QED, where the three vector particles are photons, $(V_1, V_2, V_3) = (\gamma, \gamma, \gamma)$, such diagram pairs will completely cancel. In QCD, with three gluons, $(V_1, V_2, V_3) = (g, g, g)$, the two diagrams form a colour-antisymmetric contribution to the one-loop correction of the three-gluon vertex. Two mixed cases are $(V_1, V_2, V_3) = (\gamma, \gamma, g)$, where the diagrams vanish individually because of their colour structure, and $(V_1, V_2, V_3) = (\gamma, g, g)$ where they again cancel against one another. We shall now investigate how things stand if we include the electroweak sector. In particular we have to worry about diagrams containing one or three γ^5's, since in such cases the two diagrams with oppositely-oriented fermion loops have the *same* sign.

Since we shall be interested only in the possible ultraviolet behaviour of the theory, we may assume that all fermions have the same (negligible) mass. Also, we consider the contribution of a single *fermion family*, that is a charged lepton (with charge Q_L), a neutrino (with charge Q_ν, which we alllow to be nonzero for the moment), and an up-type quark (with charge Q_U) and a down-type quark (with charge Q_D). The quarks are supposed to occur in N_c colour types. We consider the collective effect of all these fermions together in the table below, where we have a look at all the possible problematic axial-vector contributions. We shall apply the shorthand $\sigma \equiv 4{s_\mathrm{w}}^2/e$, and in the third column drop all common overall factors.

V_1, V_2, V_3	terms	total contribution
γ, γ, Z	$\sum_f {Q_f}^2 a_f$	$N_c \left({Q_U}^2 - {Q_D}^2\right) + {Q_\nu}^2 - {Q_L}^2$
γ, Z, Z	$\sum_f Q_f v_f a_f$	$N_c \Big(Q_U(1-\sigma Q_U) + Q_D(1+\sigma Q_D)\Big)$ $+ Q_\nu(1-\sigma Q_\nu) + Q_L(1+\sigma Q_L)$
γ, W, W	$\sum_f Q_f {g_\mathrm{w}}^2$	$N_c(Q_U + Q_D) + Q_\nu + Q_L$
Z, Z, Z	$\sum_f {v_f}^2 a_f$	$N_c \Big((1-\sigma Q_U)^2 - (1+\sigma Q_D)^2\Big)$ $+ (1-\sigma Q_\nu)^2 - (1+\sigma Q_L)^2$
Z, Z, Z	$\sum_f {a_f}^3$	trivially zero
Z, W, W	$\sum_f v_f {g_\mathrm{w}}^2$	$N_c \Big((1-\sigma Q_U) - (1+\sigma Q_D)\Big)$ $+ (1-\sigma Q_\nu) - (1+\sigma Q_L)$
Z, W, W	$\sum_f a_f {g_\mathrm{w}}^2$	trivially zero
Z, g, g	$\sum_f a_f {g_\mathrm{s}}^2$	trivially zero

Inspection tells us that in fact two conditions suffice to let the sum over fermions vanish :

$$\begin{aligned} N_c\left({Q_U}^2 - {Q_D}^2\right) &= {Q_L}^2 - {Q_\nu}^2 \quad , \\ N_c(Q_U + Q_D) &= -\,Q_L - Q_\nu \quad . \end{aligned} \tag{11.120}$$

The ratio of these two result in the requirement

$$Q_U - Q_D = Q_\nu - Q_L \quad (= -Q_\mathrm{w}) \quad , \tag{11.121}$$

which is nothing else than the charge conservation we already encountered in Eq.(11.30). We see that the single condition, necessary to ensure all the cancellations, is that the quark charges conform to

$$Q_U = \frac{Q_\nu - Q_L}{2} - \frac{Q_\nu + Q_L}{2N_c} \quad , \quad Q_D = \frac{Q_L - Q_\nu}{2} - \frac{Q_\nu + Q_L}{2N_c} \quad , \tag{11.122}$$

In the Standard Model, with $N_c = 3$, we have indeed

$$Q_\nu = 0 \quad , \quad Q_U = -\frac{2}{3}Q_L \quad , \quad Q_D = \frac{1}{3}Q_L \quad . \tag{11.123}$$

This is one of those curious instances where the universe appears to arrange itself to make the theory as well-behaved as possible. Whether there is a deeper reason for this I do not pretend to know.

11.7 Conclusions and remarks

We have now derived all vertices of the electroweak Standard Model. That is to say, the more usual textbook derivations arrive at precisely the set of Feynman rules that we have also obtained. There are, however, a number of differences between the treatment given here and the usual one.

- We have not invoked any symmetry principle, but rather the (underlying) $SU(2) \times U(1)$ symmetry has spontaneously emerged from our choices for the 'minimal' solution, for instance by insisting on only a single Z particle while we could have opted for more.

- Since we have not invoked any symmetry, there is also no need to explain its 'breaking' in order to arrive at massive W's and Z's. Instead, we have simply faced the observed fact of their massiveness and come to grips with it with the help of a Higgs sector.

- There is, as we have already discussed, a logical distinction between the two uses of the weak mixing angle, in which the ratio of coupling constants is logically 'prior' to the ratio $m_{\mathrm{W}}/m_{\mathrm{Z}}$.

- We have not needed to introduce any Higgs doublet, but rather only a single, physically observable H particle. This approach elegantly sidesteps the question *whether*, and if so *how* the Higgs field configuration is 'spontaneously broken'. This would indicate that the Higgs particle is also, in a sense, logically prior to a complete Higgs doublet.

11.8 Exercises for Chapter 11

Excercise 54 The width of the W
Assume that the weak coupling constant g_{W} is universal, *i.e.* independent of the fermion's flavour.

1. Compute (at the tree level) the decay width for the decay

$$W^- \quad \rightarrow \quad e^- \; \bar{\nu}_e$$

2. Assuming that all quarks and leptons are essentially massless compared to the W mass m_{W}, with of course the exception of the top quark. Determine the total W decay with Γ_W. Hint : quarks have colour !

3. Insert your result into Eq.(11.23), and verify that unitarity is not violated in this process. For the unitarity limit, see Appendix 13.13.

Excercise 55 The W width with Cabibbo mixing
In the simplest form of the standard model the W couples with universal coupling to ud and cs quark-antiquark pairs (and, of course to tb but we disregard that here). In fact, its couplings are more complicated, and we have the following pattern of couplings :

$$\begin{aligned} &Wud : g_{\rm W}\, \cos(\theta_c) \quad , \quad Wus : g_{\rm W}\, \sin(\theta_c) \quad , \\ &Wcd : -g_{\rm W}\, \sin(\theta_c) \quad , \quad Wcs : g_{\rm W}\, \cos(\theta_c) \quad . \end{aligned}$$

Here, $\theta_c \approx 13^o$ is the so-called *Cabibbo angle*. Show that this refinement does not change the total W decay width.

Excercise 56 The width of the Z
Compute the total Z decay width at the tree level, with the same approximation for the masses as in exercise 54.

Excercise 57 Landau-Yang strikes again, *sneakily*!
From the fact that there is a ZWW vertex in the electroweak model, and Eq.(11.97), you might hope to circumvent a 'weak' variation of the Landau-Yang theorem by letting $c_{\rm W}$ go to zero, so that the W mass vanishes and the Z could actually decay into two massless spin-1 particles. Show that this is impossible. Hint : this is *simple*.

Excercise 58 The width of the H
Before the Higgs was found in 2012, its mass was unknown. Let us assume that $m_{\rm H}$ is sufficiently large for decays into heavy bosons to be possible.

1. Compute the decay width of the Higgs into a fermion-antifermion pair. Show that these widths are proportional to $m_{\rm H}$.

2. Compute the widths $\Gamma(H \to W^+W^-)$ and $\Gamma(H \to ZZ)$. Show that these widths are proportional to ${m_{\rm H}}^3$.

3. Esimate the value of $m_{\rm H}$ for which $\Gamma_H \approx m_{\rm H}$. This was considered to be the largest realistic value of $m_{\rm H}$.

Excercise 59 Unitarity-violating fermions
Assume that there are some extremely heavy fermions around, called F_1 and F_2, both with mass M, much larger than m_{H}.

1. Compute the total cross section for $F_1\bar{F}_1 \rightarrow F_2\bar{F}_2$ at energies large compared to M.
2. Compare your result with the unitarity limit and find a restriction on M.

Chapter 12

Example computations

In this chapter we shall go through several actual computations of a number of tree-level processes and some loop diagrams. The theory lives in four Minkowski dimensions ; but in the loop calculations we shall use dimensional regularization throughout.

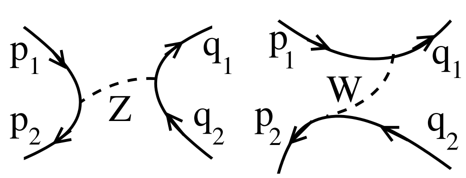
12.1 Neutrino production in e^+e^- scattering

12.1.1 The cross section

In this section we consider the process

$$e^-(p_1, \lambda)\, e^+(p_2, \lambda) \quad \to \quad \nu_j(q_1)\, \bar{\nu}_j(q_2) \quad (j = e, \mu) \;\; ,$$

where both the $e^\pm$ and the neutrinos are taken to be massless, and we have indicated the momenta and the electronic helicities.



For $j = e$ (electron neutrinos) the process is described by two diagrams at the tree level. For $j = \mu$ (muon or tau neutrinos) only the first diagram contributes.

We shall neglect the width of the Z boson[1]. Using the looser notation for

[1] Since the W is exchanged with negative invariant mass and hence cannot decay at all, its width is naturally zero in this case anyway.

spinors allowed in the massless case we can write the two diagrams as follows, where we indicate their dependence on the helicity:

$$\begin{aligned}
\mathcal{M}_1(\lambda) &= \frac{i\hbar\, a_\nu}{s - {m_\mathrm{Z}}^2}\, \overline{u}_\lambda(p_2)(v_e + a_e\gamma^5)\gamma^\alpha u_\lambda(p_1)\, \overline{u}_-(q_1)(1+\gamma^5)\gamma_\alpha u_-(q_2) \quad , \\
\mathcal{M}_2(\lambda) &= \frac{i\hbar\, {g_\mathrm{W}}^2}{t - {m_\mathrm{W}}^2}\, \overline{u}_-(q_1)(1+\gamma^5)\gamma^\alpha u_\lambda(p_1)\, \overline{u}_\lambda(p_2)(1+\gamma^5)\gamma_\alpha u_-(q_2) \quad .
\end{aligned} \tag{12.1}$$

We have introduced the Mandelstam variables

$$\begin{aligned}
&s = (p_1 + p_2)^2 = (q_1 + q_2)^2 \quad , \quad t = (p_1 - q_1)^2 = (p_2 - q_2)^2 \quad , \\
&u = (p_1 - q_2)^2 = (p_2 - q_1)^2 = -\,s - t \quad .
\end{aligned} \tag{12.2}$$

The first step is to get rid of the explicit γ^5's :

$$\begin{aligned}
\mathcal{M}_1(\lambda) &= \frac{2i\hbar\, a_\nu(v_e - \lambda a_e)}{s - {m_\mathrm{Z}}^2}\, \overline{u}_\lambda(p_2)\gamma^\alpha u_\lambda(p_1)\, \overline{u}_-(q_1)\gamma_\alpha u_-(q_2) \quad , \\
\mathcal{M}_2(\lambda) &= \frac{4i\hbar\, {g_\mathrm{W}}^2}{t - {m_\mathrm{W}}^2}\, \overline{u}_-(q_1)\gamma^\alpha u_\lambda(p_1)\, \overline{u}_\lambda(p_2)\gamma_\alpha u_-(q_2) \quad .
\end{aligned} \tag{12.3}$$

Using the Chisholm identity we can compute the explicit helicity forms :

$$\begin{aligned}
\mathcal{M}_1(+) &= \frac{4i\hbar\, a_\nu(v_e - a_e)}{s - {m_\mathrm{Z}}^2}\, s_+(p_2, q_2)\, s_-(p_1, q_1) \quad , \\
\mathcal{M}_1(-) &= \frac{4i\hbar\, a_\nu(v_e + a_e)}{s - {m_\mathrm{Z}}^2}\, s_-(p_2, q_1)\, s_+(q_2, p_1) \quad , \\
\mathcal{M}_2(+) &= 0 \quad , \\
\mathcal{M}_2(-) &= \frac{8i\hbar\, {g_\mathrm{W}}^2}{t - {m_\mathrm{W}}^2}\, s_-(q_1, p_2)\, s_+(q_2, p_1) \quad .
\end{aligned} \tag{12.4}$$

Keeping in mind the antisymmetry of the spinor products, and the Fermi minus sign, we find that up to an irrelevant overall complex phase the amplitudes are given by

$$\begin{aligned}
\mathcal{M}(+) &= 4\hbar\, \frac{a_\nu(v_e - a_e)}{s - {m_\mathrm{Z}}^2}\, t \quad , \\
\mathcal{M}(-) &= 4\hbar \left(\frac{a_\nu(v_e + a_e)}{s - {m_\mathrm{Z}}^2} + \frac{2{g_\mathrm{W}}^2}{t - {m_\mathrm{W}}^2} \right) u \quad .
\end{aligned} \tag{12.5}$$

We arrive at

$$\left\langle|\mathcal{M}|^2\right\rangle \;=\; 4\hbar^2\left\{\frac{{a_\nu}^2(v_e-a_e)^2}{(s-{m_\mathrm{Z}}^2)^2}\,t^2 \;+\; \frac{{a_\nu}^2(v_e+a_e)^2}{(s-{m_\mathrm{Z}}^2)^2}\,u^2\right.$$
$$\left. +\;\frac{4{g_\mathrm{W}}^2a_\nu(v_e+a_e)}{(s-{m_\mathrm{Z}}^2)(t-{m_\mathrm{W}}^2)}\,u^2 \;+\; \frac{4{g_\mathrm{W}}^4}{(t-{m_\mathrm{W}}^2)^2}\,u^2\right\}\quad . \tag{12.6}$$

The centre-of-mass frame is the obvious choice to work in ; in this frame

$$t = -\frac{s}{2}(1-\cos\theta)\quad , \tag{12.7}$$

where θ is the angle between $\vec{p}_1$ and $\vec{q}_1$, and the cross section has no azimuthal-angle dependence. We may therefore write the phase space as follows :

$$dV(p_1+p_2;q_1,q_2) = \frac{d\cos\theta\;d\phi}{32\pi^2}\;\to\;\frac{d\cos\theta}{16\pi} = \frac{1}{8\pi s}dt\quad , \tag{12.8}$$

with the integration interval being $t\in[-s,0]$. The various integrals are easily worked out ; we have

$$\int\limits_{-s}^{0}\frac{u^2}{(t-{m_\mathrm{W}}^2)^2} = s\left(\frac{s}{{m_\mathrm{W}}^2}+2-2\left(1+\frac{{m_\mathrm{W}}^2}{s}\right)\log\left(1+\frac{s}{{m_\mathrm{W}}^2}\right)\right)\quad ,$$
$$\int\limits_{-s}^{0}\frac{u^2}{t-{m_\mathrm{W}}^2} = s^2\left(\frac{3}{2}+\frac{{m_\mathrm{W}}^2}{s}-\left(1+\frac{{m_\mathrm{W}}^2}{s}\right)^2\log\left(1+\frac{s}{{m_\mathrm{W}}^2}\right)\right)\quad ,$$
$$\int\limits_{-s}^{0}t^2\,dt = \int\limits_{-s}^{0}u^2\,dt = \frac{1}{3}s^3\quad . \tag{12.9}$$

Putting everything together[2] we obtain for the total cross section the expression

$$\sigma(e^+e^-\to\nu_e\bar{\nu}_e) =$$
$$\frac{\hbar^2}{4\pi s}\left\{\frac{2}{3}{a_\nu}^2\left({v_e}^2+{a_e}^2\right)\left|\frac{s}{s-{m_\mathrm{Z}}^2}\right|^2\right.$$
$$+\;4{g_\mathrm{W}}^2a_\nu(v_e+a_e)\frac{s}{s-{m_\mathrm{Z}}^2}\left(\frac{3}{2}+\frac{{m_\mathrm{W}}^2}{s}-\left(1+\frac{{m_\mathrm{W}}^2}{s}\right)^2\log\left(1+\frac{s}{{m_\mathrm{W}}^2}\right)\right)$$
$$\left.+\;4{g_\mathrm{W}}^4\left(\frac{s}{{m_\mathrm{W}}^2}+2-2\left(1+\frac{{m_\mathrm{W}}^2}{s}\right)\log\left(1+\frac{s}{{m_\mathrm{W}}^2}\right)\right)\right\}\quad . \tag{12.10}$$

[2]And not forgetting the flux factor !

For muon (or tau) neutrinos only the first line remains :

$$\sigma(e^+e^- \to \nu_\mu\bar{\nu}_\mu) = \frac{\hbar^2}{6\pi s} {a_\nu}^2({v_e}^2 + {a_e}^2) \left| \frac{s}{s - {m_\mathrm{Z}}^2} \right|^2 \quad . \tag{12.11}$$

12.1.2 Unitarity considerations

The above neutrino cross section lends itself to a few interesting observations.

At the Z peak

As it stands, the cross section diverges for $s = {m_\mathrm{Z}}^2$. This is of course due to our neglecting Γ_Z. To remedy this, we may replace ${m_\mathrm{Z}}^2$ by ${m_\mathrm{Z}}^2 - i m_\mathrm{Z}\Gamma_\mathrm{Z}$, and neglect the second (W-exchange) diagram[3]. Close to the Z pole, the cross section for each neutrino type is then given by

$$\sigma(e^+e^- \to \nu\bar{\nu}) = \frac{\hbar^2\, {a_\nu}^2({v_e}^2 + {a_e}^2)}{6\pi} \frac{{m_\mathrm{Z}}^2}{(s - {m_\mathrm{Z}}^2)^2 + {m_\mathrm{Z}}^2{\Gamma_\mathrm{Z}}^2} \qquad (s \approx {m_\mathrm{Z}}^2) \ , \tag{12.12}$$

while at the very peak we have[4]

$$\sigma(e^+e^- \to \nu\bar{\nu}) = \frac{\hbar^2\, {a_\nu}^2({v_e}^2 + {a_e}^2)}{6\pi{\Gamma_\mathrm{Z}}^2} \qquad (s = {m_\mathrm{Z}}^2) \ , \tag{12.13}$$

This can be cast in an instructing form, using the fact that

$$\Gamma(Z \to e^+e^-) = \frac{\hbar\,({v_e}^2 + {a_e}^2)\, m_\mathrm{Z}}{12\pi} \quad , \quad \Gamma(Z \to \nu\bar{\nu}) = \frac{\hbar\, {a_\nu}^2\, m_\mathrm{Z}}{6\pi} \quad . \tag{12.14}$$

The cross section at the peak can therefore be written as

$$\sigma(e^+e^- \to \nu\bar{\nu}) = \frac{12\pi}{s} \left(\frac{\Gamma(Z \to e^+e^-)}{\Gamma_\mathrm{Z}}\right) \left(\frac{\Gamma(Z \to \nu\bar{\nu})}{\Gamma_\mathrm{Z}}\right) \qquad (s = {m_\mathrm{Z}}^2) \ , \tag{12.15}$$

which is exactly the form demanded by unitarity for an intermediate state (the Z) with unit spin (see section 13.13).

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[3]This is really justified ! In the sense in which $\Gamma_\mathrm{Z} \neq 0$ comes about by interactions, Γ_Z is formally of higher order in perturbation theory, and then a factor ${\Gamma_\mathrm{Z}}^{-1}$ actually *lowers* the order of such diagrams. Thus, around the Z pole, the W-exchange diagram is formally of higher order in perturbation theory.

[4]Remember, this is all strictly tree level...

Very high and very, *very* high energy

We may also consider what happens at very high energies, $s \gg m_{\rm Z}, m_{\rm W}$. In that case we may approximate the cross section by

$$\sigma(e^+e^- \to \nu_e\bar{\nu}_e) = \frac{\hbar^2}{\pi}\left\{\frac{g_{\rm W}{}^4}{m_{\rm W}{}^2} - \frac{2g_{\rm W}{}^4 + g_{\rm W}{}^2 a_\nu(v_e + a_e)}{s}\log\left(\frac{s}{m_{\rm W}{}^2}\right)\right\} \quad . \qquad (12.16)$$

At *extremely* high energies the cross section becomes very simple :

$$\sigma(e^+e^- \to \nu_e\bar{\nu}_e) = \frac{\hbar^2\, g_{\rm W}{}^4}{\pi\, m_{\rm W}{}^2} \quad (s \to \infty) \ . \qquad (12.17)$$

This is indeed a nice, simple expression — but it is a constant. Does this not conflict with unitarity, that demands a $1/s$ behaviour ? The solution appears if we realize that the unitarity behaviour is required *in each individual angular momentum channel*, or alternatively in a situation where s becomes very large together with all other momentum transfers : in this case, a fixed t/s ratio. As we can see, at high energies the process is completely dominated by t values close to zero, so the ratio t/s approaches zero[5]. Another way of putting this is to say that in the infinite-energy limit, *all* intermediate angular momentum channels contribute. A quick inspection of Eq.(12.5) shows that for $s \to \infty$ at fixed t/s the amplitudes are indeed just simple, energy-independent (but angle-dependent) quantities[6].

12.2 W pair production in e^+e^- scattering

12.2.1 Setting up the amplitude

In sec. 11.3.1 we have saved unitarity in the process $e^+e^- \to W^+W^-$ by introducing the Z particle, upon which the dangerous terms that lead to faulty high-energy behaviour are cancelled. But of course, the well-behaved remainder is also of interest, and its computation is a nice example of get-your-hands-dirty theoretical work. In detail, the process is described as

$$e^+(p_1,\lambda)\, e^-(p_2,\lambda) \quad \to \quad W^+(q_1,\epsilon_1^{\rho_1})\, W^-(q_2,\epsilon_1^{\rho_2})$$

[5]Since the $(t - m_{\rm W}{}^2)^{-1}$ propagator peaks for vanishing t. The distribution proportional to $(t - m_{\rm W}{}^2)^{-2}$, with $-s < t < 0$, gives expectation value $\langle t\rangle \approx m_{\rm W}{}^2 - m_{\rm W}{}^2\log(s/m_{\rm W}{}^2)$ so that for very large s the ratio is typically $t/s \sim -\log(s)/s$.

[6]Since $u = -s - t$, when t/s is fixed then so are u/s and u/t.

where we explicitly indicate the momenta $p_{1,2}$ (taken to be massless) and $q_{1,2}$, and the polarisation vectors $\epsilon_{1,2}$. The electron helicity is $\lambda = \pm$ and the W polarisations are denoted by $\rho_{1,2}$ that can take the values $\pm$ or 0, the latter standing for the longitudinal polarisation. There are thus 18 different amplitudes to be considered. We write the amplitude as

$$\mathcal{M}(\lambda, \rho_1, \rho_2) = i\hbar \sum_{j=1}^{2} A_j(\lambda)\, B_j(\lambda; \rho_1, \rho_2) \ , \tag{12.18}$$

where the index 1 refers to the ν_e-exchange diagram, 2 to the combined γ/Z diagrams. The A's collect the coupling constants and propagators :

$$\begin{aligned} A_1(\lambda) &= -4{g_{\mathrm{W}}}^2 \frac{1}{\Delta}\, \delta_{\lambda,-} \ , \\ A_2(\lambda) &= \frac{e^2}{s} + \frac{g_{\mathrm{WWZ}}(v_e - \lambda a_e)}{s - {m_{\mathrm{Z}}}^2} = -\frac{e^2 {m_{\mathrm{Z}}}^2}{s(s - {m_{\mathrm{Z}}}^2)} + \frac{4{g_{\mathrm{W}}}^2}{s - {m_{\mathrm{Z}}}^2}\, \delta_{\lambda,-} \end{aligned} \tag{12.19}$$

where $P = p_1 + p_2 = q_1 + q_2$ and $s = P^2$, $\Delta = (p_1 - q_1)^2$. We see that $A_2(+)$ goes asymptotically as $1/s^2$ at high energies, whereas $A_{1,2}(-)$ decreases as $1/s$. The real work of calculation is in the 'spacetime' objects

$$\begin{aligned} B_1(-; \rho_1, \rho_2) &= \overline{u}_-(p_1)\, \not{\epsilon}_1^{\rho_1}\, (\not{q}_1 - \not{p}_1)\, \not{\epsilon}_2^{\rho_2}\, u_-(p_2) \ , \\ B_2(\lambda; \rho_1, \rho_2) &= \overline{u}_\lambda(p_1)\, \gamma_\alpha\, u_\lambda(p_2)\; Y(q_1, \epsilon_1^{\rho_1}; q_2, \epsilon_2^{\rho_2}; -q_1 - q_2, \alpha) \\ &= (\epsilon_1^{\rho_1} \cdot \epsilon_2^{\rho_2})\ \overline{u}_\lambda(p_1)\, (\not{q}_1 - \not{q}_2)\, u_\lambda(p_2) \\ &\quad + 2\,(q_2 \cdot \epsilon_1^{\rho_1})\ \overline{u}_\lambda(p_1)\, \not{\epsilon}_2^{\rho_2}\, u_\lambda(p_2) \\ &\quad - 2\,(q_1 \cdot \epsilon_2^{\rho_2})\ \overline{u}_\lambda(p_1)\, \not{\epsilon}_1^{\rho_1}\, u_\lambda(p_2) \ , \end{aligned} \tag{12.20}$$

where in the last expression we have used $(q_j \cdot \epsilon_j) = 0$.

12.2.2 Momenta and polarisations

We shall, of course, work in the centre-of-mass frame, where the momenta are chosen as follows :

$$p_1^\mu = (E, E\vec{e}_p) \ , \ p_2^\mu = (E, -E\vec{e}_p) \ , \ q_1^\mu = (E, q\vec{e}) \ , \ q_2^\mu = (E, -q\vec{e}) \ . \tag{12.21}$$

The beam energy is E so that $s = 4E^2$, and the W velocities are $\beta = q/E$, where $q^2 = E^2 - {m_{\mathrm{W}}}^2$. The $\vec{e}_p$ and $\vec{e}$ are unit vectors, with $\vec{e}_p \cdot \vec{e} = c$, the cosine of the scattering angle between e^+ and W^+. In addition we define two massless vectors[7]

$$a^\mu = (1, \vec{e}) \quad , \quad b^\mu = (1, -\vec{e}) \tag{12.22}$$

For the transverse W polarisations we shall use $\epsilon^{\pm}_{1,2} = \epsilon_\pm$ and

$$(\epsilon_\pm)^\mu = \frac{1}{\sqrt{8}}\, u_\pm(a)\, \gamma^\mu\, u_\pm(b) \ \ . \tag{12.23}$$

The two longitudinal polarisations are of different : we write

$$\epsilon^0_1 = \frac{1}{m_{\mathrm{W}}\beta}\left(q_1 - \frac{2{m_{\mathrm{W}}}^2}{s}P\right) \ \ , \ \ \epsilon^0_2 = \frac{1}{m_{\mathrm{W}}\beta}\left(q_2 - \frac{2{m_{\mathrm{W}}}^2}{s}P\right) \ \ . \tag{12.24}$$

12.2.3 Working out the amplitudes

It is enlightening to see explicitly how the various elements of $\mathcal{M}$ are computed. First, we can use $\not\epsilon_\rho \not\epsilon_\rho = 0$ to write

$$\begin{aligned} B_1(-;+,+) &= -\,\overline{u}_-(p_1)\,\not\epsilon_+\,\not p_1\,\not\epsilon_+\,u_-(p_2) \\ &= -2(p_1\cdot\epsilon_+)\,\overline{u}_-(p_1)\,\not\epsilon_+\,u_-(p_2) \\ &= -\frac{1}{2}\,\overline{u}_+(a)\,\not p_1\,u_+(b)\,\overline{u}_-(p_1)\,u_+(b)\overline{u}_+(a)\,u_-(p_2) \\ &= -\frac{1}{2}\,s_-(p_1,b)^2\,s_+(a,p_1)\,s_+(a,p_2) \ \ , \\ B_1(-;-,-) &= -\frac{1}{2}\,s_-(p_1,a)^2\,s_+(b,p_1)\,s_+(b,p_2) \ \ . \end{aligned} \tag{12.25}$$

Similarly,

$$\begin{aligned} B_1(-;+,-) &= \frac{1}{2}\,\overline{u}_-(p_1)\,u_+(b)\overline{u}_+(a)\,(\not q_1 - \not p_1)\,u_+(a)\overline{u}_+(b)\,u_-(p_2) \\ &= (a\cdot q_1 - p_1)\,\overline{u}_-(p_1)\,\not b\,u_-(p_2) \\ &= \frac{\Delta + m^2}{2q}\,\frac{\overline{u}_-(p_1)\,\not q_2\,u_-(p_2)}{q} \\ &= \frac{2(\Delta+m^2)}{\beta^2 s}\,\overline{u}_-(p_1)\not q_2\,u_-(p_2) \ \ = \ \ B_1(-;-,+) \ . \end{aligned} \tag{12.26}$$

[7] Note that the definition of a and b can be made Lorentz invariant by requiring them to be massless linear combinations of q_1 and q_2.

The next one is a bit more involved :

$$\begin{aligned}
B_1(-;+,0) &= \frac{1}{m_\mathrm{W}\beta}\overline{u}_-(p_1)\,\not{\epsilon}_+\,(\not{q}_1-\not{p}_1)\,\left(\not{q}_2-\frac{2{m_\mathrm{W}}^2}{s}\not{p}_1\right)\,u_-(p_2)\\
&= \frac{1}{m_\mathrm{W}\beta}\,\overline{u}_-(p_2)\,\left(-\Delta\not{\epsilon}_+-\frac{2{m_\mathrm{W}}^2}{s}(2p_1\cdot q_1)\not{\epsilon}_+\right.\\
&\qquad\left.+\frac{2{m_\mathrm{W}}^2}{s}(2p_1\cdot\epsilon_+)\not{q}_1\right)\,u_-(p_2)\\
&= s_-(p_1,b)s_+(a,p_2)\left(-\Delta-\frac{2{m_\mathrm{W}}^2}{s}(2p_1\cdot q_1)\right)\\
&\quad+\frac{2{m_\mathrm{W}}^2}{2}s_+(a,p_1)s_-(p_1,b)\,q\,s_-(p_1,a)s_+(a,p_2)\Big\}\\
&= \frac{s_-(p_1,b)s_+(a,p_2)}{m_\mathrm{W}\beta\sqrt{2}}\left(-\Delta-\frac{2{m_\mathrm{W}}^2}{s}\Big((2p_1.q_1)-2q(a\cdot p_1)\Big)\right)\\
&= -\frac{1}{m_\mathrm{W}\beta}\,\overline{u}_-(p_1)\,\not{\epsilon}_+\,u_-(p_2)\,\,\Big(\Delta+{m_\mathrm{W}}^2(1-\beta)\Big)\quad . \qquad (12.27)
\end{aligned}$$

This is the most non-trivial of them all : notice the juggling with the spinor products. Nevertheless the result factorizes nicely, and *that* does not depend on the use of the standard form. The other such cases are

$$\begin{aligned}
B_1(-;-,0) &= -\frac{1}{m_\mathrm{W}\beta}\,\overline{u}_-(p_1)\,\not{\epsilon}_+\,u_-(p_2)\,\,\Big(\Delta+{m_\mathrm{W}}^2(1+\beta)\Big)\quad ,\\
B_1(-;0,\pm) &= -B_1(-;\pm,0)\quad . \qquad (12.28)
\end{aligned}$$

The last case for B_1 is

$$\begin{aligned}
&B_1(-;0,0) =\\
&\quad\frac{1}{{m_\mathrm{W}}^2\beta^2}\,\overline{u}_-(p_1)\,\left(\not{q}_1-\frac{2{m_\mathrm{W}}^2}{s}\not{p}_2\right)\,(\not{q}_1-\not{p}_1)\,\left(\not{q}_2-\frac{2{m_\mathrm{W}}^2}{s}\not{p}_1\right)\,u_-(p_2)\\
&= \frac{1}{{m_\mathrm{W}}^2\beta^2}\,\overline{u}_-(p_1)\left(\Delta\not{q}_2-\Delta\frac{2{m_\mathrm{W}}^2}{s}\not{p}_1+\Delta\frac{2{m_\mathrm{W}}^2}{s}\not{p}_2+\frac{4{m_\mathrm{W}}^4}{s^2}\not{p}_2\not{q}_1\not{p}_1\right)u_-(p_2)\\
&= \frac{1}{{m_\mathrm{W}}^2\beta^2}\,\overline{u}_-(p_1)\,\not{q}_2\,u_-(p_2)\,\,\left(\Delta+\frac{4{m_\mathrm{W}}^4}{s}\right)\quad . \qquad (12.29)
\end{aligned}$$

Note that we have used momentum conservation, $\overline{u}(p_1)(\not{q}_1+\not{q}_2)u(p_2)=0$. Computationally, for B_2 things are simpler. First of all, we have trivially

$$B_2(\lambda;+,+)=B_2(\lambda,-,-)=0\quad , \qquad (12.30)$$

and, almost equally trivially,

$$B_2(\lambda;+,-) = B_2(\lambda;-,+) = 2\,\overline{u}_\lambda(p_1)\,\not{q}_2\,u_\lambda(p_2) \quad , \tag{12.31}$$

owing to our choice of the polarisations. Next, by

$$\left(q_{1,2}\cdot\epsilon^0_{2,1}\right) = \frac{1}{m_{\mathrm{W}}\beta}\left((q_1q_2) - {m_{\mathrm{W}}}^2\right) = \frac{s\beta}{2m_{\mathrm{W}}} \tag{12.32}$$

we have

$$B_2(\lambda,\pm,0) = -\frac{s\beta}{m_{\mathrm{W}}}\,\overline{u}_\lambda(p_1)\,\not{\epsilon}_\pm\,u_\lambda(p_2) \quad , = -B_2(\lambda,0,\pm) \quad . \tag{12.33}$$

And, finally,

$$B_2(\lambda;0,0) = \frac{s+2{m_{\mathrm{W}}}^2}{{m_{\mathrm{W}}}^2}\,\overline{u}_\lambda(p_1)\,\not{q}_2\,u_\lambda(p_2) \quad . \tag{12.34}$$

We see that the nonzero B_2 always carry the same spinor sandwich as do the B_1. Indeed they had better if they want to show (partial) cancellation between them. As expected, the B's behave, for large energy, as E^2 if both W's are transverse, while they go as E^3 and E^4 if one or both become longitudinally polarised, respectively.

Of the various spinorial objects we can compute the absolute values :

$$\begin{aligned}
|s_-(p_1,b)^2 s_+(a,p_1)s_+(a,p_2)|^2 &= s^2(1+c)^3(1-c) \quad , \\
|s_-(p_1,a)^2 s_+(b,p_1)s_+(b,p_2)|^2 &= s^2(1-c)^3(1+c) \quad , \\
|\overline{u}_\lambda(p_1)\not{q}_2 u_\lambda(p_2)|^2 &= \frac{1}{4}\beta^2 s^2(1-c^2) \quad , \\
|\overline{u}_\lambda(p_1)\not{\epsilon}_{-\lambda} u_\lambda(p_1)|^2 &= \frac{1}{2}s(1+c)^2 \quad , \\
|\overline{u}_\lambda(p_1)\not{\epsilon}_{\lambda} u_\lambda(p_1)|^2 &= \frac{1}{2}s(1-c)^2 \quad .
\end{aligned} \tag{12.35}$$

Using these results, we can easily evaluate the absolute values of the amplitudes. Some observations are in order here. First, there are two *vanishing* amplitudes : $\mathcal{M}(+;+,+) = \mathcal{M}(+;-,-) = 0$. This is, of course, due to our choice of $\epsilon_\pm$ for both transverse polarisations. Secondly, the nonzero amplitudes *do not look nice*, what with the occurence of the two couplings g_{W} and e, two masses m_{Z} and m_{W}, the denominator Δ here and there, and β cropping up in various places. There is a message here : *real phenomenology is messy*. It is only in the more Platonic realm of very high energies that we may hope to see our results simplify, and this we shall now investigate.

12.2.4 W pair production at very high energy

As has been amply discussed in these notes, the high-energy behaviour of amplitudes often requires diagrams to partially cancel against one another. Indeed, that is what led us to introduce the Z boson in WW production in the first place. However, here we are interested in what is actually *left* after the cancellations have taken place. We shall consider the limit where E is very much larger than $m_{\mathrm{Z}}, m_{\mathrm{W}}$, at fixed c. In that limit, we may approximate $\Delta \approx -2E^2(1-c)$. We shall neglect overall complex phases in what follows. The amplitude for this $2 \to 2$ process should have no energy dimension (*cf.* sec. 6.3.3), so any contribution going as E^{-1} or lower can be neglected. Of course all contributions going as E^{+1} or higher had better cancel[8] ! This actually leaves only a few surviving amplitudes. In the first place,

$$\begin{aligned} \mathcal{M}(-;+,+) &\approx \frac{\hbar e^2}{2{s_{\mathrm{W}}}^2}\frac{(1+c)}{(1-c)}(1-c^2)^{1/2} \;\;, \\ \mathcal{M}(-;-,-) &\approx \frac{\hbar e^2}{2{s_{\mathrm{W}}}^2}\,(1-c^2)^{1/2} \;\;. \end{aligned} \tag{12.36}$$

The apparent singularity at $c = 1$ is of course due to our approximating Δ. Note also that $e^2\hbar$ is actually the properly dimensionless quantity $4\pi\alpha$, see sec. 9.2.5. Next we have

$$\mathcal{M}(+;0,0) \approx \frac{\hbar e^2}{2}\,\frac{{m_{\mathrm{Z}}}^2}{{m_{\mathrm{W}}}^2}\,(1-c^2)^{1/2} \;\;. \tag{12.37}$$

The most difficult one is

$$\begin{aligned} \mathcal{M}(-;0,0) &= \hbar\,\overline{u}_-(p_1)\not{q}_2 u_-(p_2)\,K \;\;, \\ K &= -\frac{4{g_{\mathrm{W}}}^2}{\beta^2{m_{\mathrm{W}}}^2\Delta}\left(\Delta + \frac{4{m_{\mathrm{W}}}^2}{s}\right) \\ &\quad + \frac{s+2{m_{\mathrm{W}}}^2}{{m_{\mathrm{W}}}^2}\left(-\frac{e^2{m_{\mathrm{Z}}}^2}{s(s-{m_{\mathrm{Z}}}^2)} + \frac{4{g_{\mathrm{W}}}^2}{s-{m_{\mathrm{Z}}}^2}\right) \;\;. \end{aligned} \tag{12.38}$$

Performing the cancellations inside the factor K with care we find

$$\mathcal{M}(-;0,0) \approx \frac{\hbar e^2}{2}\,(1-c^2)^{1/2}\;\left(\frac{1}{{s_{\mathrm{W}}}^2} - \frac{{m_{\mathrm{Z}}}^2}{{m_{\mathrm{W}}}^2}\left(\frac{1}{2{s_{\mathrm{W}}}^2} - 1\right)\right) \;\;. \tag{12.39}$$

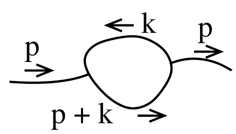
[8] This is what we did all the hard work for, after all.

We have now proven that the amplitudes surviving the high-energy limit are dimensionless and well behaved ; but they still depend on the ratio $m_{\rm Z}/m_{\rm W}$. Since we have taken the fermions to be massless, the Higgs boson is absent at the tree level here. Nevertheless (but this is, as we see, an *additional* input !) we may take $m_{\rm W}/m_{\rm Z} = c_{\rm W}$ from our discussion of the minimal Higgs sector in sec. 11.4.2, and then we find the even more pleasing-looking results

$$\begin{aligned} \mathcal{M}(-;+,+) &\approx 4\pi\alpha\,(1-c^2)^{1/2}\,\frac{1}{2{s_{\rm W}}^2}\,\frac{1+c}{1-c}\;\;, \\ \mathcal{M}(-;-,-) &\approx 4\pi\alpha\,(1-c^2)^{1/2}\,\frac{1}{2{s_{\rm W}}^2}\;\;, \\ \mathcal{M}(+;0,0) &\approx 4\pi\alpha\,(1-c^2)^{1/2}\,\frac{1}{2{c_{\rm W}}^2}\;\;, \\ \mathcal{M}(-;0,0) &\approx 4\pi\alpha\,(1-c^2)^{1/2}\,\frac{1}{4{s_{\rm W}}^2{c_{\rm W}}^2}\;\;. \end{aligned} \tag{12.40}$$

12.3 Self-energy graph in φ^3 theory

The first nontrivial one-loop example is the self-energy diagram from φ^3 theory :



The momentum flowing through the external propagators is p^μ. The diagram (excluding the external propagators) is given by

$$\Sigma(p^2) = \frac{i^2(i\lambda)^2}{(2\pi)^4}\int d^4k\,\frac{1}{(k^2-m^2+i\epsilon)((k+p)^2-m^2+i\epsilon)}\;\;. \tag{12.41}$$

Performing the Feynman trick, and replacing the number of dimensions (originally 4) by 2ω, where ω will approach 2 at the end, we write the diagram as

$$\Sigma(p^2) = \frac{\lambda^2(\mu^2)^{2-\omega}}{(2\pi)^{2\omega}}\int\limits_0^1 dx\int d^{2\omega}k\frac{1}{\Big(k^2+2x(k\cdot p)+xp^2-m^2+i\epsilon\Big)^2}\;\;, \tag{12.42}$$

where μ is the dimensionful quantity needed to keep the overall dimensionality of the diagram consistent. We now shift the loop momentum k^μ to

$k^\mu - xp^\mu$ so as to make the linear term disappear :

$$\Sigma(p^2) = \frac{\lambda^2(\mu^2)^{2-\omega}}{(2\pi)^{2\omega}} \int_0^1 dx \int_{-\infty}^{\infty} dk^0 d^{2\omega-1}d\vec{k} \frac{1}{\left((k^0)^2 - |\vec{k}|^2 + x(1-x)p^2 - m^2 + i\epsilon\right)^2} \quad , \tag{12.43}$$

where we have already singled out the timelike component of k^μ for special treatment : by the Wick rotation, we see that we may rotate[9] the k^0 integration contour

$$\text{from} \quad \int_{-\infty}^{+\infty} dk^0 \quad \text{to} \quad \int_{-i\infty}^{+i\infty} d(ik^0)$$

so that we arrive at

$$\Sigma(p^2) = \frac{i\lambda^2(\mu^2)^{2-\omega}}{(2\pi)^{2\omega}} \int_0^1 dx \int d^{2\omega}k \frac{1}{\left(k^2 - x(1-x)p^2 + m^2 - i\epsilon\right)^2} \quad , \tag{12.44}$$

where k^2 now refers to the *Euclidean* square $(k^0)^2 + |\vec{k}|^2$. Writing $k^2 = u$ the diagram then becomes

$$\Sigma(p^2) = \frac{i\lambda^2(\mu^2)^{2-\omega}}{(4\pi)^\omega\Gamma(\omega)} \int_0^1 dx \int_0^\infty du \frac{u^{\omega-1}}{\left(u - x(1-x)p^2 + m^2 - i\epsilon\right)^2} \quad . \tag{12.45}$$

We can now do the u integral using the standard formula

$$\int_0^\infty du \frac{u^\alpha}{(1+u)^\beta} = \frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha-1)}{\Gamma(\beta)} \tag{12.46}$$

so that

$$\Sigma(p^2) = \frac{i\lambda^2(\mu^2)^{2-\omega}\Gamma(2-\omega)}{(4\pi)^\omega} \int_0^1 dx \frac{1}{\left(m^2 - x(1-x)p^2 - i\epsilon\right)^{2-\omega}} \quad . \tag{12.47}$$

[9]Note that the direction of the Wick rotation does *not* depend on $|\vec{k}|^2$, p^2 or m^2 : the poles are *always* in the lower-right-hand and the upper-left-hand parts of the complex k^0 plane.

The factor $\Gamma(2-\omega)$ indicates that this diagram is logarithmically divergent[10]. At this point we may carefully take the limit $\omega \to 2$. Writing $\omega = 2-\epsilon$ (where *this* ϵ has nothing to do with the $i\epsilon$ in the propagators ! This should not lead to confusion) we have the following expansion :

$$\begin{aligned}\Gamma(2-\omega) &= \Gamma(\epsilon) = \frac{\Gamma(1+\epsilon)}{\epsilon} \\ &= \frac{1}{\epsilon}\left(\Gamma(1) + \Gamma'(1)\epsilon + \Gamma''(1)\epsilon^2/2 + \cdots\right) \\ &= \frac{1}{\epsilon} - \gamma_E + \left(\frac{\gamma_E^2}{2} + \frac{\pi^2}{12}\right)\epsilon + \cdots \quad , \end{aligned} \tag{12.48}$$

where $\gamma_E \approx 0.5772156649$ is Euler's constant[11]. Similarly,

$$A^\epsilon = \exp(\epsilon \log(A)) = 1 + \epsilon \log(A) + \epsilon^2 \log(A)^2/2 + \cdots \tag{12.49}$$

Where we should truncate the ϵ expansion depends on the loop order we are considering. For two-loop computations, the terms with ϵ^1 must be retained, but for the present one-loop calculation we can restrict ourselves to the divergent and constant terms :

$$\begin{aligned}\Sigma(p^2) &= \frac{i\lambda^2}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + \log(\mu^2) - R(s)\right) \quad , \\ R(s) &= \int\limits_0^1 dx\ \log(m^2 - x(1-x)s - i\epsilon) \quad . \end{aligned} \tag{12.50}$$

In the evaluation of the x integral, it of course becomes important to keep careful track of the logarithm's singularity structure. We can distinguish three cases, depending on the value of $s = p^2$.

1. $s < 0$. In this case the logarithm's argument is always positive. Writing $x = (1+y)/2$ we have, with $b = \sqrt{1 + 4m^2/|s|}$, and using partial

[10] In four spacetime dimensions. A factor $\Gamma(1-\omega)$ implies a *quadratic* divergence in four dimensions. For higher dimensions the divergences become more severe, as is only to be expected fos such multidimensional integrals.

[11] Lots and lots of mathematical things are called after Euler. To spread the credit somewhat, it is also called the Euler-Mascheroni constant.

integration :

$$
\begin{aligned}
R(s) &= \int_0^1 dx\ \log(m^2 + |s|x - |s|x^2) \\
&= \int_0^1 dy\ \log(m^2 + |s|/4 - |s|y^2/4) \\
&= \log(|s|) + \int_0^1 dy\ \log(b^2 - y^2) \\
&= \log(|s|) + \Big[y\log(b^2 - y^2)\Big]_0^1 + \int_0^1 dy \frac{2y^2}{b^2 - y^2} \\
&= \log(|s|) + \log(b^2 - 1) + b\log\left(\frac{b+1}{b-1}\right) - 2 \\
&= \log(m^2) - 2 + b\log\left(\frac{b+1}{b-1}\right) \quad . \qquad (12.51)
\end{aligned}
$$

2. $0 < s < 4m^2$. Proceeding in a similar way as above, but now with $\eta = \sqrt{4m^2/s - 1}$, we can write

$$
\begin{aligned}
R(s) &= \log(s/4) + \int_0^1 dy \log(y^2 + \eta^2) \\
&= \log(s/4) + \log(\eta^2 + 1) - \int_0^1 dy \frac{2y^2}{y^2 + \eta^2} \\
&= \log(m^2) - 2 + 2\eta\arctan(1/\eta) \quad . \qquad (12.52)
\end{aligned}
$$

3. $s > 4m^2$. This is the more tricky case since the argument crosses zero twice as x moves between 0 and 1. The two roots are given as $x_{0,1}$, where

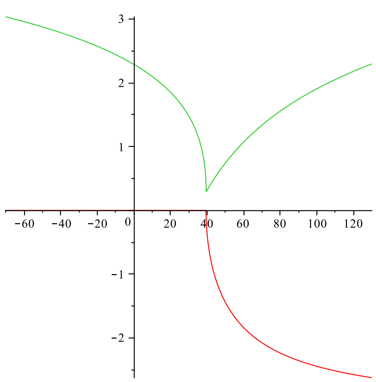
$$x_1 = (1{+}\beta{+}i\epsilon)/2 \quad , \quad x_0 = 1 - x_1 = (1{-}\beta{-}i\epsilon) \quad , \quad \beta = \sqrt{1 - 4m^2/s} \quad . \qquad (12.53)$$

Since $\beta > 0$ we shall have occasion to use

$$\log(-x_1) = \log(x_1) - i\pi \quad , \quad \log(-x_0) = \log(x_0) + i\pi \quad . \qquad (12.54)$$

We evaluate the integral as follows :

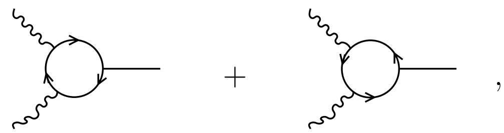
$$
\begin{aligned}
R(s) &= \log(s) + \int\limits_0^1 dx\ \Big(\log(x-x_0) + \log(x-x_1)\Big) \\
&= \log(s) + \Bigg[(x-x_0)\log(x-x_0) + (x-x_1)\log(x-x_1) - 2x\Bigg]_0^1 \\
&= \log(s) + x_1\Big(\log(x_1) + \log(-x_1)\Big) + x_0\Big(\log(x_0) + \log(-x_0)\Big) - 2 \\
&= \log(s) + 2x_1\log(x_1) + 2x_0\log(x_0) + i\pi(x_0 - x_1) \\
&= \log(m^2) - 2 + \beta\left(\log\left(\frac{1+\beta}{1-\beta}\right) - i\pi\right) \quad .
\end{aligned}
\tag{12.55}
$$



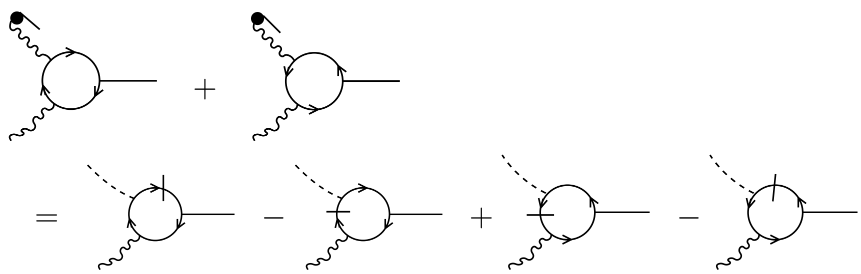
The function $R(s)$ for $m = 3.14$. It is continuous both at $s = 0$ and $s = 4m^2$. The real part of R is positive (since $\log(3.14^2) > 2$), its imaginary part is negative. At $s = 4m^2$ the imaginary part 'switches on' suddenly. This indicates that $R(s)$ has a cut along the real axis starting at that value ; and of course the 'kink' in the real part tells us the same. We also see that the real part of $\Sigma(p^2)$ only develops above the 'threshold' value $p^2 = 4m^2$, and is positive there, as required by the unitarity arguments of chapter 6.

12.4 The gluon-gluon-Higgs vertex

Although the gluon is massless and therefore has no direct coupling to the Higgs, such a coupling is effectively realized by quark loops. Since the top quark, being the heaviest, has the strongest interaction with the Higgs, we shall concentrate on this. At one-loop order, we then have two contributing diagrams :

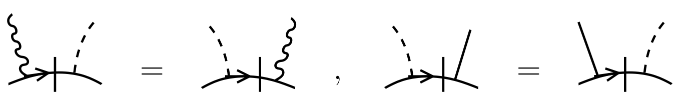


which differ in the orientation of the top quark line. If our theory is to be consistent, this amplitude must be ultraviolet-finite (since otherwise we would have to introduce a counterterm which would mean a direct gluon-Higgs coupling after all) and it must obey current conservation. We shall verify this point first, by putting a handlebar on one of the gluons :



(12.56)

By using the identities



(12.57)

we see that the third and fourth diagram are actually equal to the first and second one (flipped over), so that the sum vanishes and current conservation (gauge invariance) is assured.

The process we investigate is, more explicitly, given as

$$g(q_1, \epsilon_1, j) \quad + \quad g(q_2, \epsilon_2, \ell) \quad \to \quad H$$

where we have explicitly given the momenta, polarizations, and colours of the gluons. We denote the Higgs mass by m and the top quark mass by M, and shall make extensive use of

$$(q_i \cdot q_i) = (q_i \cdot \epsilon_i) = 0 \quad , \quad 2(q_1 \cdot q_2) = m^2 \quad . \tag{12.58}$$

Although we do not expect (or hope to see) divergences, we shall still work in 2ω dimensions for reasons that will be come clear later on. One of our diagrams[12] is given by

$$\begin{aligned}
\mathcal{M}_1 &= \frac{(-1)i^6 g^2 g_{\mathrm{ttH}}\mu^{4-2\omega}}{(2\pi)^{2\omega}}\mathrm{Tr}\left(T^j T^\ell\right)\int d^{2\omega}p\;\frac{N}{D}\quad, \\
D &= (p^2-M^2+i\epsilon)((p-q_2)^2-M^2+i\epsilon)((p+q_1)^2-M^2+i\epsilon)\quad, \\
N &= \mathrm{Tr}\left((\not{p}+M)\not{\epsilon}_2(\not{p}-\not{q}_2+M)(\not{p}+\not{q}_1+M)\not{\epsilon}_1\right)\quad. \qquad (12.59)
\end{aligned}$$

The other diagram, $\mathcal{M}_2$, is obtained by interchange of the two gluons. Using the shorthand

$$\delta x \equiv dx_1\,dx_2\,dx_3\;\delta(x_1+x_2+x_3-1) \qquad (12.60)$$

we can write, using the Feynman trick,

$$\frac{1}{D} = \int \delta x\;\frac{2}{\left(p^2+2x_1(p\cdot q_1)-2x_2(p\cdot q_2)-M^2+i\epsilon\right)^3}\quad; \qquad (12.61)$$

which, by the redefinition

$$p^\mu = k^\mu - x_1 {q_1}^\mu + x_2 {q_2}^\mu \qquad (12.62)$$

becomes

$$\frac{1}{D} \to \int \delta x\;\frac{2}{\left(k^2-M^2+x_1x_2m^2+i\epsilon\right)^3}\quad. \qquad (12.63)$$

Since the maximum value[13] of x_1x_2 is $1/4$, we see that the $i\epsilon$ can be expected to become relevant whenever $m^2 \geq 4M^2$, *i.e.* when the Higgs is sufficiently heavy to decay into two top quarks[14].

We now turn to N. Since the whole amplitude is current-conserving, we are allowed to replace the polarizations by explicitly current-conserving combinations :

$${\epsilon_1}^\mu \;\to\; {\eta_1}^\mu = {\epsilon_1}^\mu - \frac{(q_2\cdot\epsilon_1)}{(q_1\cdot q_2)}{q_1}^\mu \quad,\quad {\epsilon_2}^\mu \;\to\; {\eta_2}^\mu = {\epsilon_2}^\mu - \frac{(q_1\cdot\epsilon_2)}{(q_1\cdot q_2)}{q_2}^\mu\quad, \qquad (12.64)$$

---

[12] In fact, the second one.

[13] Reached when $x_1 = x_2 = 1/2, x_3 = 0$.

[14] Since we now know that $M > m$ this is not in order here, but for lighter quarks it may be.

with the nice properties that

$$\eta_1 \cdot q_{1,2} = \eta_2 \cdot q_{1,2} = 0 \ . \tag{12.65}$$

This simplifies the trace in N :

$$\begin{aligned} N \to \mathrm{Tr}\left((\not{p} + M)\not{\eta}_2(\not{p} - \not{q}_2 + M)(\not{p} + \not{q}_1 + M)\not{\eta}_1 \right) \\ = \ 4M\left\{ 4(p\cdot\eta_1)(p\cdot\eta_2) - p^2(\eta_1\cdot\eta_2) + M^2(\eta_1\cdot\eta_2) - m^2(\eta_1\cdot\eta_2)/2 \right\} \ . \end{aligned} \tag{12.66}$$

Performing the same shift (12.62) as before, we have

$$\begin{aligned} N \ \to \ 4M\Big\{ & 4(k\cdot\eta_1)(k\cdot\eta_2) - k^2(\eta_1\cdot\eta_2) + M^2(\eta_1\cdot\eta_2) + m^2(\eta_1\cdot\eta_2)(x_1x_2 - 1/2) \\ & + 2(k\cdot q_1)(\eta_1\cdot\eta_2) - 2(k\cdot q_2)(\eta_1\cdot\eta_2)\Big\} \ . \end{aligned} \tag{12.67}$$

Since D is even in k, the integral of the last two terms in (12.67) will vanish, and we discard them. Considering the first term, we notice that Lorentz invariance requires that

$$\int d^{2\omega}k\ k^\mu\, k^\nu\, f(k^2) = \frac{1}{2\omega}\, g^{\mu\nu} \int d^{2\omega}k\ k^2\, f(k^2) \ \ , \tag{12.68}$$

where the proportionality factor can be checked by multiplying both sides with $g_{\mu\nu}$. The effective form of N is therefore, finally,

$$N \to 4M(\eta_1\cdot\eta_2)\left(\left(\frac{2}{\omega} - 1\right) k^2 + M^2 + m^2(x_1x_2 - 1/2) \right) \ \ , \tag{12.69}$$

and the diagram is therefore

$$\begin{aligned} \mathcal{M}_1 &= 4g^2 g_{\mathrm{ttH}}\delta_{j,\ell} M(\eta_1\cdot\eta_2)\mu^{4-2\omega}\ Q \ \ , \\ Q &= \frac{1}{(2\pi)^{2\omega}} \int \delta x\, d^{2\omega}k \frac{\left(\frac{2}{\omega} - 1\right) k^2 + M^2 + m^2(x_1x_2 - 1/2)}{\left(k^2 - M^2 + x_1x_2m^2 + i\epsilon\right)^3} \end{aligned} \tag{12.70}$$

We see that in fact $\mathcal{M}_2 = \mathcal{M}_1$, owing to our use of the η's rather than the ϵ's. Performing the Wick rotation, the by-now familiar[15] techniques allow us to write

$$\begin{aligned} Q &= \frac{i}{(4\pi)^\omega \Gamma(\omega)} \int \delta x \int\limits_0^\infty ds\; s^{\omega-1} \frac{\left(\frac{2-\omega}{\omega}\right) s - M^2 - m^2(x_1x_2 - 1/2)}{\left(s + M^2 - x_1x_2m^2 - i\epsilon\right)^3} \\ &= \frac{i}{(4\pi)^\omega \Gamma(\omega)} \int \delta x \left[\left(\frac{2-\omega}{\omega}\right) \frac{\Gamma(\omega+1)\Gamma(2-\omega)}{\Gamma(3)} \frac{1}{(M^2 - x_1x_2m^2 - i\epsilon)^{2-\omega}} \right. \\ &\quad \left. - \frac{\Gamma(\omega)\Gamma(3-\omega)}{\Gamma(3)} \frac{M^2 + m^2(x_1x_2 - 1/2)}{(M^2 - x_1x_2m^2 - i\epsilon)^{3-\omega}}\right] \quad . \end{aligned} \tag{12.71}$$

From the identities

$$\Gamma(\omega+1)/\omega = \Gamma(\omega) \quad , \quad (2-\omega)\Gamma(2-\omega) = \Gamma(3-\omega) \quad , \tag{12.72}$$

we see that

$$Q = \frac{i\, m^2 \Gamma(3-\omega)}{4(4\pi)^\omega} \int \delta x \frac{1 - 4x_1x_2}{(M^2 - x_1x_2m^2 - i\epsilon)^{3-\omega}} \quad . \tag{12.73}$$

Now we can afford to let $\omega \to 2$, and find for the amplitude

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 + \mathcal{M}_2 = 2\mathcal{M}_1 \\ &= \frac{2ig^2 g_{\mathrm{ttH}} M}{(4\pi)^2}\, \delta_{j,\ell}\, (\eta_1 \cdot \eta_2)\, F(M^2/m^2) \quad , \\ F_{\mathrm{ttH}}(t) &= \int\limits_0^1 dx_1 \int\limits_0^{1-x_1} dx_2 \quad \frac{1 - 4x_1x_2}{t - x_1x_2 - i\epsilon} \quad . \end{aligned} \tag{12.74}$$

A number of remarks are in order here. For very large values of t we have

$$F_{\mathrm{ttH}}(t) \approx 1/(3t) \quad . \tag{12.75}$$

Since

$$g_{\mathrm{ttH}} = \frac{eM}{2s_{\mathrm{W}} m_{\mathrm{W}}} \tag{12.76}$$

[15] Hopefully.

the amplitude becomes independent of M as M becomes very large[16]. That this happens is due to the fact that in Eq.(12.71) the M^2 terms in the numerator cancels if we combine the two terms. But for *that* to happen, the first term has to be present. Were we to set $d = 4$ from the outset, it would be absent. Of course, the fact that the numerator would contain a k^2 term and the loop integral would be divergent if d would differ ever so slightly from 4 should give us pause. This is the reason why we have to stick to variable dimension in this calculation : if we don't, then the amplitude will be proportional to M^2 ! Furthermore we can introduce the 'field strength tensors'

$$F_j{}^{\mu\nu} \equiv \epsilon_j{}^{\mu} q_j{}^{\nu} - q_j{}^{\mu}\epsilon_j{}^{\nu} \quad , \quad j = 1, 2 \quad , \tag{12.77}$$

and realize that

$$(\eta_1 \cdot \eta_2) = m^2 \; F_1{}^{\mu\nu} \, F_{2\mu\nu} \quad . \tag{12.78}$$

With the additional definition

$$\alpha_s = g^2/(4\pi) \tag{12.79}$$

we see that the amplitude takes on the form

$$\lim_{M\to\infty} \mathcal{M} = \frac{ie\,\alpha_s}{12\pi m_{\mathrm{W}} s_{\mathrm{W}}} \; F_1{}^{\mu\nu} \, F_{2\mu\nu} \, \delta_{j,\ell} \quad . \tag{12.80}$$

For finite values of M we have

$$\mathcal{M} = \left(\lim_{M\to\infty} \mathcal{M}\right) \frac{3M^2}{m^2} \, F_{\mathrm{ttH}}(M^2/m^2) \quad . \tag{12.81}$$

We now turn to the calculation of $F(t)$ for finite t, and first rewrite

$$\begin{aligned}
F(t) &= 2 + (1-4t)H(t) \quad , \\
H(t) &= \int\limits_0^1 dx_1 \int\limits_0^{1-x_1} dx_2 \frac{1}{t - x_1 x_2 - i\epsilon} \\
&= \int\limits_0^1 dx \, \frac{-1}{x} \log\left(\frac{t - i\epsilon - x + x^2}{t - i\epsilon}\right)
\end{aligned}$$

[16]This further implies that very heavy fermions from a possible fourth generation will contribute appreciably to de decay $H \to gg$ or $H \to \gamma\gamma$!

$$
\begin{aligned}
&= \int_0^1 dx\, \frac{-1}{x} \log\left(\frac{(x_- - x)(x_+ - x)}{t - i\epsilon}\right) \\
&= \int_0^1 dx \left(-\frac{1}{x}\log\left(1 - \frac{x}{x_-}\right) - \frac{1}{x}\log\left(1 - \frac{x}{x_+}\right)\right) \quad , \qquad (12.82)
\end{aligned}
$$

where

$$ {x_\pm}^2 - x_\pm + t - i\epsilon = 0 \qquad (12.83) $$

so that

$$ x_+ + x_- = 1 \quad , \quad x_+ x_- = t - i\epsilon \qquad (12.84) $$

We now distinguish two cases. In the first place, let $t > 1/4$. Then

$$ x_\pm = \frac{1}{2}\Big(1 \pm \gamma\Big) \quad , \quad \gamma = \sqrt{4t - 1}\ . \qquad (12.85) $$

We can then use the definition of the dilogarithm function Li_2, further described in Appendix 13.15, to arrive at

$$ t > 1/4 \quad : \quad H(t) = \mathrm{Li}_2\left(\frac{1}{x_+}\right) + \mathrm{Li}_2\left(\frac{1}{x_-}\right) \quad . \qquad (12.86) $$

Since $x_- = (x_+)^*$, this expression has no imaginary part, as expected. Furthermore, the expansion

$$ \mathrm{Li}_2(z) = \sum_{n \geq 1} \frac{z^n}{n^2} \quad , \qquad (12.87) $$

valid for $|z| < 1$, leads to the correct form for $F(t)$ for large t (hence large γ).

The other case of interest[17] is $0 < t < 1/4$. We can write

$$ x_\pm = u_\pm \pm i\epsilon \quad , \quad u_\pm = \frac{1}{2}\Big(1 \pm \beta\pm\Big) \quad , \quad \beta = \sqrt{1 - 4t}\ . \qquad (12.88) $$

We now first consider

$$ H_- = \int_0^1 dx\, \frac{-1}{x} \log\left(1 - \frac{1}{x_-}\right) \quad . \qquad (12.89) $$

[17] Since $t = M^2/m^2$, the case $t < 0$ is irrelevant.

Cleverly, we first take a derivative :

$$\begin{aligned}\frac{\partial}{\partial u_-}H_- &= \frac{\partial}{\partial x_-}H_- = \int_0^1 dx \frac{1}{x_-(x-x_-)} \\ &= \int_0^1 dx \frac{1}{x}\Big(\log(1-x_-) - \log(-x_-)\Big) \quad . \end{aligned} \tag{12.90}$$

By carefully taking the limit $i\epsilon \to 0$ we see that this can be written as

$$\frac{\partial}{\partial u_-}H_- = \frac{1}{u_-}\Big(\log(1-u_-) - \log(u_-) - i\pi\Big) \quad . \tag{12.91}$$

Also realizing that $H_- = \mathrm{Li}_2(1)$ when $u_- = 1$, we arrive at

$$H_- = \frac{\pi^2}{3} - \mathrm{Li}_2(u_-) - \frac{1}{2}\log(u_-)^2 - i\pi\log(u_-) \quad . \tag{12.92}$$

We can replace u_- by u_+ so as to compute the analogous H_+ ; the only difference is in the different sign of $i\epsilon$ so that we find

$$H_+ = \frac{\pi^2}{3} - \mathrm{Li}_2(u_+) - \frac{1}{2}\log(u_+)^2 + i\pi\log(u_+) \quad ; \tag{12.93}$$

The final result is, therefore,

$$\begin{aligned} H(t) &= H_- + H_+ \\ &= \frac{2\pi^2}{3} - (\mathrm{Li}_2(u_-) + \mathrm{Li}_2(u_+)) - \frac{1}{2}\log(u_+)^2 - \frac{1}{2}\log(u_-)^2 + i\pi\log\left(\frac{u_+}{u_-}\right) \\ &= \frac{\pi^2}{2} - \frac{1}{2}\log\left(\frac{u_+}{u_-}\right)^2 + i\pi\log\left(\frac{u_+}{u_-}\right) \quad . \end{aligned} \tag{12.94}$$

Throughout, we have here used the properties of Li_2 as discussed in Appendix 13.15.

Chapter 13

Appendices

13.1 Convergence issues in perturbation theory

Let us reinspect Eq.(1.19), taking $\mu = 1$ for simplicity :

$$\begin{aligned} G_{2n} &= H_{2n}/H_0 \ , \\ H_{2n} &= \sum_{k\geq 0} \frac{(4k+2n)!}{2^{5k+n}\ 3^k\ (2k+n)!\ k!} (-\lambda_4)^k \ , \\ H_0 &= \sum_{k\geq 0} \frac{(4k)!}{2^{5k}\ 3^k\ (2k)!\ k!} (-\lambda_4)^k \ . \end{aligned} \qquad (13.1)$$

Although we have treated the expressions for the H's as if they were well-defined objects, in fact these series do not converge ! For large k and fixed n the k^{th} term in H_{2n} contains the numerical coefficient

$$\frac{(4k+2n)!}{2^{5k+n}\ 3^k\ (2k+n)!\ k!}$$

which increases *superexponentially*[1] with k : which implies that the series has a radius of convergence equal to zero. The procedure of taking the ratio H_{2n}/H_0, while it mixes terms of different order in λ_4, does not help to repair this ; a simple numerical study shows that

$$G_2 = \sum_{k\geq 0} \sigma_k (-\lambda_4)^k \quad , \quad \sigma_k \sim k!\ (2/3)^k \ , \qquad (13.2)$$

[1]This means that the coefficient increases with k faster than A^k for *any* A : roughly speaking, it grows like $(k!)$.

so that also G_2 (and, it can be checked, the higher G's) are described by series with vanishing radius of convergence. This should not come as a surprise. For, in the discussion of the perturbation expansion we have assumed the coupling constant λ_4 to be small, but positive. If, on the other hand, it was small but *negative*, perturbation theory would look very different ; in fact it would look like nothing at all since for negative λ_4 the path integral is completely undefined. Therefore, the perturbative expansion is not regular around $\lambda_4 = 0$, and in the set of all φ^4 theories the point $\lambda_4 = 0$ constitutes an *essential singularity*.

All may not be lost, however. The method of *Borel summation* sometimes[2] enables us to assign a value to a sum with vanishing radius of convergence. Suppose that a function of a positive variable x is given by the sum

$$f(x) = \sum_{k\geq 0} c_k\, x^k \quad , \tag{13.3}$$

where the coefficients c_k grow superexponentially. Clearly it is difficult to make sense of such a sum ; but it *may* be possible to make sense of a *related* sum :

$$g(x) = \sum_{k\geq 0} \frac{c_k}{k!}\, x^k \quad , \tag{13.4}$$

simply because the coefficients do not grow as rapidly. Let us suppose that this is indeed the case. We then may employ the formula

$$\int\limits_0^\infty dy\ \exp(-y)\ (xy)^n = n!\ x^n \quad , \quad n = 0, 1, 2, \ldots \tag{13.5}$$

to arrive at the rule

$$f(x) = \int\limits_0^\infty dy\ e^{-y}\ g(xy) \quad . \tag{13.6}$$

Notice that here, we have *again* interchanged summation and integration, thus in a sense repairing the damage done when we arrived at the perturbation expansion in the first place. This approach is called Borel summation. We can illustrate this in a simple example. Let us take $c_k = 1$, that is

$$f(x) = \sum_{k\geq 0} x^k = \frac{1}{1-x} \quad : \tag{13.7}$$

[2]In zero dimensions this will work. In four-dimensional Minkowski space things are not nearly as simple...

we immediately find that

$$g(x) = \sum_{k\geq 0} \frac{x^k}{k!} = e^x \quad , \tag{13.8}$$

and indeed

$$\int\limits_0^\infty dy\; e^{-y}\; e^{xy} = \frac{1}{1-x} \quad . \tag{13.9}$$

However, an important observation is to be made here. The sum for $f(x)$ converges (conditionally) for the region $|x| \leq 1$, whereas the sum for $g(x)$ converges *everywhere*, and the Borel integral converges in this case as long as $\Re(x) < 1$, thus immeasurably enlarging the region of x values where the Borel-summed version makes sense.

We now turn to a more challenging example : the sum

$$F(x) = \sum_{k\geq 0} k!\; (-x)^k \quad , \tag{13.10}$$

with x positive. In that case we find

$$G(x) = \sum_{k\geq 0} (-x)^k = \frac{1}{1+x} \tag{13.11}$$

and the Borel sum reads

$$F(x) = \int\limits_0^\infty dy\; e^{-y} \frac{1}{1+xy} = \frac{1}{x}\, e^{1/x}\; E_1\left(\frac{1}{x}\right) \tag{13.12}$$

where the function E_1, the *exponential integral*, given by

$$E_1(z) = \int\limits_z^\infty dt\; \frac{\exp(-t)}{t} \quad , \tag{13.13}$$

is a little-known but perfectly well-defined function. $F(x)$ is a function that starts (obviously) at $F(0) = 1$ and then gently decreases. Borel summation works ! But how do we actually compute the series $F(x)$? The theory of *asymptotic functions* provides an answer. Let us consider not the infinite sum $F(x)$ as given in Eq.(13.10) but its *truncated* version

$$F_K(x) = \sum_{k=0}^{K-1} k!\; (-x)^k \tag{13.14}$$

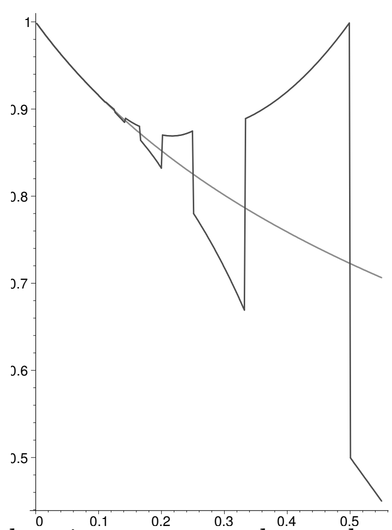
It can be shown that the *difference* between $f(x)$ and $f_K(x)$ is *of the order*[3] of the first neglected term :

$$\left|F(x) - F_K(x)\right| = \mathcal{O}\left(K!\;(-x)^K\right) \quad . \tag{13.15}$$

Taking ‘order’ to mean ‘roughly equal in magnitude, barring accidents’[4] we might therefore conclude that the *optimal* value of K is that for which the error term is minimal, that is, we truncate around $K \approx 1/x$. In that case

$$\left\lfloor K!x^K \right\rfloor_{x=1/K} = K!\;K^{-K} \;\approx e^{-K} = e^{-1/x} \quad , \tag{13.16}$$

so that the numerical error can be very small indeed for small x. As an illustration we give here the actual and asymptotically-inspired-truncated result for the function (13.10).



The exact and truncated results for the function $F(x)$ of (13.10). The smooth curve is the exact, the zigzagging one the truncated result. The approximate value oscillates around the true one ; but for small x the difference is negligible. This shows that, even if a sum is divergent, it may still be possible to make sense out of it by Borel summation.

Note that in our example we have required x to be positive, so that $(-x)^n$ oscillates in sign. That this is essential becomes clear when we try to Borel-sum

$$\overline{F}(x) = \sum_{n\geq 0}\; n!\;(x)^n \quad , \quad x > 0 \quad : \tag{13.17}$$

[3]Also this statement needs interpretation. In the theory of asymptotic series it means that the difference will go to zero at least as fast as the first neglected term goes to zero, *not* that these two numbers must be necessarily comparable in magnitude. As an example, the object $10^{12}/x^2$ is formally of the order of $1/x$ as $x \to \infty$, but x has to be *really large* for them to be of equal size. Fortunately, it often happens that the difference and the neglected term *are* of similar magnitude.

[4]Only to be justified by its success.

the Borel integral reads

$$F(x) = \int\limits_0^\infty dy\; e^{-y}\; \frac{1}{1-xy} \quad , \tag{13.18}$$

and this integral runs into problems around $y = 1/x$. One may of course extend the integral to complex y values, and then skirt around the singularity ; but it is not clear whether we should pass the point $y = 1/x$ above, or below, the real axis. The *ambiguity*, that is, the difference between the results from the alternative contours, is of course given by the number

$$\oint\limits_{y\sim 1/x} dy\; \frac{e^{-y}}{1-xy} = 2\pi i\; \frac{e^{-1/x}}{x} \, ,$$

and, since during the integration we might decide to circle around the singularity any number of times, arbitrary multiples of the ambiguity may be added. We see that the Borel integral becomes ambiguous : it may be some consolation that the ambiguity is *nonperturbative* in nature, *i.e.* it has no series expansion for infinitesimal but real and positive x. We conclude that the function $\overline{F}(x)$ is given by

$$\overline{F}(x) = -\frac{1}{x}\, e^{-1/x}\; \left(Ei\left(\frac{1}{x}\right) + (2n+1)i\pi \right) \quad , \tag{13.19}$$

where n is an undetermined integer[5].

13.2 More on symmetry factors

13.2.1 The origin of symmetry factors

In this section we shall return to the 'simple' world of zero dimensions, since symmetry factors do not depend on the dimensionality of the theory. Let us consider again the $\varphi^{3/4}$ theory, with a path integrand

$$\exp\left(-S(\varphi) + J\varphi\right) = \exp\left(-\frac{\mu}{2!}\varphi^2 - \frac{\lambda_3}{3!}\varphi^3 - \frac{\lambda_4}{4!}\varphi^4 + J\varphi\right)$$

[5]For the functions E_1 and Ei, see *e.g.* M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, ch.5.

$$= \quad e^{-\mu\varphi^2/2!} \sum_{n_{1,3,4}\geq 0} \frac{1}{n_3!}\left(\frac{-\lambda_3\varphi^3}{3!}\right)^{n_3} \frac{1}{n_4!}\left(\frac{-\lambda_4\varphi^4}{4!}\right)^{n_4} \frac{1}{n_1!}\left(J\varphi\right)^{n_1} \tag{13.20}$$

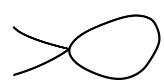
We see that a diagram with n_3 three-vertices, n_4 four-vertices and n_1 source vertices carries an *a priori* factor of

$$\frac{1}{n_1!\; n_3!\; n_4!\; (3!)^{n_3}\; (4!)^{n_4}} \quad .$$

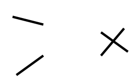
We then have to consider the number of ways in which a particular diagram can be formed by connecting external lines and the vertices in appropriate ways. The best way to learn this is, of course, to see how it is done.

13.2.2 Explicit computation of symmetry factors

Let us start with the very simple diagram

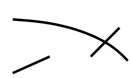


It is built up from 2 external lines (distinguishable !) and one four-vertex : the a-priori factor is therefor 1/24. We can lay out the ingredients as a tool kit[6] : We can denote this as follows :



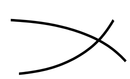
$$\frac{1}{4!}$$

To build the diagram, we first connect one of the external lines to the vertex : there are 4 possible ways to do so since all legs of the vertex are indistinguishable. We then have



$$\frac{4}{4!}$$

To attach the other external leg, there are now 3 choices :

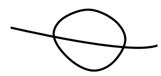


$$\frac{4\times 3}{4!}$$

[6] If you have ever played with K'NEX this may look familiar

For the remaining operation of linking the two other legs of the vertex there is of course only one possibility. The resulting symmetry factor is $4 \times 3/24 = 1/2$ as advertised.

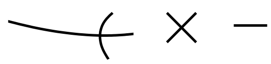
We next turn to a slightly more complicated diagram :



The tool kit is now

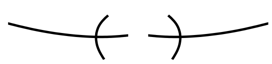
$$\frac{1}{2! \times 4! \times 4!}$$

The first external leg can now be attached in 8 ways, since the vertices and all their legs are indistinguishable :



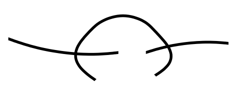
$$\frac{8}{2! \times 4! \times 4!}$$

For the other external line there are now 4 possibilities :



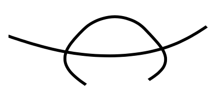
$$\frac{8 \times 4}{2! \times 4! \times 4!}$$

Now we arbitrarily consider the upper vertex leg on the left. It has to be connected to one on the right, which can be done in 3 ways :



$$\frac{8 \times 4 \times 3}{2! \times 4! \times 4!}$$

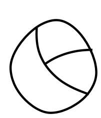
The next leg on the left now has 2 ways to go :



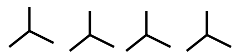
$$\frac{8 \times 4 \times 3 \times 2}{2! \times 4! \times 4!}$$

and then the diagram is closed uniquely, leading to a symmetry factor of $8.4.3.2/2!4!4! = 1/6$.

The symmetry factors of vacuum diagrams are no exception, as we can see from the construction of

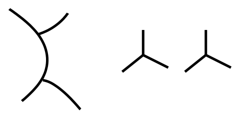


The tool kit is now



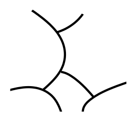
$$\frac{1}{4! \times (3!)^4}$$

We arbitrarily pick one of the vertices and attach to an arbitrarily chosen leg one of the 9 remaining vertex legs :



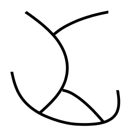
$$\frac{9}{4! \times (3!)^4}$$

The other two vertices are attached in 6 and 3 ways, respectively :



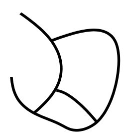
$$\frac{9 \times 6 \times 3}{4! \times (3!)^4}$$

Now take one of the outside legs. It must be attached to another outside leg that comes from *another* vertex, hence 4 possibilities :



$$\frac{9 \times 6 \times 3 \times 4}{4! \times (3!)^4}$$

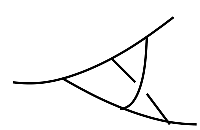
In the same way, for the next outside leg there are possibilities :



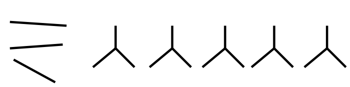
$$\frac{9 \times 6 \times 3 \times 4 \times 2}{4! \times (3!)^4}$$

and then the diagrams is closed uniquely. The symmetry factor is therefore 1/24, corresponding to the symmetry of permuting the vertices.

We finish with a truly nontrivial one :

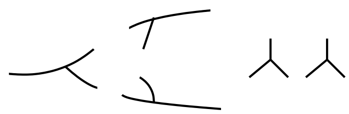


Note that the two crossing lines do not touch but form an 'overpass' : this diagram is nonplanar in the sense that an overpass remains no matter how you try to draw it. The toolkit is now even more impressive :



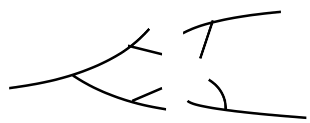
$$\frac{1}{5! \times (3!)^5}$$

We can attach the external lines to different 3-vertices in 15, 12, and 9 ways, respectively :



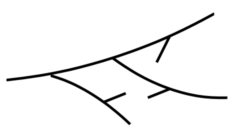
$$\frac{15 \times 12 \times 9}{5! \times (3!)^5}$$

Now the leftmost object can be attached to the two remaining 3-vertices in 6 and then in 3 ways :



$$\frac{15 \times 12 \times 9 \times 6 \times 3}{5! \times (3!)^5}$$

The topmost leg of the object on the left has to be connected to both the upper and the lower objects on the right, for which there are thus 4 and 2 options, respectively :



$$\frac{15 \times 12 \times 9 \times 6 \times 3 \times 4 \times 2}{5! \times (3!)^5}$$

By now, it ought to be obvious that there are 2 ways to finish off the diagram, leading to a symmetry factor of $(15.12.9.6.3.4.2.2)/(5!(3!)^5)$, in simpler terms $1/2$. The actual symmetry is, once again, the permutation of the two innermost vertices.

13.3 Completely solvable models in zero dimensions

13.3.1 A logarithmic action

The free theory is of course one in which we can calculate all Green's functions exactly to all orders – but that is because they are trivial. Are there less trivial actions for which we can compute *everything*? Consider, for example, the action given by

$$\begin{aligned} S(\varphi) &= -\frac{\mu}{a^2}\log(1-a\varphi) - \frac{\mu}{a}\,\varphi \\ &= \frac{\mu}{2}\varphi^2 + \frac{a\mu}{3}\varphi^3 + \frac{a^2\mu}{4}\varphi^4 + \cdots \quad . \end{aligned} \tag{13.21}$$

Here, a is some dimensionful constant, and the field is supposed to take values only on $(-\infty, 1/a)$. Since

$$S'(\varphi) = \frac{\mu}{a}\left(\frac{1}{1-a\varphi} - 1\right) = \mu\left(\varphi + a\varphi^2 + a^2\varphi^3 + \cdots\right) \quad , \tag{13.22}$$

The SDe for the path integral reads

$$\mu\left(\hbar Z' + a\hbar^2 Z'' + a^2\hbar^3 Z''' + \cdots\right) = JZ \quad . \tag{13.23}$$

Differentiating this once more and multiplying with $a\hbar$ gives

$$\mu\left(a\hbar^2 Z'' + a^2\hbar^3 Z''' + \cdots\right) = a\hbar(Z + JZ') \quad . \tag{13.24}$$

By subtraction we therefore find

$$\mu\hbar Z' = JZ - a\hbar Z - a\hbar JZ' \quad , \tag{13.25}$$

The solution to this differential equation is the path integral

$$Z(J) = \left(1 + \frac{a}{\hbar}J\right)^{-(1+\mu/a^2\hbar)} \exp\left(\frac{J}{a\hbar}\right) \quad ; \tag{13.26}$$

but, more importantly, we can simply read off $\phi(J)$ from Eq.(13.25) :

$$\phi(J) = \hbar\frac{Z'}{Z} = \frac{J - a\hbar}{\mu + aJ} \quad . \tag{13.27}$$

We have now completely solved the SDe. It appears that all loop corrections beyond one loop vanish identically ! Moreover we can write Eq.(13.27) also as

$$J = \frac{\mu\phi + a\hbar}{1 - a\phi} \quad , \tag{13.28}$$

so that the effective action is

$$\Gamma(\phi) = -\frac{\mu}{a^2}\log(1 - a\phi) - \frac{\mu}{a}\phi - \hbar\log(1 - a\phi) \quad . \tag{13.29}$$

The effective action, also, is free of corrections beyond one loop. Results such as this one can provide a powerful check on other calculations. For instance, the results of Eq.(1.93) and Eq.(1.98) for the effective action can be applied for this action, and indeed we find that, at one loop, $\Gamma_1(\phi) = -\hbar\log(1 - a\phi)$, and at two loops, $\Gamma_2(\phi) = 0$. Furthermore, the fact that if we (**a**) allow for all possible vertices, (**b**) assign the Feynman rule $-(n-1)!/\hbar$ to an n-point vertex, and $\hbar$ to each propagator, then all connected Green's functions (or their 1PI parts only) must vanish beyond one loop, is very helpful in determining whether we have forgotten some diagrams in a nontrivial calculation.

13.3.2 An exponential action

Next, we consider the action

$$S(\varphi) = \frac{\mu}{a^2}\left(e^{a\varphi} - 1 - a\varphi\right) \quad . \tag{13.30}$$

From

$$S'(\varphi) = \frac{\mu}{a}\left(e^{a\varphi} - 1\right) = \frac{\mu}{a}\left(a\varphi + \frac{1}{2!}a^2\varphi^2 + \frac{1}{3!}a^3\varphi^3 + \cdots\right) \tag{13.31}$$

we obtain the SDe in the form

$$\frac{\mu}{a}\left(a\hbar Z' + \frac{a^2\hbar^2}{2!}Z'' + \frac{a^3\hbar^3}{3!}Z''' + \cdots\right) = \frac{\mu}{a}\Big(Z(J + a\hbar) - Z(J)\Big) = JZ(J) \quad . \tag{13.32}$$

In other words,

$$Z(J + a\hbar) = \left(1 + \frac{J}{a\hbar}\right)Z(J) \quad , \tag{13.33}$$

which *functional* equation has the solution[7]

$$Z(J) = \Gamma\left(\frac{\mu}{a^2\hbar}\right)^{-1}\left(\frac{a^2\hbar}{\mu}\right)^{J/a\hbar}\Gamma\left(\frac{\mu}{a^2\hbar} + \frac{J}{a\hbar}\right) \quad . \tag{13.34}$$

The corresponding field function reads

$$\phi(J) = \frac{1}{a\hbar}\left[\log\left(\frac{a^2\hbar}{\mu}\right) + \psi\left(\frac{\mu + aJ}{a^2\hbar}\right)\right] \quad , \tag{13.35}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. This function has an asymptotic expansion for large z :

$$\psi(z) \sim \log(z) - \frac{1}{2z} - \sum_{n\geq 2}\frac{B_n}{n}z^{-n} \quad , \quad z \to \infty \quad . \tag{13.36}$$

Here, the B_n are the Bernoulli numbers, defined by their generating function as follows :

$$F(x) \equiv \frac{x\ e^x}{e^x - 1} = \sum_{n\geq 0} B_n\frac{x^n}{n!} \quad . \tag{13.37}$$

[7]Here, Γ does of course not denote any effective action, but rather the 'factorial' Gamma function.

It is easily seen that $B_0 = 1$ and $B_1 = 1/2$; but more significantly, from the fact that $F(x) - x/2$ is actually[8] even in x, we see that all B_n vanish for odd $n \geq 3$; which again means that all *odd* loop corrections beyond the first order vanish for all Green's functions ! The receipe is even simpler than in the previous case : replacing each vertex by -1 and each propagator by 1, all odd-loop Green's functions must be identically zero ; yet another powerful check on our computations.

13.4 Alternative solutions to the Schwinger-Dyson equation

13.4.1 Alternative contours in the complex plane

Alternative contours for general theories

In section 1.2.5, it was mentioned that φ^3 theory is not well-defined for real fields since the action will go to infinity whenever $\varphi \to +\infty$ or $\varphi \to -\infty$. It is instructive to lift the requirement that φ be real. In that case, we see that different integration contours become available for which the path integral is well-defined (albeit not necessarily real). Let us consider a zero-diensional theory with general action

$$S(\varphi) = \sum_{p=1}^{m} \frac{\lambda_p}{p!}\, \varphi^p \quad . \tag{13.38}$$

The requirement for the path integral to be defined is that at both endpoints (still assumed to be at infinity in some complex direction) the *real part* of the action goes to positive infinity. That is,

$$\begin{aligned} &\Re(\varphi^m) \;\to\; +\infty \quad \Rightarrow \\ &\qquad -\frac{\pi}{2m} + \frac{2\pi}{m}k \;<\; \arg(\varphi) \;<\; \frac{\pi}{2m} + \frac{2\pi}{m}k \quad , \quad k = 1,2,\ldots,m \; . \end{aligned} \tag{13.39}$$

The argument of the endpoints are restricted to certain intervals. Inside each interval the precise value of the argument is irrelevant since the path integral will be precisely the same: we may therefore say that for a theory

[8]From the way it is written, this seems unlikely — but it is true.

with highest interaction term φ^m the admissible endpoints are $\infty_n^{(m)}$, with $n = 0, 1, 2, \ldots, m-1$, where

$$\infty_n^{(m)} = \lim_{r\to\infty} re^{i\phi_n} \quad , \quad \phi_n \in \left(\frac{2\pi}{m}(n-\frac{1}{4}), \frac{2\pi}{m}(n+\frac{1}{4})\right) \quad . \tag{13.40}$$

Since the path integrand is analytic, the theory is completely determined by the endpoints. We see that for a theory with highest interaction of the form φ^m there are precisely $m-1$ independent solutions to the SDe, as necessary since the SDe is a linear differential equation of order $m-1$. We may take these as given by a contour running between $\infty_0^{(m)}$ and any of the $m-1$ other $\infty_n^{(m)}$. By suitably combining several integrals we can of course also obtain a theory based on a contour running between any two distinct $\infty_n^{(m)}$.

An interesting observation can be made on the limit of vanishing coupling. Consider an action in which the highest coupling is $\lambda_m\varphi^m/m!$, and the next highest is $\lambda_k\varphi^k/k!$. We can immediately see that the theory will remain well-defined in the limit $\lambda_m \to 0$, provided that its endpoints $\infty^{(m)}$ are chosen such as to overlap with two distinct endpoints of the subleading coupling, $\infty^{(k)}$. If this is not the case the path integral will not be defined in the limit of vanishing leading coupling constant.

Alternative contours for φ^3 theory

As an example, let us look again at φ^3 theory. There are three endpoints $\infty_{0,1,2}^{(3)}$. Since the point $-\infty$ is not inside one of the admissible edpoints, the real axis is not a valid contour as we have remarked. An interesting well-defined choice is the contour between $\infty_1^{(3)}$ and $\infty_2^{(3)}$: by symmetry we see that, as long as the action's parameters and the source are real, the path integral and $\phi(J)$ are well-defined and real. On the other hand, both endpoints overlap with the same endpoint $\infty_1^{(2)}$, which means that in the limit λ_3 the theory must become ill-defined. A quick look at the tree-level form of the theory bears this out : for the action

$$S(\varphi) = \frac{\lambda}{6}\varphi^3 + \frac{\mu}{2}\varphi^2 \tag{13.41}$$

the classical solution is given by

$$S'(\phi_c(J)) = J \quad \Rightarrow \quad \phi_c(J) = \frac{\mu}{\lambda}\left(-1 \pm \sqrt{1+\frac{2\lambda J}{\mu^2}}\right) \quad . \tag{13.42}$$

Choosing the $-$ sign we obtain a clasical tadpole $\phi_c(0) = -2\mu/\lambda$, which corresponds to the contour discussed above[9] ; and indeed it becomes ill-defined as $\lambda \to 0$. The choice of the $+$ sign gives a classical solution that has a Taylor series expansion around $\lambda = 0$. It corresponds to the contours running from $\infty_0^{(3)}$ to either $\infty^{(1)}$ or $\infty^{(2)}$; it is not possible to tell which of the two contours is intended. In fact the situation appears to be even worse. If λ, μ and J are all real, the SDe can be iteratively solved starting from the classical solution, and the perturbation series is completeley fixed as well as *real* ; whereas the fact that the two integration contours are really distinct from the real axis tells us that the path integral (and hence $\phi(J)$) ought to be complex, with the results from the two contours related by complex conjugation. We conclude that the difference between the two alternative path integral must be *non-perturbative* in nature.

Alternative contours for φ^4 theory

For φ^4 theory, with action

$$S(\varphi) = \frac{1}{4!}\lambda\varphi^4 + \frac{1}{2}\mu\varphi^2 \quad ,$$

there are three independent contours. Since $\infty_{1,3}^{(4)}$ do not overlap with any $\infty^{(2)}$, we see that only the real axis gives a theory in which the limit $\lambda_4 \to 0$ is well-defined. Another interesting contour is that running between $\infty_3^{(4)}$ and $\infty_1^{(4)}$: we may take this contour to be the imaginary axis. By the simple variable transformation $\varphi \to i\varphi'$ we see that the theory we are *actually* investigating here is that with real field φ' but action

$$S(\varphi') = \frac{1}{4!}\lambda_4\varphi'^4 - \frac{1}{2}\mu\varphi'^2 \quad ,$$

that is, a theory with the 'wrong' sign for the quadratic term. Such models are regularly studied in connection with the phenomenon of spontaneous symmetry breaking[10]. As we see, this theory does not have a standard perturbative expansion around $\lambda_4 = 0$ even though the tadpole vanishes.

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[9] It should be observed that the classical SDe allows us to construct the full classical solution from the tadpole $\phi_c(0)$, and that from the classical solution we can construct the full quantum solution — all perturbatively, and some care has to be taken if the tadpole is nonzero.

[10] In zero dimensions, spontaneous symmetry breaking does not occur.

13.4.2 Alternative endpoints

Fixed non-infinite endpoints

Those theories of φ^3 or φ^4 kind that show a regular behaviour as $\lambda \to 0$ have in common that their contour may be drawn so as to include a part that crosses the point $\varphi = 0$ along the real axis[11]. We can therefore envisage theories where the contour crosses the origin (assumed to be where the minimum of the action is) along the real axis, but where we keep the path integral wel-defined simply by letting the contour end at *finite* distance from the origin. The value of the path integral will then, of course, depend on where the endpoints are – but is that a problem ? As an example, consider the free theory with for the contour the real axis between, say, $\varphi_- < 0$ and $\varphi_+ > 0$. This contour includes $\varphi = 0$, and we may trust perturbation theory insofar as it can be trusted at all. The difference between this 'restricted' path integral and the one where the whole real axis is included is given by the error function with arguments $\varphi_\pm$, that is, terms that are of order $\exp(-{\varphi_\pm}^2/(2\hbar\mu))$. This will lead to a theory that differs from the standard free one on a nonperturbative level only, as long as $\varphi_\pm$ is not of order $\hbar$. It is easy to see that this phenomenon will persist for interacting theories as well. Our upshot is that finite endpoints are acceptable as long as we are doing perturbation theory, and as long as the origin can be crossed along the real axis in an unambiguous manner.

Moving endpoints

Finitely positioned endpoints of the integration contour will in general lead to nonperturbative inhomogeneous terms in the SDe, as we have seen. There is, however, another possibility : that of letting the contour endpoints depend on the source. To see how this is possible, let us consider the path integral over the real axis, assuming that the action diverges acceptably at $\varphi = -\infty$, and that the upper limit of the path integral resides at the source-dependent value $\varphi = c(J)$. Denoting[12] by $A(\varphi, J)$ the integrand $\exp(-S(\varphi) + J\varphi)$, we

[11]The φ^3 theory with endpoints $\infty^{(3)}_1$ and $\infty^{(3)})_2$ can also be deformed to go over the origin along the real axis — but then it has to go 'forth' and 'back' over that point, which rather spoils the idea since the contributions will cancel one another.

[12]We take $\hbar = 1$ for simplicity here.

then have the (unnormalized) path integral

$$Z(J) = \int\limits_{-\infty}^{c(J)} d\varphi\ A(\varphi, J) \quad , \tag{13.43}$$

for which we can deduce the derivatives

$$\begin{aligned}
Z'(J) &= c'(J)A(c(J), J) + \int\limits_{-\infty}^{c(J)} d\varphi\ \varphi\ A(\varphi, J) \quad , \\
Z"(J) &= \left[2c'(J)c(J) + c"(J) + c'(J)^2(J - S'(c(J))\right] A(c(J), J) \\
&\quad + \int\limits_{-\infty}^{c(J)} d\varphi\ \varphi^2\ A(\varphi, J) \quad ,
\end{aligned} \tag{13.44}$$

and so on. By suitably choosing $c(J)$ we can make sure that $Z(J)$ obeys the exact, homogeneous SDe. For the free theory, the SDe reads

$$\begin{aligned}
0 &= JZ(J) - \mu Z'(J) \\
&= -\mu c'(J)A(c(J), J) + \int\limits_{-\infty}^{c(J)} d\varphi\ (J - \mu\varphi)A(\varphi, J) \\
&= \Big(1 - \mu c'(J)\Big)\ A(c(J), J) \quad :
\end{aligned} \tag{13.45}$$

and we conclude that the theory with a restricted but J-dependent endpoint will be completely indistinguishable from the standard free theory if

$$c(J) = c(0) + J/\mu \quad . \tag{13.46}$$

By some poetic justice, the endpoint must move uniformly for the free theory (in the sense in which J stands for 'time'). We can of course also introduce a moving lower endpoint, and in fact, for any theory, we can let the two endpoints satisfy their own differential equation independently of one another. For the free theory, we find that a contour over any finite interval leads to the correct SDe, provided the interval moves along the real axis with the correct 'speed'. The extension to interacting theories we glibly leave as an excercise to the reader.

13.5 Concavity of the effective action

In the zero-dimensional case of a single field variable, the effective action is concave. Let us now investigate whether this persists in case of more fields. Let the collection of all fields be denoted by $\{\varphi\}$ as before, and the collection of all sources, one for each field, by $\{J\}$. We shall denote the combined probability density of all fields, including the effects of the sources, by $P(\{\varphi\},\{J\})$. The effective action is now that function of the collection of all field functions $\{\phi\}$ that has the correct classical equation :

$$\frac{\partial}{\partial\phi_n}\Gamma(\{\phi\}) = J_n \quad . \tag{13.47}$$

Concavity of the effective action in the many-field case means that the matrix

$$\Gamma_{nm} \equiv \frac{\partial}{\partial\phi_n}\frac{\partial}{\partial\phi_m}\Gamma(\{\phi\}) = \frac{\partial}{\partial\phi_m}J_n \tag{13.48}$$

has only *positive* eigenvalues. If this is the case, then also its inverse, the matrix

$$H_{mn} = \frac{\partial}{\partial J_m}\phi_n \tag{13.49}$$

must have only positive eigenvalues[13]. That is, for any eigenvector a of H the eigenvalue λ must be positive :

$$\sum_n H_{mn}\, a_n = \lambda\, a_m \quad , \quad \lambda > 0 \quad . \tag{13.50}$$

In turn, this is guaranteed if

$$\sum_{m,n} H_{mn}\, a_m\, a_n > 0 \tag{13.51}$$

for *any* vector a. Now, we have

$$\phi_m = \frac{\int(\prod_n d\varphi_n)\; P(\{\varphi\},\{J\})\; \varphi_m}{\int(\prod_n d\varphi_n)\; P(\{\varphi\},\{J\})} \quad , \tag{13.52}$$

[13]Since Γ_{mn} is symmetric, so is H_{mn} although this is not obvious from the form it is written here.

and therefore

$$\frac{1}{\hbar}H_{mn} = \frac{\int (\prod_n d\varphi_n)\ P(\{\varphi\},\{J\})\ \varphi_m\varphi_n}{\int (\prod_n d\varphi_n)\ P(\{\varphi\},\{J\})} - \frac{\int (\prod_n d\varphi_n)\ P(\{\varphi\},\{J\})\ \varphi_m}{\int (\prod_n d\varphi_n)\ P(\{\varphi\},\{J\})}\ \frac{\int (\prod_n d\varphi_n)\ P(\{\varphi\},\{J\})\ \varphi_n}{\int (\prod_n d\varphi_n)\ P(\{\varphi\},\{J\})} . \tag{13.53}$$

We now employ the following trick : *duplicate* the set of fields $\{\varphi\}$ by the addition of another set of fields, $\{\hat{\varphi}\}$, with the combined probability density

$$P(\{\varphi\},\{\hat{\varphi}\},\{J\}) = P(\{\varphi\},\{J\})P(\{\hat{\varphi}\},\{J\}) \ . \tag{13.54}$$

By this construction, the random variables φ and $\hat{\varphi}$ are statistically independent. We can then write the matrix H as

$$\frac{1}{\hbar}H_{mn} = \langle \varphi_m\varphi_n - \varphi_m\hat{\varphi}_n\rangle \ , \tag{13.55}$$

with the average taken with respect to the new probability density. Using the fact that this density is symmetric in $\varphi \leftrightarrow \hat{\varphi}$, we can write this as

$$\begin{aligned} \frac{1}{\hbar}H_{mn} &= \frac{1}{2}\langle \varphi_m\varphi_n - \varphi_m\hat{\varphi}_n - \hat{\varphi}_m\varphi_n + \hat{\varphi}_m\hat{\varphi}_n\rangle \\ &= \frac{1}{2}\langle (\varphi_m - \hat{\varphi}_m)(\varphi_n - \hat{\varphi}_n)\rangle \ , \end{aligned} \tag{13.56}$$

and we arrive at

$$\sum_{m,n} H_{mn}\, a_m\, a_n = \frac{\hbar}{2}\left\langle \left(\sum_n (\varphi_n - \hat{\varphi}_n)a_n\right)^2\right\rangle \ , \tag{13.57}$$

which is necessarily positive. The matrix H has, therefore, only positive eigenvalues, and the effective action is always concave. It is of course possible (and even likely in the case of *continuum* theories that have a noncountable infinity of field values) that the eigenvalue is actually infinite. In that case the effective action contains *flat* directions. So perhaps the more careful statement is that the effective action cannot be *convex* anywhere.

A final point to note is that our proof relies only on the fact that the φ values are randomly distributed over *some* nonvanishing region, no matter how small. Of course, by restricting the values that the φ are allowed to take we will change the effective action ; but it will never be convex.

13.6 Diagram counting

13.6.1 Tree graphs and asymptotics

Direct counting

An interesting and useful application of zero-dimensional field theory lies in the topic of *counting* diagrams. To the extent that we may consider every diagram as being of the same 'order of magnitude' this gives an idea, however crude, of the amplitude to be expected. Of particular interest is the behaviour of the number of graphs under extreme circumstances such as when the number of external lines becomes very large. In this section we shall consider the simplest case, that of tree-level Green's functions of a single self-interacting field.

In order to count diagrams, we can simply consider the zero-dimensional theory so that we are not bothered by summing diagrams over internal degrees of freedom. Secondly, we replace every vertex, and every propagator by unity. This reduces every Feynman diagram to just its symmetry factor. For tree diagrams, the symmetry factor is unity; for loop graphs, the symmetry factors are nontrivial and getting rid of them is quite cumbersome[14]. The appropriate action reads

$$S(\varphi) = \frac{1}{2}\varphi^2 - F(\varphi) \quad , \quad F(\varphi) = \sum_{k\geq 3} \frac{\epsilon_k}{k!}\varphi^k \quad , \tag{13.58}$$

where ϵ_k is unity for every k-point interaction proposed in the theory, otherwise zero. Since we only consider counting graphs, the fact that S may become negative infinity for infinite φ does not bother us. The number-of-diagrams generating function

$$\Phi(J) = \sum_{n\geq 0} \frac{N_n}{n!} J^n \quad , \tag{13.59}$$

where N_n is the number of tree graphs with $n+1$ external lines, is given by the classical version of the SDe :

$$\Phi = J + F'(\Phi) \quad . \tag{13.60}$$

[14]In the literature 'counting diagrams' is usually understood to mean 'counting diagrams with symmetry factors'.

There are several ways of solving for Φ. We may directly solve Eq.(13.60) as an algebraic equation and then expand in powers of J, but this is practical only in the simplest cases such as the $\varphi^{3/4}$ theory. Alternatively, we can approach the root of $\Phi = J + F'(\Phi)$ by *Lagrange expansion*[15] :

$$\Phi = J + \sum_{n\geq 1} \frac{1}{n!}\left(\frac{\partial}{\partial J}\right)^{n-1}\left(F'(J)\right)^n \quad . \tag{13.61}$$

This is useful in theories with only a single coupling, such as pure φ^4 theory. In more complicated theories, the best approach for n not too large is simply to iterate Eq.(13.60) by computer algebra. For pure φ^p theories we can explicitly work out the result of the Lagrange expansion. The counting equation is

$$\phi = J + \frac{1}{m!}\phi^m \quad , \quad m = p - 1 \quad , \tag{13.62}$$

so that Lagrange's formula gives

$$\begin{aligned} \phi &= J + \sum_{n>0} \frac{1}{n!\,(m!)^n}\left(\frac{\partial}{\partial J}\right)^{n-1} J^{mn} \\ &= \sum_{n\geq 0} \frac{(mn)!}{n!(m!)^n(mn-n+1)!}\, J^{mn-n+1} \quad . \end{aligned} \tag{13.63}$$

The nonvanishing N's are therefore

$$N_{n(m-1)+1} = \frac{(mn)!}{n!(m!)^n} \quad , \quad n = 0, 1, 2 \ldots \tag{13.64}$$

As expected, for $m > 2$ some connected Green's functions vanish identically at the tree level since no diagrams contribute.

Asymptotic methods

For asymptotically large n, we can estimate the form of N_n by realizing that these must be given by the behaviour of $\Phi(J)$ near that of its singularities that lies closest to the origin in the complex-J plane. Now, if $\Phi(J)$ is singular,

[15] This is proven in detail in section 13.15.8.

then $\Phi'(J)$ is divergent[16], so that $dJ/d\Phi$ must vanish. We therefore solve the equation

$$\frac{\partial}{\partial\Phi}J = 1 - F''(\Phi) = 0 \tag{13.65}$$

for Φ. If the highest power of interaction in the theory is φ^m, this equation has $m-2$ complex roots $\Phi_1, \Phi_2, \ldots, \Phi_{m-2}$, and

$$J_p = \Phi_p - F'(\Phi_p) \quad , \quad p = 1, 2, \ldots, m-2 \ . \tag{13.66}$$

Now, single out that J_p that has the smallest absolute value[17], which we shall call J_0, and its corresponding Φ_p will be writtten Φ_0. For J and Φ very close to the values J_0 and Φ_0, respectively, we may use Taylor expansion to write

$$J \approx J_0 - \frac{1}{2}F'''(\Phi_0)(\Phi_0 - \Phi)^2 \ , \tag{13.67}$$

since the linear term vanishes by definition. Hence

$$\Phi \approx \Phi_0 - \left(1 - \frac{J}{J_0}\right)^{1/2}\sqrt{\frac{2J_0}{F'''(\Phi_0)}} \tag{13.68}$$

close to the singularity. From the standard Taylor expansion[18]

$$1 - \sqrt{1-x} = \sum_{n\geq 0}\frac{(2n)!}{(n+1)!n!2^{2n+1}}x^{n+1} \tag{13.69}$$

we then recover the asymptotic form for N_n :

$$N_n \approx \frac{(2n-2)!}{(n-1)!}\,\frac{1}{(4J_0)^n}\,\sqrt{\frac{8J_0}{F'''(\Phi_0)}} \ . \tag{13.70}$$

This estimate grows roughly as $n!$, as ought to have been immediately obvious from the fact that $\Phi(J)$ has a *finite radius of convergence* ; the above, more careful, treatment gives an estimate that is quite good even for non-huge n. As an application, we may consider purely gluonic QCD. In this theory, the

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[16] The divergence might also show up in higher derivatives only, but in every actual case that I have studied the divergence shows up in Φ'.

[17] The case that there are several such values is discussed in the next paragraph.

[18] This can be proven by applying the Lagrange expansion to the object $u = y + u^2/2 = 1 - \sqrt{1-2y}$, and putting $y = x/2$.

only interactions are between 3 or 4 gluons, and the theory is equivalent, as far as counting is concerned, to the $\varphi^{3/4}$ theory, with

$$F(\varphi) = \frac{1}{3!}\varphi^3 + \frac{1}{4!}\varphi^4 \quad . \tag{13.71}$$

The solutions of Eq.(13.65) and the corresponding J values are

$$\Phi_1 = -1 + \sqrt{3} \ , \ J_1 = -\frac{4}{3} + \sqrt{3} \ ; \quad \Phi_2 = -1 - \sqrt{3} \ , \ J_2 = -\frac{4}{3} - \sqrt{3} \ , \tag{13.72}$$

so that $J_0 = \sqrt{3} - 4/3$, $\Phi_0 = \sqrt{3} - 1$, and $F'''(\Phi_0) = \sqrt{3}$. In the table we give the exact number N_n, and its asymptotic estimate. The approximation is better than one per cent for $n \geq 3$. The non-polynomial (that is, $n!$) growth of the number of diagrams with n can be seen as an immediate indication of the failure of perturbation theory as a convergent series, as discussed in Appendix 1.

n	N_n (exact)	N_n(asymptotic)
1	1	0.85
2	1	1.07
3	4	4.01
4	25	25.17
5	220	220.94
6	2485	2493.60
7	34300	34397.35
8	559405	560754.85
9	10525900	10547973.57

Coarse-graining effects

In the above we have assumed that there is only a single J_0. This is indeed usually the case ; for pure φ^p theories, however, Eq.(13.65) reads

$$\frac{1}{q!}\varphi^q = 1 \quad , \quad q = p - 2 \ , \tag{13.73}$$

and this has solutions

$$\phi_n = (q!)^{1/q} \exp\left(2i\pi\frac{n}{q}\right) \quad , \quad n = 1, 2, \ldots, q \ ; \tag{13.74}$$

the corresponding values for J are

$$J_n = 1 - \frac{1}{(q+1)!}{\phi_n}^{q+1} = \frac{q}{q+1}\,\phi_n \quad , \quad n = 1, 2, \ldots, q \ , \tag{13.75}$$

and these have all the same absolute value. The thing to do is therefore to take the asymptotic contributions from all these q singular points into account, and sum them. We then obtain

$$\begin{aligned} N_k &\approx \sum_{n=1}^{q} \frac{(2k-2)!}{(k-1)!}(4J_n)^{-k}\sqrt{\frac{8(q-1)!J_n}{{\phi_n}^{q-1}}} \\ &= \frac{(2k-2)!}{(k-1)!}\left(\frac{q+1}{4q}\right)^k \sqrt{\frac{8}{q}} \ \sum_{n=1}^{q} {\phi_n}^{-(k-1)} \ . \end{aligned} \tag{13.76}$$

The sum over the n values of ϕ will vanish completely, except when $k-1$ is a multiple of q, and then it evaluates to $q/(q!)^{k-1}$; this is exactly the behaviour we found using Lagrange expansion.

We *might* have proceeded otherwise, by simply taking the single real solution $\phi_q = (q!)^{1/q}$ as the only singular point. The number of diagrams N_k will then be nonvanishing for *every* k value, while in the asymptotic expression (13.76) the sum over n ϕ's is replaced by ${\phi_q}^{-(k-1)}$, that is precisely q times smaller than the nonvanishing sums of Eq.(13.76). We see that the taking into account of only the single, real solution causes the asymptotic values of N_k to be 'smeared out' ; N_k is then never zero anymore, but its *average* value[19] is still correct.

13.6.2 Counting one-loop diagrams

The SDe approach to counting diagrams has a number of interesting or useful applications, one of which we discuss here. We can extend the treatment of the previous section as follows. For the case of purely gluonic QCD the number of one-loop diagrams *including their symmetry factors* can be counted by iterating the appropriate Schwinger-Dyson equation :

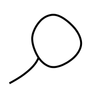
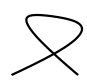
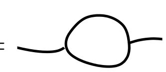
$$\Phi(J) = J + \frac{1}{2}\Phi^2 + \frac{1}{6}\Phi^3 + \frac{\hbar}{2}\,(1+\Phi)\,\Phi' \tag{13.77}$$

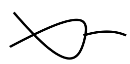
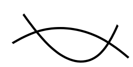
[19] For the correct definition of 'average'.

and taking care to discard terms of order $\hbar^2$ or higher. As an example, the gluonic 20-point function is given by

$$\begin{aligned} N(19) &= N_0(19) + \hbar\, N_1(19) + \mathcal{O}\left(\hbar^2\right) \ , \\ N_0(19) &= 11081983532721088487500 \ , \\ N_1(19) &= 2900013601350201168582750 \ . \end{aligned} \tag{13.78}$$

The number $N_0(19)$ is the actual number of diagrams since tree diagrams always have unit symmetry factor ; but the number $N_1(19)$ underestimates the actual number of diagrams since the symmetry factors are not trivial. We can see, however, that the only possible nonntrivial symmetry factor *at the one-loop level* is $1/2$, as evidenced by the factor $\hbar/2$ in Eq.(13.77). Inspection tells us that in this theory the only elementary Feynman diagrams that have symmetry factor $1/2$ are

$E_1 =$  , $E_2 =$  , $E_3 =$  ,

$E_4 =$  , $E_5 =$  .

All diagrams that contain one of these elementaries as a subgraph will have a symmetry factor $1/2$, and it will suffice to determine *their* number and multiply it by two[20]. Alternatively, we may get rid of all such diagrams, and work with the difference. This is the more useful approach ; and it illustrates how we may go about using counterterms to impose constraints on the structure of Feynman diagrams. The procedure is best explained by going through it step by step. In the first place, it will become necessary to again distinguish betwee three- and four-point vertices. We therefore modify Eq.(13.77) be reinserting labels for these couplings:

$$\Phi(J) = J + \frac{g_3}{2}\Phi^2 + \frac{g_4}{6}\Phi^3 + \frac{\hbar}{2}\left(g_3 + g_4\Phi\right)\Phi' \tag{13.79}$$

Iterating this gives for the first N :

$$N(0) \;=\; \frac{\hbar}{2} g_3 \ ,$$

[20]This relies, of course, on the fact that there can be no diagrams containing two (or more) of the elementaries, since that would be a two-loop diagram (or even higher).

$$\begin{aligned} N(1) &= 1+\hbar\left(\frac{1}{-}g_4+{g_3}^2\right) \quad , \\ N(2) &= g_3+\hbar\left(4{g_3}^2+\frac{7}{2}g_4g_3\right) \quad , \\ N(3) &= g_4+3{g_3}^2+\hbar\left(\frac{7}{2}{g_4}^2+24{g_3}^4+\frac{59}{2}g_4{g_3}^2\right) \quad . \end{aligned} \tag{13.80}$$

We can now start to remove graphs. We shall get rid of all diagrams with a tadpole by introducing a tadpole counterterm $\hbar T$ in the SDe:

$$\Phi(J) = J + \frac{g_3}{2}\Phi^2 + \frac{g_4}{6}\Phi^3 + \frac{\hbar}{2}\left(g_3+g_4\Phi\right)\Phi' - \hbar T \tag{13.81}$$

We see that this amounts to replacing J by $J-\hbar T$, and the N's become

$$\begin{aligned} N(0) &= \hbar\left(\frac{1}{2}g_3-T\right) \quad , \\ N(1) &= 1+\hbar\left(\frac{1}{-}g_4+{g_3}^2-g_3T\right) \quad , \\ N(2) &= g_3+\hbar\left(4{g_3}^2+\frac{7}{2}g_4g_3-g_4T-3{g_3}^2T\right) \quad , \\ N(3) &= g_4+3{g_3}^2+\hbar\left(\frac{7}{2}{g_4}^2+24{g_3}^4+\frac{59}{2}g_4{g_3}^2-10g_4g_3T-15{g_3}^3T\right) \quad . \end{aligned} \tag{13.82}$$

The tadpole $N(0)$ is removed by choosing $T = g_3/2$; and by the recursive structure of the SDe all diagrams containing the elementariy E_1 are removed as well. The remaining low-order Ns are now

$$\begin{aligned} N(1) &= 1+\hbar\left(\frac{1}{2}g_4+\frac{1}{2}{g_3}^2\right) \quad , \\ N(2) &= g_3+\hbar\left(3g_4g_3+\frac{5}{2}{g_3}^3\right) \quad , \\ N(3) &= g_4+3{g_3}^2+\hbar\left(\frac{7}{2}{g_4}^2+\frac{49}{2}g_4{g_3}^2+\frac{33}{2}{g_3}^4\right) \quad . \end{aligned} \tag{13.83}$$

Next, we want to get rid of the two self-energy bubbles E_2 and E_3. To this end, we again modify the SDe:

$$\Phi(J) = J + \frac{g_3}{2}\Phi^2 + \frac{g_4}{6}\Phi^3 + \frac{\hbar}{2}\left(g_3+g_4\Phi\right)\Phi' - \hbar T + \frac{\hbar B}{1+\hbar B}\Phi \quad , \tag{13.84}$$

where the strange-looking form of the counterterm is justified by the fact that we can rewrite Eq.(13.84) into

$$\Phi(J) = \left(J + \frac{g_3}{2}\Phi^2 + \frac{g_4}{6}\Phi^3 + \frac{\hbar}{2}\left(g_3 + g_4\Phi\right)\Phi' - \hbar T\right)\left(1 + \hbar B\right) \quad , \quad (13.85)$$

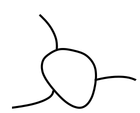
which lends itself better to the purpose of iteration. We then obtain

$$\begin{aligned} N(1) &= 1 + \hbar\left(\frac{1}{2}g_4 + \frac{1}{2}{g_3}^2 + B\right) \quad , \\ N(2) &= g_3 + \hbar\left(3g_4g_3 + \frac{5}{2}{g_3}^3 + 3Bg_3\right) \quad , \\ N(3) &= g_4 + 3{g_3}^2 + \hbar\left(\frac{7}{2}{g_4}^2 + \frac{49}{2}g_4{g_3}^2 + \frac{33}{2}{g_3}^4 + 4Bg_4 + 15B{g_3}^2\right) \quad . \end{aligned} \quad (13.86)$$

Requiring $N(1) = 1$ leads to $B = -(g_4 + {g_3}^2)/2$, and we are left with

$$\begin{aligned} N(2) &= g_3 + \hbar\left(\frac{3}{2}g_4g_3 + {g_3}^3\right) \quad , \\ N(3) &= g_4 + 3{g_3}^2 + \hbar\left(\frac{3}{2}g_4^2 + 15g_4{g_3}^2 + 9{g_3}^4\right) \quad . \end{aligned} \quad (13.87)$$

Now, the one-loop contribution to the three-point function $N(2)$ must *not* be completely cancelled, since it contains the diagram



which has symmetry factor 1 and must be retained. We therefore add a counterterm to the three-point coupling in the SDe:

$$\Phi(J) = \left(J + \frac{(g_3 - \hbar\delta_3)}{2}\Phi^2 + \frac{g_4}{6}\Phi^3 + \frac{\hbar}{2}\left(g_3 + g_4\Phi\right)\Phi' - \hbar T\right)\left(1 + \hbar B\right) \quad , \quad (13.88)$$

where the counterterm is needed only at one place since we are working to one-loop accuracy. The result of the iteration is

$$\begin{aligned} N(2) &= g_3 + \hbar\left(\frac{3}{2}g_4g_3 + {g_3}^3 - \delta_3\right) \quad , \\ N(3) &= g_4 + 3{g_3}^2 + \hbar\left(\frac{3}{2}g_4^2 + 15g_4{g_3}^2 + 9{g_3}^4 - 6\delta_3g_3\right) \quad . \end{aligned} \quad (13.89)$$

The condition now is that

$$N(2) = g_3 + \hbar {g_3}^3 \quad , \tag{13.90}$$

which requires $\delta_3 = 3g_4g_3/2$ to remove all elementaries E_4, and leads to

$$N(3) = g_4 + 3{g_3}^2 + \hbar\left(\frac{3}{2}{g_4}^2 + 6g_4{g_3}^2 + 9{g_3}^4\right) \quad . \tag{13.91}$$

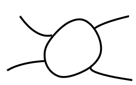
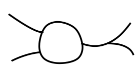
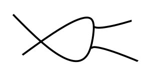
The same trick can be applied to the four-point coupling: the SDe is then

$$\Phi(J) = \left(J + \frac{(g_3 - \hbar\delta_3)}{2}\Phi^2 + \frac{(g_4 - \hbar\delta_4)}{6}\Phi^3 + \frac{\hbar}{2}\left(g_3 + g_4\Phi\right)\Phi' - \hbar T\right)\left(1 + \hbar B\right) \quad , \tag{13.92}$$

which gives

$$N(3) = g_4 + 3{g_3}^2 + \hbar\left(\frac{3}{2}{g_4}^2 + 6g_4{g_3}^2 + 9{g_3}^4 - \delta_4\right) \quad . \tag{13.93}$$

For the four point coupling, we only want to retain the diagrams

 ,  , and  ,

which occur respectively 3,6, and 3 times. Therefore, $\delta_4 = 3{g_4}^2/2$ removes all occurrences of E_5. With these choices, the SDe Eq.(13.92) can be iterated (and truncated to one-loop order!) to give all diagrams that do *not* contain any of the elementaries $E_{1,\ldots,5}$ as subdiagrams[21].

For the 20-point gluonic amplitude we find that the number of diagrams with symmetry factor unity is given by

$$\hat{N}(19) = N_0(19) + \hbar M_1(19) \quad , \quad M_1(19) = 2013070318716871853439000 \quad . \tag{13.94}$$

The total number of one-loop diagrams is therefore given by

$$\hat{N}_1(19) = M_1(19) + 2\left(N_1(19) - M_1(19)\right) = 3786956883983530483726500 \quad . \tag{13.95}$$

[21]The actual implementation of the approach described here in computer algebra may-have to be somewhat modified in the interest of speed : simply iterating Eq.(13.92) as it stands may lead to unwieldily large expressions.

In the table we give the results for the amplitudes from two to twenty external lines. It is seen that the ratio of one-loop to tree diagrams increases with the number of external legs ; while the *average* symmetry factor per one-loop diagram seems to slowly approaches unity. It can indeed be proven that asymptotically it does do so.

$n+1$	$N_0(n)$	$\hat{N}_1(n)/N_0(n)$	avg.symm.
2	1.	3.	0.5000
3	1.	14.	0.5357
4	4.	24.75	0.5758
5	25.	37.88	0.6066
6	220.	52.09	0.6309
7	2485.	67.47	0.6506
8	34300.	83.86	0.6672
9	$5.594 \quad 10^{5}$	101.2	0.6813
10	$1.053 \quad 10^{7}$	119.4	0.6936
11	$2.244 \quad 10^{8}$	138.5	0.7044
12	$5.349 \quad 10^{9}$	158.3	0.7140
13	$1.409 \quad 10^{10}$	178.9	0.7226
14	$4.064 \quad 10^{12}$	200.2	0.7305
15	$1.274 \quad 10^{14}$	222.2	0.7376
16	$4.315 \quad 10^{15}$	244.9	0.7441
17	$1.569 \quad 10^{17}$	268.2	0.7502
18	$6.101 \quad 10^{18}$	292.1	0.7558
19	$2.525 \quad 10^{20}$	316.6	0.7609
20	$1.108 \quad 10^{22}$	341.7	0.7658

The above strategy can of course be applied to other problems as well. For instance, we may remove *all* one-loop three- and four point elementaries instead of just those with symmetry factor one-half : in that case we are essentially renormalising the theory. It should also be clear that in that case, in which we just want to remove subdiagrams rather than count them, it is easy to go to more loops in an order-by-order approach.

13.7 Frustrated and unusual actions

13.7.1 Frustrating your neighbours

The one-dimensional action we have studied was based on 'nearest-neighbour' interactions. We can, of course, extend this treatment to include 'next-to-nearest-neighbour' interactions as well. Let us take

$$S(\{\varphi\}) \;=\; \sum_n \Delta\left[\frac{1}{2}\mu{\varphi_n}^2 - \gamma_1\varphi_n\varphi_{n+1} - \gamma_2\varphi_n\varphi_{n+2}\right]$$

$$
\begin{aligned}
&= \sum_n \Delta \left[\frac{1}{2}(\mu - 2\gamma_1 - 2\gamma_2){\varphi_n}^2 - \frac{1}{2}(\gamma_1 + 4\gamma_2)(\varphi_{n+1} - \varphi_n)^2\right. \\
&\qquad \left. - \frac{1}{2}\gamma_2(\varphi_{n+2} - 2\varphi_{n+1} + \varphi_n)^2\right] \quad , \qquad (13.96)
\end{aligned}
$$

with the continuum behaviour of μ, γ_1 and γ_2 to be determined. We disregard any other interactions since we shall only be interested in the propagator. Setting up the SDe for the discrete propagator is trivial: we have

$$
\begin{aligned}
\Pi(n) &= \frac{\hbar}{\mu}\delta_{n,0} + \gamma_1 \Big(\Pi(n+1) + \Pi(n-1)\Big) \\
&\quad + \gamma_2 \Big(\Pi(n+2) + \Pi(n-2)\Big) \quad , \qquad (13.97)
\end{aligned}
$$

so that Fourier transformation gives us

$$
\begin{aligned}
\Pi(n) &= \frac{\hbar}{2i\pi} \oint\limits_{|u|=1} du \; \frac{u^{n-1}}{f(u)} \quad , \\
f(u) &= \mu - \gamma_1 \left(u + \frac{1}{u}\right) - \gamma_2 \left(u^2 + \frac{1}{u^2}\right) \quad . \qquad (13.98)
\end{aligned}
$$

In the continuum limit, the only relevant poles of the integrand are those at values of u such that $|u| = 1 - \mathcal{O}(\Delta)$. Let u_j $(j = 1, 2, \ldots)$ be these poles: then

$$
\Pi(x) = \hbar \sum_j \frac{{u_j}^{|x|/\Delta}}{f'(u_j)} \quad . \qquad (13.99)
$$

Writing $u = 1 - v\Delta$, we can approximate

$$
\begin{aligned}
f(u) &= (\mu - 2\gamma_1 - 2\gamma_2) - (\gamma_1 + 4\gamma_2)(v^2\Delta^2 + v^3\Delta^3) \\
&\quad - (\gamma_1 + 5\gamma_2)v^4\Delta^4 + \mathcal{O}\left(\Delta^2\right) \quad . \qquad (13.100)
\end{aligned}
$$

There are now two possible continuum limits. In the first case, we can assume that $\gamma_1 + 4\gamma_2$ does not vanish. In that case, we can take $\gamma_1 + 4\gamma_2 \sim 1/\Delta$, and the resulting continuum limit is indistinguishable from the nearest-neighbour case. For later reference we shall denote this propagator by

$$
P_1(x) = \frac{\hbar}{2m} \exp(-m|x|) \quad . \qquad (13.101)
$$

The more curious solution is provided by the special choice $\gamma_1 = -4\gamma_2$. The only sensible continuum limit in that case is to take

$$\gamma_1 + 5\gamma_2 \sim \frac{1}{\Delta^3} \quad \to \quad \gamma_1 \sim \frac{4}{\Delta^3}\ ,\ \gamma_2 \sim -\frac{1}{\Delta^3}\ , \tag{13.102}$$

and

$$\mu = m^4\Delta + 2\gamma_1 + 2\gamma_2 \sim m^2\Delta + \frac{6}{\Delta^3}\ . \tag{13.103}$$

The poles of the integrand are therefore approximately given by

$$f(u) = \Delta\left(m^4 + v^4\right) + \mathcal{O}\left(\Delta^2\right) = 0\ , \tag{13.104}$$

so that the solutions are

$$u_k \approx 1 - \Delta m\left(\frac{1+i}{\sqrt{2}}\right)^{2k-3}\quad ,\ \ k = 1,2,3,4\ . \tag{13.105}$$

Only u_1 and u_2 are inside the unit circle, and we obtain the propagator

$$\Pi(x) \quad = \quad \frac{\hbar}{m^3\sqrt{8}}\exp\left(\frac{-m|x|}{\sqrt{2}}\right)\left(\cos\left(\frac{m|x|}{\sqrt{2}}\right) + \sin\left(\frac{m|x|}{\sqrt{2}}\right)\right) \tag{13.106}$$

which we shall denote by $P_2(x)$: it has the interesting property that $\Pi_2(x)$ is negative for mx between $3\pi/4$ and $7\pi/4$, modulo 2π. An discrete action such as the one belonging to this continuum limit, in which nearest-neighbour and next-to-nearest-neighbour couplings have opposite sign, are called *frustrated*[22]. The continuum limit of the propagator can also be written as

$$P_2(x) = \frac{\hbar}{2\pi}\int \frac{\exp(ikx)}{k^4 + m^4}\,dk\ , \tag{13.107}$$

and that of the action reads

$$S[\varphi] = \int \left[\frac{1}{2}m^4\varphi(x)^2 + \frac{1}{2}\varphi''(x)^2\right]\ . \tag{13.108}$$

[22]Frustrated in the sense that 'not all couplings can have it their own way'.

13.7.2 Increasing frustration

It is quite possible to construct even more frustrated actions, as follows. Let us suppose that the action is given by

$$S(\{\varphi\}) = \sum_n \left[\frac{1}{2}\mu{\varphi_n}^2 - \sum_{j=1}^{p}\gamma_j\varphi_n\varphi_{n+j}\right] \quad . \tag{13.109}$$

The propagator is given by Eq.(13.99), where now

$$f(u) = \mu - \sum_{j=1}^{p}\gamma_j\left(u^j + \frac{1}{u^j}\right) \quad . \tag{13.110}$$

We shall now arrange for the only the highest possible power of $1-u$ to survive in this expression. We first put $u = \exp(ik\Delta)$, so that the function $f(u)$ becomes

$$\begin{aligned} f(u) &= \mu - \sum_{j=1}^{p} 2\gamma_j\cos(jk\Delta) = \mu - \sum_{r\geq 0}(k\Delta)^{2r}\,B_r \quad , \\ B_r &\equiv \sum_{j=1}^{p}\frac{2(-)^r}{(2r)!}j^{2r}\gamma_j \quad . \end{aligned} \tag{13.111}$$

We now seek to find the γ's such that

$$B_1 = B_2 = \cdots = B_{p-1} = 0 \quad , \quad B_p = -\frac{1}{\Delta^{2p-1}} \quad . \tag{13.112}$$

In that case, we can take arbitrary constants c_r, with $c_p = 1$, and always have

$$\sum_{r=1}^{p} c_r B_r = \sum_{j=1}^{p}\gamma_j Q(j) = B_p \quad , \tag{13.113}$$

with

$$Q(j) = \sum_{r=1}^{p}\frac{2(-)^r}{(2r)!}c_r j^{2r} \quad . \tag{13.114}$$

The polynomial $Q(j)$ is even and of degree $2p$ in j, and $Q(0) = 0$. We can now, for any preassigned q with $1 \leq q \leq p$, choose the numbers c_r such that

$$Q(0) = \cdots = Q(q-1) = Q(q+1) = \cdots = Q(p) = 0 \quad , \quad Q(q) \neq 0 \quad , \tag{13.115}$$

upon which

$$\gamma_q = B_p/Q(q) \quad . \tag{13.116}$$

Obviously, the polynomial $Q(j)$ is given by

$$Q(j) = \frac{2(-)^p}{(2p)!} \prod_{\substack{0 \le n \le p \\ n \neq q}} \left(j^2 - n^2\right) \quad , \tag{13.117}$$

from which we derive

$$\gamma_q = \frac{(-)^{q-1}(2p)!}{\Delta^{2p-1}(p-q)!(p+q)!} \quad , \quad 1 \le q \le p \quad . \tag{13.118}$$

The continuum limit of the propagator is, then

$$\Pi_p(x) = \frac{\hbar}{2\pi} \int dk \, \frac{\exp(ikx)}{k^{2p} + m^{2p}} \tag{13.119}$$

The poles of the integrand are located at $k = m\omega_j$, where

$$\omega_j = \exp\left(i\pi \frac{2j+1}{2p}\right) \quad , \quad j = 0, 1, 2, \ldots, 2p \quad , \tag{13.120}$$

so that Cauchy integration gives

$$\Pi_p(x) = \frac{-i\hbar}{2pm^{2p-1}} \sum_{j=0}^{p} \omega_j \, \exp(i\omega_j m|x|) \quad . \tag{13.121}$$

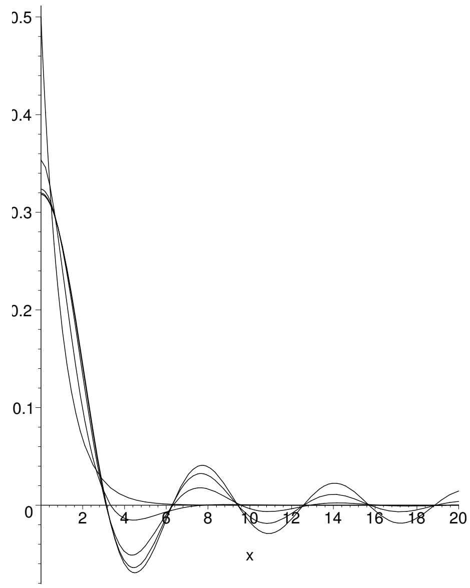
We may even investigate the limit $p \to \infty$: in that case we may approximate

$$\frac{1}{k^{2p} + m^{2p}} \approx \begin{cases} m^{-2p} & \text{if } -m < k < m \\ 0 & \text{elsewhere} \end{cases} \tag{13.122}$$

so that the propagator takes the form

$$\Pi_p(x) \approx \frac{\hbar}{2\pi m^{2p}} \int\limits_{-m}^{m} dk \exp(ikx) = \frac{1}{m^{2p-1}\pi} \, \frac{\sin(mx)}{mx} \quad . \tag{13.123}$$

The propagators $\Pi_p(x)$ for $\hbar = m = 1$, as a function of x. The values of p are 1,2,5,10, and also the asymptotic form of Eq.(13.123) is plotted. For large p the asymptotic form is approximated smoothly.



The higher the value of p, the more frustrated the lattice is, and the more difficult it becomes for momentum modes with high wave number to propagate through the lattice, as is evident from the Fourier form (13.119). For the totally frustrated lattice, all wave numbers smaller than m propagate equally, and all wave nubers larger than m do not propagate *at all*.

13.8 Newton's First Law revisited

13.8.1 Introduction : the matter of sources

In our discussion of Newton's first law in section 5.3.3 we have used a particular expression for the shape of the time-dependent part of the source, motivated by mathematical convenience. Here we shall redo the analysis of 5.3.3 but with several different time dependences of the source. The response of the field function to the source is

$$\phi(x^0, \vec{x}) = \frac{1}{(2\pi)^4} \int dk^0\, d^3k\, \frac{e^{-ik^0x^0 + i\vec{k}\cdot\vec{x}}}{k^{0\,2} - \omega^2 + i\epsilon}\, J_t(k^0)\, J_s(\vec{k}) \quad , \tag{13.124}$$

with $\omega = \sqrt{\vec{k}^2 + m^2}$ as usual. The *space* part of the source will be Gaussian : in position language it reads

$$J_s(\vec{x}) = (2\pi\sigma^2)^{-3/4}\ \exp\left(-\frac{\vec{x}^2}{4\sigma^2} + \frac{i}{\hbar}\vec{p}\cdot\vec{x}\right) \tag{13.125}$$

which corresponds to unit strength,

$$\int d^3x \; |J_s(\vec{x})|^2 = 1 \quad . \tag{13.126}$$

In momentum language, we have

$$J_s(\vec{k}) = \int d^3x \; J_s(\vec{x}) \; e^{-i\vec{k}\cdot\vec{x}} = (8\pi\sigma^2)^{3/4} \; \exp\left(-\sigma^2\left(\vec{k} - \frac{1}{\hbar}\vec{p}\right)^2\right) \quad . \tag{13.127}$$

For the time dependence of the source we examine three alternatives, which may be called *slow*, *fast*, and *abrupt*, respectively : in position language,

$$\begin{aligned} J_t^{(1)}(x^0) &= \frac{1}{(a_1)^{1/2}} \; \exp\left(-\frac{|x^0|}{a_1} - \frac{i}{\hbar}p^0x^0\right) \quad , \\ J_t^{(2)}(x^0) &= \frac{1}{(2\pi a_2^2)^{1/4}} \; \exp\left(-\frac{x^{02}}{4a_2^2} - \frac{i}{\hbar}p^0x^0\right) \quad , \\ J_t^{(3)}(x^0) &= \frac{1}{(2a_3)^{1/2}} \, \theta\left(-a_3 < x^0 < a_3\right) \; \exp\left(-\frac{i}{\hbar}p^0x^0\right) \quad . \end{aligned} \tag{13.128}$$

These three sources are all normalised to unit strength :

$$\int dx^0 \; |J_t^{(j)}(x^0)|^2 = 1 \quad , \quad j = 1, 2, 3 \quad . \tag{13.129}$$

To compare the *spread* of the sources in the time domain we can use

$$\left\langle x^{02} \right\rangle = \int dx^0 \; (x^0)^2 \; |J_t^{(j)}(x^0)|^2 = \frac{{a_1}^2}{2} = {a_2}^2 = \frac{{a_3}^2}{3} \quad . \tag{13.130}$$

In momentum language we have

$$\begin{aligned} J_t^{(1)}(k^0) &= \frac{2{a_1}^{1/2}}{\Delta^2 + 1/{a_1}^2} \quad , \\ J_t^{(2)}(k^0) &= (8\pi a_2^2)^{1/4} \, e^{-a_2^2\Delta^2} \quad , \\ J_t^{(3)}(k^0) &= \frac{-i}{(2a_3)^{1/2}\Delta} \left(e^{ia_3\Delta} - e^{-ia_3\Delta}\right) \quad , \end{aligned} \tag{13.131}$$

with $\Delta = k^0 - p^0/\hbar$. The response of the field to the timelike part of the source is

$$\psi_j \equiv \int dk^0 \; \frac{e^{-ik^0x^0}}{k^{02} - \omega^2 + i\epsilon} \; J_t^{(j)}(k^0) \quad , \tag{13.132}$$

and this is what we now investigate for positive times : $x^0 > 0$.

13.8.2 Slow, fast and abrupt

Slow source: exponential behaviour

The case $j = 1$ is what we have already investigated. Recalling the discussion of how to close the contour in the k^0 plane, we have

$$\begin{aligned}\psi_1 &= -2i\pi\sqrt{a_1}\omega\left\{\frac{\exp(-i\omega x^0)}{(\omega - p^0/\hbar)^2 + 1/{a_1}^2}\right.\\ &\qquad\left.+\frac{ia_1\omega\ \exp(-x^0/a_1 - ip^0x^0/\hbar)}{(p^0/\hbar - i/a_1)^2 - \omega^2}\right\} \qquad (13.133)\end{aligned}$$

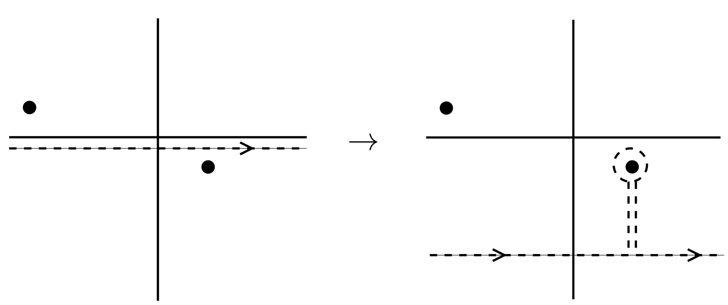
As we have remarked before, the second term dies out with the source, so that for large enough times $x^0 \gg a_1$ we can disregard it.

Fast source : Gaussian behaviour

The second, 'moderate' time dependence has to be treated more carefully. This is due to the fact that the exponential $\exp(-{k^0}^2)$ diverges when $k^0 \to \infty$ if the argument of k^0 lies in $(\pi/4, 3\pi/4)$ or $(5\pi/4, 7\pi/4)$, so we cannot close the contour simply as in the previous case. Instead, we write

$$\begin{aligned}\psi_2 &= a_2(8\pi a_2)^{1/4}\ \exp\left(-\frac{{x^0}^2}{4{a_2}^2} - ix^0p^0/\hbar\right)\ \mathcal{A}_2\ \ ,\\ \mathcal{A}_2 &= \int\limits_{-\infty}^{\infty} dy\ \frac{\exp\left(-(y + i\tau)^2\right)}{(y - b_+ + i\epsilon)(y - b_- - i\epsilon)}\ \ ,\\ &\quad b_\pm = a_2(\pm\omega - p^0/\hbar)\ \ ,\quad \tau = x^0/(2a_2)\ \ , \qquad (13.134)\end{aligned}$$

and we have chosen $y = a_2(k^0 - p^0/\hbar)$. The y integral runs over the real axis ; we may shift it downwards by an amount τ provided we include a contour integral around the point $b_+ - i\epsilon$, as indicated in the figures below.



The shift of the contour to make it run from $-\infty - i\tau$ to $+\infty - i\tau$. The poles at $b_\pm \mp i\epsilon$ are indicated.

The result of this operation on $\mathcal{A}_2$ is

$$
\begin{aligned}
\mathcal{A}_2 &= -\oint_{y\sim b_+-i\epsilon} dy\ \frac{\exp\left(-(y+i\tau)^2\right)}{(y-b_++i\epsilon)(y-b_--i\epsilon)} \\
&\quad + \int\limits_{-\infty}^{\infty} dz\ \frac{\exp(-z^2)}{(z-b_-i\tau)(z-b_--i\tau)} \\
&= -i\frac{2\pi}{b_+-b_-}\exp\left(-(b_++i\tau)^2\right) \\
&\quad + \int\limits_{-\infty}^{\infty} dz\ \frac{\exp(-z^2)}{b_+-b_-}\left[\frac{1}{z-b_--i\tau}-\frac{1}{z-b_+-i\tau}\right]\ . \qquad (13.135)
\end{aligned}
$$

For large times ($\tau \to \infty$) the two integral terms in the second both behave as $\sim 1/\tau$ so, as in the previous case, they give rise to a contribution that dies out with the source. We therefore have

$$
\lim_{\tau\to\infty}\psi_2 \ \sim\ a_2(8\pi a_2)^{1/4}\ \exp\left(-a_2^2(\omega-p^0/\hbar)^2-i\omega x^0\right)\ . \qquad (13.136)
$$

As before, the on-shell condition implied by $\omega \approx p^0/\hbar$ is enforced.

Abrupt source : Heavyside behaviour

This case is the easiest one to analyse once we realise that $J_t^{(3)}$ is perfectly regular at $\Delta = 0$ notwithstanding the denominator. So for $x^0 > 0$ we may simply close the contour in the lower half complex k^0 plane to find

$$
\psi_3 = \frac{-\pi}{\omega\sqrt{2}}\ \exp(-i\omega x^0)\ \frac{e^{ia_3(\omega-p^0/\hbar)}-e^{-ia_3(\omega-p^0/\hbar)}}{\sqrt{a_3}(\omega-p^0/\hbar)}\ . \qquad (13.137)
$$

The numerator in the last factor remains bounded in absolute value. Therefore, as long as $\omega \neq p^0/\hbar$, ψ_3 goes to zero as $1/\sqrt{a_3}$ for large a_3, while at $\omega = p^0/\hbar$ it approaches infinity as $\sqrt{a_3}$: yet another situation in which the on-shell condition is enforced.

13.8.3 Conclusion : general effect of the sources

We have seen that all three type of sources exhibit a large-time behaviour

$$
\psi_j \sim e^{-i\omega x^0}\ d_j(a_j;\omega-p^0/\hbar)\ , \qquad (13.138)
$$

characterised by a plane wave $\exp(-i\omega x^0)$, with the contributing values of ω governed by a distribution d_j such that for a_j becoming large (hence the source becoming broader in time) the energies ω are closely clustered around the central value $p^0/\hbar$. The reasoning leading to the mass-shell condition $E^2 \approx |\vec{p}|^2 c^2 + M^2 c^4$, as well as the motion along straight trajectories, $\vec{x} \approx \vec{p}\,t/p^0$, follow in each case as discussed in section 5.3.3.

13.9 Some techniques for one-loop diagrams

13.9.1 The 'Feynman trick'

Consider n positive real numbers a_j, $j = 1..n$. We can write

$$\prod_{j=1}^{n} \frac{1}{a_j} = \int_0^\infty dz_1\, dz_2 \cdots dz_n\ \exp(-z_1 a_1 - z_2 a_2 - \cdots - z_n a_n) \tag{13.139}$$

In this integral, we may define s as the sum of the z's, and define x_j as z_j/s, as follows:

$$\begin{aligned}\prod_{j=1}^{n} \frac{1}{a_j} = \int_0^\infty & dz_1\, dz_2 \cdots dz_n\, ds\, dx_1\, dx_2 \cdots dx_n \\ & \times\ \exp(-z_1 a_1 - z_2 a_2 - \cdots - z_n a_n) \\ & \times\, \delta(z_1 + z_2 + \cdots + z_n - s) \\ & \times\, \delta\left(x_1 - \frac{z_1}{s}\right)\, \delta\left(x_2 - \frac{z_2}{s}\right) \cdots \delta\left(x_n - \frac{z_n}{s}\right)\ . \end{aligned} \tag{13.140}$$

We can now eliminate the z's in favor of the x's:

$$\begin{aligned}\prod_{j=1}^{n} \frac{1}{a_j} = \int_0^\infty & dx_1\, dx_2 \cdots dx_n\, ds \\ & \times\, s^{n-1}\ \exp\Big(- s(x_1 a_1 + x_2 a_2 + \cdots + x_n a_n)\Big) \\ & \times\, \delta(x_1 + x_2 + \cdots + x_n - 1)\ \ . \end{aligned} \tag{13.141}$$

A last integral over s then gives us the formula known as the *Feynman trick*:

$$\begin{aligned}\prod_{j=1}^{n} \frac{1}{a_j} = \Gamma(n) \int_0^1 & dx_1\, dx_2 \cdots dx_n\ \Big(x_1 a_1 + x_2 a_2 + \cdots + x_n a_n\Big)^{-n} \\ & \times\, \delta(x_1 + x_2 + \cdots + x_n - 1)\ \ . \end{aligned} \tag{13.142}$$

For example,

$$\frac{1}{a_1\ a_2} = \int\limits_0^1\ dx\ \frac{1}{\Big(xa_1 + (1-x)a_2\Big)^2}\quad . \tag{13.143}$$

13.9.2 A general one-loop integral

We shall compute the integral

$$I = \int \frac{d^Dq}{(2\pi)^D}\ \frac{|\vec{q}|^n}{(|\vec{q}|^2 + a^2)^m} \tag{13.144}$$

in the spirit of dimensional regularization. That is, we shall assume that D, n and m are such that the integral converges: where it does not, we *define* the integral by analytical continuation from the convergence region. The number a^2 is not necessarily a positive real number, but again we shall reach other values for a^2 by analytical continuation from positive real values.

In the first place, by scaling the vector $\vec{q}$ by a factor $\sqrt{a^2}$ we find that

$$I = a^{D+n-2m} I' \quad , \quad I' = \int \frac{d^Dq}{(2\pi)^D}\ \frac{|\vec{q}|^n}{(|\vec{q}|^2 + 1)^m}\quad . \tag{13.145}$$

Next, we compute $W_D(t)$, the number of D-dimensional Euclidean vectors $\vec{q}$ of a given length t, as follows:

$$\begin{aligned}
W_D(t) &= \int d^Dq\ \delta(|\vec{q}| - t) \\
&= 2t \int d^Dq\ \delta(|\vec{q}|^2 - t^2) \\
&= 2t \int dq^1\ dq^2 \cdots dq^D\ \delta\Big((q^1)^2 + (q^2)^2 + \cdots + (q^D)^2 - t^2\Big) \\
&= (2t)2^D \int\limits_0^\infty dq^1\ dq^2 \cdots dq^D\ \delta\Big((q^1)^2 + (q^2)^2 + \cdots + (q^D)^2 - t^2\Big) \\
&= 2t^{D+1} \int\limits_0^\infty dy_1 \cdots dy_D\ {y_1}^{-1/2} \cdots {y_D}^{-1/2}\ \delta\Big(t^2(y_1 + \cdots y_D - 1)\Big) \\
&= 2t^{D-1}\,\frac{\Gamma(1/2)^D}{\Gamma(D/2)} = 2t^{D-1}\frac{\pi^{D/2}}{\Gamma(D/2)}\quad ,
\end{aligned} \tag{13.146}$$

where we have written $q^j = {y_j}^{1/2} t$, and used Euler's formula of sect.(13.15.4). Hence,

$$I' = \frac{1}{(4\pi)^{D/2}\Gamma(D/2)} I'' \quad , \quad I'' = \int\limits_0^\infty du \, \frac{u^{(D+n)/2-1}}{(u+1)^m} \quad , \tag{13.147}$$

where we have used $u = t^2$. Another application of Euler's formula gives us

$$\begin{aligned} I'' &= \int\limits_1^\infty du \; u^{-m} \; (u-1)^{(D+n)/2-1} = \int\limits_0^1 du \; u^{m-2} \; \left(\frac{1}{u} - 1\right)^{(D+n)/2-1} \\ &= \int\limits_0^1 du \; u^{m-1-(D+n)/2} \; (1-u)^{(D+n)/2-1} \\ &= \frac{\Gamma\left(m - (D+n)/2\right) \Gamma\left((D+n/2\right)}{\Gamma(m)} \quad . \end{aligned} \tag{13.148}$$

We arrive at the general formula

$$\int \frac{d^D q}{(2\pi)^D} \, \frac{|\vec{q}|^n}{(|\vec{q}|^2 + a^2)^m} = a^{D+n-2m} \, \frac{\Gamma\left(m - \frac{D+n}{2}\right) \; \Gamma\left(\frac{D+n}{2}\right)}{(4\pi)^{D/2} \; \Gamma\left(\frac{D}{2}\right) \; \Gamma(m)} \quad . \tag{13.149}$$

In the special case $m = 2$, $n = 0$ and $D = 4 - 2\epsilon$, with infinitesimally small ϵ, we find

$$\begin{aligned} \int \frac{d^D q}{(2\pi)^D} \, \frac{1}{(|q|^2 + a^2)^2} \quad &= \quad \frac{a^{-2\epsilon} \; \Gamma(\epsilon)}{(4\pi)^{2-\epsilon} \; \Gamma(2)} \\ = \quad \frac{1}{(4\pi)^2} \Big(1 - \epsilon \log(a^2) + \cdots\Big) &\left(1 - \epsilon \log(4\pi) + \cdots\right) \left(\frac{1}{\epsilon} - \gamma_E + \cdots\right) \\ = \quad \frac{1}{(4\pi)^2} \; &\left(\frac{1}{\epsilon} - \gamma_E - \log(4\pi) - \log(a^2) + \mathcal{O}\left(\epsilon\right)\right) \quad , \end{aligned} \tag{13.150}$$

where we have used

$$\Gamma(\epsilon) = \frac{1}{\epsilon}\Gamma(1+\epsilon) = \frac{1}{\epsilon}\Big(1 - \epsilon\gamma_E + \mathcal{O}\left(\epsilon^2\right)\Big) \quad , \tag{13.151}$$

and $\gamma_E \approx 0.577216$ is Euler's constant.

Another curious feature of dimensional regularization is that of $a \to 0$. For $D + n - 2m > 0$, we find that the integral vanishes: for instance,

$$\int d^{4-2\epsilon} q = \int d^{4-2\epsilon} q \; |\vec{q}|^2 = d^{4-2\epsilon} q \; \frac{1}{|\vec{q}|^2} = 0 \quad , \tag{13.152}$$

whereas in particular the last integral appears to be divergent both for small and large values of $|\vec{q}|$.

13.10 The fundamental theorem for Dirac matrices

13.10.1 Proof of the fundamental theorem

In this appendix we prove the following statement : if we have two sets of four matrices, γ^μ and $\hat{\gamma}^\mu$ $(\mu = 0, 1, 2, 3)$, satisfying Dirac's anticommutation relation

$$\gamma^\mu\,\gamma^\nu + \gamma^\nu\,\gamma^\mu = 2\,g^{\mu\nu} \quad , \quad \hat{\gamma}^\mu\,\hat{\gamma}^\nu + \hat{\gamma}^\nu\,\hat{\gamma}^\mu = 2\,g^{\mu\nu} \quad , \tag{13.153}$$

then there is a matrix S such that

$$\hat{\gamma}^\mu = S\,\gamma^\mu\,S^{-1} \quad . \tag{13.154}$$

To this end, we first set up a basis of the Clifford algebra as follows :

$$\begin{aligned}
&\Gamma_0 = 1 \quad , \quad \Gamma_1 = \gamma^0 \quad , \quad \Gamma_2 = i\gamma^1 \quad , \quad \Gamma_3 = i\gamma^2 \quad , \quad \Gamma_4 = i\gamma^3 \quad , \\
&\Gamma_5 = \gamma^0\gamma^1 \quad , \quad \Gamma_6 = \gamma^0\gamma^2 \quad , \quad \Gamma_7 = \gamma^0\gamma^3 \quad , \quad \Gamma_8 = i\gamma^1\gamma^2 \quad , \\
&\Gamma_9 = i\gamma^1\gamma^3 \quad , \quad \Gamma_{10} = i\gamma^2\gamma^3 \quad , \quad \Gamma_{11} = i\gamma^0\gamma^1\gamma^2 \quad , \quad \Gamma_{12} = i\gamma^0\gamma^1\gamma^3 \quad , \\
&\Gamma_{13} = i\gamma^0\gamma^2\gamma^3 \quad , \quad \Gamma_{14} = \gamma^1\gamma^2\gamma^3 \quad , \quad \Gamma_{15} = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad ,
\end{aligned} \tag{13.155}$$

which we denote by Γ_k, $k = 0, 1, 2, \ldots, 15$; and using the $\hat{\gamma}^\mu$ we construct an analogous set $\hat{\Gamma}_k$ in the same way. These have a few interesting properties. In the first place, ${\Gamma_k}^2 = 1$ for all k. Secondly, for every pair j and k there are numbers n and c_n such that

$$\Gamma_j\,\Gamma_k = c_n\,\Gamma_n \quad , \quad c_n = 1, -1, i \text{ or } -i. \tag{13.156}$$

From these properties it follows that simultaneously

$$\Gamma_k\,\Gamma_j = \frac{1}{c_n}\,\Gamma_n \tag{13.157}$$

We can thus construct the multiplication table given below[23] , where the possible values of j define the rows, and those for k the columns: the corresponding entry is then the value of n. For instance,

$$\Gamma_6\,\Gamma_4 = \Gamma_{13}$$

(in this case c_{13} happens to be 1).

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	5	6	7	2	3	4	11	12	13	8	9	10	15	14
2	2	5	0	8	9	1	11	12	3	4	14	6	7	15	10	13
3	3	6	8	0	10	11	1	13	2	14	4	5	15	7	9	12
4	4	7	9	10	0	12	13	1	14	2	3	15	5	6	8	11
5	5	2	1	11	12	0	8	9	6	7	15	3	4	14	13	10
6	6	3	11	1	13	8	0	10	5	15	7	2	14	4	12	9
7	7	4	12	13	1	9	10	0	15	5	6	14	2	3	11	8
8	8	11	3	2	14	6	5	15	0	10	9	1	13	12	4	7
9	9	12	4	14	2	7	15	5	10	0	8	13	1	11	3	6
10	10	13	14	4	3	15	7	6	9	8	0	12	11	1	2	5
11	11	8	6	5	15	3	2	14	1	13	12	0	10	9	7	4
12	12	9	7	15	5	4	14	2	13	1	11	10	0	8	6	3
13	13	10	15	7	6	14	4	3	12	11	1	9	8	0	5	2
14	14	15	10	9	8	13	12	11	4	3	2	7	6	5	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Note that in this table every row and every column contains each of the numbers from 0 to 15 precisely once. Hence, if we keep j fixed and let k run from 0 to 15, the value of n will also take on all values from 0 to 15 (although generally in a different order). Obviously, for the set $\hat{\Gamma}$ exactly the same multiplication table holds.

We are now ready to prove the theorem. Let A be an arbitrary matrix, and define S by

$$S \equiv \sum_{k=0}^{15} \hat{\Gamma}_k \; A \; \Gamma_k \quad . \tag{13.158}$$

[23]Kids! Don't do this at home, since constructing this multiplication table is extremely tedious. The numbers c_n are not given: they are anyhow only defined up to a sign, since we can always replace Γ_j by $-\Gamma_j$ (using $\gamma^2\gamma^0$ instead of $\gamma^0\gamma^2$, say) without changing the Dirac anticommutation relation.

This has the desired property since

$$\begin{aligned}\hat{\Gamma}_j\; S\; \Gamma_j &= \sum_{k=0}^{15} \hat{\Gamma}_j\; \hat{\Gamma}_k\; A\; \Gamma_k\; \Gamma_j \\ &= \sum_{n=0}^{15} c_n\; \hat{\Gamma}_n\; A\; \frac{1}{c_n}\Gamma_n = S \quad , \end{aligned} \tag{13.159}$$

in other words,

$$\hat{\Gamma}_j\; S = S\; \Gamma_j \quad . \tag{13.160}$$

It remains to ensure that the matrix S actually has an inverse. Since A can be chosen at will (except $A = 0$) this is not a problem. Let us pick another matrix B and construct

$$T = \sum_{k=0}^{15} \Gamma_k\; B\; \hat{\Gamma}_k \quad . \tag{13.161}$$

For this matrix we obviously have

$$\Gamma_j\; T = T\; \hat{\Gamma}_j \quad . \tag{13.162}$$

Combining Eq.(13.160) and (13.162) we see that the product TS commutes with Γ_j (and the product ST commutes with $\hat{\Gamma}_j$). Therefore TS is proportional to the unit matrix and we can adjust the elements of B such that $T = S^{-1}$.

It is an interesting observation that the dimensionality of the γ^μ and that of the $\hat{\gamma}^\mu$ does *not* have to be the same. In that case the matrices A and B are simply not square matrices but have different numbers of rows and columns.

13.10.2 The charge conjugation matrix

An application of the fundamental theorem is the following. The anticommutation relation, if satisfied by the Dirac matrices γ^μ, is automatically also satisfied by the matrices $-(\gamma^\mu)^T$ where T stands for the transpose. There exists, therefore, a matrix C such that

$$\hat{\gamma}^\mu = C\; \gamma^\mu\; C^{-1} = -(\gamma^\mu)^T \quad . \tag{13.163}$$

This is called the *charge conjugation matrix*. In the representation given in section 7.3.1, we have

$$\hat{\gamma}^0 = -\gamma^0 \quad , \quad \hat{\gamma}^1 = \gamma^1 \quad , \quad \hat{\gamma}^2 = -\gamma^2 \quad , \quad \hat{\gamma}^3 = \gamma^3 \;\; ; \tag{13.164}$$

and we see that a good choice is

$$C = C^{-1} = \gamma^0\gamma^2 \quad . \tag{13.165}$$

Since this form is not proof against change of representation, the use of the charge conjugation matrix in arguments and derivations lacks somewhat in elegance.

13.11 Dirac projection operators

13.11.1 Dirac projection operators

Formulation of the problem

The challenge discussed in this section is the following: given an element Π of the Clifford algebra that satisfies

$$\overline{\Pi} = \Pi \quad , \quad \Pi^2 = \Pi \quad , \tag{13.166}$$

what is its generic form ? In addition, can we find several such elements Π_j, $j = 1, 2, \ldots, n$ that decompose unity, that is,

$$\Pi_j\Pi_k = \delta_{j,k}\,\Pi_j \quad , \quad \sum_{j=1}^{n} \Pi_j = 1 \quad ? \tag{13.167}$$

If we can find solutions, then we see that the smallest possible size of the Dirac matrices is $n \times n$: also, we may be able to construct an operator that can serve as the numerator of the Dirac propagator, with the understanding that it will be **(a)** a projection operator of the type (13.166) on the mass shell, and **(b)** dependent *only* on the particle's momentum, in order to ensure that all degrees of freedom propagate in the same manner. It is evident that any uniqueness of the possible solutions corresponds directly to the uniqueness of the Dirac equation.

The equivalence transform

It must be remembered that we may discuss the propagator of a free Dirac particle without reference to any of its interactions whatsoever. Therefore we may encounter the situation where two or more different forms of the propagator are possible, that result in exactly the same physics simply because the different alternatives can be transformed into one another by a change in the particle's interactions. We adopt the following position: if there are two projection operators of the type (13.166), Π and Π', say, that can be transformed into one another by means of a Clifford element Σ :

$$\Pi' = \Sigma\,\Pi\,\overline{\Sigma} \quad , \quad \Sigma\overline{\Sigma} = 1 \ , \tag{13.168}$$

where Σ depends *only* on the particle momentum[24], the two alternatives Π and Π' will be deemed equivalent.

13.11.2 The first regular case

We may write a putative solution in the general form

$$\Pi = \frac{1}{4}\left((2-S) + \not{p} + \gamma^5\not{q} + iP\gamma^5 + T_{\alpha\beta}\sigma^{\alpha\beta}\right) \ , \tag{13.169}$$

where S, p^μ, q^μ and P are real, and $T^{\mu\nu}$ is real and antisymmetric. The requirement is now that $N \equiv \Pi^2 - \Pi$ vanish, and so its trace with any Clifford element must also vanish. We can immediately find

$$2\mathrm{Tr}\left(\gamma^5\not{p}N\right) = (p\cdot q)S \quad , \quad 2\mathrm{Tr}\left((\gamma^5\not{q} - \not{p})N\right) = (p^2+q^2)S \ . \tag{13.170}$$

There are now several possibilities, the first of which is the regular case : it is the case where $S \neq 0$ and $p^2 \neq 0$. We see that it implies that $q^2 = -p^2$ and $p\cdot q = 0$, so that p and q are linearly independent and one of them must be timelike. In that case we may form a Vierbein by finding two additional vectors ${e_{1,2}}^\mu$ with

$$p\cdot e_{1,2} = q\cdot e_{1,2} = e_1\cdot e_2 = 0 \quad , \quad {e_{1,2}}^2 = -1 \ , \tag{13.171}$$

so that the tensor T can be decomposed[25] as follows:

$$T^{\alpha\beta} = c_{pq}p^{[\alpha}q^{\beta]} + c_{12}{e_1}^{[\alpha}{e_2}^{\beta]} + \sum_{j=1,2}\left(c_{pj}p^{[\alpha}{e_j}^{\beta]} + c_{qj}q^{[\alpha}{e_j}^{\beta]}\right) \ , \tag{13.172}$$

---

[24] In order to avoid the situation where the different degrees of freedom propagate differently after all.

[25] No matter that the vectors $e_{1,2}$ are not unambiguous : the point is that a decompisition is *possible*.

where the coefficients are all real and the square brackets indicate antisymmetrization over the indices. We can now find two more conditions:

$$\begin{aligned} \frac{1}{p^2 S}\mathrm{Tr}\left((c_{p1}p^\mu {e_1}^\nu - c_{q2}q^\mu {e_2}^\nu)\sigma_{\mu\nu}N\right) &= {c_{p1}}^2 + {c_{q2}}^2 \ , \\ \frac{1}{p^2 S}\mathrm{Tr}\left((c_{p2}p^\mu {e_2}^\nu - c_{q1}q^\mu {e_1}^\nu)\sigma_{\mu\nu}N\right) &= {c_{p2}}^2 + {c_{q1}}^2 \ , \end{aligned} \tag{13.173}$$

which tells us that $c_{p1} = c_{p2} = c_{q1} = c_{q2} = 0$. The tensorial part can therefore only consist of $\not{p}\not{q}$ and $\not{p}\gamma^5\not{q}$, and we may write

$$\Pi = \frac{1}{4}\left((2-S) + \not{p} + \gamma^5\not{q} + iP\gamma^5 + ia\not{p}\not{q} + b\not{p}\gamma^5\not{q}\right) \tag{13.174}$$

with a and b real. Then, the results

$$-\frac{2}{p^2}\mathrm{Tr}\left(\not{p}N\right) = S - p^2 b \quad , \quad 2i\mathrm{Tr}\left(\gamma^5 N\right) = SP + p^4 ab \tag{13.175}$$

fix the values of $a = -P/p^2$ and $b = S/p^2$. Continuing, we evaluate

$$-\frac{1}{p^2}\epsilon_{\alpha\beta\mu\nu}p^\alpha q^\beta \mathrm{Tr}\left(\sigma^{\mu\nu}N\right) = S^2 + P^2 - p^2 \ \ , \tag{13.176}$$

which proves that p^μ must actually be the timelike vector, and fixes $|P|$. Using all the relations obtained, we finally have

$$\mathrm{Tr}\left(N\right) = S^2 - 1 \ \ , \tag{13.177}$$

which tells us that if $S \neq 0$ we can take $S = 1$ (without loss of generality since both Π and $1 - \Pi$ satisfy Eq.(13.166)), and we must have $p^2 \geq 1$. The generic form of Π in the regular case can be written as follows. We have an angle χ such that $p^2 = \cosh(\chi)^2$ and $P = \sinh(\chi)$, and two vectors k^μ and s^μ such that $k \cdot k = 1$, $s \cdot s = -1$ and $k \cdot s = 0$; then $p^\mu = \cosh(\chi)k^\mu$ and $q^\mu = \cosh(\chi)s^\mu$, and

$$\begin{aligned} \Pi(\alpha,\beta) &= \frac{1}{4}\Big(1 + \alpha\beta\not{k}\gamma^5\not{s} + \alpha\left[\cosh(\chi)\not{k} + i\sinh(\chi)\gamma^5\right] \\ &\qquad + \beta\left[\cosh(\chi)\gamma^5\not{s} - i\sinh(\chi)\not{k}\not{s}\right]\Big) \ . \end{aligned} \tag{13.178}$$

The two parameters α and β satisfy $\alpha, \beta = \pm 1$, and we have introduced them here since the set of four elements $\Pi(1,1)$, $\Pi(1,-1)$, $\Pi(-1,1)$ and $\Pi(-1,-1)$

satisfy Eq.(13.167). The situation can be simplified further by the use of the equivalence transform based on

$$\Sigma = \cosh(\chi/2) - i\sinh(\chi/2)\gamma^5 \not{k} \quad : \tag{13.179}$$

the equivalent form is then given by the simpler

$$\Pi(\alpha,\beta) = \frac{1}{4}\left(1+\alpha\not{k}\right)\left(1+\beta\gamma^5\not{s}\right) \quad . \tag{13.180}$$

The only possible way to relate this projection operator to a massive on-shell Dirac particle of mass m and momentum p^μ is to choose $k^\mu = p^\mu/m$, while s^μ then embodies the remaining (spin) degree of freedom. The final result is the well-known Dirac form

$$\begin{aligned} &\Pi(\alpha,\beta) = \frac{1}{4m}\left(m+\alpha\not{p}\right)\left(1+\beta\gamma^5\not{s}\right) \quad , \\ &p\cdot p = m^2\ ,\ s\cdot s = -1\ ,\ p\cdot s = 0\ , \alpha,\beta = \pm \quad . \end{aligned} \tag{13.181}$$

Obviously, the sum of any two or three of the above projection operators is also a resolution to our quest.

13.11.3 Irregular cases

First irregular case

Let us now assume that, in Eq.(13.170), $S \neq 0$ and $p^\mu \neq 0$ but $p^2 = 0$. In that case q^μ must be proportional to p^μ, and we write $q^\mu = c\, p^\mu$. Now the trace

$$-2\mathrm{Tr}\left(\gamma^\mu N\right) = Sp^\mu + c\epsilon_{\rho\mu\alpha\beta}p^\rho T^{\alpha\beta} \tag{13.182}$$

proves that both T and c must be nonzero. Then, the relation

$$-\left(Sg^{\mu\kappa}g^{\nu\lambda} + P\epsilon^{\mu\nu\kappa\lambda}\right)\mathrm{Tr}\left(\sigma_{\mu\nu}N\right) = T^{\kappa\lambda}\left(S^2+P^2\right) \tag{13.183}$$

shows that no solution is possible in this case since T must vanish.

Second irregular case

Let us now assume $S \neq 0$ and $p^\mu = 0$. From

$$2\mathrm{Tr}\left(\gamma^5\gamma^\mu N\right) = S\, q^\mu \tag{13.184}$$

we find that also q must vanish. Eq.(13.183) then says that the tensorial term must also be absent, upon which

$$2i\mathrm{Tr}\left(\gamma^5 N\right) = SP \tag{13.185}$$

proves that also $P = 0$. The only possibilities left are the trivial ones $\Pi = 1$ and $\Pi = 0$.

13.11.4 The second regular case

We have now examined all consequences of the assumption $S \neq 0$. The remaining case $S = 0$ gives a projection operator that can be written as

$$\Pi = \frac{1}{2}\left(1 + \not{p} + \gamma^5\not{q} + iP\gamma^5 + T_{\alpha\beta}\sigma^{\alpha\beta}\right) \quad . \tag{13.186}$$

The relation

$$-\frac{1}{8}\epsilon^{\mu\nu\kappa\lambda}\,\mathrm{Tr}\left(\sigma_{\mu\nu}N\right) = PT^{\kappa\lambda} - \frac{1}{2}\left(q^\kappa p^\lambda - p^\kappa q^\lambda\right) \tag{13.187}$$

allows us to distinguish two cases, $P = 0$ and $P \neq 0$.

The case $P \neq 0$

In this case the vectors p and q are not necessarily related to one another. The projection operator reads

$$\Pi = \frac{1}{2}\left(1 + \not{p} + \gamma^5\not{q} - \frac{i}{2P}(\not{p}\not{q} - \not{q}\not{p}) + iP\gamma^5\right) \quad , \tag{13.188}$$

under the single condition (from $\mathrm{Tr}\,(N)$) that

$$\frac{1}{P^2}\left(p^2q^2 - (p\cdot q)^2\right) + p^2 - q^2 - P^2 = 1 \quad . \tag{13.189}$$

Now, we can always find a vector r^μ with $p\cdot r = q\cdot r = 0$ and $r^2 = -1$. The equivalence transform

$$\Sigma = \frac{1}{\sqrt{2}}\left(1 - i\gamma^5\not{r}\right) \tag{13.190}$$

will then eliminate both the axial-vector and the pseudoscalar term, so that we actually arrive at a special case of the situation for $P = 0$.

The case $P = 0$

In this case p and q must be proportional to one another. The projection operator then takes the form[26]

$$\Pi = \frac{1}{2}\left(1 + a\not{k} + b\gamma^5\not{k} + i\not{k}\not{r}\right) \quad , \tag{13.191}$$

with $k^2 = \pm 1$ or 0, $k \cdot r = 0$, and a, b real. The single condition that can be found is

$$k^2(a^2 - b^2 + r^2) = 1 \quad , \tag{13.192}$$

so that k^2 cannot vanish.

Now, assume that $k^2 = +1$. The equivalence transforn

$$\Sigma = \frac{1}{\sqrt{2}}\left(1 - i\not{k}\right) \tag{13.193}$$

then eliminates the axial-vector and tensorial term at the cost of introducing a pseudoscalar one, and we find the equivalent form[27]

$$\Pi = \frac{1}{2}\left(1 + c\not{k} + iP\gamma^5\right) \quad , \quad k^2 = 1 \quad , \quad c^2 = 1 + P^2 \quad , \tag{13.194}$$

in other words, there is an angle α such that

$$\Pi = \frac{1}{2}\left(1 + \cosh(\alpha)\not{k} + i\sinh(\alpha)\gamma^5\right) \quad . \tag{13.195}$$

The equivalence transform

$$\Sigma = \cosh(\alpha/2) - i\sinh(\alpha/2)\gamma^5\not{k} \tag{13.196}$$

then suffices to produce the equivalent form

$$\Pi = \frac{1}{2}\left(1 + \not{k}\right) \quad , \tag{13.197}$$

which we recognize as the combination $\Pi(1,1) + \Pi(1,-1)$ of the first regular case.

---

[26]This is most easily imagined by letting q become parallel to p as P diminishes towards zero.

[27]Here, k^μ has be redefined, but still $k^2 = +1$.

The remaining alternative is that the vector k^μ of Eq.(13.191) obeys $k^2 = -1$. Now the equivalence transform

$$\Sigma = \frac{1}{\sqrt{2}}\left(1 - i\gamma^5 \not{k}\right) \tag{13.198}$$

gives[28]

$$\Pi = \frac{1}{2}\left(1 + \cosh(\alpha)\gamma^5 \not{k} + i\sinh(\alpha)\gamma^5\right) \quad , \quad k^2 = -1 \tag{13.199}$$

The next equivalence transform,

$$\Sigma = \cosh(\alpha/2) + i\sinh(\alpha/2)\gamma^5 \not{k} \tag{13.200}$$

produces the final form

$$\Pi = \frac{1}{2}\left(1 + \gamma^5 \not{k}\right) \quad , \tag{13.201}$$

that is included in the first regular case as $\Pi(1,1) + \Pi(-1,1)$.

13.11.5 Conclusions

We have established the following results:

- The *finest* decomposition of the unity in Clifford space is that into the four projection operators given in Eq.(13.181);
- Consequently, the smallest possible size of the Dirac matrices is 4×4;
- The Dirac equation in its well-known form is in fact the *only possible* one, up to equivalence transforms that may obscure, but cannot change, the physics since the interaction vertices can always compensate.

It must be noticed that, in the 'second regular case' we have been cavalier in accepting equivalence transformations without determining that they depend only on the particle momentum. In fact, since in that case we have $S = 0$ the unity is decomposed into two sectors, Π and $1 - \Pi$, and so we may feel confident that, whatever degrees of freedom are propagating, they will do so identically. The real requirement of momentum-only dependence resides in the 'first regular case'.

[28]Again, under redefinition of k with $k^2 = -1$.

13.12 States of higher integer spin

13.12.1 The spin algebra for integer spins

In this Appendix we shall consider systems of spinning particles with arbitrary integer spin. Such particles states can be represented, in the Feynman rules, as tensors of some rank r :

$$|s, m\rangle^{\mu_1\mu_2\mu_3\cdots\mu_r}$$

where s stands for the *total* spin of the particle, and m denotes the spin along some quantization axis, for which we shal take the z direction here. That is, once we have found the correct operators of the spin algebra

$$(S_{x,y,z})^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_r} \quad \text{and} \quad (S^2)^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_r}$$

Then we have, by definition,

$$\begin{aligned} (S^2)^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_r}\ |s, m\rangle^{\nu_1\nu_2\cdots\nu_r} &= \hbar^2\, s(s+1)\ |s, m\rangle^{\mu_1\mu_2\cdots\mu_r} \quad , \\ (S_z)^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_r}\ |s, m\rangle^{\nu_1\nu_2\cdots\nu_r} &= \hbar\, m\ |s, m\rangle^{\mu_1\mu_2\cdots\mu_r} \quad . \end{aligned} \tag{13.202}$$

It is easy to see that the spin algebra is correctly constructed once we have raising and lowering operators

$$(S_\pm)^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_r} \quad , \quad S_- = (S_+)^\dagger \quad ,$$

with

$$[[S_+, S_-], S_+] = 2\hbar^2\, S_+. \tag{13.203}$$

We can then find the other algebra elements via

$$\begin{aligned} & S_x = \frac{1}{2}\Big(S_+ + S_-\Big) \quad , \quad S_y = \frac{1}{2i}\Big(S_+ - S_-\Big) \quad , \quad S_z = \frac{1}{2\hbar}[S_+, S_-] \quad , \\ & S^2 = \frac{1}{2}\Big\{S_+, S_-\Big\} + (S_z)^2 \quad . \end{aligned} \tag{13.204}$$

We will start with particles in their rest frame[29]. The spin representations are built using four unit vectors, with obvious notation, as t^μ, x^μ, y^μ and z^μ, which obey

$$t\cdot t = 1 \quad , \quad x\cdot x = y\cdot y = z\cdot z = -1 \quad , \quad t\cdot x = t\cdot y = t\cdot z = x\cdot y = x\cdot z = y\cdot z = 0 \quad . \tag{13.205}$$

[29]This implies that the particles are *massive*. For massless partices, see later on.

Things will become easier if we also define

$$x_\pm{}^\mu = \frac{1}{\sqrt{2}}\Big(x^\mu \pm i\, y^\mu\Big) \tag{13.206}$$

so that

$$x_\pm \cdot x_\pm = 0 \;\;,\;\; x_+ \cdot x_- = -1 \;\;,\;\; x_\pm \cdot z = x_\pm \cdot t = 0 \;\;. \tag{13.207}$$

Since the spin of a particle informs us about its behaviour under rotations in the three-dimensional *spacelike* part of Minkowski space, we always require, for particles in their rest frame,

$$|s,m\rangle^{\mu_1\mu_2\cdots\mu_r}\; t_{\mu_j} = 0 \;\;,\;\; j = 1,2,\ldots,r \;\;. \tag{13.208}$$

This means that the appropriate tensors in fact contain only the three vectors x_+, x_-, and z ; for instance the rank-4 tensor $|s,m\rangle^{\mu_1\mu_2\mu_3\mu_4}$ may contain a term $x_+{}^{\mu_1}x_-{}^{\mu_2}z^{\mu_3}x_+{}^{\mu_4}$. In general, the particle's tensor is a linear combination of such terms : which precise linear combination it is depends on s and m, and this is what we want to look into.

13.12.2 Rank one for spin one

The simplest nontrivial case is that of a rank-1 tensor, that is, a vector. We have already considered these in Chapter 8. We can define

$$|1,1\rangle^\mu = x_+{}^\mu \;\;,\;\; |1,0\rangle^\mu = z^\mu \;\;,\;\; |1,-1\rangle^\mu = -x_-{}^\mu \;\;, \tag{13.209}$$

so that

$$\langle 1,1|^\mu = x_-{}^\mu \;\;,\;\; \langle 1,0|^\mu = z^\mu \;\;,\;\; \langle 1,-1|^\mu = -x_+{}^\mu \;\;. \tag{13.210}$$

For brevity, we shall use the easily interpretable notation

$$|1,1\rangle = |+\rangle \;\;,\;\; |1,0\rangle = |0\rangle \;\;,\;\; |1,-1\rangle = -\,|-\rangle \;\;. \tag{13.211}$$

These states are properly normalized, since

$$\langle 1,m_1|1,m_2\rangle = \langle 1,m_1|_\mu\; |1,m_2\rangle^\mu = -\,\delta_{m_1,m_2} \;\;. \tag{13.212}$$

In addition, the states are complete in the sense that

$$\sum_{\lambda=+,-,0} |1,\lambda\rangle^\mu\, \langle 1,\lambda|_\nu = t^\mu t_\nu - \delta^\mu{}_\nu \;\;\equiv\;\; \Delta^\mu{}_\nu \;\;. \tag{13.213}$$

Note that

$$\Delta^{\mu\alpha}\,\Delta_{\alpha\nu} = -\Delta^{\mu}{}_{\nu} \quad , \quad \Delta^{\mu}{}_{\mu} = -3 \ . \tag{13.214}$$

We now proceed to set up the spin algebra. A general raising operator can always be written in the form

$$S_+ = \sqrt{2}\hbar \left(a\ |+\rangle\langle 0| + b\ |0\rangle\langle -| \right) \ , \tag{13.215}$$

where a and b are some complex numbers ; and so

$$S_- = \sqrt{2}\hbar \left(a^*\ |0\rangle\langle +| + b^*\ |-\rangle\langle 0| \right) \ . \tag{13.216}$$

From

$$\begin{aligned} S_+S_- &= -2\hbar^2 \left(|a|^2\ |+\rangle\langle +| + |b|^2\, |0\rangle\langle 0| \right) \ , \\ S_-S_+ &= -2\hbar^2 \left(|a|^2\ |0\rangle\langle 0| + |b|^2\, |-\rangle\langle -| \right) \ , \end{aligned} \tag{13.217}$$

we find that to get the correct form of S_z we have to take $|a| = |b| = 1$, since only then[30]

$$S_z = -\hbar \left(|+\rangle\langle +| - |-\rangle\langle -| \right) \ ; \tag{13.218}$$

furthermore, we find automatically

$$S^2 = -2\hbar^2 \left(|+\rangle\langle +| + |0\rangle\langle 0| + |-\rangle\langle -| \right) \ , \tag{13.219}$$

which shows that we have here indeed a spin-one system. For reasons that will become clear later on we shall choose $a = -1$ and $b = 1$. Thus,

$$S_+\,|+\rangle = 0 \ , \ S_+\,|0\rangle = \sqrt{2}\hbar\ |+\rangle \ , \ S_+\,|-\rangle = -\sqrt{2}\hbar\ |0\rangle \ . \tag{13.220}$$

[30]Do not be confused with the overall minus signs emerging here ! Remember that the states are normalized to *minus* unity. This is a consequence of our dealing with spacelike objects in an essentially Minkowski space.

In more explicit tensorial language, we have the following matrix forms :

$$
\begin{aligned}
S_+{}^{\mu}{}_{\nu} &= \sqrt{2}\hbar\left(-x_+{}^{\mu}z_{\nu} + z^{\mu}x_{+\nu}\right) \quad , \\
S_-{}^{\mu}{}_{\nu} &= \sqrt{2}\hbar\left(-z^{\mu}x_{-\nu} + x_-{}^{\mu}z_{\nu}\right) \quad , \\
S_z{}^{\mu}{}_{\nu} &= \hbar\left(-x_+{}^{\mu}x_{-\nu} + x_-{}^{\mu}x_{+\nu}\right) \quad , \\
S^{2\mu}{}_{\nu} &= -2\hbar^2\left(x_+{}^{\mu}x_{-\nu} + x_-{}^{\mu}x_{+\nu} + z^{\mu}z_{\nu}\right) \quad .
\end{aligned}
\tag{13.221}
$$

13.12.3 Rank-2 tensors

By taking tensor products of vectors we can build more complicated systems. Let us attempt rank-2 tensors. We can easily construct the spin algebra for this system as follows :

$$
\Sigma_j{}^{\mu\nu}{}_{\alpha\beta} = S_j{}^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} + \delta^{\mu}{}_{\alpha}S_j{}^{\nu}{}_{\beta} \quad , \quad j = +,-,z \quad , \tag{13.222}
$$

and it is easily checked that these also obey the correct commutation relations

$$
[\Sigma_+, \Sigma_-] = 2\hbar\,\Sigma_z \quad , \quad [\Sigma_z, \Sigma_+] = \hbar\,\Sigma_+ \quad ; \tag{13.223}
$$

the operator for the total spin is of course

$$
\Sigma^{2\mu\nu}{}_{\alpha\beta} = S^{2\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} + \delta^{\mu}{}_{\alpha}S^{2\nu}{}_{\beta} + S_+{}^{\mu}{}_{\alpha}S_-{}^{\nu}{}_{\beta} + S_-{}^{\mu}{}_{\alpha}S_+{}^{\nu}{}_{\beta} + 2S_z{}^{\mu}{}_{\alpha}S_z{}^{\nu}{}_{\beta} \quad . \tag{13.224}
$$

There is precisely one rank-2 tensor with a spin $2\hbar$ along the z axis : it is the tensor product

$$
|2,2\rangle^{\mu\nu} = |1,1\rangle^{\mu}\,|1,1\rangle^{\nu} = x_+{}^{\mu}x_+{}^{\nu} \quad \equiv \quad |{+}{+}\rangle \quad , \tag{13.225}
$$

with obvious notation. It is straightforward to check that the total spin of this object is, indeed, equal to $2\hbar$. By applying the lowering operator as given in Eq.(13.222), and normalizing, we can immediately recover the other states in the spin-2 sector :

$$
\begin{aligned}
|2,2\rangle &= |{+}{+}\rangle \quad , \\
|2,1\rangle &= \Big(|{+}0\rangle + |0{+}\rangle\Big)/\sqrt{2} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
|2,0\rangle &= \Big(-|{+}{-}\rangle - |{-}{+}\rangle + 2\,|00\rangle\Big)/\sqrt{6}\ , \\
|2,-1\rangle &= \Big(-|{-}0\rangle - |0{-}\rangle\Big)/\sqrt{2}\ , \\
|2,-2\rangle &= |{-}{-}\rangle\ . \qquad (13.226)
\end{aligned}
$$

These five objects are totally symmetric. They are also traceless in the sense that $|2,m\rangle^{\mu\nu}\, g_{\mu\nu} = 0$; this is due to our choice for the constants a and b made above. The one object made up from $|{+}0\rangle$ and $|0{+}\rangle$ that is orthonormal to $|2,1\rangle$ is $|{+}0\rangle - |0{+}\rangle$, which forms the basis of a spin-1 sector :

$$
\begin{aligned}
|1,1\rangle &= \Big(|{+}0\rangle - |0{+}\rangle\Big)/\sqrt{2}\ , \\
|1,0\rangle &= \Big(|{-}{+}\rangle - |{+}{-}\rangle\Big)/\sqrt{2}\ , \\
|1,-1\rangle &= \Big(|{-}0\rangle - |0{-}\rangle\Big)/\sqrt{2}\ . \qquad (13.227)
\end{aligned}
$$

Finally, one single state is left :

$$
|0,0\rangle = \Big(|{+}{-}\rangle + |{-}{+}\rangle + |00\rangle\Big)/\sqrt{3}\ , \qquad (13.228)
$$

which upon inspection is seen to have zero spin. The orthonormality of these nine states is easily checked. Some simple algebra also tells us that

$$
\begin{aligned}
\sum_{m=-2}^{2} |2,m\rangle^{\mu\nu}\ \langle 2,m|_{\alpha\beta} &= \frac{1}{2}\Delta^{\mu}{}_{\alpha}\,\Delta^{\nu}{}_{\beta} + \frac{1}{2}\Delta^{\mu}{}_{\beta}\,\Delta^{\nu}{}_{\alpha} - \frac{1}{3}\Delta^{\mu\nu}\,\Delta_{\alpha\beta}\ , \\
\sum_{m=-1}^{1} |1,m\rangle^{\mu\nu}\ \langle 1,m|_{\alpha\beta} &= \frac{1}{2}\Delta^{\mu}{}_{\alpha}\,\Delta^{\nu}{}_{\beta} - \frac{1}{2}\Delta^{\mu}{}_{\beta}\,\Delta^{\nu}{}_{\alpha}\ , \\
|0,0\rangle^{\mu\nu}\ \langle 0,0|_{\alpha\beta} &= \frac{1}{3}\Delta^{\mu\nu}\,\Delta_{\alpha\beta}\ , \qquad (13.229)
\end{aligned}
$$

so that there is a completeness relation of the form

$$
\sum_{s=0}^{2}\sum_{m=-s}^{s} |s,m\rangle^{\mu\nu}\,\langle s,m|_{\alpha\beta} = \Delta^{\mu}{}_{\alpha}\,\Delta^{\nu}{}_{\beta}\ . \qquad (13.230)
$$

This confirms that no states have been overlooked.

13.12.4 Rank-3 tensors

For the sake of illustration we also give the complete set of rank-3 tensorial states. These fall apart in one spin-3, two spin-2, three spin-1 and a single spin-0 sector, giving the correct total of 27 possible orthonormal states, listed below. For reasons of typography I have left out the normalizing denominators ; these can of course be trivially recovered.

$$
\begin{array}{rcl}
 & & \text{spin-3 :} \\
|3,3\rangle & = & |+++\rangle \\
|3,2\rangle & = & |++0\rangle + |+0+\rangle + |0++\rangle \\
|3,1\rangle & = & 2\,|+00\rangle + 2\,|0+0\rangle + 2\,|00+\rangle \\
 & & -\,|++-\rangle - |+-+\rangle - |-++\rangle \\
|3,0\rangle & = & 2\,|000\rangle - |+0-\rangle - |0-+\rangle - |-+0\rangle \\
 & & -\,|-0+\rangle - |+-0\rangle - |0+-\rangle \\
|3,-1\rangle & = & |+--\rangle + |-+-\rangle + |--+\rangle \\
 & & -2\,|-00\rangle - 2\,|0-0\rangle - 2\,|00-\rangle \\
|3,-2\rangle & = & |--0\rangle + |-0-\rangle + |0--\rangle \\
|3,-3\rangle & = & -\,|---\rangle \\
 & & \text{spin-2(1) :} \\
|2,2\rangle & = & |+0+\rangle + |0++\rangle - 2\,|++0\rangle \\
|2,1\rangle & = & 2\,|00+\rangle - |+-+\rangle - |-++\rangle \\
 & & -\,|+00\rangle - |0+0\rangle + 2\,|++-\rangle \\
|2,0\rangle & = & |+0-\rangle + |0+-\rangle - |-0+\rangle - |0-+\rangle \\
|2,-1\rangle & = & 2\,|00-\rangle - |+--\rangle - |-+-\rangle \\
 & & -\,|0-0\rangle - |-00\rangle + 2\,|--+\rangle \\
|2,-2\rangle & = & 2\,|--0\rangle - |0--\rangle - |-0-\rangle \\
 & & \text{spin-2(2) :} \\
|2,2\rangle & = & |+0+\rangle - |0++\rangle \\
|2,1\rangle & = & |+00\rangle - |0+0\rangle - |+-+\rangle + |-++\rangle \\
|2,0\rangle & = & -\,|0-+\rangle + |-0+\rangle - |+0-\rangle \\
 & & +\,|0+-\rangle - 2\,|+-0\rangle + 2\,|-+0\rangle \\
|2,-1\rangle & = & |+--\rangle - |-+-\rangle + |-00\rangle - |0-0\rangle
\end{array}
$$

$$\begin{aligned}
|2,-2\rangle &= |0--\rangle - |-0-\rangle \\
&\text{spin-1(1)}: \\
|1,1\rangle &= 6\,|++-\rangle + 3\,|0+0\rangle + 3\,|+00\rangle \\
&\quad + |+-+\rangle + |-++\rangle - 2\,|00+\rangle \\
|1,0\rangle &= 3\,|0+-\rangle + 3\,|+0-\rangle + 3\,|-0+\rangle \\
&\quad +3\,|0-+\rangle - 2\,|+-0\rangle - 2\,|-+0\rangle + 4\,|000\rangle \\
|1,-1\rangle &= 2\,|00-\rangle - 3\,|-00\rangle - 3\,|0-0\rangle \\
&\quad - |+--\rangle - |-+-\rangle - 6\,|--+\rangle \\
&\text{spin-1(2)}: \\
|1,1\rangle &= |+00\rangle - |0+0\rangle + |+-+\rangle - |-++\rangle \\
|1,0\rangle &= |0+-\rangle - |+0-\rangle + |0-+\rangle - |-0+\rangle \\
|1,-1\rangle &= |+--\rangle - |-+-\rangle + |0-0\rangle - |-00\rangle \\
&\text{spin-1(3)}: \\
|1,1\rangle &= |+-+\rangle + |-++\rangle + |00+\rangle \\
|1,0\rangle &= |+-0\rangle + |-+0\rangle + |000\rangle \\
|1,-1\rangle &= |+--\rangle + |-+-\rangle + |00-\rangle \\
&\text{spin-0}: \\
|0,0\rangle &= |+-0\rangle + |-0+\rangle + |0+-\rangle \\
&\quad - |0-+\rangle - |+0-\rangle - |-+0\rangle
\end{aligned} \tag{13.231}$$

Note that the spin-0 state is totally antisymmetric : obviously, this is the only possible such state in three space dimensions. We can also compute the ‘partial’ completeness relations pertaining to each spin sector. Some algebra teaches us that these are the following set of mutually orthogonal projection operators :

$$\begin{aligned}
&\text{spin-3}: \sum_{m=-3}^{3} |3,m\rangle^{\mu\nu\rho}\,\langle 3,m|_{\alpha\beta\gamma} = \\
&\quad \frac{1}{6}\Big(\Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\beta}\Delta^{\rho}{}_{\gamma} + \Delta^{\mu}{}_{\beta}\Delta^{\nu}{}_{\gamma}\Delta^{\rho}{}_{\alpha} + \Delta^{\mu}{}_{\gamma}\Delta^{\nu}{}_{\alpha}\Delta^{\rho}{}_{\beta} \\
&\qquad + \Delta^{\mu}{}_{\beta}\Delta^{\nu}{}_{\alpha}\Delta^{\rho}{}_{\gamma} + \Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\gamma}\Delta^{\rho}{}_{\beta} + \Delta^{\mu}{}_{\gamma}\Delta^{\nu}{}_{\beta}\Delta^{\rho}{}_{\alpha}\Big) \\
&\quad -\frac{1}{15}\Big(\Delta^{\mu\nu}\left(\Delta^{\rho}{}_{\alpha}\Delta_{\beta\gamma} + \Delta^{\rho}{}_{\beta}\Delta_{\gamma\alpha} + \Delta^{\rho}{}_{\gamma}\Delta_{\alpha\beta}\right)
\end{aligned}$$

$$+ \Delta^{\nu\rho}\left(\Delta^{\mu}{}_{\alpha}\Delta_{\beta\gamma} + \Delta^{\mu}{}_{\beta}\Delta_{\gamma\alpha} + \Delta^{\mu}{}_{\gamma}\Delta_{\alpha\beta}\right)$$
$$+ \Delta^{\rho\mu}\left(\Delta^{\nu}{}_{\alpha}\Delta_{\beta\gamma} + \Delta^{\nu}{}_{\beta}\Delta_{\gamma\alpha} + \Delta^{\nu}{}_{\gamma}\Delta_{\alpha\beta}\right)\Bigg)$$

$$\text{spin-2(1)}: \sum_{m=-2}^{2} |2,m\rangle^{\mu\nu\rho}\,\langle 2,m|_{\alpha\beta\gamma} =$$
$$\frac{1}{3}\left(\Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\beta} + \Delta^{\nu}{}_{\alpha}\Delta^{\mu}{}_{\beta}\right)\Delta^{\rho}{}_{\gamma}$$
$$-\frac{1}{6}\left(\left(\Delta^{\mu}{}_{\beta}\Delta^{\nu}{}_{\gamma} + \Delta^{\nu}{}_{\beta}\Delta^{\mu}{}_{\gamma}\right)\Delta^{\rho}{}_{\alpha} + \left(\Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\gamma} + \Delta^{\nu}{}_{\alpha}\Delta^{\mu}{}_{\gamma}\right)\Delta^{\rho}{}_{\beta}\right)$$
$$+\frac{1}{6}\Delta^{\mu\nu}\left(\Delta^{\rho}{}_{\alpha}\Delta_{\beta\gamma} + \Delta^{\rho}{}_{\beta}\Delta_{\alpha\gamma}\right)$$
$$+\frac{1}{6}\left(\Delta^{\mu\rho}\Delta^{\nu}{}_{\gamma} + \Delta^{\nu\rho}\Delta^{\mu}{}_{\gamma}\right)\Delta_{\alpha\beta}$$
$$-\frac{1}{12}\left(\Delta^{\mu\rho}\Delta^{\nu}{}_{\alpha}\Delta_{\beta\gamma} + \Delta^{\mu\rho}\Delta^{\nu}{}_{\beta}\Delta_{\alpha\gamma} + \Delta^{\nu\rho}\Delta^{\mu}{}_{\alpha}\Delta_{\beta\gamma} + \Delta^{\nu\rho}\Delta^{\mu}{}_{\beta}\Delta_{\alpha\gamma}\right)$$
$$-\frac{1}{3}\Delta^{\mu\nu}\Delta^{\rho}{}_{\gamma}\Delta_{\alpha\beta}$$

$$\text{spin-2(2)}: \sum_{m=-2}^{2} |2,m\rangle^{\mu\nu\rho}\,\langle 2,m|_{\alpha\beta\gamma} =$$
$$\frac{1}{3}\left(\Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\beta} - \Delta^{\nu}{}_{\alpha}\Delta^{\mu}{}_{\beta}\right)\Delta^{\rho}{}_{\gamma}$$
$$+\frac{1}{6}\left(\Delta^{\mu}{}_{\gamma}\Delta^{\nu}{}_{\beta}\Delta^{\rho}{}_{\alpha} - \Delta^{\mu}{}_{\gamma}\Delta^{\nu}{}_{\alpha}\Delta^{\rho}{}_{\beta} - \Delta^{\nu}{}_{\gamma}\Delta^{\mu}{}_{\beta}\Delta^{\rho}{}_{\alpha} + \Delta^{\nu}{}_{\gamma}\Delta^{\mu}{}_{\alpha}\Delta^{\rho}{}_{\beta}\right)$$
$$+\frac{1}{4}\left(\Delta^{\mu\rho}\Delta^{\nu}{}_{\alpha}\Delta_{\beta\gamma} - \Delta^{\mu\rho}\Delta^{\nu}{}_{\beta}\Delta_{\alpha\gamma} - \Delta^{\nu\rho}\Delta^{\mu}{}_{\alpha}\Delta_{\beta\gamma} + \Delta^{\nu\rho}\Delta^{\mu}{}_{\beta}\Delta_{\alpha\gamma}\right)$$

$$\text{spin-1(1)}: \sum_{m=-1}^{1} |1,m\rangle^{\mu\nu\rho}\,\langle 1,m|_{\alpha\beta\gamma} =$$
$$\frac{1}{15}\Delta^{\mu\nu}\Delta^{\rho}{}_{\gamma}\Delta_{\alpha\beta}$$
$$-\frac{1}{10}\Delta^{\mu\nu}\left(\Delta^{\rho}{}_{\alpha}\Delta_{\beta\gamma} + \Delta^{\rho}{}_{\beta}\Delta_{\alpha\gamma}\right)$$

$$
\begin{aligned}
&-\frac{1}{10}\left(\Delta^{\mu\rho}\Delta^{\nu}{}_{\gamma}+\Delta^{\nu\rho}\Delta^{\mu}{}_{\gamma}\right)\Delta_{\alpha\beta}\\
&+\frac{3}{20}\left(\Delta^{\mu\rho}\Delta^{\nu}{}_{\alpha}\Delta_{\beta\gamma}+\Delta^{\mu\rho}\Delta^{\nu}{}_{\beta}\Delta_{\alpha\gamma}+\Delta^{\nu\rho}\Delta^{\mu}{}_{\alpha}\Delta_{\beta\gamma}+\Delta^{\nu\rho}\Delta^{\mu}{}_{\beta}\Delta_{\alpha\gamma}\right)\\
\text{spin-1(2)}:&\ \sum_{m=-1}^{1}|1,m\rangle^{\mu\nu\rho}\,\langle 1,m|_{\alpha\beta\gamma}=\\
&-\frac{1}{4}\left(\Delta^{\mu\rho}\Delta^{\nu}{}_{\alpha}\Delta_{\beta\gamma}-\Delta^{\mu\rho}\Delta^{\nu}{}_{\beta}\Delta_{\alpha\gamma}-\Delta^{\nu\rho}\Delta^{\mu}{}_{\alpha}\Delta_{\beta\gamma}+\Delta^{\nu\rho}\Delta^{\mu}{}_{\beta}\Delta_{\alpha\gamma}\right)\\
\text{spin-1(3)}:&\ \sum_{m=-1}^{1}|1,m\rangle^{\mu\nu\rho}\,\langle 1,m|_{\alpha\beta\gamma}=\\
&\frac{1}{3}\Delta^{\mu\nu}\Delta^{\rho}{}_{\gamma}\Delta_{\alpha\beta}\\
\text{spin-0}:&\ \ |0,0\rangle^{\mu\nu\rho}\,\langle 0,0|_{\alpha\beta\gamma}=\\
&\frac{1}{6}\Big(\Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\beta}\Delta^{\rho}{}_{\gamma}+\Delta^{\mu}{}_{\beta}\Delta^{\nu}{}_{\gamma}\Delta^{\rho}{}_{\alpha}+\Delta^{\mu}{}_{\gamma}\Delta^{\nu}{}_{\alpha}\Delta^{\rho}{}_{\beta}\\
&\quad-\Delta^{\nu}{}_{\alpha}\Delta^{\mu}{}_{\beta}\Delta^{\rho}{}_{\gamma}-\Delta^{\nu}{}_{\beta}\Delta^{\mu}{}_{\gamma}\Delta^{\rho}{}_{\alpha}-\Delta^{\nu}{}_{\gamma}\Delta^{\mu}{}_{\alpha}\Delta^{\rho}{}_{\beta}\Big)\quad .
\end{aligned}
\tag{13.232}
$$

The total completeness relations is also valid :

$$
\sum_{s=0}^{3}\sum_{m=-s}^{s}|s,m\rangle^{\mu\nu\rho}\,\langle s,m|_{\alpha\beta\gamma}=\Delta^{\mu}{}_{\alpha}\,\Delta^{\nu}{}_{\beta}\,\Delta^{\rho}{}_{\gamma}\quad , \tag{13.233}
$$

provided we sum over all sectors with the same s.

13.12.5 Massless particles : surviving states

So far, we have taken our particles to be at rest, with a momentum p for which

$$
p^{\mu}=m\,t^{\mu}\quad .
$$

For *moving* particles, we can obtain the correct states by simply performing the appropriate Lorentz boost. As already indicated, we shall take the motion of the particles to be along the z axis ; our states have been prepared for this by taking z as the spin quantization axis. The momentum of the particle will then be

$$
p^{\mu}=m\,t^{\mu}\quad\to\quad p^{\mu}=p^{0}\,t^{\mu}+|\vec{p}|\,z^{\mu}\quad , \tag{13.234}
$$

and the vector z^μ becomes, under the same boost

$$z^\mu \quad \to \quad \left(\frac{|\vec{p}|}{m}\right) t^\mu + \left(\frac{p^0}{m}\right) z^\mu \quad . \tag{13.235}$$

The vectors $x_\pm$ are not affected by the boost. It is therefore sufficient to replace, in Eqns.(13.209),(13.226), (13.227),(13.228), and (13.231), z by its boosted form.

Let us now consider the extreme case : that of a massless particle. We can view this as the limit $p^0/m \to \infty$ of a massive particle. In that limit, z^μ diverges badly, and we must again adopt the point of view presented in chapter 8 : **the theory wil only be viable if those tensors that diverge in the massless limit decouple completely**. That is, the only *observable* states must be those that do not diverge, *i.e.* those that contain x_+'s and x_-'s but not any trace of a z. A quick inspection in our inventory of states reveals that only a handful of states are left :

$$\begin{array}{ll} \text{rank-1, spin-1 :} & |1,1\rangle = |+\rangle \quad , \quad |1,-1\rangle = -\,|-\rangle \\ \text{rank-2, spin-2 :} & |2,2\rangle = |{+}{+}\rangle \quad , \quad |2,-2\rangle = |{-}{-}\rangle \\ \text{rank-2, spin-1 :} & |1,0\rangle = (|{+}{-}\rangle - |{-}{+}\rangle)/\sqrt{2} \\ \text{rank-3, spin-3 :} & |3,3\rangle = |{+}{+}{+}\rangle \quad , \quad |3,-3\rangle = -\,|{-}{-}{-}\rangle \end{array} \tag{13.236}$$

With the exception of the rank-2, spin-1 state, the so-called *Kalb-Ramond* state, all the surviving states have $m = \pm s$ and are totally symmetric. Is this general ? In other words, how do we know that there is no rank-31, spin-17 state that is built up from *only* x_+'s and x_-'s ? We can answer this question by the following pleasing argument. Since the ladder operators $\Sigma_\pm$ transform physical states into one another, any physical state must be an eigenstate of $\Sigma_+\Sigma_-$ or $\Sigma_-\Sigma_+$[31]. Disregarding, for simplicity, minus signs and factors $\sqrt{2}$, the effect of Σ_+ is $0 \to +$, $- \to 0$, and that of Σ_- is $+ \to 0$, $0 \to -$. We can therefore write

$$\Sigma_+\Sigma_- \,|{+}{-}\rangle \to \Sigma_+ \,|0{-}\rangle \to |{+}{-}\rangle + |00\rangle \quad . \tag{13.237}$$

[31] It is of course possible that Σ_+ acting on our state, say, will give zero, and then it is an eigenstate of $\Sigma_-\Sigma_+$ with eigenvalue zero. We may avoid this trivial case by choosing, instead, $\Sigma_+\Sigma_-$, under which our state will have a nonzero eigenvalue.

Let us now consider a hypothetical massless-particle candidate state. It will be a linear combination of kets with lots of $+$'s and $-$'s. Among these we concentrate on three kets in particular :

$$T_1 = |\cdots + + - \cdots\rangle \;\;,\;\; T_2 = |\cdots + - + \cdots\rangle \;\;,\;\; T_3 = |\cdots - + + \cdots\rangle \;\;. \tag{13.238}$$

The rest of the content of the kets (indicated by the ellipses, and consisting of some sequences of $+$'s and $-$'s) is identical for the three kets. The candidate state contains these T's in some linear combination :

$$C_1T_1 + C_2T_2 + C_3T_3 + \text{lots of other terms}$$

Let us now consider what happens if we let $\Sigma_+\Sigma_-$ work on these kets. T_1 will turn into a lot of terms, among which we can recognize two important ones :

$$T_1 \to |\cdots + 00 \cdots\rangle + |\cdots 0 + 0 \cdots\rangle + \cdots \;\;. \tag{13.239}$$

Similarly, we find for T_2 and T_3 :

$$\begin{aligned} T_2 &\to |\cdots 00 + \cdots\rangle + |\cdots + 00 \cdots\rangle + \cdots \;\;, \\ T_3 &\to |\cdots 00 + \cdots\rangle + |\cdots 0 + 0 \cdots\rangle + \cdots \;\;. \end{aligned} \tag{13.240}$$

We now note a few things. In the first place, a resulting ket like $|\cdots 0 + 0 \cdots\rangle$ can *only* come from the T's (in this case, from T_1 and T_3). In the second place, our candidate state can*not* contain this ket by itself, since it must be free of 0's. In the third place, such unwanted kets must drop out because our state is an eigenstate of Σ^2. We must therefore rely on cancellations between the T's. In fact, we need simultaneously

$$C_1 = -C_2 \;\;,\;\; C_2 = -C_3 \;\;,\;\; C_3 = -C_1 \;\;. \tag{13.241}$$

Obviously, $C_{1,2,3} = 0$: our three T's do not occur at all[32] ! But of course we can repeat the same argument for any other such three kets. We see that the only possibilities to have admissible massless-particle states are twofold:

- *Only* $+$'s, or *only* $-$'s, occur. These are precisely the rank-s, spin-s states such as we have found, and this persists also for $s > 3$. Note that these states are totally symmetric — not for some deep field-theoretical reason, but because they can't help it.

- Precisely *one* $+$ and *one* $-$ occur. This is the Kalb-Ramond state, which now stands revealed as a lone exception.

[32] A three-cornered argument such as this, in which all T's disappear, deserves to be called a *Bermuda triangle*.

13.12.6 Massless propagators

For massless states, the spin sums cannot be built up from objects like ${\Delta^\mu}_\alpha$ since these diverge. An often-used recipe is the following. For a massless particle of momentum p^μ, define

$$p^\mu = (p^0, \vec{p}) \quad , \quad \bar{p}^\mu = (p^0, -\vec{p}) \ . \tag{13.242}$$

Obviously, this is not a Lorentz-invariant definition, but as we shall see that is not a problem. The point is that a $\bar{p}$ can be found whatever the Lorentz frame is. We can now write

$$\begin{aligned} |+\rangle^\mu \langle+|_\nu + |-\rangle^\mu \langle-|_\nu &= {x_+}^\mu x_{-\nu} + {x_-}^\mu x_{+\nu} \\ = \quad \frac{1}{p\cdot\bar{p}}&\left(p^\mu\bar{p}_\nu + \bar{p}^\mu p_\nu\right) - {\delta^\mu}_\nu \quad \equiv \quad {\nabla^\mu}_\nu \ . \end{aligned} \tag{13.243}$$

In analogy to Eq.(13.214) we now have

$$\nabla^{\mu\alpha}\,\nabla_{\alpha\nu} = -{\nabla^\mu}_\nu \quad , \quad {\nabla^\mu}_\mu = -2 \ . \tag{13.244}$$

If, as we must promise ourselves, massless states only couple to conserved sources (on which the handlebar operation gives zero), the terms containing $\bar{p}$ will always drop out. We can now write the spin sums for the surviving massless states as follows :

$$\begin{aligned}
&\text{rank-1, spin-1 :} && {\nabla^\mu}_\alpha \ , \\
&\text{rank-2, spin-2 :} && \frac{1}{2}\left({\nabla^\mu}_\alpha{\nabla^\nu}_\beta + {\nabla^\mu}_\beta{\nabla^\nu}_\alpha\right) - \frac{1}{2}\nabla^{\mu\nu}\nabla_{\alpha\beta} \ , \\
&\text{rank-2, spin-1 :} && \frac{1}{2}\left({\nabla^\mu}_\alpha{\nabla^\nu}_\beta - {\nabla^\mu}_\beta{\nabla^\nu}_\alpha\right) \ , \\
&\text{rank-3, spin-3 :} && \frac{1}{6}\Big({\nabla^\mu}_\alpha{\nabla^\nu}_\beta{\nabla^\rho}_\gamma + {\nabla^\mu}_\beta{\nabla^\nu}_\gamma{\nabla^\rho}_\alpha + {\nabla^\mu}_\gamma{\nabla^\nu}_\alpha{\nabla^\rho}_\beta \\
& && \quad + {\nabla^\mu}_\beta{\nabla^\nu}_\alpha{\nabla^\rho}_\gamma + {\nabla^\mu}_\alpha{\nabla^\nu}_\gamma{\nabla^\rho}_\beta + {\nabla^\mu}_\gamma{\nabla^\nu}_\beta{\nabla^\rho}_\alpha\Big) \\
& && -\frac{1}{12}\Big(\nabla^{\mu\nu}\left({\nabla^\rho}_\alpha\nabla_{\beta\gamma} + {\nabla^\rho}_\beta\nabla_{\gamma\alpha} + {\nabla^\rho}_\gamma\nabla_{\alpha\beta}\right) \\
& && \quad + \nabla^{\nu\rho}\left({\nabla^\mu}_\alpha\nabla_{\beta\gamma} + {\nabla^\mu}_\beta\nabla_{\gamma\alpha} + {\nabla^\mu}_\gamma\nabla_{\alpha\beta}\right) \\
& && \quad + \nabla^{\rho\mu}\left({\nabla^\nu}_\alpha\nabla_{\beta\gamma} + {\nabla^\nu}_\beta\nabla_{\gamma\alpha} + {\nabla^\nu}_\gamma\nabla_{\alpha\beta}\right)\Big)
\end{aligned} \tag{13.245}$$

Compared to the massive case, some coefficients are different : -1/2 rather than -1/3 in the spin-2 case, and -1/12 instead of -1/15 for spin-3. This is due, of course, to the different traces of Δ and ∇. The spin sum for the massless vector particle (rank-1, spin-1) is in fact that of the axial gauge discussed in Chapter 8, with the gauge vector r chosen to be $\bar{p}$. Note that, *whatever* r^μ, we can always move to the centre-of-mass frame of p^μ and r^μ, and in that frame we have precisely $r^\mu = \bar{p}^\mu$.

13.12.7 Spin of the Kalb-Ramond state

Concerning the Kalb-Ramond (KR) state, there may be some controversy. For a massless particle in this state, the spin along the axis of motion must, under measurement, always come out zero. It is not easy to see how such a particle can be distinguished from a scalar one. Indeed, in string theory where the KR state comes up naturally, it is considered to describe a (pseudo)scalar particle called the *axion*. In order to talk sensibly about the spin of the KR state it is useful to consider how it may be measured, for instance using fermions. We therefore consider the coupling of a rank-2, spin-1 state to fermions. The interaction vertex must have the properties that **(a)** it is an antisymmetric rank-2 tensor, and **(b)** it is current-conserving, in order to make sense in the massless limit. Denoting the two fermions by ψ and $\overline{\psi}$ the simplest choice appears to be

$$\overline{\psi}\,\epsilon^{\mu\nu\rho\sigma}\,p_\rho\,(A + B\gamma^5)\,\gamma_\sigma\,\psi$$

where p is the momentum of the antisymmetric tensor state, and A and B are constants. This interaction vertex vanishes trivially under the handlebar operation. For the process

$$\bar{f}(p_1)\,f(p_2) \quad\to\quad f(p_3)\,\bar{f}(p_4)$$

by the exchange of a KR state of mass M, we then have the amplitude

$$\begin{aligned}
\mathcal{M} \;&=\; i\hbar\,\overline{v}(p_1)\,\epsilon^{\mu\nu\rho\sigma}\,p_\rho\,(A + B\gamma^5)\,\gamma_\sigma u(p_2) \\
&\quad\times\,\frac{\Delta_{\mu\alpha}\Delta_{\nu\beta} - \Delta_{\mu\beta}\Delta_{\nu\alpha}}{2(s - M^2)} \\
&\quad\times\,\overline{u}(p_3)\,\epsilon^{\alpha\beta\kappa\lambda}\,p_\kappa\,(A' + B'\gamma^5)\,\gamma_\lambda v(p_4)\ \ , \\
s = p\cdot p \quad &, \quad p = p_1 + p_2 = p_3 + p_4\ \ . \qquad (13.246)
\end{aligned}$$

Because of the current conservation and the antisymmetry of the vertices, we may replace $\Delta_{\mu\alpha}\Delta_{\nu\beta} - \Delta_{\mu\beta}\Delta_{\nu\alpha}$ by $2g_{\mu\alpha}g_{\nu\beta}$. Furthermore, since

$$\epsilon^{\mu\nu\rho\sigma}\, p_\rho\, \epsilon_{\mu\nu}{}^{\kappa\lambda}\, p_\kappa = 2\left(p^\sigma\, p^\lambda - s\, g^{\sigma\lambda}\right) \tag{13.247}$$

we have

$$\begin{aligned}
\mathcal{M} \;=\; & -2i\hbar\,\frac{1}{s-M^2} \\
& \left(\left(\overline{v}(p_1)\left(A(m_2-m_1)+B(m_1+m_2)\gamma^5\right)u(p_2)\right.\right. \\
& \qquad \left.\times\;\overline{u}(p_3)\left(A'(m_3-m_4)-B'(m_3+m_4)\gamma^5\right)\overline{v}(p_4)\right) \\
& -s\left(\overline{v}(p_1)\left(A+B\gamma^5\right)\gamma^\mu\, u(p_2)\right. \\
& \qquad \left.\left.\times\;\overline{u}(p_3)\left(A'+B'\gamma^5\right)\gamma_\mu\, v(p_4)\right)\right)\quad .
\end{aligned} \tag{13.248}$$

Here m_j is the mass of momentum p_j. Note that, in contrast to *e.g.* the case of QED, $m_1 = m_2$ or $m_3 = m_4$ is not necessary for current conservation. We can now investigate several situations. In the first place, if $M \neq 0$ the amplitude has a pole for some nonzero s value, which we may take as the signal of a particle. The second term in brackets in Eq.(13.248) then tells us that, indeed, a spin-1 particle has been exchanged[33]. The occurrence of the first term is, then, not surprising : a similar contribution is found in *e.g.* the W exchange in muon decay. Secondly, we may take $M = 0$. In that case, the second term no longer has a pole. It can therefore not survive a truncation argument, and must not be counted as coming from any particle propagation. The first term *does* survive ; if we also assume flavour conservation so that $m_1 = m_2$ and $m_3 = m_4$, the only degree of freedom that propagates is, indeed, that of a pseudoscalar.

[33]We can measure this, for instance by looking at the angular distribution of the produced fermion-antifermion pair ; see also Appendix **??**.

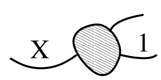
13.13 Unitarity bounds

13.13.1 Resonances

In this appendix we shall establish bounds on total cross sections as implied by the unitarity of the theory. We are interested in *upper bounds* on cross sections, that is we want to investigate the most *efficient* way to get rid of the initial state in favour of some final state. Now, as is known from the elementary theory of coupled oscillators, the most efficient way to pump energy (*i.e.* the energy content of the initial-state particles) into another state is by resonance. In our language, this means that we shall consider two initial-state particles colliding and coupling to another particle with just the right energy to put that particle on its mass shell. Unavoidably, if the new particle can be made in such a way it can also decay, and it therefore must have a nonzero decay width which protects its propagator from exploding. We shall investigate this process in some detail.

13.13.2 Preliminaries : decay widths

We shall investigate the unitarity bound on the cross section for a given initial two-particle state 1 to evolve into a given n-particle state 2 by way of a resonant particle X of rest mass M and total decay width Γ. This means that particle X must couple both to 1 and to 2. There is therefore a possible decay X$\to$1, given by the Feynman diagram



The corresponding matrix element can be written as

$$\mathcal{M}_{\mathrm{X}\to 1} = i\,\overline{A}_k \cdot u_j \quad . \tag{13.249}$$

In this admittedly abstract expression, u stands for the external-line factor[34] for the incoming X particle that has, in addition to energy and momentum, a discrete quantum number j denoting its angular momentum (for brevity we shall use the smaller word 'spin' throughout this section). We shall assume that j runs from 1 to N, so that there are in total N spin states : for a spin-J particle, therefore, $N = 2J+1$. Similarly the final state is characterized by a

[34]This might be just a number, or a spinor, or a polarization vector,... take your pick.

discrete quantum number k alongside the continuous energy and momentum variables, and k is assumed to run from 1 to K. For instance, if 1 stands for an electron-positron state, $K = 4$ since there are two spin states for the electron and two for the positron. Thus, A_k denotes the total of the connected diagrams (the blob) and any external-line factors for a final state with discrete quantum number k. The total decay width Γ_1 for X to go into the two-particle state 1 is given by

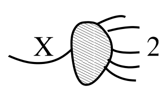
$$\Gamma_1 = \frac{1}{2M}\frac{1}{N}\sum_{j,k} \int \overline{A}_k \cdot u_j \, \overline{u}_j \cdot A_k \, \frac{1}{(2\pi)^2}\frac{d\Omega}{8} \, \frac{\lambda(M^2, m^2, m'^2)^{1/2}}{M^2} \, S_1 \quad , \tag{13.250}$$

where m and m' are the masses of the two particles in 1. The symmetry factor S_1 equals 1 if the particles are distinguishable, and 1/2 if they are not. Ω is of course the solid angle of one of the particles in the rest frame of X. The angle- and spin-averaged transition rate is therefore

$$\frac{1}{K}\sum_{j,k} \int \frac{d\Omega}{4\pi} \, \overline{A}_k \cdot u_j \, \overline{u}_j \cdot A_k = \frac{16\pi M \Gamma_1 N}{S_1 K} \, \frac{M^2}{\lambda^{1/2}} \quad , \tag{13.251}$$

with $\lambda^{1/2} = \lambda(M^2, m^2, m'^2)^{1/2}$.

The process X$\to$2 is described by the Feynman diagrams contained in



and is written

$$\mathcal{M}_{\mathrm{X}\to 2} = i \, \overline{B}_l \cdot u_j \quad , \tag{13.252}$$

where l denotes the discrete quantum numbers in the state 2. The width for the process is given by

$$\Gamma_2 = \frac{1}{2M}\frac{1}{N}\sum_{j,l} \int \overline{B}_l \cdot u_j \, \overline{u}_j \cdot B_l \, dV_n \, S_2 \quad , \tag{13.253}$$

where dV_n is the n-particle phase space factor going with the state 2, and S_2 is the appropriate symmetry factor.

13.13.3 The rôle of angular momentum conservation

Let us consider the process X→2 in some greater detail. It is easy to conceive of a final state 2 that couples *only* to a particle of spin J and to no other spin. Now, our important supposition : if the initial particle is at rest, and *if space is isotropic so that there is no preferred direction*, this does not only mean that angular momentum is conserved but also that the various $2J+1$ spin states of the X particle are to be treated on the same footing, so that each spin state must have the same decay width. This in its turn implies that the integrated-over final state must form a projection onto the pure spin-J state :

$$\sum_l B_l\,\overline{B}_l = \mathcal{B}(M^2)\,\sum_n u_n\overline{u}_n \qquad (13.254)$$

where n runs, of course, from 1 to N. Obviously, under the isotropy assumption $\mathcal{B}$ can only depend on M^2. We find that

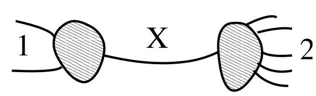
$$\int \sum_l \overline{B_l}\cdot u_j\,\overline{u}_{j'}\cdot B_l = \mathcal{B}(M^2)\,\sum_n \overline{u}_n\cdot u_j\,\overline{u}_{j'}\cdot u_n \propto \delta_{j,j'}\;\;, \qquad (13.255)$$

or, in other words,

$$\sum_l \int \overline{B}_l\cdot u_j\,\overline{u}_{j'}\cdot B_l = 2\,M\,\Gamma_2\,\delta_{j,j'}\;\;. \qquad (13.256)$$

13.13.4 The unitarity bound

We now consider the process 1→2 by X exchange. For total scattering invariant mass $\sqrt{s}$, it is given by the diagram



and the amplitude reads

$$\mathcal{M} = \frac{-i}{s-M^2+iM\Gamma}\,\overline{B}_l\cdot\Pi\cdot A_k\;\;. \qquad (13.257)$$

Here, Π is the numerator of the X propagator : on the mass shell, therefore, we must have

$$\Pi\rfloor_{s=M^2} = \sum_n u_j\,\overline{u}_j\;\;. \qquad (13.258)$$

For the total cross section we therefore have

$$\begin{aligned} \sigma &= \frac{1}{2\lambda(s,m^2,m'^2)^{1/2}}\,\frac{1}{(s-M^2)^2+m^2\Gamma^2}\,\frac{1}{K} \\ &\quad\times \sum_{k,l}\int (\overline{B}_l\cdot\Pi\cdot A_k)\,(\overline{A}_k\cdot\Pi\cdot B_l)\,dV_n\;S_2\;\;. \end{aligned} \tag{13.259}$$

On the X mass shell, we can write, with the help of Eq.(13.256),

$$\begin{aligned} \sigma &= \frac{1}{2\lambda^{1/2}}\frac{1}{M^2\Gamma^2}\frac{1}{K}\sum_{k,l,j,j'}\int (\overline{B}_l\cdot u_j\,\overline{u}_j\cdot A_k)\,(\overline{A}_k\cdot u_{j'}\,\overline{u}_{j'}\cdot B_l)\,dV_n\;S_2 \\ &= \frac{1}{2\lambda^{1/2}}\frac{1}{M^2\Gamma^2}\frac{1}{K}\sum_{k,l,j,j'}\int (\overline{B}_l\cdot u_j\,\overline{u}_{j'}\cdot B_l)\,(\overline{A}_k\cdot u_{j'}\,\overline{u}_j\cdot A_k)\,dV_n\;S_2 \\ &= \frac{1}{2\lambda^{1/2}}\frac{1}{M^2\Gamma^2}\,\frac{2M\Gamma_2}{K}\sum_{k,j,j'}\int\frac{d\Omega}{4\pi}\,\overline{A}_k\cdot u_{j'}\,\overline{u}_j\cdot A_k\,\delta_{j,j'}\;\;, \end{aligned} \tag{13.260}$$

where it must be realized that we have rewritten the integral over B-*cum*-A by the integral over B times the *average* over A. Due to angular-momentum conservation we can now write, using Eq.(13.251),

$$\sigma\rfloor_{s=M^2} = \left(\frac{\Gamma_1}{\Gamma}\right)\left(\frac{\Gamma_2}{\Gamma}\right)\,\frac{N}{S_1 K}\,\frac{16\pi\;s}{\lambda(s,m^2,m'^2)}\;\;. \tag{13.261}$$

Now, the factor Γ_2/Γ is understandable since the X particle has only a fractional probability to decay into state 2 (there may be other decay channels available, in fact at least the decay X$\to$1), and then symmetry between the reactions 1$\to$2 and 2$\to$1 requires also the presence of the factor Γ_1/Γ. We conclude that the cross section for the initial state 1 to go into *any* final state with spin J is bounded by the unitarity limit

$$\sigma_{\rm UL} = \frac{2J+1}{S_1 K}\,\frac{16\pi\;s}{\lambda(s,m^2,m'^2)}\;\;, \tag{13.262}$$

where as mentioned before S_1 is $1/2$ for indistinguishable particles and 1 for distinguishable ones, and K is the total number of possible discrete quantum numbers for the initial state[35].

[35] For example, for an initial e^+e^- state we have $S_1 = 1$, $K = 4$: for an initial state of two photons $S_1 = 1/2$, $K = 4$, and for an initial state of two gluons $S_1 = 1/2$, $K = 256$ since gluons come with 2 possible spin states and 8 different colour states.

13.14 The CPT theorem

In this appendix, we shall discuss the very fundamental CPT theorem[36] for theories with interacting particles. This theorem deals with what happens (or ought to happen) to scattering amplitudes when we relate various physical scattering processes[37]. As usual, we shall start by looking at Dirac particles.

13.14.1 Transforming spinors

In Chapter 7, we defined the standard form for the various spinors corresponding to an on-shell (anti)-particle with mass m and momentum p^μ. We recapitulate them here :

$$\begin{aligned} u_\pm(p) &= N(p)\,(\not{p}+m)\,u_\mp(k_0) \quad , \\ v_\pm(p) &= N(p)\,(\not{p}-m)\,u_\mp(k_0) \quad , \\ u_+(k_0) &= \not{k}_1\,u_-(k_0) \;\;,\;\; u_-(k_0)\overline{u}_-(k_0) = \omega_-\not{k}_0 \;\;, \\ N(p) &= 1/\sqrt{2(pk_0)} \;\;,\;\; {k_0}^2 = (k_0k_1) = 0 \;\;,\;\; {k_1}^2 = -1 \;\;. \end{aligned} \tag{13.263}$$

This is, of course, only a *phase convention*, where the phase choice is not explicit but implied by the choice of k_0, k_1 and the complex phase of $u_-(k_0)$. Now, let us apply γ^5 to these states. It is easy to see that

$$\begin{aligned} &\gamma^5\,u_+(p) = v_+(p) \quad , \quad \gamma^5\,u_-(p) = -\,v_-(p) \;\;, \\ &\overline{u}_+(p)\,\gamma^5 = -\overline{v}_+(p) \quad , \quad \overline{u}_-(p)\,\gamma^5 = \overline{v}_-(p) \;\;. \end{aligned} \tag{13.264}$$

In words, what this transformation does is *to change an incoming, right(left)-handed fermion into an outgoing, left(right)-handed antifermion* (and *vice versa*). Thus we have (**a**) the interchange of particle and anti-particle (charge conjugation, C), (**b**) the interchange of right- and left-handedness[38] (parity inversion, P), and (**c**) the interchange of initial and final state (time reversal T), which goes by the name of CPT transformation[39]. Applied to Feynman diagrams, we can depict this as follows (where we have indicated the

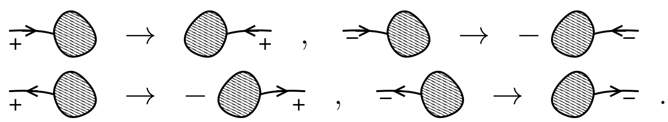
[36] Also known as the CTP theorem, the TCP theorem, the TPC theorem, the PTC theorem, or the PCT theorem.

[37] Recall that, in these notes, we concentrate on the (perturbative)*processes* that are going on, that is, scattering described by diagrams and amplitudes.

[38] Recall that for a particle + means right-handed, but for an *anti*particle it means left-handed (*cf* section 7.6.6)

[39] There is a slight subtlety here. An ingoing particle with three-momentum $\vec{p}$ is transformed into an outgoing antiparticle with the same momentum $\vec{p}$. Under P, momenta are

helicity) :



(13.265)

As we can see, the effect of CPT on any diagram is not only to interchange initial and final states, but also *to reverse the arrows on intermediate fermion lines*.

13.14.2 CPT transformation on sandwiches

Let us consider the scalar current for two on-shell momenta $p_{1,2}$, with respective masses $m_{1,2}$:

$$J_{\lambda_1\lambda_2} = \overline{u}_{\lambda_1}(p_1)u_{\lambda_2}(p_2) \ \ . \tag{13.266}$$

Under CPT, this scalar current behaves as follows :

$$\begin{aligned} J_{++} &\to \hat{J}_{++} = -\overline{v}_+(p_1)v_+(p_2) \ \ , \\ J_{+-} &\to \hat{J}_{+-} = \overline{v}_+(p_1)v_-(p_2) \ \ . \end{aligned} \tag{13.267}$$

At first sight, these CPT transforms look nothing like the original. Note, however, that using the standard form we can write them as traces :

$$\begin{aligned} J_{++} &= N(p_1)N(p_2)\,\mathrm{Tr}\,(\omega_-\not{k}_0(\not{p}_1+m_1)(\not{p}_2+m_2)) \ \ , \\ J_{+-} &= N(p_1)N(p_2)\,\mathrm{Tr}\,(\omega_-\not{k}_0(\not{p}_1+m_1)(\not{p}_2+m_2)\not{k}_1) \ \ , \end{aligned} \tag{13.268}$$

whereas

$$\begin{aligned} \hat{J}_{++} &= -\,N(p_1)N(p_2)\,\mathrm{Tr}\,(\omega_-\not{k}_0(\not{p}_1-m_1)(\not{p}_2-m_2)) \ \ , \\ \hat{J}_{+-} &= N(p_1)N(p_2)\,\mathrm{Tr}\,(\omega_-\not{k}_0(\not{p}_1-m_1)(\not{p}_2-m_2)\not{k}_1) \ \ , \end{aligned} \tag{13.269}$$

Keeping track of which terms in these traces actually survive[40], we see that, appearances notwithstanding,

$$\hat{J}_{\lambda_1\lambda_2} = J_{\lambda_1\lambda_2} \ \ . \tag{13.270}$$

---

inverted so that $\vec{p}$ becomes $-\vec{p}$: but under T the *velocities* are *again* inverted. The same holds, of course, for spin vectors. It is only the fact that "+" means right-handed for particles and left-handed for antiparticles that ensures that the net result is just a change of *handedness*.

[40]For $J_{\pm\pm}$, these are the terms that contain an *odd* number of masses, for $J_{\pm\mp}$ those with *even* numbers of masses survive.

Similar (almost trivial) trace arguments show that, under CPT,

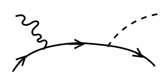
$$
\begin{aligned}
J_{\lambda_1\lambda_2}{}^{\mu} &= \overline{u}_{\lambda_1}(p_1)\,\gamma^{\mu}\,u_{\lambda_2}(p_2) &&\to\ -\,J_{\lambda_1\lambda_2}{}^{\mu}\ , \\
J_{\lambda_1\lambda_2}{}^{\mu\nu} &= \overline{u}_{\lambda_1}(p_1)\,\gamma^{\mu}\gamma^{\nu}\,u_{\lambda_2}(p_2) &&\to\ +\,J_{\lambda_1\lambda_2}{}^{\mu\nu}\ , \\
J_{\lambda_1\lambda_2}{}^{\mu\nu\alpha} &= \overline{u}_{\lambda_1}(p_1)\,\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\,u_{\lambda_2}(p_2) &&\to\ -\,J_{\lambda_1\lambda_2}{}^{\mu\nu\alpha}\ ,
\end{aligned}
\tag{13.271}
$$

and so on.

13.14.3 CPT transformation on diagrams

Consider a nontrivial but very simple diagram, for simplicity taken from the electroweak process

$$
e^-(p_1)\gamma(q_1) \to e^-(p_2)Z^0(q_2) \quad :
$$



(13.272)

Leaving out overall constants and denominators, this can be written as

$$
\begin{aligned}
\mathcal{M} &= A_{\mu\nu}\quad \bar{\epsilon}^{\mu}_{\lambda_Z}(q_2)\,\epsilon_{\lambda_\gamma}{}^{\nu}(q_1)\ , \\
A_{\mu\nu} &= \overline{u}_{\lambda_2}(p_2)\,\omega\gamma_\mu\,(\not{q}+m)\,\gamma_\nu\,u_{\lambda_1}(p_1)\ , \\
q &= p_1+q_1 = p_2+q_2 \quad , \quad \omega = g_v + g_a\gamma^5\ ,
\end{aligned}
\tag{13.273}
$$

where we have indicated the handedness (helicity) of the external particles. For the polarization vectors we take the representation given in Eq.(8.35), and for $\not{q}$ we may, if we wish, use Eq.(7.69) to write

$$
\not{q} = \frac{1}{2}\gamma_\alpha\ \ \overline{u}_+(q)\gamma^\alpha u_+(q)\ . \tag{13.274}
$$

Let us now see what happens if we apply CPT. In the first place,

$$
\not{q}\ \ \to\ \ -\not{q}\ , \tag{13.275}
$$

following immediately from Eq.(13.274)[41] Therefore, $A_{\mu\nu}$ transforms as

$$
A_{\mu\nu}\ \ \to\ \ -\lambda_1\lambda_2\ \ \overline{v}_{\lambda_1}(p_2)\,\omega\gamma_\mu\,(-\not{q}+m)\,\gamma_\nu\,v_{\lambda_2}(p_1)\ . \tag{13.276}
$$

[41]Antoher approach might be to find a set of timelike, positive-energy momenta $k_{1,2,3,\ldots}$ with masses $m_{1,2,3,\ldots}$, and a set of constants $c_{1,2,3,\ldots}$ such that $\sum_j c_j k_j{}^\alpha = q^\alpha$ and $\sum_j c_j m_j = m$. Obviously, this is always possible. We can then write $\not{q}+m = \sum_j c_j(u_+(k_j)\overline{u}_+(k_j) + u_-(k_j)\overline{u}_-(k_j))$, which under CPT are transformed into $\sum_j c_j(-v_+(k_j)\overline{v}_+(k_j) - v_-(k_j)\overline{v}_-(k_j)) = -\not{q}+m$.

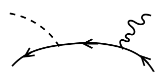
The arguments given in the previous section show that this evaluates again to $A_{\mu\nu}$ itself. Finally, for the polarization vectors we have, for instance,

$$\epsilon_{\lambda_\gamma}{}^{\nu} \ \ \to \ \ -\epsilon_{\lambda_\gamma}{}^{\nu} = -\bar{\epsilon}^{\nu}_{-\lambda_\gamma} \ \ , \tag{13.277}$$

so that the CPT transform of an *incoming, left(right)-handed* photon can be interpreted as that of an *outgoing, right(left)-handed* photon with the same momentum, up to an overall minus sign. The same goes of course for $\epsilon_{\lambda_Z}{}^{\mu}$. We see that, under CPT, the amplitude $\mathcal{M}$ remains unchanged[42] : but the *interpretation* is now that of the process

$$e^+(p_2)Z^0(q_2) \to e^+(p_1)\gamma(q_1) \ \ ,$$

with the understanding that left(right)-handed particles have been replaced by right(left)-handed ones. The corresponding Feynman diagram is now



, (13.278)

which may help you to understand the replacing of $\not{q}$ by $-\not{q}$: in diagrammatic terms, it comes from the fact that now q runs *against* the propagator's arrow.

It is now easy to see that we can perform similar operations on every conceivable Feynman diagram in our theory[43], and we shall always find that it transforms into itself. We say that our theory is *CPT-invariant* : if we (**a**) replace every external particle by its antiparticle (and *vice versa*), (**b**) interchange the initial and final states, and (**c**) interchange right- and left-handed, then all amplitudes remain the same. This is the CPT theorem.

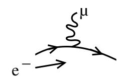
13.14.4 How to kill CPT, and what it costs

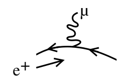
Like all such theorems, the CPT theorem can only be valid under a number of circumstances. Here, we mention the most important of these.

[42] You might think that the fact that the two minus signs coming from the polarization vector cancel so nicely is suspicious : but you should realize that if three external bosons were involved there would be two internal fermion propagators instead of one.

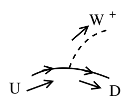
[43] If push comes to shove, we can always write *every* vector quantity in the diagram with spinors : we then end up with a massively complicated object containing loads of (anti)spinors and their conjugates, but for the rest only fixed numbers or matrices ; for such structures, we have already proven everything that is needed.

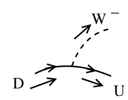
In the first place, comparing the diagrams (13.272) and (13.278) we see that we have implicitly assumed that the vertices of the theory are insensitive to what is the 'incoming', and what the 'outgoing' particle : for instance, the two vertices



and


are both assigned the value $iQ\gamma^\mu/\hbar$. More poignantly, in the electroweak sector we use the *same* vertex for



and


It *is*, of course, possible to let the vertex depend on the 'orientation' of the (sub)process : such theories, which as we see are not easily expressed diagrammatically[44], are called *non-Hermitian.* A non-Hermitian action would ruin CPT.

In the second place, and more subtly, we have assumed that there is, at least, the very possibility of a vacuum state through which particles can move ; in the literature, this means that *there is a state with lowest energy.* If the spectrum of the theory is not bounded from below, CPT is ruined : but, again, it is not easy to see how any ordinary particle physics could be alive under such circumstances[45], whether CPT invariant or not.

In the last place, there is the issue of Lorentz invariance. We have assumed that every vector h^μ will, under CPT, turn into $-h^\mu$, and this is very important for proving the CPT invariance of amplitudes. Suppose, now, that we introduce into our theory a *fixed* vector[46] f^μ, simply a set of four universally defined[47] numbers which enter nontrivially into the Feynman rules. Such a vector would, under CPT, *not* turn into its opposite ; but neither would it change under Lorentz transformations, it would simply remain f^μ. CPT would be ruined together with Lorentz invariance. A theory violating CPT will therefore manifest itself in being Lorentz-*non*invariant. You might

[44] At least in the way we have formulated things.

[45] In these notes, we take the existence of particles with a perturbative description for granted.

[46] We speak of a 'vector' here in the sense that it has four components, *not* in the sense of its behavior under coordinate transformations : indeed, the whole *point* is that it doesn't transform at all.

[47] Think of having some inspiration, or a voice from heaven engraving these numbers on stone tablets.

hope to avoid this by having, built into the fabric of the universe, some physically meaningful vector quantity f^μ, that *does* change with Lorentz transformations[48]. Still, CPT would be ruined, but we must also conclude that the 'vacuum' state is itself simply not Lorentz invariant since there is a 'preferred momentum'.

Note that it is, in principle, possible to violate Lorentz invariance without destroying CPT. For instance we can use a fixed 'tensor' $f^{\mu\nu}$ rather than a vector f^μ. Such a tensor does not change sign under CPT, exactly as it should. We can then construct theories where Lorentz invariance is violated but CPT invariance is not[49].

We see that the conditions under which CPT symmetry holds are very plausible and general, but they are *not* unavoidable. CPT may be ruined, but we can see that by the concomitant violation of Lorentz invariance, either in the interactions of the theory or in the structure of the vacuum itself !

13.15 Mathematical Miscellanies

In this section some collected mathematical issues are discussed which are usueful in the main text, or maybe just of some interest.

13.15.1 The Gaussian doubling trick

The Gaussian integral

$$G = \int\limits_{-\infty}^{\infty} dx\ \exp\left(-x^2\right) \tag{13.279}$$

is not easily computed in the standard manner. However, there is a 'once seen, never forgotten' way of doing it, by doubling the integral and going over to polar variables :

$$G^2 \;=\; \int\limits_{-\infty}^{\infty} dx\,dy\ \exp\left(-\left(x^2+y^2\right)\right) \;=\; \int\limits_{0}^{2\pi} d\phi \int\limits_{0}^{\infty} dr\ r\ \exp\left(-r^2\right)$$

[48]Such a thing would be, for instance, the 'momentum of the æther'.

[49]As an example, we can use, for the kinetic part of a Lagrangian, the object $f^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi$ rather than the usual $g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi$.

$$= \; 2\pi \int\limits_0^\infty dr\; r\; \exp\left(-r^2\right) \;=\; \pi \int\limits_0^\infty ds\; \exp(-s) \;=\; \pi \;. \tag{13.280}$$

13.15.2 The Dirac delta distribution

The Kronecker delta, as introduced in chapter 0, is defined for integer arguments and reads

$$\delta_{m,n} = \theta(m = n) \;, \tag{13.281}$$

so that

$$\delta_{n,m} = 0 \quad \text{for } m \neq n \quad , \quad \sum_{n=-\infty}^{\infty} \delta_{m,n} = 1 \;. \tag{13.282}$$

The Dirac delta distribution[50] is the continuum variant of this. Being a distribution, it is really defined in the context of integration with a test function[51]. The Dirac delta is commonly denoted by (surprise !) $\delta(x)$ and its definition is

$$\int\limits_{-\infty}^{\infty} dx\; \delta(x-a)\; f(x) = f(a) \tag{13.283}$$

for all test functions $f(x)$. Viewed as some kind of function it therefore has properties analogous to those in Eq.(13.282) :

$$\delta(x) = 0 \quad \text{for } x \neq 0 \quad , \quad \int\limits_{-\infty}^{\infty} dx\; \delta(x) = 1 \;. \tag{13.284}$$

Applying partial integration (and assuming cavalierly that this is allowed !) we also find properties of its derivatives :

$$\int\limits_{-\infty}^{\infty} dx\; \delta'(x-a)\; f(x) = -f'(a) \quad , \quad \int\limits_{-\infty}^{\infty} dx\; \delta''(x-a)\; f(x) = +f''(a) \;, \tag{13.285}$$

and so on. The Dirac delta can be viewed as the limit of a set of nonnegative functions with unit integral, that are increasingly narrow and more and more

[50] Colloquially, the Dirac delta *function*, but it is really a distribution in the sense of Schwartz.

[51] A test function has compact support and is inifinitely many times differentiable : simplistically, it is a *nice* function.

peaked at zero.

An important result that we use extensively in the text is

$$\int_{-\infty}^{\infty} dx\ \exp(ixz) = 2\pi\ \delta(z)\ \ , \tag{13.286}$$

which we now 'prove'. The above integral is not absolultely convergent since $|\exp(izx)| = 1$ and the integrand keeps oscillating between -1 and 1 forever. To bring this under control, we introduce a small but positive number ϵ and write

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\ \exp(ixz) &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx\ \exp(-\epsilon x^2 + ixz) \\
&= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx\ \exp\left(-\epsilon\left(x - \frac{iz}{2\epsilon}\right)^2 - \frac{z^2}{4\epsilon}\right) \\
&= \lim_{\epsilon\to 0} \sqrt{\frac{\pi}{\epsilon}}\ \exp\left(-\frac{z^2}{4\epsilon}\right) \\
&= 2\pi \lim_{\epsilon\to 0} \frac{1}{\sqrt{4\pi\epsilon}}\ \exp\left(-\frac{z^2}{4\epsilon}\right)\ \ ,
\end{aligned} \tag{13.287}$$

and this is precisely a limit as discussed above. The use of the vanishingly small but positive parameter ϵ here is, in fact, the same as its rôle in the regularization of the path integral in section 5.2.2.

13.15.3 Generating the Bell numbers

In order to arrive at the generating function for the Bell number $B(n)$, we start with a more basic concept. By $B_n(k)$ we denote the number of ways to divide n distinct objects into k non-empty groups: we shall then have $B(n) = \sum_{k\geq 0} B_n(k)$. For zero objects, there is obviously only one way to divide them, namely in zero groups:

$$B_0(k) = \delta_{k,0}\ \ . \tag{13.288}$$

If we have $n-1$ objects distributed into k groups, we can let the nth object form its own group, or add to one of the existing groups in k different ways.

This gives us the recursion

$$B_n(k) = B_{n-1}(k-1) + k\,B_{n-1}(k) \quad , \quad n \geq 1 \quad . \tag{13.289}$$

Let now form the set of generating functions

$$\phi_k(z) = \sum_{n\geq 0} \frac{z^n}{n!} B_n(k) \quad , \quad k = 0, 1, 2, \ldots \tag{13.290}$$

From Eq.(13.288) we have that $\phi_0(z) = 1$, and from Eq.(13.289)

$$\phi'_k(z) = k\,\phi_k(z) + \phi_{k-1}(z) \quad , \quad \phi_0(z) = 1 \quad (k \geq 1) \quad . \tag{13.291}$$

It is easily checked that the unique solution to these inhomogeneous first-order differential equations is

$$\phi_k(z) = \frac{1}{k!}\left(e^z - 1\right)^k \quad , \tag{13.292}$$

so that

$$\sum_{n\geq 0} \frac{z^n}{n!} B(n) = \sum_{k\geq 0} \phi_k(z) = e^{(e^z-1)} \quad . \tag{13.293}$$

13.15.4 Euler's formula

Consider the following identity:

$$\prod_{j=1}^{n} \Gamma(m_j+1) \quad = \int_0^\infty \prod_{j=1}^{n} \left(dz_j\; {z_j}^{m_j}\; e^{-z_j}\right) \quad . \tag{13.294}$$

In this integral, we employ the same technique as in sect.(13.9.1):

$$\begin{aligned}\prod_{j=1}^{n} \Gamma(m_j+1) \quad = & \\ & \int_0^\infty \prod_{j=1}^{n} \left(dz_j\; {z_j}^{m_j}\right)\; \exp\left(-\sum_{j=1}^{n} z_j\right) \\ & \times\, ds\; \delta\left(s - \sum_{j=1}^{n} z_j\right)\; \prod_{j=1}^{n}\left(dx_j\; \delta\left(x_j - \frac{z_j}{s}\right)\right) \quad .\end{aligned} \tag{13.295}$$

Eliminating the z's in favor of the x's gives

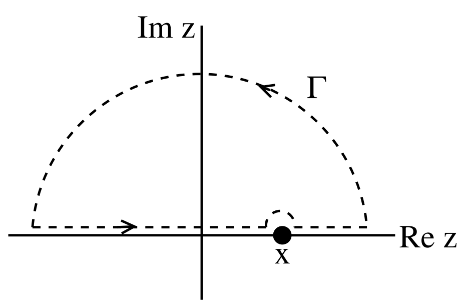
$$\prod_{j=1}^{n} \Gamma(m_j + 1) \quad = \quad \int_0^\infty ds\, dx_1 \cdots dx_n \; s^{m_1+\cdots+m_n+n-1}\, e^{-s} \times {x_1}^{m_1} \cdots {x_n}^{m_n}\; \delta(x_1 + \cdots + x_n - 1) \;\; , \tag{13.296}$$

and the final integral over s results in *Euler's formula*:

$$\int_0^1 dx_1 \cdots dx_n \; {x_1}^{m_1} \cdots {x_n}^{m_n} \delta(x_1 + \cdots + x_n - 1) = \frac{\Gamma(m_1+1)\Gamma(m_2+1)\cdots\Gamma(m_n+1)}{\Gamma(m_1+m_2+\cdots+m_n+n)} \;\; . \tag{13.297}$$

13.15.5 The Kramers-Kronig relation

We consider a function $f(z)$ that is analytic for $\Im(z) > 0$, and goes to zero sufficiently fast as $|z| \to \infty, \Im(z) > 0$. We may then construct a contour Γ as indicated below :



The contour runs along the real axis from $-\infty$ to $+\infty$. At the point x it circles around it, and a big half-circle then leads back from $+\infty$ to $-\infty$. By Cauchy's theorem, we have

$$\oint_\Gamma \frac{f(z)}{z - x}\, dz = 0 \;\; ,$$

since also $f(z)/(z-x)$ is analytic on, and inside, Γ. Splitting the integral into its various contributions, we therefore have

$$0 = \int_{-\infty}^{x-\epsilon} \frac{f(z)}{z-x}\,dz + \int_{x+\epsilon}^{+\infty} \frac{f(z)}{z-x}\,dz - \frac{1}{2}\oint_{z\sim x} \frac{f(z)}{z-x}\,dz \quad , \tag{13.298}$$

where we have assumed that the big half-circle does not contribute since $f(z)/(z-x)$ vanishes fast enough. The number ϵ is infinitesimal, and the sum of the first two terms is called the *principal value integral* :

$$\mathcal{P}\int_{-\infty}^{+\infty} \frac{f(z)}{z-x}\,dz \equiv \lim_{\epsilon\to 0}\left\{\int_{-\infty}^{x-\epsilon} \frac{f(z)}{z-x}\,dz + \int_{x+\epsilon}^{+\infty} \frac{f(z)}{z-x}\,dz\right\} \quad . \tag{13.299}$$

We therefore have the following equality:

$$\mathcal{P}\int_{-\infty}^{+\infty} \frac{f(z)}{z-x}\,dz = i\pi\, f(x) \quad ; \tag{13.300}$$

and by inspecting the real and imaginary parts separately we arrive at the *Kramers-Kronig relations*

$$\begin{aligned} \Re f(x) &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{+\infty} \frac{\Im f(z)}{z-x}\,dz = i\pi\, f(x) \\ \Im f(x) &= \frac{-1}{\pi}\mathcal{P}\int_{-\infty}^{+\infty} \frac{\Re f(z)}{z-x}\,dz = i\pi\, f(x) \quad . \end{aligned} \tag{13.301}$$

13.15.6 The dilogarithm function

The dilogarithm function $\mathrm{Li}_2(z)$ is defined by the following integral :

$$\mathrm{Li}_2(z) = -\int_0^z du \frac{1}{u}\log(1-u) \quad , \tag{13.302}$$

where the integration contour should *not* cross the cut in the logarithm (this is usually chosen to be the real axis at z values larger than 1). By expanding the logarithm for small values, we see immediately that

$$\mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2} \quad , \quad |z| < 1 \quad , \tag{13.303}$$

which is handy for evaluating the dilogarithm for small arguments. We also see immediately that

$$\mathrm{Li}_2(1) = \sum_{n\geq 1} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \quad . \tag{13.304}$$

There are a number of useful identities for the dilogarithm, of which we give a number below. You can prove them by differentiating the left-hand and right-hand sides with respect to z, and additionally checking them for some special value such as $z = 0$ or $z = 1$ or so.

$$\begin{aligned}
\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) &= \frac{1}{2}\mathrm{Li}_2(z^2) \quad , \\
\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) &= \frac{\pi^2}{6} - \log(z)\log(1-z) \quad , \\
\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) &= -\frac{\pi^2}{6} - \frac{1}{2}\Big(\log(-z)\Big)^2 \quad , \\
\mathrm{Li}_2(1-z) + \mathrm{Li}_2\left(1-\frac{1}{z}\right) &= -\frac{1}{2}\Big(\log(z)\Big)^2 \quad , \\
\mathrm{Li}_2(-z) - \mathrm{Li}_2(1-z) + \frac{1}{2}\mathrm{Li}_2\Big(1-z^2\Big) &= -\frac{\pi^2}{12} - \log(z)\log(1+z) \quad .
\end{aligned} \tag{13.305}$$

Some other special values can also be derived using the above identitites :

$$\begin{aligned}
&\mathrm{Li}_2(-1) = -\frac{\pi^2}{12} \quad , \quad \mathrm{Li}_2(0) = 0 \quad , \\
&\mathrm{Li}_2(1/2) = \frac{\pi^2}{12} - \frac{1}{2}\Big(\log(2)\Big)^2 \quad , \quad \mathrm{Li}_2(2) = \frac{\pi^2}{4} - i\pi\log(2)
\end{aligned} \tag{13.306}$$

13.15.7 On values of the Zeta function

The value of $\zeta(2)$ given in the previous section, and more, can be found using a beautiful almost-rigorous method due to Euler, which I cannot resist including here. Consider the function $\sin(x)/x$. This is an analytic function that equals 1 for $x = 0$ and has zeroes whenever $x = n\pi$, $n = \pm1, \pm2, \pm3, \ldots$. We therefore have

$$\begin{aligned}
\frac{\sin(x)}{x} &= \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right) \\
&= 1 - \frac{x^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} + \frac{x^4}{\pi^4}\sum_{n,m=1}^{\infty}\frac{\theta(n<m)}{n^2m^2} - \cdots
\end{aligned} \tag{13.307}$$

Also, by Taylor expanding we have

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots \tag{13.308}$$

By comparing the various powers of x in both expressions, we can immediately see that $\zeta(2) = \pi^2/6$. Also $\zeta(4)$ can be inferred :

$$\begin{aligned} \zeta(4) &= \sum_{n=1}^{\infty} \frac{1}{n^4} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^2 - \sum_{n,m=1}^{\infty} \frac{\theta(n \neq m)}{n^2 m^2} \\ &= \left(\frac{\pi^2}{6}\right)^2 - 2\left(\frac{\pi^4}{120}\right) = \frac{\pi^4}{90} \quad ; \end{aligned} \tag{13.309}$$

and you can go much higher by hand — if you are Euler.

13.15.8 The Lagrange expansion

Here we deal with the solution of the equation

$$\xi = x + f(\xi) \quad . \tag{13.310}$$

We shall assume that $f(0) = 0$ so that x and ξ 'tend to be close', especially for small x. The task is to express ξ as a function of x and $f(x)$ only. First we note that ξ is given as the root of the equation

$$\phi(y) = 0 \quad , \quad \phi(y) = y - x - f(y) \quad , \tag{13.311}$$

which by our assumption of smallishness has only a single simple root in a sufficiently small neighbourhood of $y = 0$. The function $\phi'(y)/\phi(y)$ therefore only has a simple pole at $y = \xi$, and we may write

$$\xi = \frac{1}{2\pi i}\oint dy\; y\,\frac{\phi'(y)}{\phi(y)} = \frac{1}{2\pi i}\oint dy\; \frac{y\big(1 - f'(y)\big)}{y - x - f(y)} \quad , \tag{13.312}$$

where the contour is taken inside the region that contains only the single pole. Since $f(y)$ is small if x and y are in the neighbourhood of zero, we may expand

$$\begin{aligned} \xi &= \frac{1}{2\pi i}\oint dy \sum_{n\geq 0} \frac{1}{(y-x)^{n+1}}\, y\big(1 - f'(y)\big)\, f(y)^n \\ &= \sum_{n\geq 0} \frac{1}{n!}\left(\frac{\partial}{\partial x}\right)^n \big(x\, f(x)^n - x\, f'(x)\, f(x)^n\big) \end{aligned} \tag{13.313}$$

The first term in the second line reads

$$x + \sum_{n\geq 1} \frac{1}{n!} \left(\frac{\partial}{\partial x}\right)^n \Big(x\, f(x)^n\Big)$$

which we can rewrite as

$$x + \sum_{n\geq 1} \frac{1}{n!} \left(\frac{\partial}{\partial x}\right)^{n-1} f(x)^n + \sum_{n\geq 1} \frac{1}{(n-1)!} \left(\frac{\partial}{\partial x}\right)^{n-1} \Big(x\, f'(x)\, f(x)^{n-1}\Big) \quad :$$

and the last term of this expression cancels against the last term in Eq.(13.313). We are left with the following :

$$\xi \;=\; x + \sum_{n\geq 1} \frac{1}{n!} \left(\frac{\partial}{\partial x}\right)^{n-1} f(x)^n \quad . \tag{13.314}$$

A note is in order here about the computational properties of this relation. One might simply *iterate* Eq.(13.310) to arrive at the result, in the following manner :

$$\begin{aligned} \xi &= x \;\; , \\ \xi &= x + f(x) \;\; , \\ \xi &= x + f(x + f(x)) \;\;\approx\;\; x + f(x) + f'(x)f(x) \;\; , \end{aligned} \tag{13.315}$$

and so on, assuming $f(x)$ and its derivatives to be small enough to warrant Taylor expansion. This reproduces the Lagrange expansion.