Dynamics of fast rotating neutron stars: Time evolution of linear perturbations in full general relativity

Christian J. Krüger & Kostas D. Kokkotas

Institut für Astronomie und Astrophysik
University of Tübingen

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Neutron Stars — laboratories for matter under extreme conditions

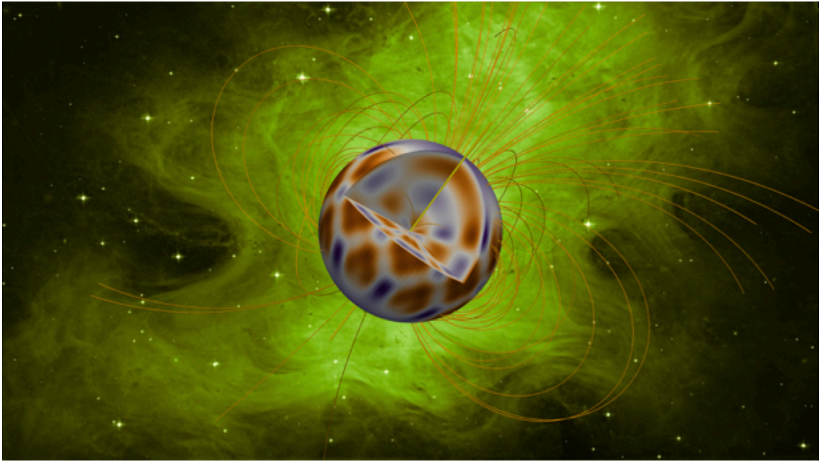

Illustration: Gaertig, Kokkotas (2019)

Equations of State - A Zoo

What's inside a neutron star?

- Neutron stars are the final stage of the lives of massive stars ($M \gtrsim 8\,M_\odot$).
- About $1.4 - 2.0$ solar masses are compressed into a ball of radius $r \approx 11 - 14\,\text{km}$.
- Density reaches more than $10^{15}\mathrm{g/cm^3}$, several times that of an atomic nucleus.
- Matter is described by an **equation of state** (EOS).

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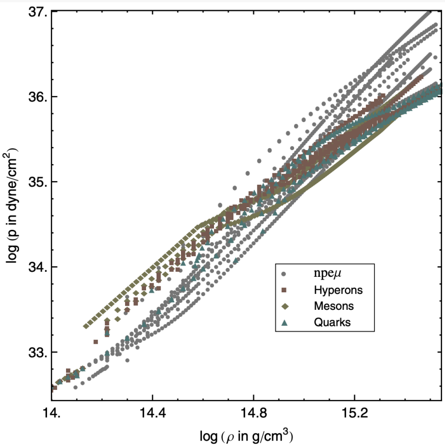

Read et al. (2009)

- EOS is a function $p = p(\rho)$.
- While the low density EOS is well constrained, it has large uncertainties at high densities (in the NS core, $\rho \gtrsim \rho_{\rm nuc}$).
- Numerous EOS have been proposed.
- Various models (variational method, Hartree-Fock, RMF, ...), various particles ($npe\mu$, π, K, quarks, ...), phase transition, ...

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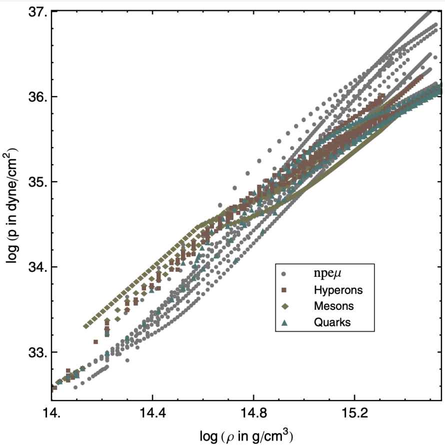

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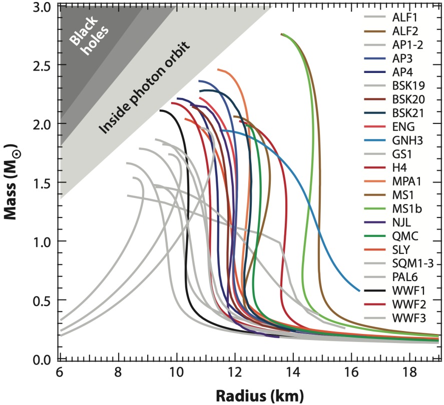

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- EOS is typically visualised via the resulting M-R curve for NSs.
- Some EOS are already ruled out by observation:
    - 2 $M_\odot$ NS,
    - Tidal deformability in binary mergers,
    - NICER observations, ...

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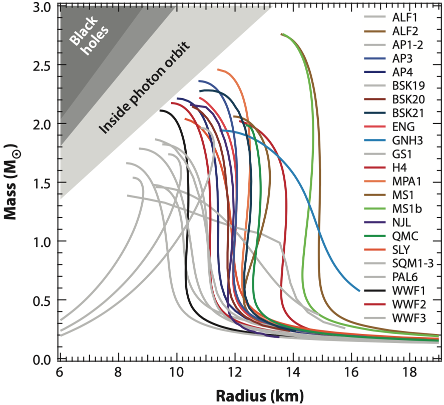

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How can we probe the interior of an NS

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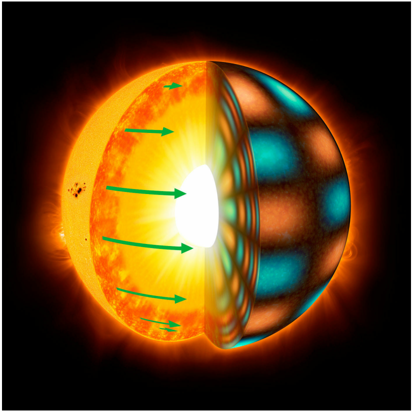
Helioseismology



scitechdaily.com / MPS (2018)

- Seismology extremely successful in the sun
  → Helioseismology.
- Thousands of individual oscillation modes observed → detailed knowledge of the Sun's internal structure.

Asteroseismology

NS oscillations are extremely difficult to observe.

- via gravitational waves,
- perhaps in magnetar flares or by modulation of other e/m signals,
- via impact on dynamic system, e.g., binary inspirals.

For the moment

- calculate frequencies based on proposed models,
- unveil patterns and universal relations.

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Neutron Star Oscillations - Formulating the Problem

- Einstein equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

and (implied) conservation of energy-momentum

$$\nabla_\mu T^{\mu\nu} = 0$$

- General metric for a rotating neutron star

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi} r^2 \sin^2\theta \left(d\phi - \omega dt\right)^2 + e^{2\mu}\left(dr^2 + r^2 d\theta^2\right)$$

(as used by the `rns`-code (Friedman & Stergioulas (1995)))

- Neutron star matter modelled as perfect fluid

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}$$

These equations need to be solved – but it's quite complicated...

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Perturbation Equations

Simplification I

- Assume that the star is "almost" in equilibrium.
- Employ *first order perturbation theory*, i.e. write every quantity as "background + perturbation":

$$X = \bar{X} + \delta X$$

- Then throw away all terms that are quadratic (or higher) in the perturbations.

$\rightarrow$ Required numerical methods are *a lot* easier and computational costs are incomparably lower!

Perturbation Equations

Simplification II

- Most general form of the equations:

$$\frac{\partial \mathcal{Q}}{\partial t} = A_r \frac{\partial \mathcal{Q}}{\partial r} + A_\theta \frac{\partial \mathcal{Q}}{\partial \theta} + A_\phi \frac{\partial \mathcal{Q}}{\partial \phi} + A\mathcal{Q}$$

- $\mathcal{Q}$ is a vector containing all evolved perturbation quantities and the A, A_r, A_θ and A_ϕ are coefficient matrices that depend on the background quantities only.

Perturbation Equations

Simplification II

- Assume axisymmetry:

$$Q(t, r, \theta, \phi) = \tilde{Q}(t, r, \theta)e^{im\phi}$$

- This removes the azimuthal derivatives from the equations:

$$\frac{\partial}{\partial \phi} \to im$$

- Number of dimensions reduced by 1.

Numerical Implementation

Methods for solving the hyperbolic PDEs: Method of Lines

- Finite differences of 2nd order for spatial derivatives.
- Runge-Kutta 3rd order for time stepping.
- Use Kreiss-Oliger dissipation to stabilise time evolution. Coefficients are of the order 10^{-5} to 10^{-7}.

$$\frac{\partial \mathcal{Q}}{\partial t} = A_r \frac{\partial \mathcal{Q}}{\partial r} + A_\theta \frac{\partial \mathcal{Q}}{\partial \theta} + imA_\phi \mathcal{Q} + A\mathcal{Q} + \alpha \nabla^2 \mathcal{Q}$$

- Derivatives of Kreiss-Oliger term are evaluated on grid coordinates rather than physical coordinates.

Perturbation equations

Time Evolution of Perturbation Equations

- **Perturbed** (linearised) Einstein Equations
  & Conservation of Energy-Momentum

$$\delta G_{\mu\nu} = 8\pi\delta T_{\mu\nu},$$
$$\delta\left(\nabla_\nu T^{\mu\nu}\right) = 0,$$

with the typical metric perturbations

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}.$$

- Metric perturbations require choice of gauge:
  Choose the Hilbert Gauge:

$$\nabla^\mu h_{\mu\nu} = 0.$$

$\rightarrow$ 10 coupled wave equations for the metric components.

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$$\frac{(e^{\mu})^2\left(\frac{\partial^2}{\partial t^2}PP\right)}{(e^{\nu})^2} = \left(\frac{2\,\mathrm{Im}\,(\psi_\theta-\nu_\theta+\cot(\theta))}{\sin(\theta)\,r^2}+\frac{\mathrm{Im}\omega\sin(\theta)\left(e^{\psi}\right)^2\omega_\theta}{(e^{\nu})^2}\right)BB+\left((2\kappa1-\kappa)\,(e^{\mu})^2-\frac{2\left(\nu_r\mu_r r^2+r\nu_r-\nu_\theta\mu_\theta+2\nu_\theta^2+\nu_{\theta,\theta}\right)}{r^2}+\frac{\left(e^{\psi}\right)^2\left(\sin^2(\theta)\right)\omega_\theta^2}{(e^{\nu})^2}\right)HH$$
$$+\left((e^{\mu})^2\kappa-\frac{2\left(r^2\psi_r\nu_r-\mu_r^2r^2+r\nu_r+\psi_\theta\nu_\theta-\mu_\theta^2+2\nu_\theta\mu_\theta-2\mu_r r+\cot(\theta)\,\nu_\theta-1\right)}{r^2}-\frac{\left(\sin^2(\theta)\right)\left(r^2\omega_r^2+\omega_\theta^2\right)\left(e^{\psi}\right)^2}{2\,(e^{\nu})^2}\right)KK-\frac{\mathrm{I}\omega_\theta mMM}{(e^{\nu})^2\,r}$$
$$+\left((-2\kappa1+\kappa)\,(e^{\mu})^2+\frac{-2\psi_r\mu_r r^2+2\left(\cot^2(\theta)\right)+2\cot(\theta)\,\mu_\theta-4\cot(\theta)\,\nu_\theta-2\mu_r r+2\psi_\theta\mu_\theta-4\psi_\theta\nu_\theta-2\psi_r r-2\psi_{\theta,\theta}}{r^2}-\frac{\left(e^{\psi}\right)^2\left(\sin^2(\theta)\right)\omega_\theta^2}{(e^{\nu})^2}\right)WW$$
$$+\left(2\sin(\theta)\,(e^{\mu})^2\left(e^{\psi}\right)^2 r\omega1\kappa1+\frac{\sin(\theta)\left(\omega_r\mu_r r^2+2\psi_\theta\omega_\theta-\omega_\theta\mu_\theta+2\cot(\theta)\,\omega_\theta+r\omega_r+\omega_{\theta,\theta}\right)\left(e^{\psi}\right)^2}{r}\right)YY+8\,(e^{\mu})^4\,\pi Q6$$
$$+\left(\left(-m^2g\_phph-\kappa\right)(e^{\mu})^2-\frac{2\left(\mu_r^2r^2+\psi_\theta^2-2\psi_\theta\nu_\theta+2\psi_\theta\cot(\theta)+\mu_\theta^2-2\nu_\theta\mu_\theta+2\mu_r r-\nu_\theta^2-2\cot(\theta)\,\nu_\theta+\cot^2(\theta)+1\right)}{r^2}+\frac{\left(e^{\psi}\right)^2\left(\sin^2(\theta)\right)\omega_\theta^2}{(e^{\nu})^2}\right)PP+\frac{(r\nu_r+\psi_r r+2)\left(\frac{\partial}{\partial r}PP\right)}{r}+\frac{(\psi_\theta+5\nu_\theta+\cot(\theta))\left(\frac{\partial}{\partial\theta}PP\right)}{r^2}$$
$$+\frac{(4\nu_\theta\mu_r r-2\nu_r\mu_\theta r-2\mu_\theta\psi_r r+2\nu_r\nu_\theta r+2\nu_\theta\psi_r r+\nu_{r,\theta}r+\psi_{r,\theta}r-2\mu_\theta+6\nu_\theta)\,QQ}{r^2}-\frac{2\,(-\nu_\theta+\mu_\theta)\left(\frac{\partial}{\partial r}QQ\right)}{r}+\frac{2\,(\mu_r r+1)\left(\frac{\partial}{\partial\theta}QQ\right)}{r^2}+\frac{\partial^2}{\partial r^2}PP-\frac{2\,\mathrm{I}\,(e^{\mu})^2\,\omega\left(\frac{\partial}{\partial t}PP\right)m}{(e^{\nu})^2}+\frac{\sin(\theta)\left(e^{\psi}\right)^2\omega_\theta\left(\frac{\partial}{\partial t}BB\right)}{(e^{\nu})^2}+\frac{\frac{\partial^2}{\partial\theta^2}PP}{r^2}$$

$$\frac{\partial}{\partial t}Q1=\left(\mathrm{I}\omega m\kappa2-\frac{\mathrm{Im}\omega\,(p+\rho)}{2\,(e^{\nu})^2}\right)HH+\left(2\,\mathrm{I}\Omega m\kappa2-\frac{\mathrm{Im}\omega\,(p+\rho)}{2\,(e^{\nu})^2}\right)KK+\left(2\,\mathrm{I}\Omega m\kappa2-\frac{\mathrm{Im}\omega\,(p+\rho)}{2\,(e^{\nu})^2}\right)PP+\left(\mathrm{I}\,(\omega+2\omega1)\,m\kappa2+\frac{\mathrm{Im}\omega\,(p+\rho)}{2\,(e^{\nu})^2}\right)WW-\mathrm{I}\sin(\theta)\,mr\kappa2\,\omega1^2\left(e^{\psi}\right)^2YY+\left(\kappa2+\frac{-\frac{\rho}{2}-\frac{p}{2}}{(e^{\nu})^2}\right)\left(\frac{\partial}{\partial t}HH\right)+\left(2\kappa2+\frac{-\frac{\rho}{2}-\frac{p}{2}}{(e^{\nu})^2}\right)\left(\frac{\partial}{\partial t}KK\right)$$
$$+\left(2\kappa2+\frac{-\frac{\rho}{2}-\frac{p}{2}}{(e^{\nu})^2}\right)\left(\frac{\partial}{\partial t}PP\right)+\left(\kappa2+\frac{\frac{p}{2}+\frac{\rho}{2}}{(e^{\nu})^2}\right)\left(\frac{\partial}{\partial t}WW\right)-\frac{\mathrm{Im}Q2}{\sin(\theta)\,r}+\left(-\frac{2\mu_r r+3r\nu_r+\psi_r r+2}{r}-\frac{\left(\sin^2(\theta)\right)\left(e^{\psi}\right)^2r^2\omega_r\omega1}{(e^{\nu})^2}\right)Q3-\frac{\partial}{\partial r}Q3+\left(-\frac{3\nu_\theta+2\mu_\theta+\psi_\theta+\cot(\theta)}{r}-\frac{\left(\sin^2(\theta)\right)\left(e^{\psi}\right)^2r\omega_\theta\omega1}{(e^{\nu})^2}\right)Q4-\frac{\frac{\partial}{\partial\theta}Q4}{r}$$

$$(e^{\mu})^2\left(\frac{\partial}{\partial t}Q3\right)=\mathrm{I}\sin(\theta)\,\Omega m\omega1r\kappa2\left(e^{\psi}\right)^2AA+\left(-\nu_r\kappa2\,(e^{\nu})^2+\left(\sin^2(\theta)\right)\kappa2\omega1r\,(-\psi_r\omega1r+r\omega_r-\omega1)\left(e^{\psi}\right)^2\right)HH+\left(\nu_r\kappa2\,(e^{\nu})^2+\left(\sin^2(\theta)\right)\kappa2\omega1r\,(-\psi_r\omega1r+r\omega_r-\omega1)\left(e^{\psi}\right)^2\right)KK-\mathrm{Im}\,(e^{\mu})^2\,Q3\Omega$$
$$+\left(\nu_r\kappa2\,(e^{\nu})^2+\left(\sin^2(\theta)\right)\kappa2\omega1r\,(-\psi_r\omega1r+r\omega_r-\omega1)\left(e^{\psi}\right)^2\right)PP+\left(-(e^{\nu})^2\,\nu_r+\left(\sin^2(\theta)\right)r\left(-2\psi_r\omega\omega1r-\psi_r\omega1^2r+\omega_r\omega r-2\omega\omega1-\omega1^2\right)\left(e^{\psi}\right)^2\right)Q1-\sin(\theta)\,(-2\psi_r\omega1r+r\omega_r-2\omega1)\left(e^{\psi}\right)^2Q2$$
$$-\mathrm{I}\Omega m\kappa2LL+\left(-\nu_r-2\mu_r-\frac{\left(\sin^2(\theta)\right)\omega1^2r\,(\psi_r r+1)\left(e^{\psi}\right)^2}{(e^{\nu})^2}\right)(e^{\mu})^2\,Q6+\left(\nu_r\kappa2\,(e^{\nu})^2-\left(\sin^2(\theta)\right)\kappa2\omega1r\,(-\psi_r\omega1r+r\omega_r-\omega1)\left(e^{\psi}\right)^2\right)WW-\kappa2\sin(\theta)\,(2\nu_r\omega1r+2\psi_r\omega1r-r\omega_r+\omega1)\left(e^{\psi}\right)^2(e^{\nu})^2\,YY$$
$$+\sin(\theta)\,\kappa2\omega1r\left(e^{\psi}\right)^2\left(\frac{\partial}{\partial t}AA\right)+\left(-\kappa2\,(e^{\nu})^2+\frac{p}{2}+\frac{\rho}{2}\right)\left(\frac{\partial}{\partial r}HH\right)+\left(\frac{p}{2}+\frac{\rho}{2}\right)\left(\frac{\partial}{\partial r}KK\right)-\kappa2\left(\frac{\partial}{\partial t}LL\right)+\left(\frac{p}{2}+\frac{\rho}{2}\right)\left(\frac{\partial}{\partial r}PP\right)-(e^{\mu})^2\left(\frac{\partial}{\partial r}Q6\right)+\left(\kappa2\,(e^{\nu})^2-\frac{\rho}{2}-\frac{p}{2}\right)\left(\frac{\partial}{\partial r}WW\right)-\sin(\theta)\,\kappa2\omega1r\left(e^{\psi}\right)^2(e^{\nu})^2\left(\frac{\partial}{\partial r}YY\right)$$

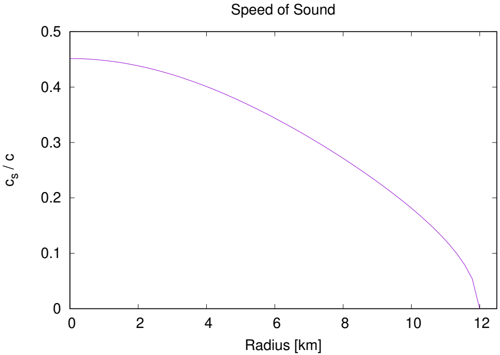
$$(e^{\mu})^2\left(\frac{\partial}{\partial t}Q4\right)=\mathrm{I}\sin(\theta)\,\Omega m\omega1r\kappa2\left(e^{\psi}\right)^2BB+\left(-\frac{\kappa2\nu_\theta\,(e^{\nu})^2}{r}-\left(\sin^2(\theta)\right)\kappa2\omega1r\,(\cot(\theta)\,\omega1+\psi_\theta\omega1-\omega_\theta)\left(e^{\psi}\right)^2\right)HH+\left(\frac{\kappa2\nu_\theta\,(e^{\nu})^2}{r}-\left(\sin^2(\theta)\right)\kappa2\omega1r\,(\cot(\theta)\,\omega1+\psi_\theta\omega1-\omega_\theta)\left(e^{\psi}\right)^2\right)KK-\mathrm{I}\Omega m\kappa2MM$$
$$+\left(\frac{\kappa2\nu_\theta\,(e^{\nu})^2}{r}-\left(\sin^2(\theta)\right)\kappa2\omega1r\,(\cot(\theta)\,\omega1+\psi_\theta\omega1-\omega_\theta)\left(e^{\psi}\right)^2\right)PP+\left(-\frac{(e^{\nu})^2\,\nu_\theta}{r}-\left(\sin^2(\theta)\right)r\left(2\cot(\theta)\,\omega\omega1+\cot(\theta)\,\omega1^2+2\psi_\theta\omega\omega1+\psi_\theta\omega1^2-\omega_\theta\omega\right)\left(e^{\psi}\right)^2\right)Q1+\sin(\theta)\,(2\cot(\theta)\,\omega1+2\psi_\theta\omega1-\omega_\theta)\left(e^{\psi}\right)^2Q2$$
$$-\mathrm{Im}\,(e^{\mu})^2\,Q4\Omega+\left(-\frac{\nu_\theta+2\mu_\theta}{r}-\frac{\left(\sin^2(\theta)\right)\omega1^2r\,(\psi_\theta+\cot(\theta))\left(e^{\psi}\right)^2}{(e^{\nu})^2}\right)(e^{\mu})^2\,Q6+\left(\frac{\kappa2\nu_\theta\,(e^{\nu})^2}{r}+\left(\sin^2(\theta)\right)\kappa2\omega1r\,(\cot(\theta)\,\omega1+\psi_\theta\omega1-\omega_\theta)\left(e^{\psi}\right)^2\right)WW-\sin(\theta)\,\kappa2\,(\cot(\theta)\,\omega1+2\nu_\theta\omega1+2\psi_\theta\omega1-\omega_\theta)\left(e^{\psi}\right)^2(e^{\nu})^2\,YY$$
$$+\sin(\theta)\,\kappa2\omega1r\left(e^{\psi}\right)^2\left(\frac{\partial}{\partial t}BB\right)+\left(-\frac{\kappa2\,(e^{\nu})^2}{r}+\frac{p+\rho}{2r}\right)\left(\frac{\partial}{\partial\theta}HH\right)+\frac{(p+\rho)\left(\frac{\partial}{\partial\theta}KK\right)}{2r}-\kappa2\left(\frac{\partial}{\partial t}MM\right)+\frac{(p+\rho)\left(\frac{\partial}{\partial\theta}PP\right)}{2r}-\frac{(e^{\mu})^2\left(\frac{\partial}{\partial\theta}Q6\right)}{r}+\left(\frac{\kappa2\,(e^{\nu})^2}{r}-\frac{p+\rho}{2r}\right)\left(\frac{\partial}{\partial\theta}WW\right)-\sin(\theta)\,\kappa2\omega1\left(e^{\psi}\right)^2(e^{\nu})^2\left(\frac{\partial}{\partial\theta}YY\right)$$

$$\frac{\partial}{\partial t}Q2=\left(\mathrm{Im}\kappa2r\sin(\theta)\,\Omega^2-\frac{\mathrm{Im}\,(p+\rho)}{2r\sin(\theta)\left(e^{\psi}\right)^2}-\frac{\mathrm{Im}\,\omega^2\,(p+\rho)\,r\sin(\theta)}{2\,(e^{\nu})^2}\right)HH+\left(2\,\mathrm{Im}\kappa2r\sin(\theta)\,\Omega^2+\frac{\mathrm{Im}\,(p+\rho)}{2r\sin(\theta)\left(e^{\psi}\right)^2}-\frac{\mathrm{Im}\,\omega^2\,(p+\rho)\,r\sin(\theta)}{2\,(e^{\nu})^2}\right)KK+\left(2\,\mathrm{Im}\kappa2r\sin(\theta)\,\Omega^2+\frac{\mathrm{Im}\,(p+\rho)}{2r\sin(\theta)\left(e^{\psi}\right)^2}-\frac{\mathrm{Im}\,\omega^2\,(p+\rho)\,r\sin(\theta)}{2\,(e^{\nu})^2}\right)PP$$
$$+\left(\mathrm{Im}\kappa2r\sin(\theta)\,\Omega^2+\frac{\mathrm{Im}\,(p+\rho)}{2r\sin(\theta)\left(e^{\psi}\right)^2}+\frac{\mathrm{Im}\,\omega^2\,(p+\rho)\,r\sin(\theta)}{2\,(e^{\nu})^2}\right)WW+\left(\mathrm{I}\omega m\kappa2\,(e^{\nu})^2-\mathrm{I}\omega m\left(\sin^2(\theta)\right)\kappa2\,\omega1^2r^2\left(e^{\psi}\right)^2\right)YY+\left(\sin(\theta)\,\kappa2r\,(\omega+2\omega1)-\frac{\omega\,(p+\rho)\,r\sin(\theta)}{2\,(e^{\nu})^2}\right)\left(\frac{\partial}{\partial t}HH\right)$$
$$+\left(2\kappa2r\sin(\theta)\,\Omega-\frac{\omega\,(p+\rho)\,r\sin(\theta)}{2\,(e^{\nu})^2}\right)\left(\frac{\partial}{\partial t}KK\right)+\left(2\kappa2r\sin(\theta)\,\Omega-\frac{\omega\,(p+\rho)\,r\sin(\theta)}{2\,(e^{\nu})^2}\right)\left(\frac{\partial}{\partial t}PP\right)+\left(\kappa2r\sin(\theta)\,\omega+\frac{\omega\,(p+\rho)\,r\sin(\theta)}{2\,(e^{\nu})^2}\right)\left(\frac{\partial}{\partial t}WW\right)$$
$$+\kappa2\,(e^{\nu})^2\left(\frac{\partial}{\partial t}YY\right)-\frac{\mathrm{I}Q5m}{\sin(\theta)\,r}+\left(-\sin(\theta)\,(2\omega\mu_r r+3\omega\nu_r r+\psi_r r\omega+2\mu_r\omega1r+\nu_r\omega1r+3\psi_r\omega1r-r\omega_r+2\omega+4\omega1)-\frac{\left(\sin^3(\theta)\right)\left(e^{\psi}\right)^2r^3\omega\omega_r\omega1}{(e^{\nu})^2}\right)Q3$$
$$-\sin(\theta)\,r\left(\frac{\partial}{\partial r}Q3\right)\Omega+\left(-\sin(\theta)\,(\cot(\theta)\,\omega+3\cot(\theta)\,\omega1+2\omega\mu_\theta+2\mu_\theta\omega1+3\nu_\theta\omega+\nu_\theta\omega1+\psi_\theta\omega+3\psi_\theta\omega1-\omega_\theta)-\frac{\left(\sin^3(\theta)\right)\left(e^{\psi}\right)^2r^2\omega\omega_\theta\omega1}{(e^{\nu})^2}\right)Q4-\sin(\theta)\left(\frac{\partial}{\partial\theta}Q4\right)\Omega$$

Numerical Implementation

Choosing a Grid

- Characteristic speed of the neutron star fluid is the speed of sound c_s.



- A uniform grid enforces a small time step at the center of the star (due to the CFL criterion).

Numerical Implementation

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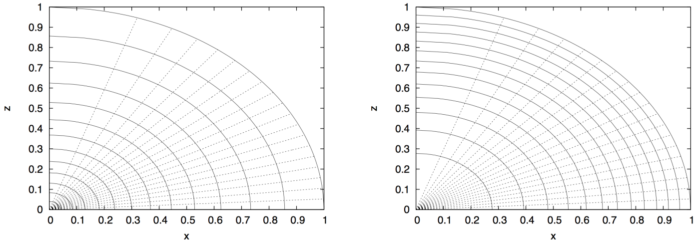
- Employ grid (roughly) according to speed of sound:



(Left: grid used in `rns`-code. Right: grid used in (Cowling) time evolution.)

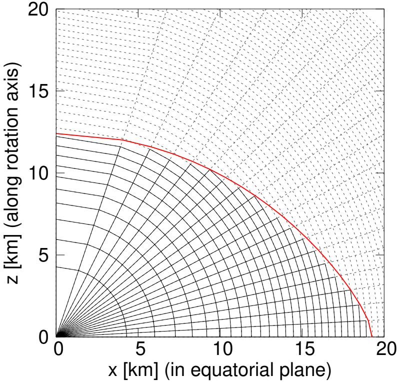
→ Time step Δt can be 10-20 times (or more) larger!

Side effects:
→ Kreiss-Oliger coefficients can be smaller by 2 orders of magnitude.
→ Deformed surface of neutron star is better resolved.

Numerical Implementation

Choosing a Grid

When spacetime is dynamic, the grid needs to extend up to a few hundred star radii.



Construct grid with similar properties:

- Grid lines squeezed close to surface of star.
- But with increased spacing toward the outer edge.

Numerical Implementation

Slicing the grid to use MPI

- Productive simulation runs on 3000x50 grid (radial $\times$ polar resolution)
- Grid can easily be sliced along polar grid lines $\rightarrow$ MPI.
- Only little communication between threads at slice boundaries necessary.
- All threads have very similar workload.

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Slicing the grid to use MPI

- Walltime of single-threaded simulations: $\sim$ 2 days.
- Local cluster in Tübingen BinAC has nodes with 28 CPUs.
- $\rightarrow$ Walltime $\sim$ 1.5 - 2.5 hours.

Numerical Implementation

Alternative: Multi-grid implementation

- Characteristic speeds of neutron star fluid and spacetime are quite different (c_s vs $c = 1$).
- Introduce individual grids for each "fluid".
  - Neutron star fluid gets grid shown in the beginning.
  - Spacetime grid is almost uniform with slightly increasing spacing toward the outer edge.
- Need to interpolate between those grids at each (intermediate) time step.
- $\rightarrow$ Time step Δt can be chosen $\sim$ 5 times larger.
  $\rightarrow$ Walltime for single-threaded run now $\sim$ 5-6 hours.

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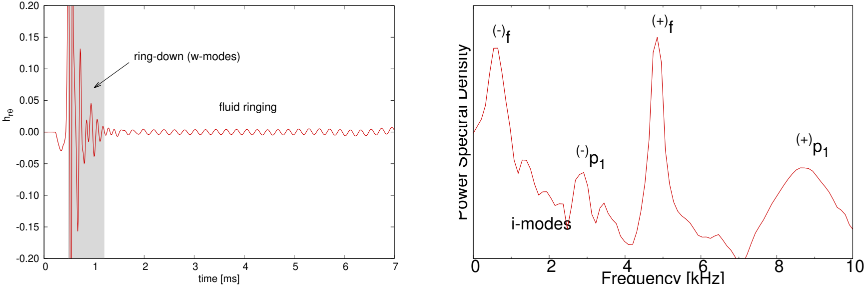
Characteristic example of time signal



EoS SLy	$M = 2.02\, M_\odot$
$\epsilon_c = 1.2e15$ g/cm^3	$\Omega = 1.3\,$kHz
$r_e/r_p = 0.56$	$\Omega/\Omega_K = 0.98$

The fundamental mode of a NS

The f-mode (fundamental) mode of a NS

- Fundamental oscillation mode of a NS; present also in constant-density models.
- Typical frequency: $1 - 3\,\text{kHz}$.
- The quadrupolar ($l = |m| = 2$) f-mode is potentially a strong emitter of GWs.
- Could be excited during late binary inspiral and impact the phase of the waveform.

Spectrum of NSs is very rich and features various other modes: p-modes, w-modes, s-modes, i-modes, ...

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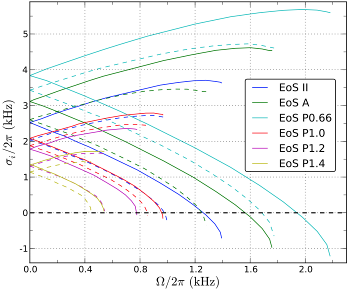
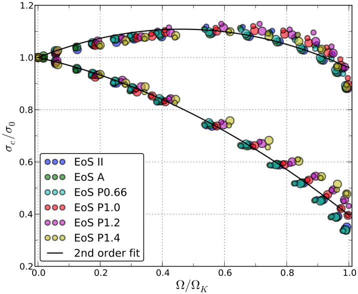
f-mode frequency in the Cowling Approximation



Gaertig, Kokkotas (2008, 2011, 2011)



Doneva, Gaertig, Kokkotas, Krüger (2013)

Fitting formulae:

$$\frac{\sigma_c^s}{\sigma_0} = 1 - 0.235\left(\frac{\Omega}{\Omega_K}\right) - 0.358\left(\frac{\Omega}{\Omega_K}\right)^2$$

$$\frac{\sigma_c^u}{\sigma_0} = 1 + 0.402\left(\frac{\Omega}{\Omega_K}\right) - 0.406\left(\frac{\Omega}{\Omega_K}\right)^2$$

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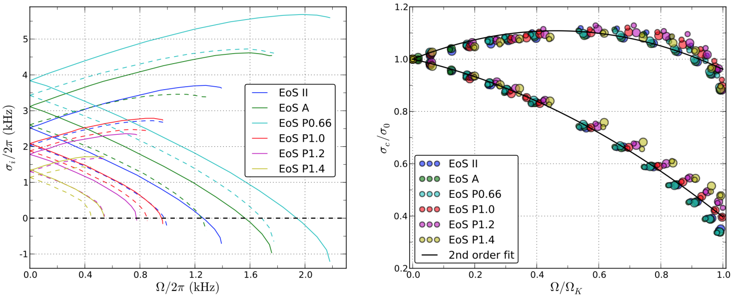

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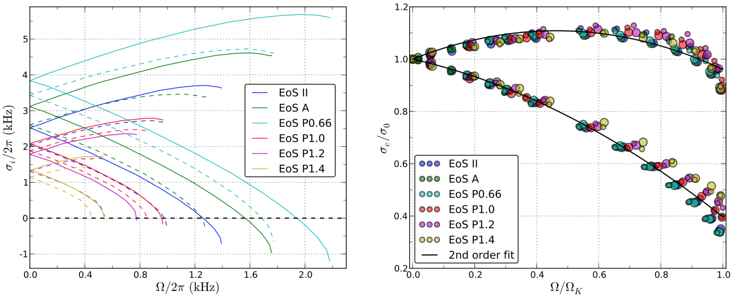

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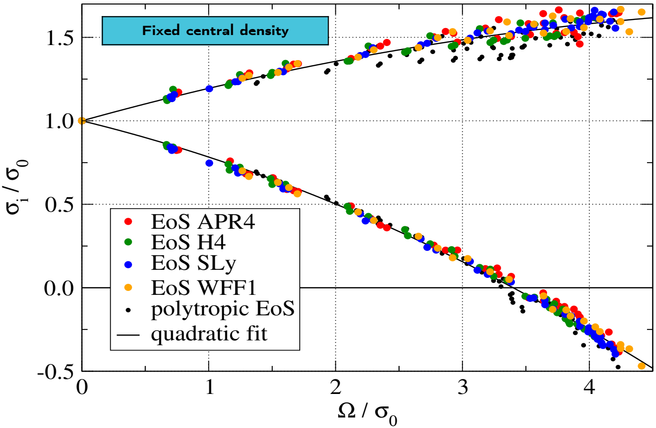
20-40% error in frequencies

(due to neglecting dynamics of spacetime)

Fitting formulae – σ/σ_0 vs. Ω/σ_0



Zeeman-like splitting of f-mode, here shown for $l = |m| = 2$.

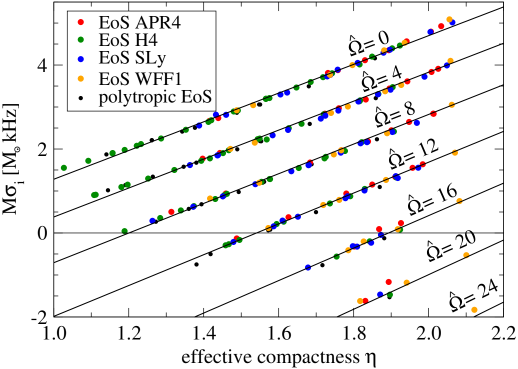
$$\frac{\sigma^u}{\sigma_0} = 1 - 0.193\left(\frac{\Omega}{\sigma_0}\right) - 0.0294\left(\frac{\Omega}{\sigma_0}\right)^2$$

$$\frac{\sigma^s}{\sigma_0} = 1 + 0.220\left(\frac{\Omega}{\sigma_0}\right) - 0.0170\left(\frac{\Omega}{\sigma_0}\right)^2$$

f-mode "changes sign", i.e., becomes CFS-unstable when $\Omega \gtrsim 3.4\sigma_0$.

Krüger, Kokkotas. PRL (2020)

Fitting formulae – $M\sigma$ vs. $\hat{\Omega}$ vs η

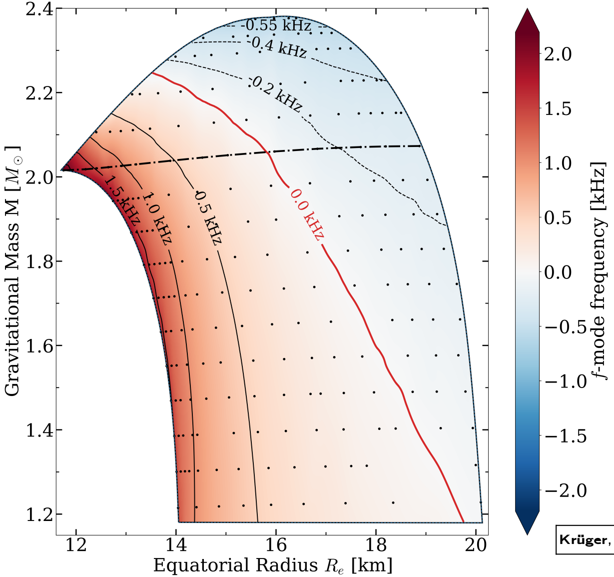


$$M\sigma_i^u = \left(-2.14 - 0.201\hat{\Omega} - 7,68 \cdot 10^{-3}\hat{\Omega}^2\right) + \left(3.42 + 1.75 \cdot 10^{-3}\hat{\Omega}^2\right)\eta \qquad \hat{\Omega} = M\Omega$$

$$M\sigma_i^s = \left(-2.14 + 0.220\hat{\Omega} - 14.6 \cdot 10^{-3}\hat{\Omega}^2\right) + \left(3.42 + 6.86 \cdot 10^{-3}\hat{\Omega}^2\right)\eta \qquad \eta = \sqrt{M^3/I}$$

Non-rotating case: **Lau, Leung, Lin (2010)** Cowling case: **Doneva, Kokkotas (2015)**

Overview of Entire (Cold) EoS H4



Krüger, Kokkotas. PRL (2020)

Summary

- Written a time evolution code from scratch to evolve perturbations of *fast* rotating neutron stars in time.
- Evolution equation are supposed to be contributed to the ETK at some point.
- With MPI parallelisation or multi-grid approach, walltime is conveniently small (few hours).
- Determined f-mode frequency and onset of CFS-instability of rapidly rotating NSs in full GR without approximation.
- Provided asteroseismological relation for f-mode frequency.
- Relevant for various astrophysical scenarios: continuous sources, inspiral + post-merger phase in binary mergers, ...
- Universal relation can be used in cheap EOS inference codes and numerous other applications, ...


Thank you for your attention!
Questions?

References

- Friedman & Stergioulas (1995): ApJ 444, 306 (1995)
- Gaertig & Kokkotas (2008): PRD 78, 064063 (2008)
- Read et al. (2009): PRD 79, 124032 (2009)
- Lau, Leung & Lin (2010): ApJ 714, 1234 (2010)
- Gaertig & Kokkotas (2011): PRD 83, 064031 (2011)
- Gaertig et al. (2011): PRL 107, 101102 (2011)
- Doneva et al. (2013): PRD 88, 044052 (2013)
- Doneva & Kokkotas (2015): PRD 92, 124004 (2015)
- Özel & Freire (2016): A&A 54, 401 (2016)
- Gaertig & Kokkotas (2019): in Tipler et al. Physik, 8. Aufl., Springer (2019)
- Krüger & Kokkotas (2020): PRL 125, 111106 (2020)