Introduction to Numerical Relativity.

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Outline

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- BSSN gauge conditions
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For Relativistic Hydrodynamics see Bruno Giacomazzo at 10:00.

For numerical methods in General Reativity see Isabel Cordero-Carrión at 11:30.

The Einstein Field Equations.

Spacetime tells matter how to move and matter tells spacetime how to curve:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}.$$

Simple, right?

The Einstein Field Equations.

Not so fast...

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu, \\
\Gamma^\mu_{\nu\alpha} &= \frac{g^{\mu\beta}}{2}\left[\frac{\partial g_{\nu\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\beta}}{\partial x^\nu} - \frac{\partial g_{\nu\alpha}}{\partial x^\beta}\right], \\
R^\mu{}_{\nu\alpha\beta} &:= \frac{\partial \Gamma^\mu_{\nu\beta}}{\partial x^\alpha} - \frac{\partial \Gamma^\mu_{\nu\alpha}}{\partial x^\beta} + \Gamma^\mu_{\rho\alpha}\Gamma^\rho_{\nu\beta} + \Gamma^\mu_{\rho\beta}\Gamma^\rho_{\nu\alpha}, \\
R_{\mu\nu} &:= R^\alpha{}_{\mu\alpha\nu}, \\
R &:= R^\mu{}_\mu, \\
G_{\mu\nu} &:= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}.
\end{aligned}
$$

The Einstein Field Equations are ten coupled, non-linear, second order partial differential equations for the full 4-dimensional spacetime.

Ingredients in a Numerical Relativity Simulation

- Formulation of the field equations:
  ADM, BSSN, CCZ4, Z4C, generalized harmonic, characteristic, conformal, $\cdots$
- Coordinates:
  Maximal slicing, 1+log, minimal distortion, Γ-driver, gauge drivers, $\cdots$
- Constraint handling:
  Free evolution, constrained evolution, constraint damping.
- Initial data corresponding to an astrophysical system.
- Handling of black holes: Excision, moving punctures.
- Boundary conditions: Constraint preserving, outgoing radiation, characteristic extraction, characteristic matching, $\cdots$
- Numerical method: Finite differences, spectral, discontinuous Galerkin, $\cdots$
- Horizon finding.
- Wave extraction.

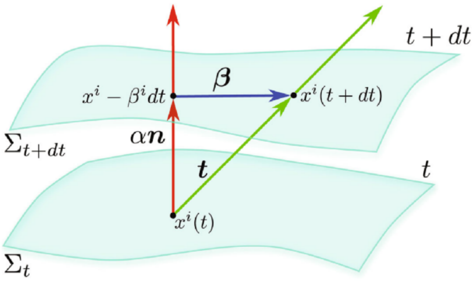
The 3+1 Split of Spacetime.

One way of splitting up the spacetime is the 3+1 split, i.e. split it into 3-dimensional hypersurfaces with unit normal vector

$$n^\mu = (1/\alpha, -\beta^i/\alpha), \quad n_\mu = (-\alpha, 0)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

$$t^\mu = \alpha n^\mu + \beta^\mu$$



$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} \left(dx^i + \beta^i dt\right) \left(dx^j + \beta^j dt\right) .$$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}, \quad g^{\mu\nu} \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^j/\alpha^2 & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}$$

The 3+1 Split of Spacetime.

The 3-metric γ_{ij} determines the curvature intrinsic to the hypersurface, however this does not account for the full 4-dimensional curvature.
The remaining curvature is encoded in how the normal vector, n^μ, is parallel transported from one point in the hypersurface to another.
With the definition of the projection operator:

$$P^\mu_\nu := \delta^\mu_\nu + n^\mu n_\nu$$

it turns out that the extrinsic curvature can be defined as:

$$K_{\mu\nu} := -P^\alpha_\mu \nabla_\alpha n_\nu = -(\nabla_\mu n_\nu + n_\nu n^\alpha \nabla_\alpha n_\nu)$$

The extrinsic curvature is purely spatial, i.e. $n^\mu K_{\mu\nu} = n^\nu K_{\mu\nu} = 0$ and symmetric $K_{ij} = K_{ji}$.

The 3+1 split can now be performed by contracting the Einstein field equations with the normal vector n^μ and P^μ_ν:

$$
\begin{aligned}
\mathcal{H} = n^\mu n^\nu G_{\mu\nu} &= 8\pi n^\mu n^\nu T_{\mu\nu} = 8\pi\rho \\
\mathcal{M}_\mu = -n^\nu P^\alpha_\mu G_{\nu\alpha} &= -8\pi n^\nu P^\alpha_\mu T_{\alpha\nu} = 8\pi j_\mu \\
\mathcal{E}_{\mu\nu} = \mathcal{P}^\alpha_\mu \mathcal{P}^\beta_\nu \mathcal{G}_{\alpha\beta} &= 8\pi P^\alpha_\mu P^\beta_\nu T_{\alpha\beta} = 8\pi S_{\mu\nu}
\end{aligned}
$$

After a long derivation, where the 4-d curvature variables are written in terms of the 3-d curvature variables and extrinsic curvature, these equations become the constraint and ADM evolution equations:

$$
\begin{aligned}
\mathcal{H} &= {}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \\
\mathcal{M} &= D_j(K^{ij} - \gamma^{ij}K) = 8\pi j^i \\
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \\
\partial_t K_{ij} &= -D_iD_j\alpha + \alpha\left[{}^{(3)}R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j\right] + 4\pi\alpha\left[\gamma_{ij}(S-\rho) - 2S_{ij}\right] \\
&\quad +\beta^k\partial_k K_{ij} + K_{ki}\partial_j\beta^k K_{kj}\partial_i\beta^k.
\end{aligned}
$$

The 3+1 Split of Spacetime.

Some comments are in order:

- The hamiltonian and momentum constraints does not contain second time derivatives of the metric and form a set of four elliptical equations.
- The evolution equations are six non-linear, second order, partial differential equations of hyperbolic type.
- The gauges, lapse and shift, are not included in the evolution equations and have to be prescribed separately.
- If the constraint equations are satisifed initially, the evolution equations guarantees, in the absence of numerical error, that they remain satisfied.
- If the constraint equations are not satisfied, the solution is not physical.
- The constraints can be added to the evolution equations and since they contain second derivatives of the metric will change the principal part of the equations.
- How is the black hole singularity to be handled?
- What are the boundary conditions?

Brief Historical Interlude.

Pioneering work in 1964 (Hahn & Lindquis) and 1975 (Smarr, Eppley) set the stage by simulationg the head on collision of two non-rotating black holes in axisymmetry.

In 1993 the Binary Black Hole Grand Challenge Alliance was formed. A collaboration among 9 different US institutions.

PROJECT TASK	COG	COMPLETION DATE
Event Horizon Locator 1d	Winicour,Seidel, Suen,Shapiro	07/06/94
Spherical adaptive parallel data structure	Browne	12/20/94
Apparent Horizon Boundary Condition 1d	Choptuik,Marsa	01/01/95
Event Horizon Locators 2d	Winicour	04/15/95
Initial Conformal Data	Cook	04/15/95
Characteristic Codes 2d	Winicour	04/15/95
Finite Elements (Elliptic Equations/ Implicit Evolution) 2d/3d	Saied,Saylor	06/01/95
1d, 2d Toolkit	Browne	10/24/95
3d Toolkit	Browne	01/01/96
3d Adaptive parallel data structure	Browne	01/01/96
Apparent Horizon Trackers (Robust 3d Computational Data)	Seidel,Cook,Huq	06/30/96
Algorithms: Elliptic Solvers, Constraint Enforcers(DAE)	Saied,Saylor	12/17/96
Apparent Horizon Boundary Condition 3d	Cook,Scheel,Huq	12/31/96
Characteristic Scalar/ Spacelike Interior 3d	Bishop et al	01/01/97
Event Horizon Locator (Characteristic)	Winicour,Marsa	01/01/97
3d Spacelike Rectangular Codes Single Grid or Fixed Refinement	Klasky,Huq,Scheel,Cook	01/01/97
Event Horizon Locator 3d	Winicour,Seidel, Suen,Shapiro	01/01/97
Outgoing Boundary Conditions/ Wave Extraction	Abrahams,Rupright	10/3/97
Physical Content of Initial Data	Finn	01/01/99
Software Librarian	Fox,Haupt	01/01/99
Newsletter Editor	Saied	01/01/99
WaveForm Detectability	Finn	01/01/99
Advisory	Thorne	01/01/99
Hyperbolic Formalism UNC	York	01/01/99
Validation	Shapiro,Teukolsky	01/01/99
Coordinates	Matzner,Huq,Winicour	01/01/99
Moving BBH data	Matzner,Huq,Cook	01/01/99
Hardware Evaluation	All	01/01/99
WAVEFORM GENERATOR ALLIANCE CODE	Alliance	01/01/99

Brief Historical Interlude.

Towards the end of the Grand Challenge Alliance, it became clear that one main problem might be with the ADM equations themselves.

In the late 1980's Nakamura, Oohara and Kojima first proposed a formulation using conformal decomposition of the metric and the tracefree part of the extrinsic curvature, Nakamura, Oohara & Kojima, 1987 and Nakamura & Oohara, 1987.

This was first modified and used for gravitational waves and neutron stars in the mid 1990's, Shibata & Nakamura, 1999 and Nakamura & OOhara, 1999 and Shibata, 1999.

Tested for weak gravitational waves by Baumgarte and Shapiro, 1999.

The Cactus framework was developed at the AEI for the purpose of allowing large collaborations to work together and to test different formulations and methods with the same code: Bona, Masso, Seidel & Walker, 1998.
An implementation of BSSN in Cactus quickly followed, Alcubierre et al. 2000

Well Posedness.

To give a flavor of the concept of well posedness, Consider a first order in space and time system of evolution equations with N variables $\boldsymbol{u}$ of the form

$$\partial_t \boldsymbol{u} + \boldsymbol{M}^i \partial_i \boldsymbol{u} = 0,$$

where $\boldsymbol{M}^i$ are $N \times N$ matrices (one for each direction).

If we choose an arbirary direction with unit vector $\boldsymbol{n}_i$ we can then construct the principal symbol $\boldsymbol{P}(\boldsymbol{n}_i) := \boldsymbol{M}^i \boldsymbol{n}_i$.
The evolution system is well posed if the principal symbol has N real eigenvalues and a complete set of eigenvectors (strongly hyperbolic) for all directionsn_i.
Applying such an analysis to the ADM equations shows that they, in general, does not have a full set of eigenvectors, though they do have N real eigenvalues.
Thus the ADM equations are only weakly hyperbolic and **NOT** well posed.
It is therefore not surprising that all simulations using the ADM equations turned out to be unstable.

The BSSN Formulation.

Introduce a conformal rescaling of the three metric

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}.$$

We choose $\psi = \gamma^{1/12}$ such that the determinant of $\tilde{\gamma}_{ij}$ is 1. In addition we introduce a trace decomposition of the extrinsic curvature.

$$\begin{aligned} K &= \gamma^{ij} K_{ij}, \\ A_{ij} &= K_{ij} - \frac{1}{3}\gamma_{ij} K. \end{aligned}$$

We then promote the following variables to evolution variables

$$\begin{aligned} \phi &= \ln\psi = \frac{1}{12}\ln\gamma \text{ or } W = 1/\psi^2 \text{ or } \chi = 1/\psi^4, \\ K &= \gamma_{ij} K^{ij}, \\ \tilde{\gamma}_{ij} &= e^{-4\phi}\gamma_{ij}, \\ \tilde{A}_{ij} &= e^{-4\phi} A_{ij}. \end{aligned}$$

The BSSN Formulation.

We finally, in addition, promote the conformal connection functions

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i{}_{jk} = -\partial_j \tilde{\gamma}^{ij}, \tag{1}$$

to evolved variables as well. The final set of evolution variables are ϕ, K, $\tilde{\gamma}_{ij}$, $\tilde{A}_{ij}$ and $\tilde{\Gamma}^i$.

Note that the momentume constraint has to be used to eliminate some troublesome terms in the evolution equation for $\tilde{\Gamma}^i$.

It is also necessary to actively enforce the constraints $\tilde{A}^i_j = 0$ and $\det(\tilde{\gamma}_{ij}) = 1$.

Finally it has proven beneficial to replace the evolved $\tilde{\Gamma}^i$ with values $\tilde{\Gamma}^i_{(\mathsf{n})} = \tilde{\gamma}^{jk}\tilde{\Gamma}^i{}_{jk}$ recalculated from the current conformal metric $\tilde{\gamma}_{ij}$, wherever derivatives of them are not needed.

It can now be proven that BSSN is in fact strongly hyperbolic, Sarbach et al., 2002.

The BSSN Formulation.

The evolution equation for all the BSSN variables are

$$\begin{aligned}
\partial_t \tilde{\gamma}_{ij} &= -2\alpha \tilde{A}_{ij} + \beta^k \partial_k \tilde{\gamma}_{ij} + \tilde{\gamma}_{ik} \partial_j \beta^k + \tilde{\gamma}_{jk} \partial_i \beta^k - \frac{2}{3} \tilde{\gamma}_{ij} \partial_k \beta^k, \\
\partial_t \phi &= -\frac{1}{6} \alpha K + \beta^k \partial_k \phi + \frac{1}{6} \partial_k \beta^k, \\
\partial_t \tilde{A}_{ij} &= e^{-4\phi} \left[-D_i D_j \alpha + \alpha R_{ij} + 4\pi\alpha \{ \gamma_{ij} (S - \rho) - 2 S_{ij} \} \right]^{TF} + \alpha \left(K \tilde{A}_{ij} - 2 \tilde{A}_{ik} \tilde{A}^k{}_j \right) \\
&\quad + \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik} \partial_j \beta^k + \tilde{A}_{jk} \partial_i \beta^k - \frac{2}{3} \tilde{A}_{ij} \partial_k \beta^k, \\
\partial_t K &= -D^i D_i \alpha + \alpha \left(\tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{3} K^2 \right) + 4\pi\alpha (\rho + S) + \beta^k \partial_k K, \\
\partial_t \tilde{\Gamma}^i &= \tilde{\gamma}^{jk} \partial_j \partial_k \beta^i + \frac{1}{3} \tilde{\gamma}^{ij} \partial_j \partial_k \beta^k + \beta^j \partial_j \tilde{\Gamma}^i - \tilde{\Gamma}^j_{(\mathrm{n})} \partial_j \beta^i + \frac{2}{3} \tilde{\Gamma}^i_{(\mathrm{n})} \partial_j \beta^j \\
&\quad - 2 \tilde{A}^{ij} \partial_j \alpha + 2\alpha \left(\tilde{\Gamma}^i{}_{jk} \tilde{A}^{jk} + 6 \tilde{A}^{ij} \partial_j \phi - \frac{2}{3} \tilde{\gamma}^{ij} \partial_j K - 8\pi e^{4\phi} j^i \right).
\end{aligned}$$

Here $R_{ij} = \tilde{R}_{ij} + R^{\phi}_{ij}$, where

$$\begin{aligned}
R^{\phi}_{ij} &= -2 \tilde{D}_i \tilde{D}_j \phi - 2 \tilde{\gamma}_{ij} \tilde{D}^k \tilde{D}_k \phi + 4 \tilde{D}_i \phi \, \tilde{D}_j \phi - 4 \tilde{\gamma}_{ij} \tilde{D}^k \phi \, \tilde{D}_k \phi, \\
\tilde{R}_{ij} &= -\frac{1}{2} \tilde{\gamma}^{lm} \partial_l \partial_m \tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i} \partial_{j)} \tilde{\Gamma}^k_{(\mathrm{n})} + \tilde{\Gamma}^k \tilde{\Gamma}_{(ij)k} + \tilde{\gamma}^{lm} \left(2 \tilde{\Gamma}^k{}_{l(i} \tilde{\Gamma}_{j)km} + \tilde{\Gamma}^k{}_{im} \tilde{\Gamma}_{klj} \right).
\end{aligned}$$

Puncture Initial Data.

In the York-Lichnerowicz conformal decomposition of the constraint equations the physical metric and extrinsic curvature is given by

$$\begin{aligned}
\gamma_{ij} &= \psi^4 \bar{g}_{ij} \\
K^{ij} &= \psi^{-10} \bar{A}^{ij} + \frac{1}{3}\gamma^{ij} K.
\end{aligned} \tag{2}$$

Under the assumption of vacuum, conformal flatness and maximal slicing, it is remarkable that the there is a simple analytical solution for $\bar{A}^{ij}$ as

$$\bar{A}_{ij} = \frac{3}{2r^2}\left[n_i P_j + n_j P_i + n_k P^k (n_i n_j - \delta_{ij}\right] - \frac{3}{r^3}\left(\epsilon_{ilk} n_j + \epsilon_{jlk} n_i\right) n^l S^k.$$

Here P^i and S^i are the ADM linear and angular momenta.
Writing the conformal factor as

$$\psi = u + \psi_{\rm BL} = u + \sum_i^N \frac{m_i}{2|r - r_i|}.$$

Then a single elliptical equation can be solved for u that is regular everywhere.

BSSN Gauge Conditions.

The simplest choice for the lapse would be $\alpha = 1$, i.e. geodesic slicing, where the coordinates are in free fall. Not a good idea when black holes are present.
A popular early lapse condition was maximal slicing which is an elliptical equation for the lapse

$$D^2\alpha = \alpha \left[K_{ij}K^{ij} + 4\pi(\rho + S)\right]$$

This condition has the important property of singularity avoidance where the lapse goes to zero fast enough that the singularity is never reached.
A popular alternative is the Bona-Masso family of slicing conditions

$$\partial_t \alpha = -\alpha^2 f(\alpha) K$$

Here $f(\alpha) = 1$ is harmonic slicing, while $f(\alpha) = 2/\alpha$ is called 1+log slicing. Harmonic slicing is only mildly singularity avoiding, while 1+log slicing is almost as singularity avoiding as maximal slicing.
All of these slicing conditions sufferes from the problem that the slices gets stretched as proper time elapses differently away from the singularity.

BSSN Gauge Conditions.

Counteracting slice stretching can be achieved with a suitable shift condition.
It turns out that $\tilde{\Gamma}^i$, introduced in BSSN, provides a convenient way to define an effective shift condition. The Gamma-driver condition

$$\begin{aligned} \partial_t \beta^i &= F B^i + \beta^j \partial_j \beta^i \\ \partial_t B^i &= \partial_t \tilde{\Gamma}^i + \beta^j \partial_j (B - \tilde{\Gamma}^i) - \eta B^i \end{aligned}$$

Originally F was chosen to keep the shift equal to zero at the punctures and there was no advection terms

$$F = F(\alpha, \psi_{\mathrm{BL}}) = \frac{3}{4} \alpha^n \psi_{\mathrm{BL}}^{-N}$$

whereas for moving punctures we want the shift to be non-zero at the punctures

$$F = \frac{3}{4}.$$

Alternatively it can be expressed in first order form

$$\partial_t \beta^i = \frac{3}{4} \tilde{\Gamma}^i + \beta^j \partial_j \beta^i - \eta \beta^i.$$

The Generalized Harmonic Formulation.

Generalized Harmonic coordinates satisfy the inhomogeneous wave equations

$$H_\mu(x,g) = g_{\mu\nu}\nabla_\alpha\nabla^\alpha x^\nu = -\Gamma_\mu,$$

where $H_\mu(x,g)$ is an arbitrary but fixed algebraic function of the coordinates x^μ and the metric $g_{\mu\nu}$.
In these coordinates the Einstein vacuum field equations become

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = -2\nabla_{(\mu}H_{\nu)} + 2g^{\alpha\beta}g^{\rho\sigma}\left(\partial_\rho g_{\alpha\mu}\partial_\sigma g_{\beta\nu} - \Gamma_{\mu\alpha\rho}\Gamma_{\nu\beta\sigma}\right).$$

All second derivatives of the metric appears on the left hand side, so the principal part is manifestly hyperbolic.
This system can be kept as a second order system or rewritten as a first order symmetric hyperbolic system.

Goddard 2005 Workshop.

November 2-4, 2005 a historical numerical relativity workshop was held at NASA Goddard. URL: https://asd.gsfc.nasa.gov/archive/astrogravs/conf/numrel2005/

AEI-LSU-FAU: Punctures, BSSN, 1+log slicing, Gamma-driver shift, co-rotating shift, drift correct, simple lego excision, fixed mesh refinement, evolved for more than an orbit, no waveforms.

Brownsville-Goddard: Moving puncture, 1+log slicing, Gamma driver shift, QC0, not a complete orbit, clean waveforms.

Pretorius: ID: Boosted scalar field collapse+Cook-Pfeiffer, Generalized Harmonic second order formulation, constraint damping, dynamic excision, about 1.5 orbits, wave extraction.

Cornell-Caltech: KST, Cook-Pfeiffer initial data, multi-domain spectral, excision, use shift to control horizon shape, constraint blowup. Had started to look at Generalized Harmonic.

The Einstein Toolkit.

Some of the important physics capabilities of the Einstein Toolkit are:

- Spacetime evolution: `McLachlan`, `Lean`, `Proca`, `Canuda`, `Baikal`.
- Relativistic Hydrodynamics: `GRHydro`,`IllinoisGRMHD`, `GiRaFFE`.
- Gravitational Wave Extraction and analysis: `Extract`, `WeylScal4`, `Multipole`, `PITTNullCode`.
- Apparent Horizon Analysis: `AHFinder`, `AHFinderDirect`, `QuasiLocalMeasures`.
- Initial Data: `Exact`, Brill Wave data, Lorene data importers, `TwoPunctures`, `FLRWSolver`, FUKA data importers, SGRID importer, more on the way...
- Adaptive Mesh Refinement: `Carpet`, `CarpetX`
- Multipatch: `Llama`.

Concluding Remarks.

- This is of course not the full story.
- I only mentioned puncture initial data, but there are many other valuable efforts going on in order to provide the best initial data possible.
- I did not cover other fomulations of the Einstein field equations that have proven useful, such as Z4c and CCZ4.
- There has been an impressive improvement in the computational infrastructures used since 2005, such as adaptive mesh refinement, multipatch infrastructures and adapted coordinates. High mass ratios, high spins.
- We have a much better understanding of the trumpet solution that punctures evolve to when 1+log and Gamma-driver shifts are used.
- The Generalized Harmonic Formulation implemented in `SPEC` benefited from impressive advancements such as the ability to control the horizon shape for excision, the development of the dual frame approach to corotating coordinates. Very long and accurate waveforms.
- More to come e.g. `CarpetX`, `SpECTRE` and other...