Initial data for binary systems

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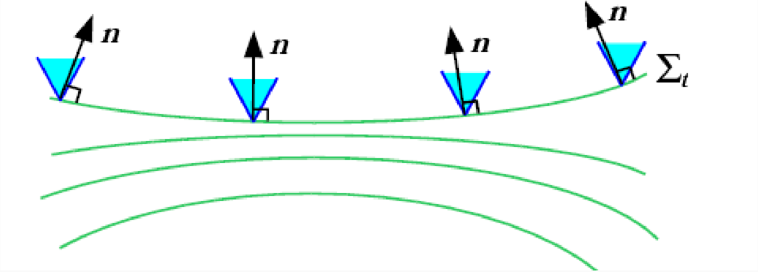
3+1 formalism

Foliation of spacetime

The 3+1 formalism is the most widely used way to write Einstein equations for NR. It makes explicit the split between space and time (see for instance Gourgoulhon (2012)).

Spacetime is foliated by a family of spatial hypersurfaces Σ_t

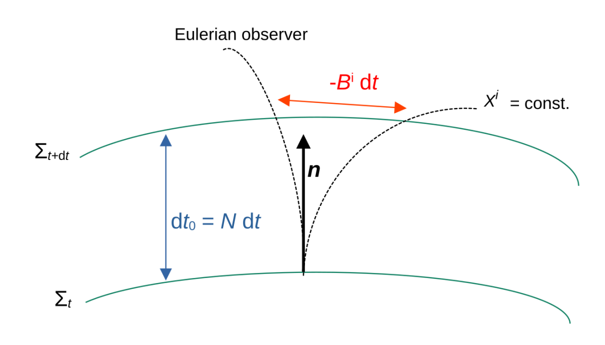


- Coordinate system of Σ_t : (x_1, x_2, x_3).
- Coordinate system of spacetime : (t, x_1, x_2, x_3).

Greek indices 4D $(0, 1, 2, 3)$ and Latin 3D $(1, 2, 3)$.

Unit normal

The unit normal can be written as $n^\mu = \left(\frac{1}{N}, -\frac{B^i}{N}\right)$.



- N, the lapse, defines the choice of time coordinate (other notation α).
- B^i, the shift, defines the choice of spatial coordinates (other notations β^i, N^i).

Projections

- Projection on the normal of a vector $\vec{V}$ is given by $n_\mu V^\mu$.
- Projection operator on the hypersurfaces $\gamma^\nu_\mu = g^\nu_\mu + n_\mu n^\nu$.
- Projection on the slices of a vector $\vec{V}$ is given by $\gamma^\nu_\mu V^\mu$.
- Projection the 4D metric $g_{\mu\nu}$ is

$$\gamma^\nu_\alpha \gamma^\mu_\beta g_{\mu\nu} = \gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta$$

- The induced metric γ_{ij} is the first fundamental form.

The 4D line-element reads

$$\mathrm{d}s^2 = -\left(N^2 - B^i B_i\right) \mathrm{d}t^2 + 2B_i \mathrm{d}t \mathrm{d}x^i + \gamma_{ij} \mathrm{d}x^i \mathrm{d}x^j$$

Extrinsic curvature K_{ij}

- Describes the part of the geometry not accounted for by the induced metric.
- It describes the variation of normal projected on the hypersurface.

$$K_{ij} = -\gamma_i^\mu \gamma_j^\nu \nabla_\mu n_\nu$$

- In the 3+1 framework, it is given by

$$\left(\partial_t - \mathcal{L}_{\vec{B}}\right) \gamma_{ij} = -2NK_{ij}$$

- It is known as the second fundamental form.

Projection of Einstein's equations

The 3+1 equations are obtained by projecting $G_{\mu\nu} = 8\pi T_{\mu\nu}$ on n^α and on the hypersurfaces.

- projections onto $n^\mu n^\nu$, the Hamiltonian constraint:
$$R + K^2 - K_{ij}K^{ij} = 16\pi E.$$
- projection onto $n^\mu \gamma_i^\nu$, the momentum constraint:
$$D_j K_i^j - D_i K = 8\pi P_i.$$
- projection onto $\gamma_i^\nu \gamma_j^\mu$, the evolution equation:
$$\begin{aligned}\left(\frac{\partial}{\partial t} - \mathcal{L}_{\vec{B}}\right) \quad & K_{ij} = \\ -D_i D_j N \quad & +N\left(R_{ij} + K K_{ij} - 2K_{ik}K_j^k + 4\pi\left[(S-E)\gamma_{ij} - 2S_{ij}\right]\right).\end{aligned}$$
- E, P_i and S_{ij} are the various projections of $T_{\mu\nu}$.

Constraint equations

- The constraints are 4 equations, that do not contain ∂_t.
- Such equations are absent in Newtonian dynamics but not for Maxwell.

Type	Einstein	Maxwell
Constraints	Hamiltonian $R + K^2 - K_{ij}K^{ij} = 0$	$\nabla \cdot \vec{E} = 0$
	Momentum : $D_j K^{ij} - D^i K = 0$	$\nabla \cdot \vec{B} = 0$
Evolution	$\frac{\partial \gamma_{ij}}{\partial t} - \mathcal{L}_{\vec{B}} \gamma_{ij} = -2NK_{ij}$	$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \mu_0} \left(\vec{\nabla} \times \vec{B} \right)$
	$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\vec{B}} K_{ij} = -D_i D_j N + N \left(R_{ij} - 2K_{ik}K^k_j + KK_{ij} \right)$	$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$

A two steps problem

Evolution problem

- Given initial value of $\gamma_{ij}\left(t=0\right)$ and $K_{ij}\left(t=0\right)$ use the evolution equations to determine the fields at later times.
- Similar to writing Newton's equation as $\partial_t x = v; \partial_t v = f/m$.
- Must ensure stability and accuracy.
- Must choose the lapse and shift in a clever way.

Initial data

- $\gamma_{ij}\left(t=0\right)$ and $K_{ij}\left(t=0\right)$ are not arbitrary but subject to the constraint equations.
- It is a set of four elliptic coupled equations.
- Needs to make the link between a given physical situation and the mathematical objects γ_{ij} and K_{ij}

Static solutions

Simplest assumptions

Hypothesis

- Conformal flatness : $\gamma_{ij} = \Psi^4 f_{ij}$, where f_{ij} is the flat metric.
- Initially static : $K_{ij} = 0$ (equivalent to no initial velocity).

Equations

- The momentum constraint is trivially verified.
- The Hamiltonian constraint reduces to $\Delta\Psi = 0$.

Brill-Lindquist data (Brill and Lindquist (1963))

- The presence of two holes is enforced by imposing the divergence of Ψ at two points :

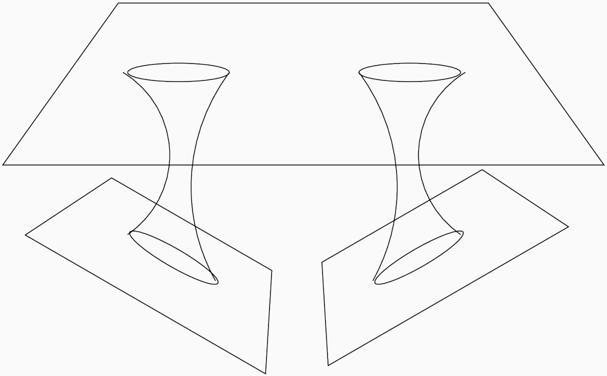
$$\Psi = 1 + \frac{\alpha_1}{||\vec{r} - \vec{c}_1||} + \frac{\alpha_2}{||\vec{r} - \vec{c}_2||}.$$

- A first asymptotic flat region is $r \to \infty$.
- It defines the mass of the binary : $m = 2\,(\alpha_1 + \alpha_2)$.

Behavior close to the singularities

- Define $\vec{r_1} = \vec{r} - \vec{c_1}$ and make $r_1 \to 0$
- New radius coordinate : $\bar{r}_1 = \dfrac{\alpha_1^2}{r_1}$
- One can show that region $\bar{r}_1 \to \infty$ is also asymptotically flat.
- It defines the mass of an individual black hole $m_1 = 2\alpha_1 \left(1 + \dfrac{\alpha_2}{c_{12}}\right)$ with $c_{12} = ||\vec{c}_1 - \vec{c}_2||$.

A topology with three sheets



Misner-Lindquist data (Misner (1963); Lindquist (1963))

- Objective : ensures that the two lower-sheets coincide.
- Define two throats at $\vec{c}_i$ with radii a_i.
- The inversion wrt to a throat is defined as

$$M\left(r_i, \theta_i, \varphi_i\ \right) \to J_i\left(M\right)\left(\frac{a_i^2}{r_i}, \theta_i, \varphi_i\right).$$

- Construct a solution that gives the same result when J_1 or J_2 is applied.
- Involves an operator constructed with combinations of J_i.

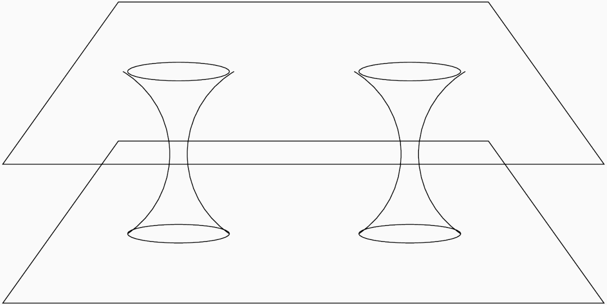
Explicit form of Misner-Lindquist data

$$\Psi = 1 + \sum_{n=1}^{\infty} c_n \left(\frac{1}{\left|\left|\vec{r} - \vec{d}_n\right|\right|} + \frac{1}{\left|\left|\vec{r} + \vec{d}_n\right|\right|} \right).$$

with $\vec{d}_n = d_n \vec{u}$, $\vec{u}$ the unit vector between the throats.

- Explicit expressions for c_n and d_n are known.
- Only two asymptotic regions with the same mass : $m = 4 \sum_{n=1}^{\infty} c_n$.
- Individual masses can be obtained, assuming that the expressions obtained in the Brill-Lindquist case are valid.

A topology with two sheets



Two classes of black hole computations

Misner-Lindquist as an elliptic problem with boundaries

- Solve $\Delta\ \Psi = 0$, outside the throats (i.e. for $r_1 > a_1$ and $r_2 > a_2$).
- The inversion with respect to the throats enforces that

$$\left.\frac{\partial_\Psi}{\partial r_i} + \frac{1}{2a_i}\Psi\right|_{r_i=a_i} = 0.$$

- Equivalent to demanding that the throats are horizons.

Two techniques

- Brill-Lindquist : the puncture method.
- Misner-Lindquist : the excision method.

Non-static solutions

Single black hole case

- One considers solutions with $K_{ij} \neq 0$.
- Analytic solutions of $D_j K^{ij} - D^i K = 0$ are known.
- $\hat{K}_{ij} = \Psi^2 K_{ij}$.
- Some are associated with a linear momentum : $\hat{K}_{ij}\left(\vec{P}\right)$.
- Some are associated with an angular momentum : $\hat{K}_{ij}\left(\vec{S}\right)$.
- As the momentum constraint is linear in K_{ij}, one can combine such solutions.

Puncture method (Brandt and Bruegmann (1997); Baumgarte (2000))

- Extension of Brill-Lindquist data.
- Set the extrinsic curvature tensor to

$$\hat{K}_{ij} = \hat{K}_{1ij}\left(\vec{P}\right) + \hat{K}_{2ij}\left(-\vec{P}\right) + \hat{K}_{1ij}\left(\vec{S}_1\right) + \hat{K}_{2ij}\left(\vec{S}_2\right).$$

- Assume that the conformal factor diverges at two points :

$$\Psi = b^{-1} + u \quad \text{with} \quad b^{-1} = \frac{\alpha_1}{||\vec{r} - \vec{c}_1||} + \frac{\alpha_2}{||\vec{r} - \vec{c}_2||}.$$

- Solve the Hamiltonian constraint for the regular part u (Poisson-like equation):

$$\Delta u + \frac{b^7}{8}\hat{K}_{ij}\ \hat{K}^{ij}\frac{1}{(1+bu)^7} = 0$$

Excision method (Kulkarni et al. (1983))

- Extension of Misner-Lindquist data.
- Start with the extrinsic curvature tensor

$$\hat{K}_{ij}^{\text{init}} = \hat{K}_{1ij}\left(\vec{P}\right) + \hat{K}_{2ij}\left(-\vec{P}\right) + \hat{K}_{1ij}\left(\vec{S}_1\right) + \hat{K}_{2ij}\left(\vec{S}_2\right).$$

- Construct from that a solution that is isometric with respect to the inversion wrt the throats.
- It makes use of combinations of J_1 and J_2.
- Analytic expressions for $\hat{K}_{ij}$ can be found.
- The conformal factor is obtained from the Hamiltonian constraint equation:

$$\Delta\Psi = -\frac{1}{8}\Psi^{-7}\hat{K}_{ij}\ \hat{K}^{ij}.$$

- That equation is solved outside the throats with

$$\frac{\partial_\Psi}{\partial r_i} + \frac{1}{2a_i}\Psi\bigg|_{r_i=a_i} = 0.$$

Kerr-Schild black hole

Single black hole

- $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + 2H\ell_\mu\ell_\nu\mathrm{d}x^\mu\mathrm{d}x^\nu$.
- H and ℓ_μ are explicitly known.

Superposition

- Boosted solutions can be constructed by applying Lorentz boosts to H and ℓ_μ.
- Initial data are obtained by superposing two boosted holes.

Kerr-Schild method for binaries (Marronetti et al. (2000))

Decomposition

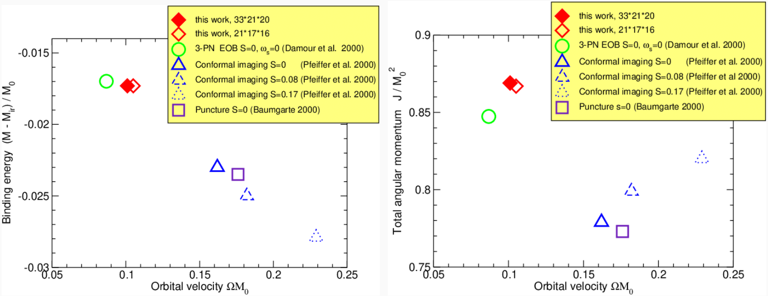
- $\gamma_{ij} = \Phi^4 \tilde{\gamma}_{ij}$.
- $K^{ij} = \Phi^{-10} \left(\tilde{A}^{ij} + (Lw)^{ij} \right) + \frac{1}{3} \gamma_{ij} \bar{K}$.
- $(Lw)^{ij}$ is the conformal Killing operator.
- Set the quantities $\tilde{\gamma}_{ij}$, $\bar{K}$ and $\tilde{A}_{ij}$ to sums of boosted Kerr-Schild.

Solving the constraints

- Both the Hamiltonian and the moment constraints must be solved numerically.
- This is done by looking for the values of w^i and Ψ.
- It produces initial data that are not conformally flat.

Limitations of those methods

- ID did not perform well in evolution codes (maybe other problems...)
- Global quantities are very different from post-Newtonian results.
- Probably comes from the explicit choice made the extrinsic curvature tensor.



from Grandclement et al. (2002)

The extended thin-sandwich method

Conformal traceless decomposition

- Define the conformal factor as $\Psi = \left(\frac{\det\gamma_{ij}}{\det f_{ij}}\right)^{1/12}$.
- $\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}$ so that $\det\tilde{\gamma}_{ij} = \det f_{ij}$.
- $K_{ij} = \Psi^4 \tilde{A}_{ij} + \frac{1}{3} K \gamma_{ij}$ with $\tilde{\gamma}^{ij} \tilde{A}_{ij} = 0$.
- Different powers of Ψ may be used, especially in defining $\tilde{A}_{ij}$.

Extrinsic curvature
Given the definition of the extrinsic curvature tensor, one gets

$$\tilde{A}^{ij} = \frac{1}{2N} \left((LB)^{ij} + \tilde{u}^{ij} \right).$$

- The indices are manipulated using $\tilde{\gamma}$.
- $\tilde{u}^{ij} = \partial_t \tilde{\gamma}^{ij}$.
- $(LB)^{ij} = \tilde{D}^i B^j + \tilde{D}^j B^i - \frac{2}{3} \tilde{D}_k B^k \tilde{\gamma}^{ij}$.

Original thin-sandwich method

Freely specifiable variables
K, $\tilde{\gamma}_{ij}$, $\tilde{u}^{ij}$ and N.

The unknowns

- The shift B^i.
- The conformal factor Ψ.

The equations

- The Hamiltonian constraint.
- The momentum constraints.

Difficulty in choosing the lapse in a meaningfully manner.

Extended thin-sandwich method (Gourgoulhon et al. (2002); Caudill et al. (2006))

Add an equation

- Consider the trace of the evolution equation.
- It becomes an equation on $\partial_t K$ because $\gamma^{ij}\partial_t K_{ij} = \partial_t K + K_{ij}\tilde{u}^{ij}$.

New set of freely specifiable variables
K, $\tilde{\gamma}_{ij}$, $\tilde{u}^{ij}$ and $\partial_t K$.

- The unknowns are N, B^i and Ψ.
- The equations are the constraints and the trace of the evolution equation.

Quasi-equilibrium configurations

One assumes that the binary is almost on a circular orbit : existence of an helical Killing vector.

Inertial coordinates

- Killing vector : $k^\alpha = (t, 0, 0, \Omega)$.
- Killing equation for a spatial tensor : $\partial_t T_i^j + \Omega \partial_\varphi T_i^j = 0$.

Corotating coordinates coordinates

- Variable change : $\varphi_{\rm cor} = \varphi_{\rm inert} - \Omega t_{\rm inert}$.
- Killing vector : $k^\alpha = (t, 0, 0, 0)$.
- Killing equation for a spatial tensor : $\partial_t T_i^j = 0$.

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Asymptotics

Link between the two coordinate systems

- In terms of metric quantities the only change is in the shift :

$$B^i_{\rm cor} = B^i_{\rm inert} + \Omega \partial^i_\varphi.$$

- One can show that the extrinsic curvature tensor is the same.

Asymptotics

- In the corotating frame : $B^i_{\rm cor} \to \Omega \partial^i_\varphi$.
- Difficulty : infinite at spatial infinity.
- Work with $B^i_{\rm inert}$ instead, maintaining $\partial_t = 0$.
- It translates in a modification of the inner boundary condition.

Choice of the freely specifiable variables

- Maximal slicing : $K = 0$.
- Stationnarity : $\partial_t K = 0$ and $\tilde{u}_{ij} = \partial_t \tilde{\gamma}_{ij} = 0$.
- Conformal flatness : $\tilde{\gamma}_{ij} = f_{ij}$.
    - It is only true at 1PN order.
    - It is not true for Kerr for instance.
    - It simplifies the equations.
    - It kills the GW emission.
    - It is consistent with the helical symmetry.

Resulting bulk equations

- trace of the evolution equation :

$$D_i D^i N = -2\frac{D_i \Psi D^i N}{\Psi} + N\Psi^4 \tilde{A}_{ij}\tilde{A}^{ij}.$$

- Hamiltonian constraint

$$D_i D^i \Psi = -\frac{\Psi^5}{8}\tilde{A}_{ij}\tilde{A}^{ij}.$$

- Momentum constraints

$$D_j D^j B^i + \frac{1}{3} D^i D_j B^j = 2\tilde{A}^{ij}\left(D_j N - 6N\frac{D_j \Psi}{\Psi}\right).$$

with $\tilde{A}^{ij} = \frac{1}{2N}\left(D^i B^j + D^j B^i - \frac{2}{3} D_k B^k f_{ij}\right)$.

The indices are manipulated with f_{ij} and the covariant derivatives are also taken with respect to f_{ij}.

Inner boundary conditions

Apparent horizons

- Excision method : solve the equation outside of two spheres.
- Impose that they are apparent horizons in equilibrium.
- Let s_a^i be the unit spatial normal to the hole a.

Resulting BCs

- $N = \frac{1}{2}$: coordinate choice, other possibilities are available.
- $\frac{\partial \Psi}{\partial_r} + \frac{\Psi}{2r} + \frac{\Psi^3}{4} \tilde{A}_{ij} s_a^i s_a^j = 0$: apparent horizon condition.
- $B^i = \frac{N}{\Psi^2} s_a^i - \Omega \partial_\varphi^i$. : horizon in equilibrium and global rotation (corotating case).

Outer boundary conditions

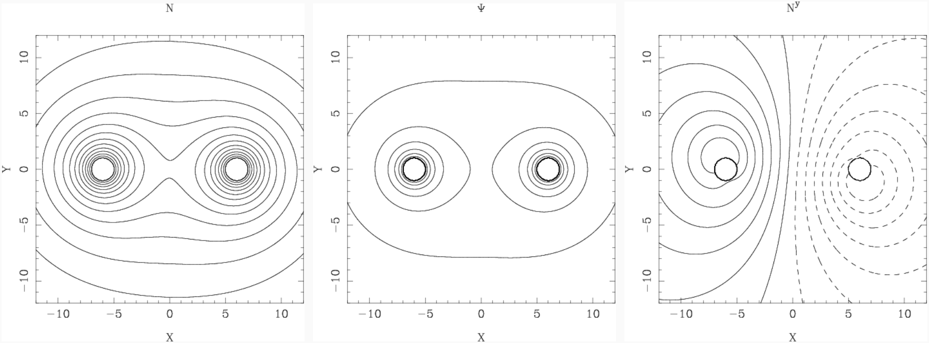
Asymptotic flatness

- $N = 1$
- $\Psi = 1$
- $B^i = 0$: work with the inertial shift.

Orbital angular velocity

- Ω appears as a parameter in the equations (BC on the shift).
- Must be fixed by the separation and sizes of the holes.
- The general relativistic theorem states that the Komar $M_{\rm K}$ and AMD $M_{\rm ADM}$ masses must coincides.
- Both quantities are given as surface integrals at infinity.
- The condition $M_{\rm K} = M_{\rm ADM}$ fixes Ω.
- It is equivalent to minimizing some kind of binding energy of the binary.

Some examples of fields



from Grandclement (2010)

Advanced topics (1/2)

Unequal mass binaries

- The local masses of the holes can be measured by surface integrals on the horizons.
- It can be used to choose the mass ratio.
- The location of the rotation axis of the binary is unknown.
- It is fixed by demanding that the linear momentum P^i_{ADM} vanishes.

Spinning black holes

- Additional contribution on the shift inner BC.
- $B^i = \frac{N}{\Psi^2} s^i_a - \Omega \partial^i_\varphi + \Omega_a \partial^i_{\varphi_a}$.
- φ^i_a is associated to the hole a.
- Surface integrals can be used to measure the individual spins and to fix Ω_a for each hole.
- Non-aligned spins are possible in principle but more difficult numerically.

Advanced topics (2/2)

Eccentricity reduction

- Initial data usually exhibit spurious eccentricity in evolution codes.
- It can be cured by slightly modifying the boundary condition on the shift :

$$B^i = \frac{N}{\Psi^2} s_a^i - (\Omega + \delta\Omega)\, \partial_\varphi^i + \dot{a} r_a^i$$

- The quantities $\delta\Omega$ and $\dot{a}$ can be obtained from PN calcultations or iteratively from numerical evolutions of the data.

Matter terms

Neutron stars (Gourgoulhon et al. (2001))

- Cold neutron stars are well described by perfect fluids :

$$T^{\mu\nu} = (e + p)\, u^{\mu} u^{\nu} + p g^{\mu\nu},$$

  where e is energy density, p the pressure and u^{μ} the four-velocity.
- Other quantities ρ the rest-mass density, h the specific enthalpy.
- The matter quantities relate to each other via an equation-of-state.
- The various 3+1 matter contributions can be obtained as projections of $T^{\mu\nu}$.
- The fluid must obey :
  - the conservation of the stress-energy momentum $\nabla_{\mu} T^{\mu\nu} = 0$.
  - the rest-mass conservation $\nabla_{\mu} (\rho u^{\mu}) = 0$.

Corotation case

- u^μ is colinear to the helical Killing vector.
- $\nabla_\mu T^{\mu\nu} = 0$ reduces to a first integral.
- $\nabla_\mu (\rho u^\mu) = 0$ is automatically satisfied.
- One unknown field only, for instance h.
- The value h_c at the center of each star is used to control to total baryonic mass.
- Once h_c is known, the first integral can be used to get h everywhere.
- The surface of the star is defined with $p = 0$ and must be found numerically.

Irrotational case

- It is supposed to be more realistic for BNS.
- Existence of a potential such that $hu_i = D_i \Phi$.
- $\nabla_\mu T^{\mu\nu} = 0$ reduces to a first integral.
- $\nabla_\mu \left(\rho u^\mu\right) = 0$ gives an elliptic equation on Φ.
- Two unknown fields : h and Φ.
- The value h_c at the center of each star is used to control to total baryonic mass.
- Once h_c is known, the first integral can be used to get h everywhere.
- The surface of the star is defined with $p = 0$ and must be found numerically.

Advanced topics for NS

Spinning NS

- Possibility to add an arbitrary spin : $hu_i = D_i \Phi + \omega \partial^i_{\varphi_a}$
- The first integral of motion is only approximate.

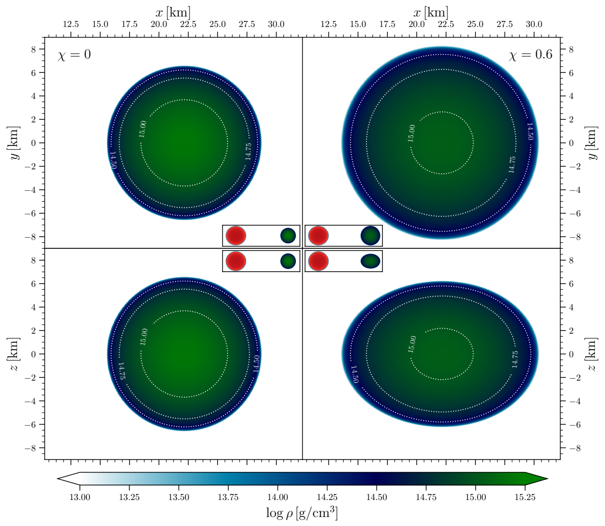
Equation of state

- Cold matter EOS relate p, ρ and h.
- Can be analytic or tabulated.

Surface fitting coordinates

- The surface is an unknown of the problem.
- Numerical coordinates can adapt to the surface.

Examples of matter profiles

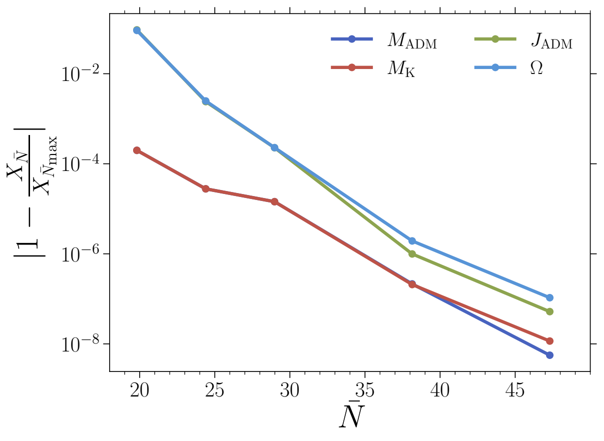


from Papenfort et al. (2021)

Some results

Numerical precision

Initial data obtained by the library Kadath with spectral methods (Grandclement (2010)).



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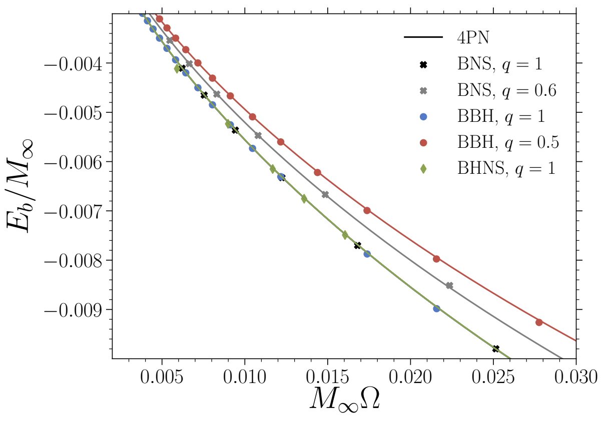
Binding energy



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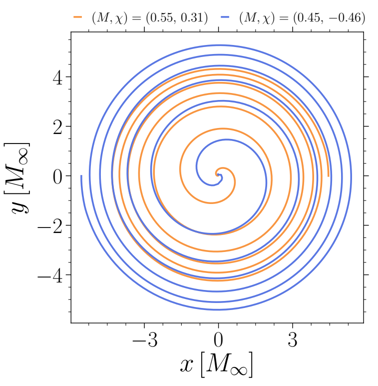
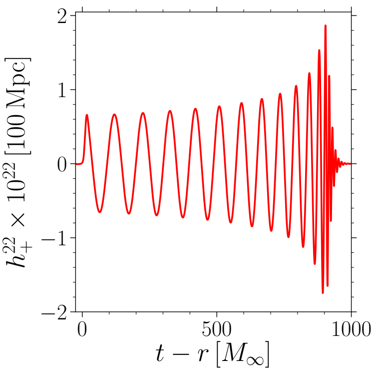
Good agreement with PN results.

Evolution of the ID for BBH (GW150914)

Evolution with the puncture method : need to “fill” the holes of the initial data.





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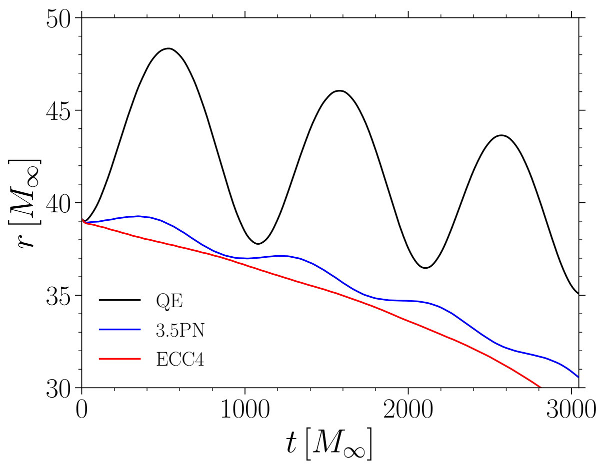
Eccentricity reduction effect



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Conclusions

- The initial value problem for binaries has a very long history.
- Since the 2000s, the problem is relatively under control.
- Big improvement in using the extended thin sandwich approach.
- It enables a better choice for the freely specifiable variables.
- Additional advanced techniques : spins, realistic EOS, eccentricity reduction.
- Possible extensions : remove the conformal flatness approximation, exotic objects.

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