

# Five parton scattering in massless QCD and its high energy limit

Based on: Phys.Rev.D 109 (2024) 9, 094025, arXiv:2311.09870 with B. Agarwal, F. Buccioni, G. Gambuti, A. Von Manteuffel, L. Tancredi  
+ ongoing work with F. Buccioni, F. Caola, G. Gambuti

# Outline

- Review of calculation of full color two-loop five-point scattering amplitudes in massless QCD

Why

How

- Five parton scattering in the high energy limit

Regge and Multi-Regge kinematics

Regge poles vs Regge cuts

Factorization beyond NLL



“Cut contamination”

2-loop central emission vertex

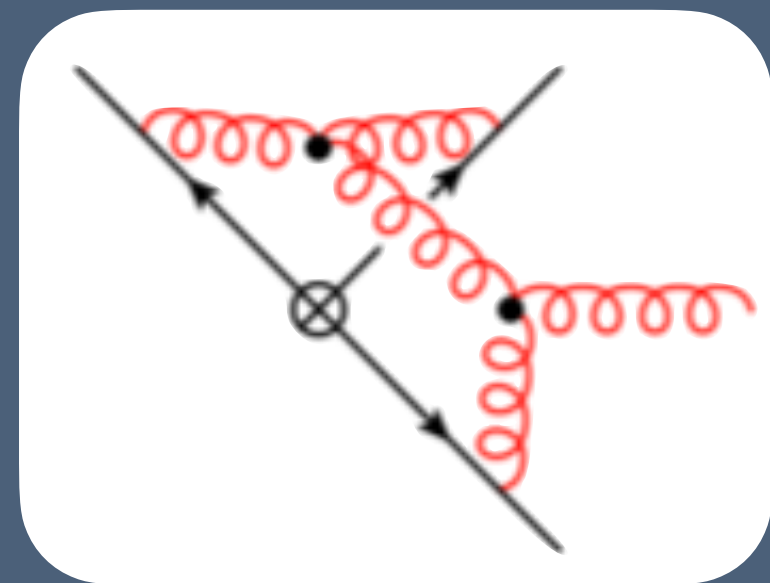
Why

“Pain is inevitable, suffering is optional”



Worth it!

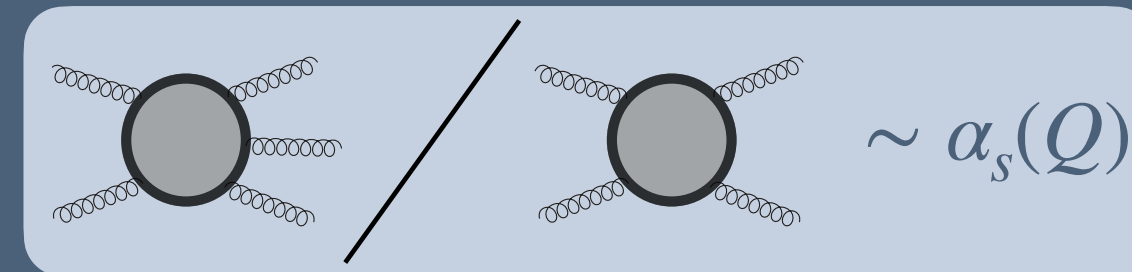
Simplest playground to study non planar sectors of QCD



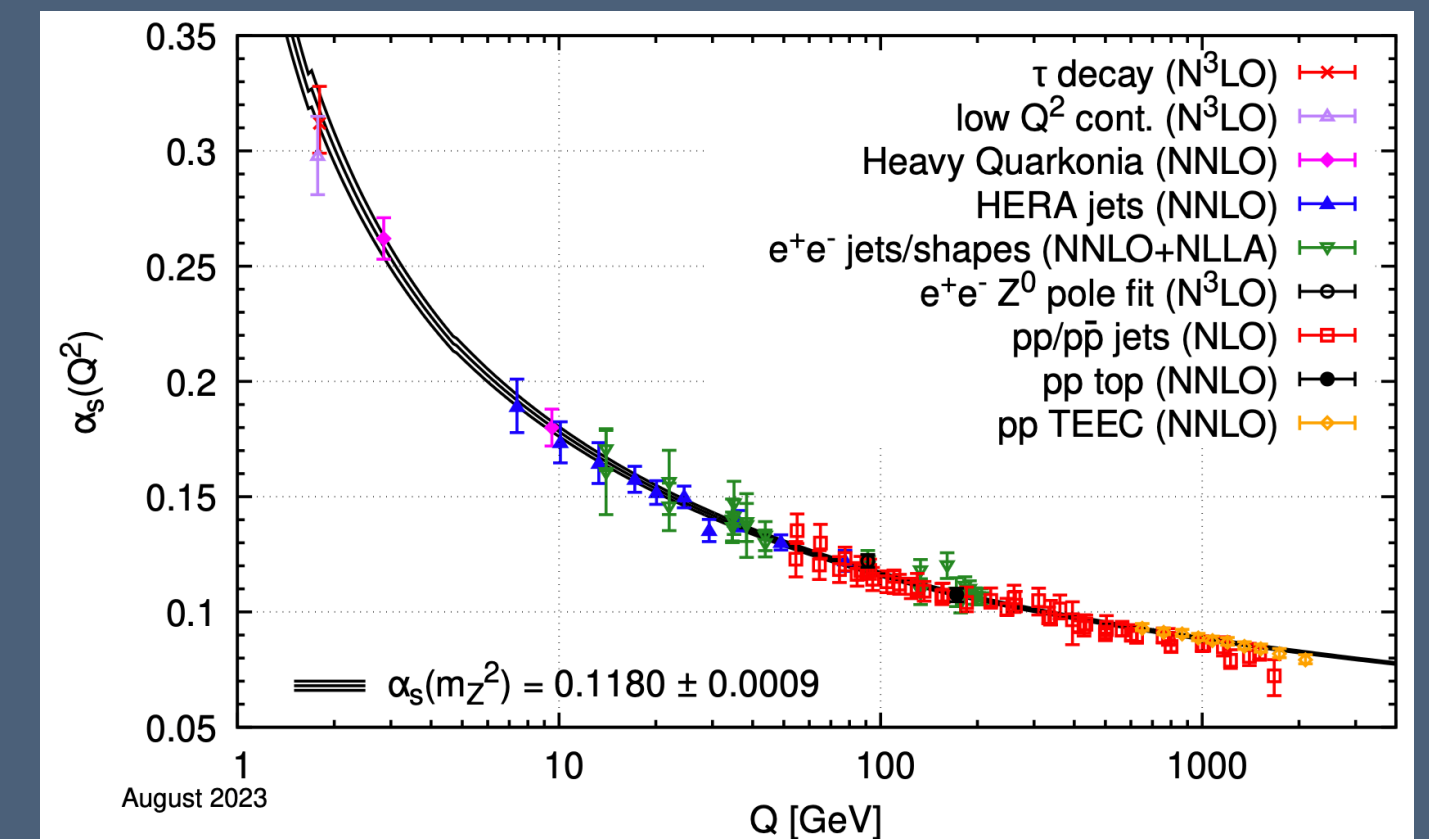
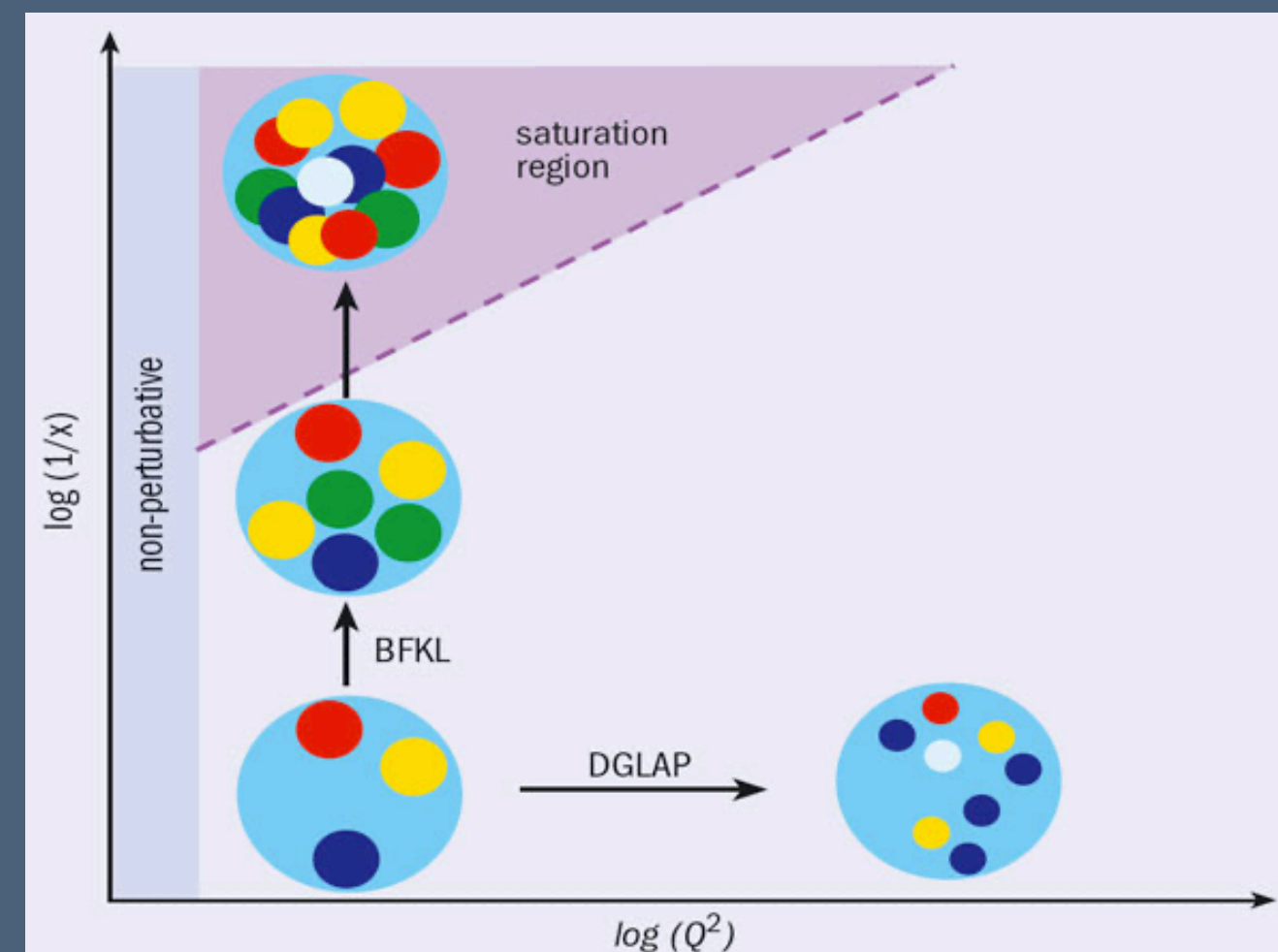
Infrared

[Dixon et al 1912.09370]

Testing running of strong coupling at TeV scale



High energy: Regge limit and factorization, BFKL evolution, PDFs@small x etc..

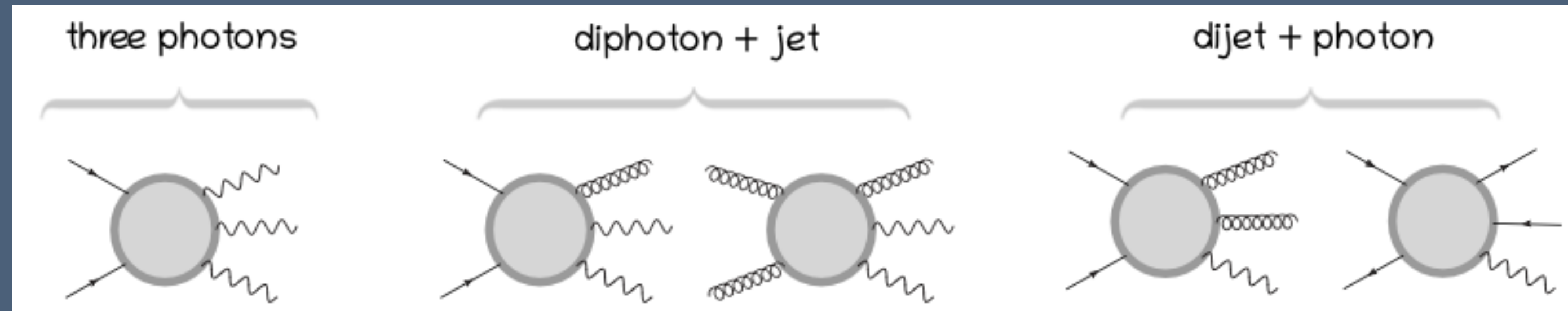


[PDG]

Input for higher order jet cross sections

e.g. 3-jet XS [Czakon, Mitov, Poncelet 2106.05331]

# Brief history of 5-point 2-loop QCD amplitudes

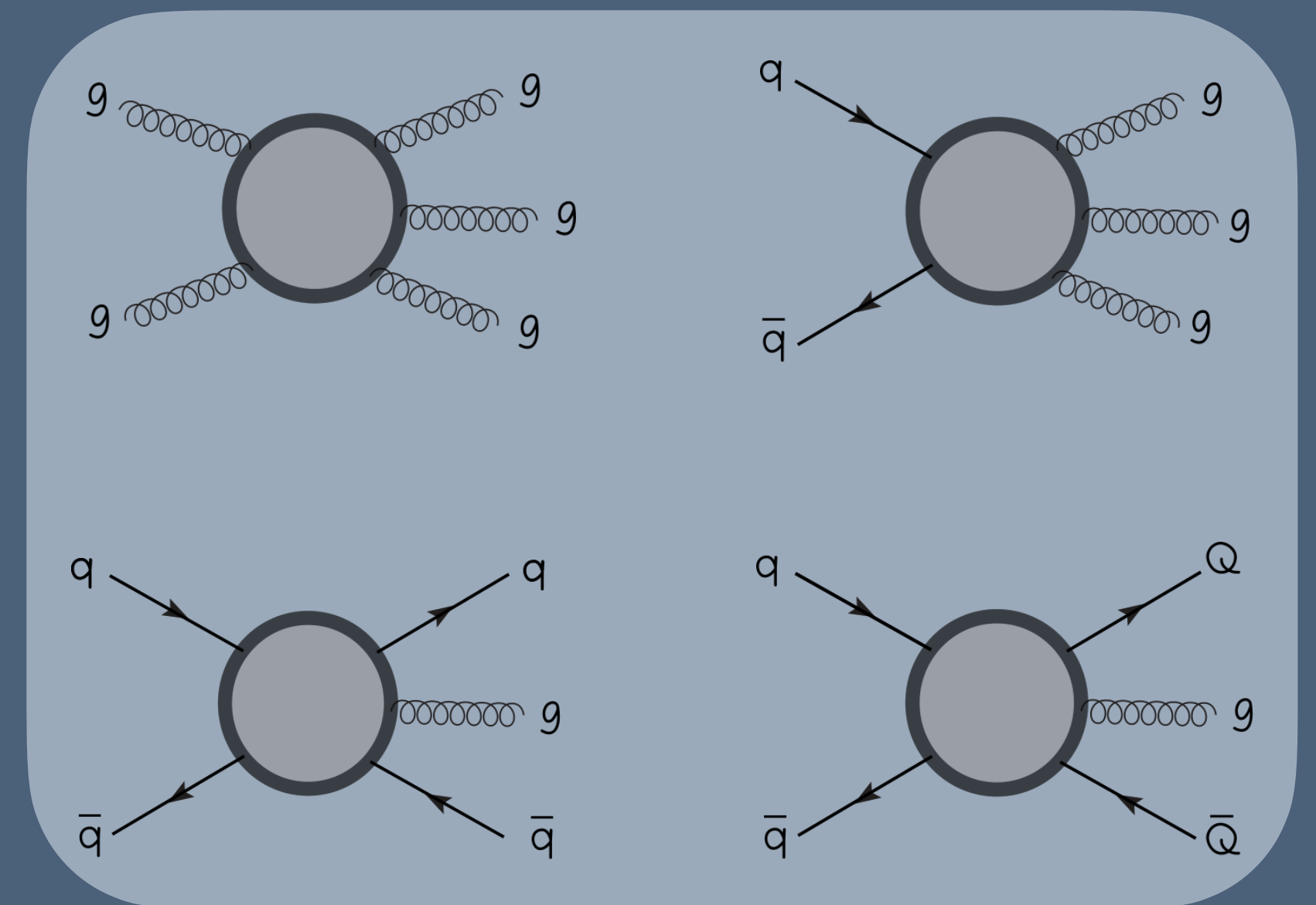


Contributors: [Abreu, Agarwal, Badger, Buccioni, Chawhdry, Chicherin, Czakon, Cordero Febres, De Laurentis, Gehrmann, Brønnum-Hansen, Hartanto, Henn, Ita, Klinkert, Kryś, Marcoli, Mitov, Moodie, Page, Pascual, Peraro, Poncelet, Sotnikov, Tancredi, Manteuffel von, Zoia, ...]

Leading color by Abreu et al in [2102.13609]



This talk



Full color: A. Agarwal, F. Buccioni, FD, G. Gambuti, A. von Manteuffel and L. Tancredi [2311.09870] + G. De Laurentis, H. Ita, M. Klinkert, V. Sotnikov in [2311.10086] & [2311.18752]

How

Old but gold

$$\mathcal{A} = \sum \text{Feynman diagrams}$$

> 40k

Helicity projection

Tancredi, Peraro:  
1906.03298 & 2012.00820

$$\mathcal{A} = \sum H_h T_h$$

5 gluons:  $\mathcal{A} = A^{\mu_1 \dots \mu_5} \epsilon_{\mu_1}^{h_1} \dots \epsilon_{\mu_5}^{h_5} = \sum_j H_j T_j^{\mu_1 \dots \mu_5} \epsilon_{\mu_1}^{h_1} \dots \epsilon_{\mu_5}^{h_5}$

$$T^{\mu_1 \dots \mu_5} \supset \begin{cases} p_i^{\mu_1} p_j^{\mu_2} \dots p_k^{\mu_5} & i, j, k = 1, \dots, 5 \\ g^{\mu_1 \mu_2} p_i^{\mu_3} p_j^{\mu_4} p_k^{\mu_5} & i, j, k = 1, \dots, 5 \\ g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} p_k^{\mu_5} & k = 1, \dots, 5 \end{cases}$$

1724 structures in d dimensions!

$$\begin{aligned} \epsilon_i \cdot p_i &= 0 \\ \epsilon_i \cdot p_{i+1} &= 0 \end{aligned}$$

142 structures

In the end we care about 4-dimensional helicity amplitudes (tHV scheme): these structures are not all independent in 4 dimensions

$$g^{\mu\nu} = g_{d=4}^{\mu\nu} + g_{d=-2\epsilon}^{\mu\nu}$$

Left with  $2^5 = 32$  combinations = number of helicity configurations!

$$\propto p_i^{\mu_1} p_j^{\mu_2} \dots p_k^{\mu_5}$$

Does not contribute to 4-dimensional helicity amplitudes, can be neglected

4 quarks (1 gluon): even worse! Dirac algebra doesn't close in d dimensions

$$T_1 = \bar{u}(p_2) \gamma_{\mu_1} u(p_1) \bar{u}(p_4) \gamma^{\mu_1} u(p_3)$$

$$T_2 = \bar{u}(p_2) \not{p}_3 u(p_1) \bar{u}(p_4) \not{p}_1 u(p_3)$$

$$T_3 = \bar{u}(p_2) \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} u(p_1) \bar{u}(p_4) \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} u(p_3)$$

$$T_4 = \bar{u}(p_2) \gamma_{\mu_1} \not{p}_3 \gamma_{\mu_2} u(p_1) \bar{u}(p_4) \gamma^{\mu_1} \not{p}_1 \gamma^{\mu_2} u(p_3)$$

∴ grows w/ # loops!

# of independent structures =  
# helicity configurations!

Key idea: rotate away the  $-2\epsilon$  subspace

[Peraro, Tancredi 1906.03298, 2012.00820]

Only  $T_1, T_2$  are left in  $d=4$ !

$$T_1 = \bar{u}(p_2) \gamma_{\mu_1} u(p_1) \bar{u}(p_4) \gamma^{\mu_1} u(p_3)$$

$$T_2 = \bar{u}(p_2) \not{p}_3 u(p_1) \bar{u}(p_4) \not{p}_1 u(p_3)$$

$$\cancel{T_3} = \cancel{T_3} + (d-4) T_3^{-2\epsilon}$$

$$\cancel{T_4} = \cancel{T_4} + (d-4) T_4^{-2\epsilon}$$

Old but gold

$$\mathcal{A} = \sum \text{Feynman diagrams} > 40k$$

Helicity projection

Tancredi, Peraro:  
1906.03298 & 2012.00820

$$\mathcal{A} = \sum H_h T_h$$

$$H = \sum_c H_c C_c$$

Color decomposition



Old but gold

$$\mathcal{A} = \sum \text{Feynman diagrams} > 40k$$

Helicity projection

Tancredi, Peraro:  
1906.03298 & 2012.00820

$$\mathcal{A} = \sum H_h T_h$$

Color decomposition

$$H = \sum_c H_c C_c$$

Polynomial in  $N_c, n_f$

$C_c$	$ggggg$	$q\bar{q}ggg$	$q\bar{q}Q\bar{Q}g$
Tree level	$\text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}) - \text{Tr}(T^{a_5}T^{a_4}T^{a_3}T^{a_2}T^{a_1})$ + permutations	$(T^{a_1}T^{a_2}T^{a_3})_{ji}$ + permutations	$T_{ij}^a \delta_{kl}$ $T_{ik}^a \delta_{jl}$
Beyond tree	$\text{Tr}(T^{a_1}T^{a_2}) \times (\text{Tr}(T^{a_3}T^{a_4}T^{a_5}) - \text{Tr}(T^{a_5}T^{a_4}T^{a_3}))$ + permutations	$\text{Tr}(T^{a_1}T^{a_2})T_{ij}^{a_3}$ $(\text{Tr}(T^{a_3}T^{a_4}T^{a_5}) - \text{Tr}(T^{a_5}T^{a_4}T^{a_3}))\delta_{ij}$ $(\text{Tr}(T^{a_3}T^{a_4}T^{a_5}) + \text{Tr}(T^{a_5}T^{a_4}T^{a_3}))\delta_{ij}$	Same as tree

Optimal choice of color basis depends on problem at hand (see later in MRK)

$$H^{c_1 \dots c_5} = \sum_i^n H_i |c_1 \dots c_5 \rangle_i \equiv \sum_i^n H_i |C_i \rangle$$

↓
↓  
 Partial amplitudes                      Color basis

$$H_i = b_i^{(2,0)} N_c^2 + b_i^{(0,0)} 1 + b_i^{(-2,0)} N_c^{-2} + b_i^{(1,1)} N_c^2 \frac{n_f}{N_c} + \dots$$

Leading color

Subleading color

Leading vs subleading?!

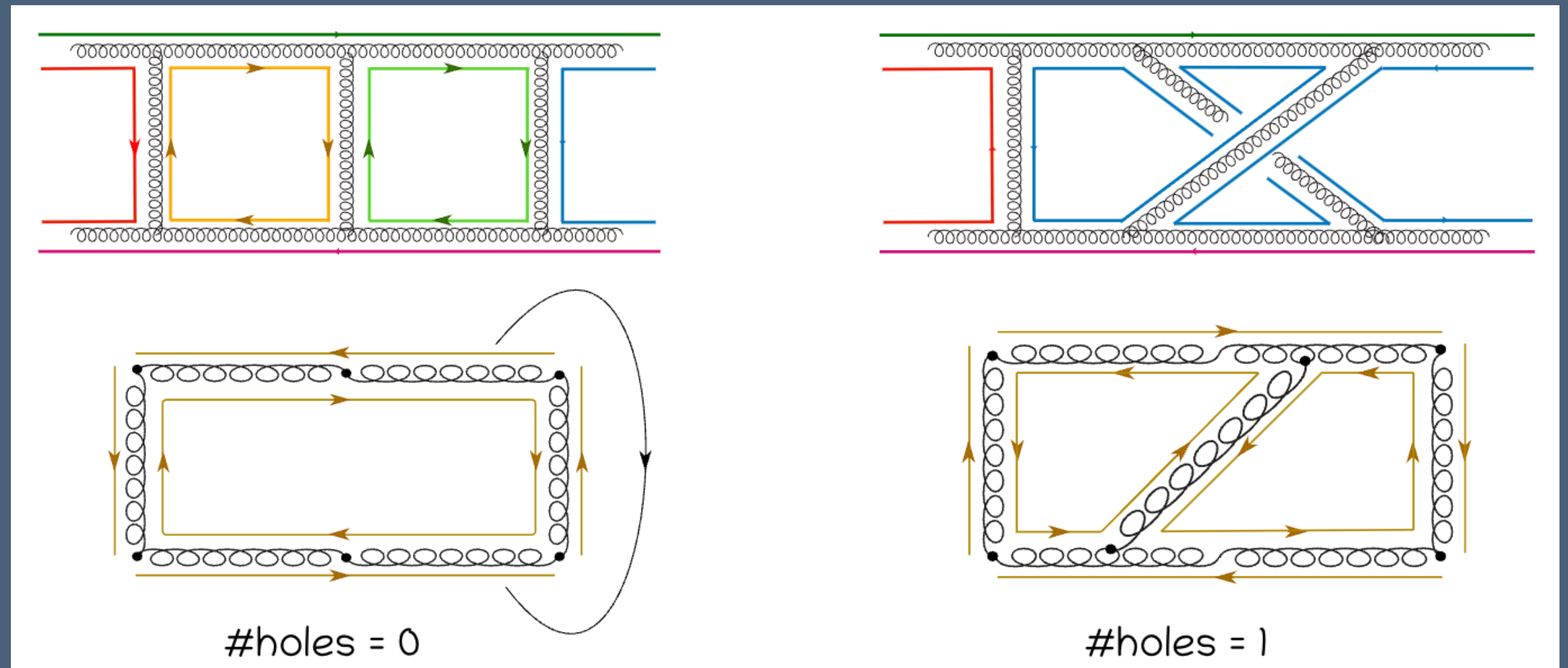
# Leading color and planar diagrams

Consider U(N) (or SU(N)) gauge theory in the 't Hooft limit:  $N \rightarrow \infty$  at  $\lambda = g^2 N$  fixed [t Hooft '73]

$$\text{Diagram} \sim g^{V_3+2V_4} N^I \sim \lambda^{\frac{1}{2}V_3+V_4} N^\chi$$

$$\chi = 2 - 2H, H = \# \text{ of "holes"}$$

$$\text{Sphere: } \chi = 2 \quad \text{Torus: } \chi = 0$$



Old but gold

$$\mathcal{A} = \sum \text{Feynman diagrams} > 40k$$

Helicity projection  
 Tancredi, Peraro:  
 1906.03298 & 2012.00820

$$\mathcal{A} = \sum H_h T_h$$

$$H = \sum_c H_c C_c$$

Color decomposition

Integration by parts identities

$$H = \sum_{m,c} R^{mc} M_m C_c$$

Master integrals & Pentagon functions

Chicherin, Sotnikov:  
 2009.07803

$$\frac{s_{23}s_{51}s_{45} + (d-4)s_{12}s_{13}s_{25}}{s_{12}s_{23}s_{34}}$$

# Integration by parts: infinitesimal Feynman integral shift symmetry

Preserved by dimreg!

$$I(k) = \int \frac{d^d k}{(2\pi)^d} f(k + \alpha q) = I(k) + \alpha \int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k^\mu} (q^\mu f(k)) + \mathcal{O}(\alpha^2)$$



$$\int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k^\mu} [q^\mu f(k)] = 0$$

$\mathcal{O}(10^6)$  Feynman integrals



$\mathcal{O}(10^3)$  master integrals

(Only ~500 independent functions)

Finite fields reconstruction  
Syzygy techniques

FinRed  
(von Manteuffel)

Price to pay: deal with rational functions resulting from IBP reduction procedure...

Old but gold

$$s_{23}s_{51}s_{45} + (d - 4)s_{12}s_{13}s_{25}$$

$$s_{12}s_{23}s_{34}$$

$\mathcal{A} = \sum$  Feynman diagram  $> 40k$

$$H = \sum H_c C_c$$

Instagram

Integration by parts identities

```
id INT(TB,7,247,7,4,1,1,-4,1,1,1,0,0) = ((-19440*s12+18468*q8*s12^2+9720*s23+45036*q1*s23^2+47628*q9*s23^2+16200*q1*q5*s23^3+47628*q1*q9*s23^3+29160*s34-16524*q8*s12*s34-36936*q8^2*s12^2*s34-9720*q1*s23*s34-9720*q1*s34^2-12636*q8*s34^2+91368*q8^2*s12*s34^2+18468*q8^3*s12^2*s34^2+10692*q1*q8*s34^3-71928*q8^2*s34^3-55404*q8^3*s12*s34^3+17496*q1*q8^2*s34^4+55404*q8^3*s34^4-18468*q1*q8^3*s34^5+184680*s45-104976*q8*s12*s45+30132*q8^2*s12^2*s45+233928*q1*s23*s45+208656*q9*s23*s45+64152*q1*q5*s23^2*s45+156168*q1*q9*s23^2*s45+135756*q5*q9*s23^2*s45-39528*q9^2*s23^2*s45-174960*q1*s34*s45-13608*q8*s34*s45+69984*q8^2*s12*s34*s45-30132*q8^3*s12^2*s34*s45-39528*q9^2*s23*s34*s45+108864*q1*q8*s34^2*s45-191484*q8^2*s34^2*s45+34992*q8^3*s12*s34^2*s45+91368*q1*q8^2*s34^3*s45+20412*q8^3*s34^3*s45-25272*q1*q8^3*s34^4*s45+148716*q1*s45^2+411480*q8*s45^2+60444*q9*s45^2-99792*q8^2*s12*s45^2+46332*q8^3*s12^2*s45^2-9504*q1*q5*s23*s45^2+99684*q1*q9*s23*s45^2+64440*q5*q9*s23*s45^2-97020*q9^2*s23*s45^2-226800*q1*q8*s34*s45^2-10368*q8^2*s34*s45^2-113868*q9^2*s34*s45^2-48600*q8^3*s12*s34*s45^2-34992*q5*q9^2*s23*s34*s45^2-14580*q9^3*s23*s34*s45^2+90720*q1*q8^2*s34^2*s45^2+13608*q8^3*s34^2*s45^2-14580*q9^3*s34^2*s45^2-11340*q1*q8^3*s34^3*s45^2-10368*q1*q5*s45^3+414720*q1*q7*s45^3-414720*q1*q8*s45^3-3240*q8^2*s45^3+84456*q5*q9*s45^3+37080*q9^2*s45^3-156816*q17^2*q7*s12*s45^3+156816*q17^2*q8*s12*s45^3+156816*q17^2*q8^2*s12*s45^3+62208*q8^3*s12*s45^3+67068*q17^3*q7*s12^2*s45^3-67068*q17^3*q8*s12^2*s45^3-67068*q17^3*q8^2*s12^2*s45^3-67068*q17^3*q8^3*s12^2*s45^3-13608*q8^3*s34*s45^3-37080*q5*q9^2*s34*s45^3+9720*q9^3*s34*s45^3-48600*q5*q9^3*s34^2*s45^3-16848*q17^2*q7*s45^4+60480*q1*q5*q7*s45^4+16848*q17^2*q8*s45^4+16848*q17^3*q8^2*s45^4+13608*q8^3*s45^4+92664*q17^3*q7*s12*s45^4-8424*q17^2*q7^2*s12*s45^4-92664*q17^3*q8*s12*s45^4-84240*q17^2*q8^2*s12*s45^4-75816*q17^3*q8^3*s12*s45^4-75816*q17^4*q7*s12^2*s45^4+25272*q17^3*q7^2*s12^2*s45^4+75816*q17^4*q8*s12^2*s45^4+50544*q17^3*q8^2*s12^2*s45^4+25272*q17^2*q8^3*s12^2*s45^4+13608*q17^3*q7*s12^2*s45^4-13608*q17^2*q8^2*s45^4-13608*q17^3*q8^3*s45^4-40824*q17^4*q7*s12*s45^4+13608*q17^3*q7^2*s12*s45^4+40824*q17^4*q8*s12*s45^4+27216*q17^3*q8^2*s12*s45^4+13608*q17^2*q8^3*s12*s45^4+27216*q17^3*q7*s12^2*s45^4-13608*q17^4*q7^2*s12^2*s45^4-4536*q17^3*q7^3*s12^2*s45^4-27216*q17^5*q8*s12^2*s45^4-13608*q17^4*q8^2*s12^2*s45^4-4536*q17^3*q8^3*s12^2*s45^4+116136*s51-4032*q4*s12*s51-22212*q8*s12*s51+9072*q1*s23*s51+16128*q4*s23*s51+377568*q9*s23*s51-65448*q1*q9*s23^2*s51-9072*q9^2*s23^2*s51+108864*q1^2*q19*s23^3*s51+108864*q1^2*q9*s23^3*s51-4536*q1*q9^2*s23^3*s51-12960*q1*s34*s51-54648*q8*s34*s51-193536*q9*s34*s51+8064*q4*q8*s12*s34*s51-33192*q8^2*s12*s34*s51-4536*q9^2*s23*s34*s51-6156*q1*q8*s34^2*s51+193536*q11*q8*s34^2*s51-16128*q4*q8*s34^2*s51+39600*q8^2*s34^2*s51+193536*q11*q9*s34^2*s51-4032*q4*q8^2*s12*s34^2*s51+55404*q8^3*s12*s34^2*s51-26568*q1*q8^2*s34^3*s51+16128*q1*q8^3*s34^3*s51-110808*q8^3*s34^3*s51+55404*q1*q8^3*s34^4*s51+100440*q1*s45*s51+360864*q8*s45*s51+71172*q9*s45*s51-23976*q8^2*s12*s45*s51-154872*q1*q9*s23*s45*s51-134784*q9^2*s23*s45*s51-162648*q1*q9^2*s23^2*s45*s51-14580*q9^3*s23^2*s45*s51-229392*q1*q8*s34*s45*s51-193536*q11*q8*s34*s45*s51+182880*q8^2*s34*s45*s51-193536*q11*q9*s34*s45*s51-205200*q9^2*s34*s45*s51-34992*q8^3*s12*s34*s45*s51-29160*q9^3*s23*s34*s45*s51+120960*q11^2*q8*s34^2*s45*s51-152280*q1*q8^2*s34^2*s45*s51+24192*q11*q8^2*s34^2*s45*s51-40824*q8^3*s34^2*s45*s51+120960*q11^2*q9*s34^2*s45*s51+96768*q11*q9^2*s34^2*s45*s51-145800*q9^3*s34^2*s45*s51-24192*q11^3*q8*s34^3*s45*s51-12096*q11^2*q8^2*s34^3*s45*s51+75816*q1*q8^3*s34^3*s45*s51-24192*q11^3*q9*s34^3*s45*s51-12096*q11^2*q9^2*s34^3*s45*s51+6912*q1*q7*s45^2*s51+436176*q17^2*q7*s45^2*s51+223128*q1*q8*s45^2*s51-436176*q17^2*q8*s45^2*s51+56016*q8^2*s45^2*s51+113868*q9^2*s45^2*s51-34272*q17^2*q7*s12*s45^2*s51+34272*q17^2*q8*s12*s45^2*s51+34272*q17^2*q8^2*s12*s45^2*s51+48600*q8^3*s12*s45^2*s51+145800*q1*q9^2*s23*s45^2*s51-192024*q1*q8^2*s34*s45^2*s51-27216*q8^3*s34*s45^2*s51+29160*q9^3*s34*s45^2*s51+34020*q1*q8^3*s34^2*s45^2*s51-14256*q1*q17^2*q7*s45^3*s51+66816*q17^2*q7*s45^3*s51-13824*q17^2*q7^2*s45^3*s51+14256*q1*q17^2*q8*s45^3*s51-66816*q17^2*q8*s45^3*s51+10584*q1*q8^2*s45^3*s51-52992*q17^2*q8^2*s45^3*s51+27216*q8^3*s45^3*s51-48600*q9^3*s45^3*s51-14400*q17^3*q7*s12*s45^3*s51+38304*q17^2*q7^2*s12*s45^3*s51+14400*q17^3*q8*s12*s45^3*s51-23904*q17^2*q8^2*s12*s45^3*s51-62208*q17^3*q8^3*s12*s45^3*s51+10584*q1*q17^2*q7*s45^4*s51-21168*q17^3*q7*s45^4*s51+24192*q17^2*q7^2*s45^4*s51-10584*q1*q17^2*q8*s45^4*s51+21168*q17^3*q8^2*s45^4*s51-10584*q1*q17^3*q8^3*s45^4*s51-3024*q17^2*q8^2*s45^4*s51-27216*q17^3*q8^3*s45^4*s51-13608*q17^3*q7^2*s12*s45^4*s51+13608*q17^2*q7^3*s12*s45^4*s51+13608*q17^3*q8^2*s12*s45^4*s51+13608*q17^2*q8^3*s12*s45^4*s51+12096*q4*s51^2+84888*q8*s51^2+12528*q9*s51^2-9072*q1*q9*s23*s51^2+80640*q4*q9*s23*s51^2-126468*q9^2*s23*s51^2-90720*q1^2*q19*s23^2*s51^2-90720*q1^2*q9^2*s23^2*s51^2-70308*q1^2*q9^2*s23^2*s45*s51^2-14580*q9^3*s23^2*s51^2-90720*q1^3*q19*s23^3*s51^2-18144*q1^2*q19^2*s23^3*s51^2-90720*q1^3*q9^2*s23^3*s51^2-72576*q1^2*q9^2*s23^3*s51^2-48600*q1*q9^3*s23^3*s51^2-7128*q1*q8*s34*s51^2-296352*q11*q8*s34*s51^2-12096*q4*q8*s34*s51^2+32328*q8^2*s34*s51^2-296352*q11*q9*s34*s51^2-36288*q4*q9*s34*s51^2-47772*q9^2*s34*s51^2-14580*q9^3*s23*s34*s51^2+36288*q11*q4*q8*s34^2*s51^2+648*q1*q8^2*s34^2*s51^2-16128*q4*q8^2*s34^2*s51^2+55404*q8^3*s34^2*s51^2+36288*q11*q4*q9*s34^2*s51^2-48600*q9^3*s34^2*s51^2-55404*q1*q8^3*s34^3*s51^2+2592*q1*q7*s45*s51^2+151740*q17^2*q7*s45*s51^2+119232*q1*q8*s45*s51^2+193536*q11*q8*s45*s51^2-151740*q17^2*q8*s45*s51^2+8604*q8^2*s45*s51^2+193536*q11*q9*s45*s51^2+205200*q9^2*s45*s51^2+29160*q9^3*s23*s45*s51^2-241920*q11^2*q8*s34*s45*s51^2+30456*q1*q8^2*s34*s45*s51^2-48384*q11*q8^2*s34*s45*s51^2+20412*q8^3*s34*s45*s51^2-241920*q11^2*q9*s34*s45*s51^2-193536*q11*q9^2*s34*s45*s51^2+29160*q9^3*s34*s45*s51^2+72576*q11^3*q8*s34^2*s45*s51^2+36288*q11^2*q8^2*s34^2*s45*s51^2-75816*q1*q8^3*s34^2*s45*s51^2+72576*q11^3*q9*s34^2*s45*s51^2+36288*q11^2*q9^2*s34^2*s45*s51^2-4968*q1*q17^2*q7*s45^2*s51^2+37800*q17^2*q7*s45^2*s51^2-5400*q1*q7^2*s45^2*s51^2+7848*q17^2*q7^2*s45^2*s51^2+4968*q1*q17^2*q8*s45^2*s51^2-37800*q17^2*q8*s45^2*s51^2+111888*q1*q8^2*s45^2*s51^2-45648*q17^2*q8^2*s45^2*s51^2+13608*q8^3*s45^2*s51^2-14580*q9^3*s45^2*s51^2-34020*q1*q8^3*s34*s45^2*s51^2+10584*q1*q17^2*q7*s45^3*s51^2-21168*q17^3*q7*s45^3*s51^2+10584*q1*q17^2*q7^2*s45^3*s51^2+10584*q17^2*q7^2
```

$$H = \sum_{m,c} R^{mc} M_m C_c$$

Reality

# Master integrals and differential equations

Canonical form,  $I_j$  vector of UT master integrals

$$dI_i(\mathbf{s}) = \epsilon dA_{ij}(\mathbf{s}) I_j(\mathbf{s}) \quad dA_{ij}(\mathbf{s}) = \sum_{n=1} a_{ij}^n d \log(W_n)$$

“Letters” of alphabet: algebraic functions of invariants

Need **boundary conditions**

Solution can be obtained systematically: iterated integrals

$$I(\vec{s}) = P \exp \left[ \epsilon \int^{\vec{s}} A(\vec{x}) d\vec{x} \right] I(\vec{s}_0)$$

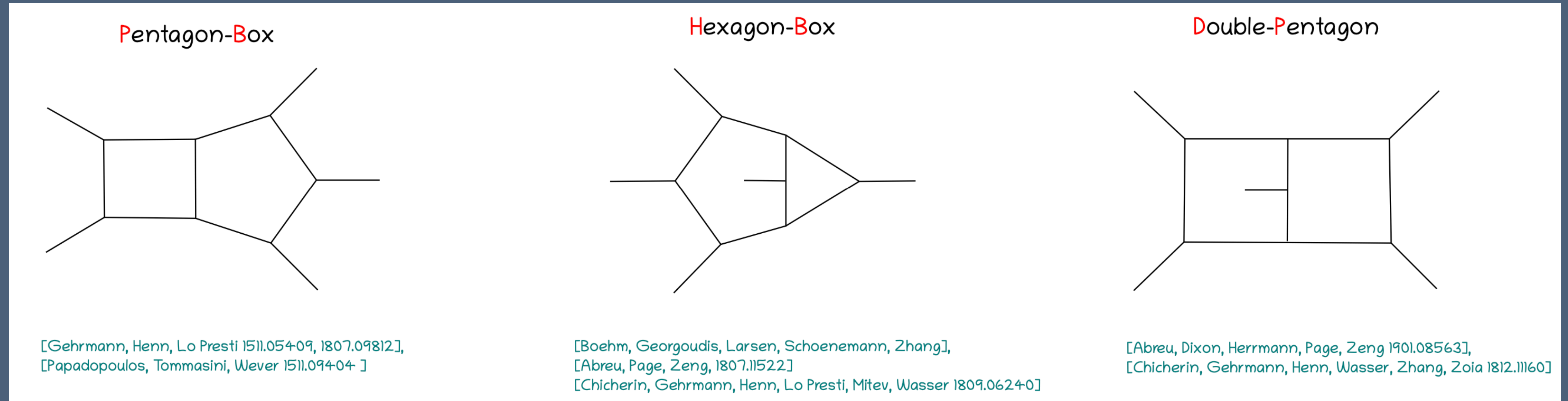
$$I^{(\omega)}(\vec{s}) = \int_{\gamma} d \log W_{i_1} \dots d \log W_{i_n}$$

$\omega$  integrations

Typical difficulties:

- Find appropriate canonical basis
- Solution might be dependent on kinematic region
- Boundary conditions can be tricky to compute
- **Non planar integrals** much more complicated (cut structure, alphabet, larger # of independent functions)

# Master integrals for $2 \rightarrow 3$ scattering and Pentagon Functions



Expressed as Chen iterated integrals, full set available. Results in the whole physical region

[Chicherin, Sotnikov 2009.07803]

$$f^{(\omega)}(\mathbf{x}) = \int_{\gamma} d \log W_{i_1} \dots d \log W_{i_n}$$

Evaluation time:  $\sim 1s$



Old but gold

$$\mathcal{A} = \sum \text{Feynman diagrams} > 40k$$

Helicity projection

Tancredi, Peraro:  
1906.03298 & 2012.00820

$$\mathcal{A} = \sum H_h T_h$$

$$H = \sum_c H_c C_c$$

Color decomposition

Integration by parts identities

$$H = \sum_{m,c} R^{mc} M_m C_c$$

This is where the magic happens!

Final result

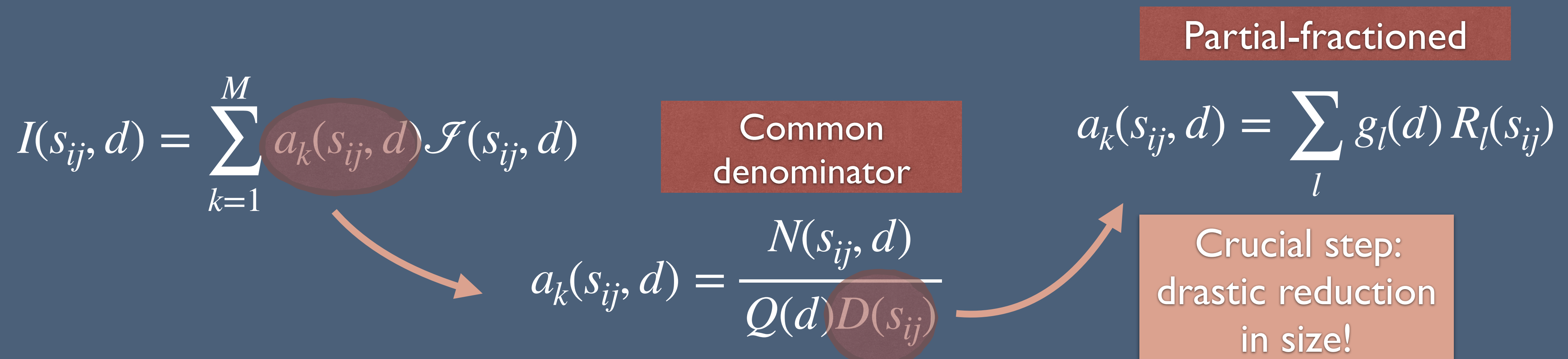
# It's a kind of magic



We employ MultivariateApart for multivariate partial fractioning:

[Heller, von Manteuffel 2101.08283]

- Avoids spurious denominators
- Produces unique results when applied to terms of a sum separately



# It's a kind of magic



We employ `MultivariateApart` for multivariate partial fractioning:

[Heller, von Manteuffel 2101.08283]

- Avoids spurious denominators
- Produces unique results when applied to terms of a sum separately

Drastic simplifications occur:

	CD		PF
<b>PB:</b> <code>INT[TA,8,255,8,5,{1,1,1,1,1,1,1,1,-5,0,0}]</code>	162 MB	→	3.9 MB
<b>HB:</b> <code>INT[TB,8,255,8,5,{1,1,1,1,1,1,1,1,-4,0,-1}]</code>	513 MB	→	9.9 MB
<b>DP:</b> <code>INT[TB,8,510,8,5,{0,1,1,1,1,1,1,1,-5,0}]</code>	2.9 GB	→	24 MB
			~ $\mathcal{O}(100)$ !!

Final results available at <https://zenodo.org/records/10227683>

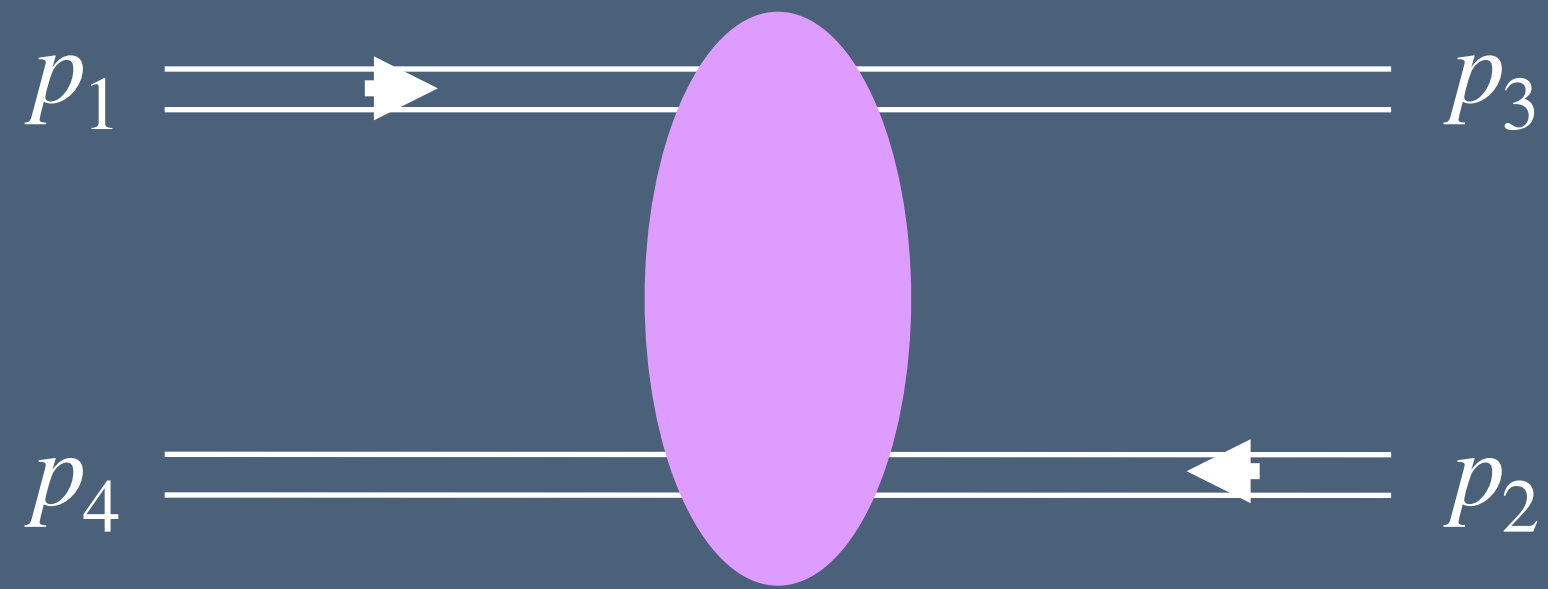
Go have a look and play with it :)

# High energy limit

In collaboration with



# Warm-up: $2 \rightarrow 2$ amplitudes in Regge kinematics



$$s \sim |u| \gg -t$$

+ Large rapidity gap

$$x = \frac{-t}{s}$$

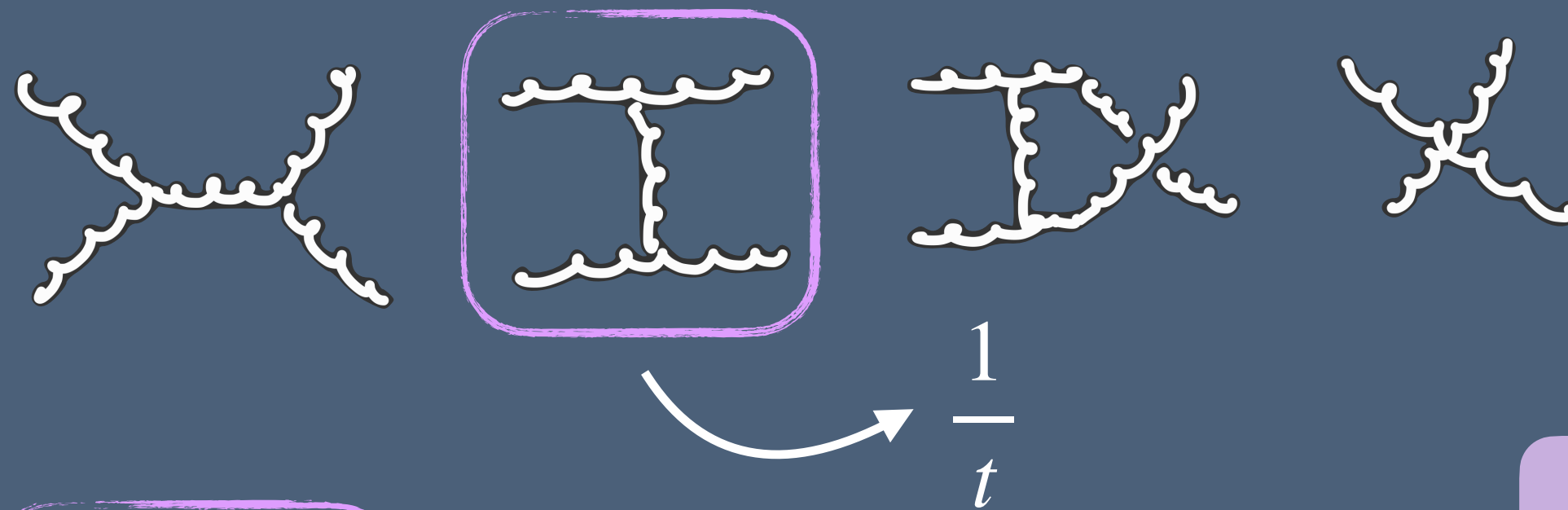
Light-cone components

$$p_1^\mu \sim p_1^+ \quad p_2^\mu \sim p_2^-$$

$$p_3^+ \gg |p_3^\perp| \quad p_4^- \gg |p_4^\perp|$$

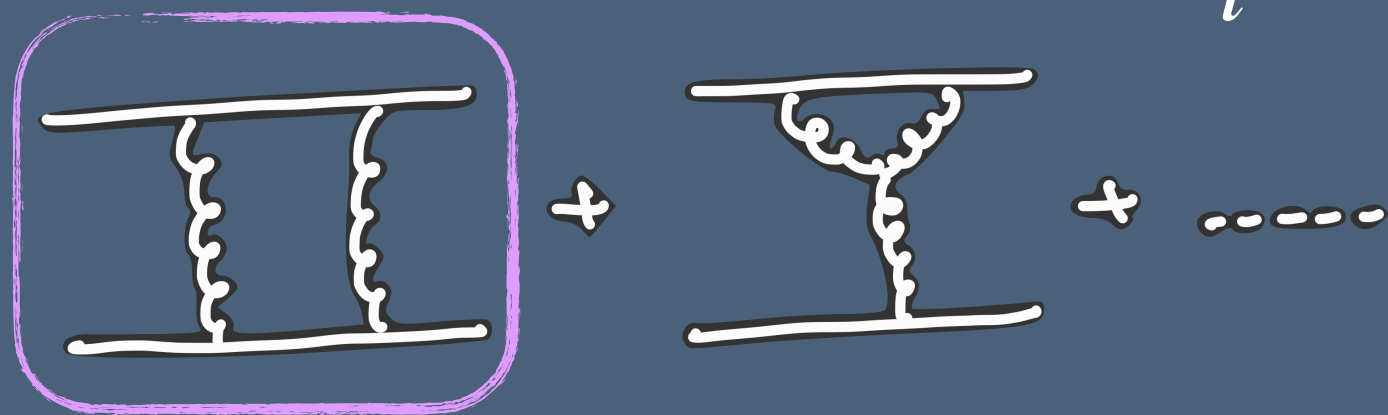
$$s_{12} = p_1^+ p_2^- \quad t \sim -k_\perp^2$$

Tree level:



$$\frac{1}{t}$$

One loop:



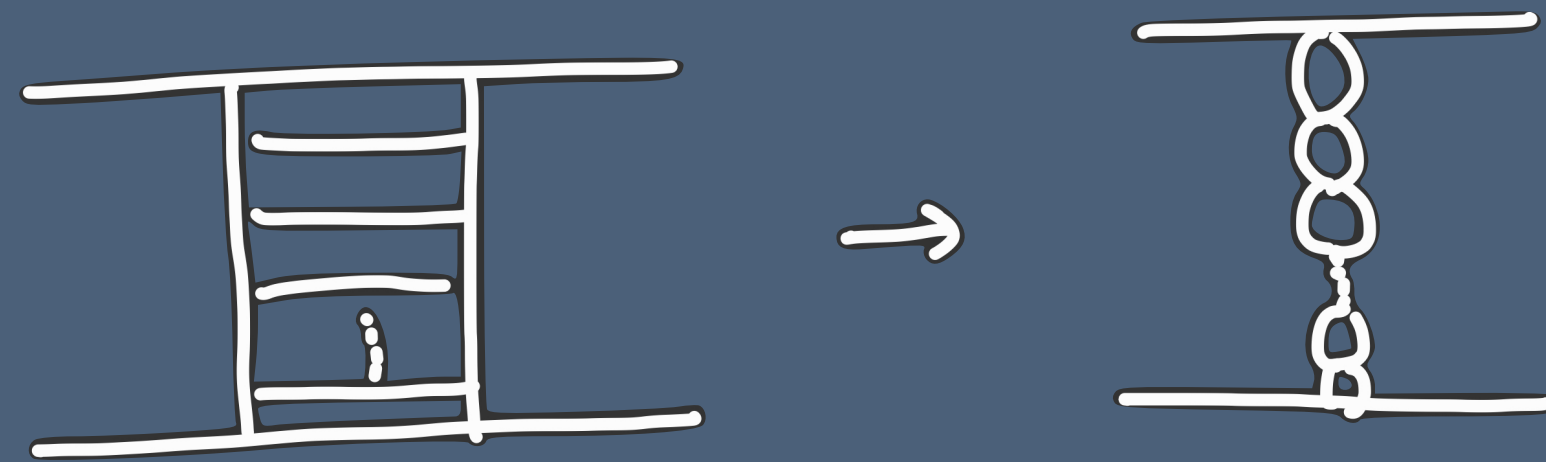
Small  $x$

$$\times \frac{\log x}{x}$$

$$\text{Dominant contribution (LL): } \mathcal{A} \sim \tau_g(t) \frac{\log x}{x}$$

# Warm-up: $2 \rightarrow 2$ amplitudes in Regge kinematics

Structure repeated to all orders: **generalized** ladder topologies

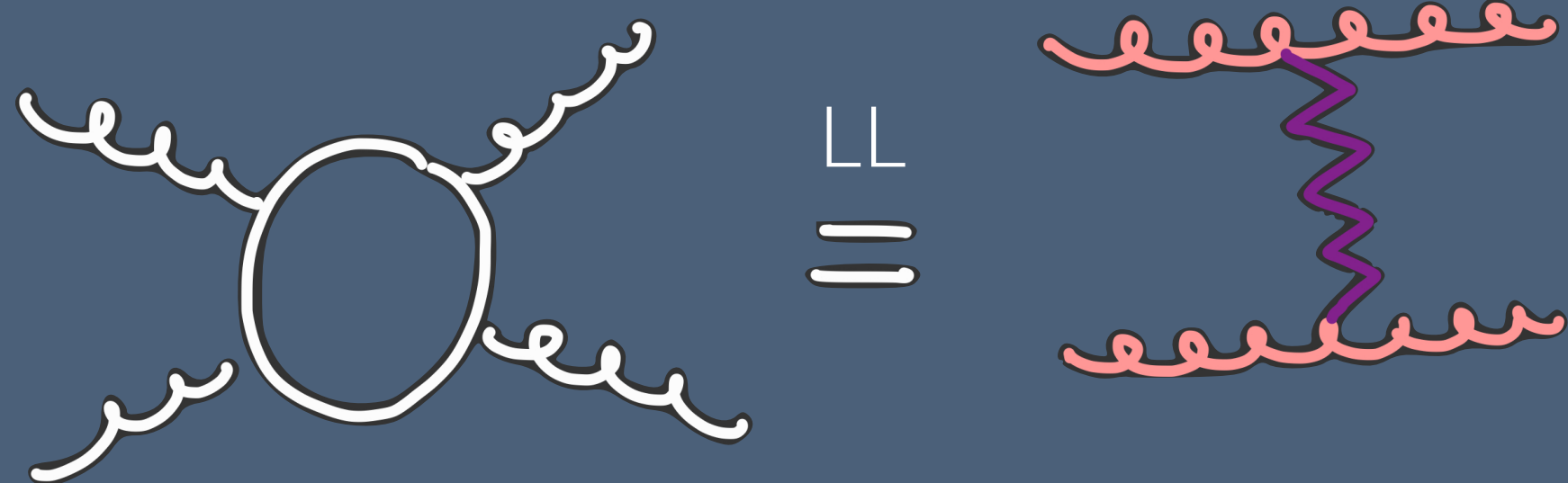


NB: this picture is schematic, in QCD things are more complicated

Factorization at LL

$$\mathcal{A}(s, t) \simeq \mathcal{A}^{(0)}(s, t) \left( \frac{s}{-t} \right)^{C_A \alpha_s \tau_g(t)}$$

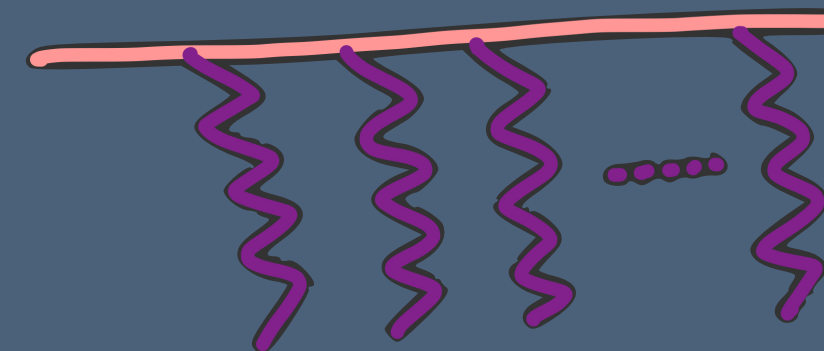
Gluon "reggeization" at LL



$\tau_g$  gluon Regge trajectory

Universal, does not depend on partonic nature of projectiles

Beyond LL: effective theory of "reggeons"



Regge pole is in the “odd amplitude”: define signature eigenstates

$$\mathcal{A}^\pm = \frac{1}{2} \left( \mathcal{A}(s, t) \pm \mathcal{A}(-s - t, t) \right)$$

$\mathcal{A}^+$ : “even”

$\mathcal{A}^-$ : “odd”



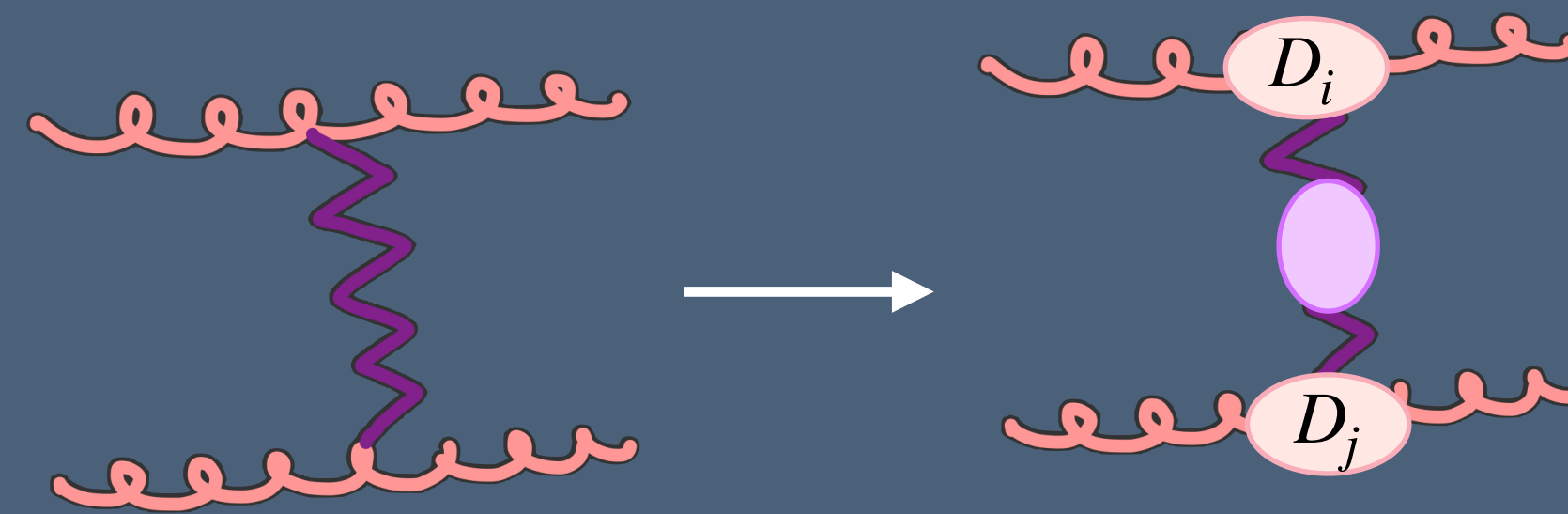
Starts with cut @1-loop

Contains Regge pole

NLL factorization [Fadin, Lipatov hep-ph/9802290]

$$\text{LL} \sim \left( \frac{\alpha_s}{2\pi} \right)^n \log^n x$$

$$\text{NLL} \sim \left( \frac{\alpha_s}{2\pi} \right)^n \log^{n-1} x$$



$$\mathcal{A}(s, t) \simeq \mathcal{A}^{(0)}(s, t) D_i(t) D_j(t) \left( \frac{s}{-t} \right)^{C_A \alpha_s \tau_g(t, \alpha_s)}$$

Impact factors

Corrections to Regge trajectory

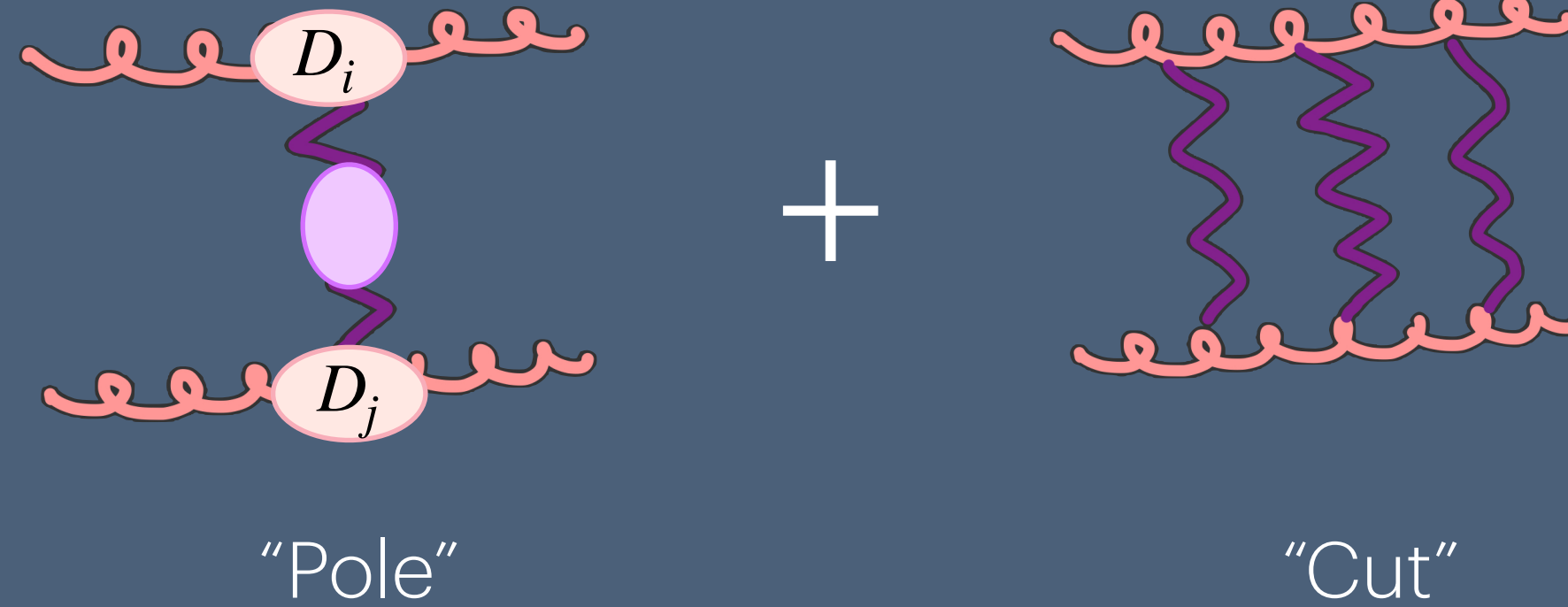
Factorization still holds (in  $\mathcal{A}^-$ )



## NNLL factorization

$$\text{NNLL} \sim \left( \frac{\alpha_s}{2\pi} \right)^n \log^{n-2} x$$

Regge cuts responsible for violation of factorization at NNLL



Factorization is restored once “cut contamination” is removed

How?

Wilson line approach

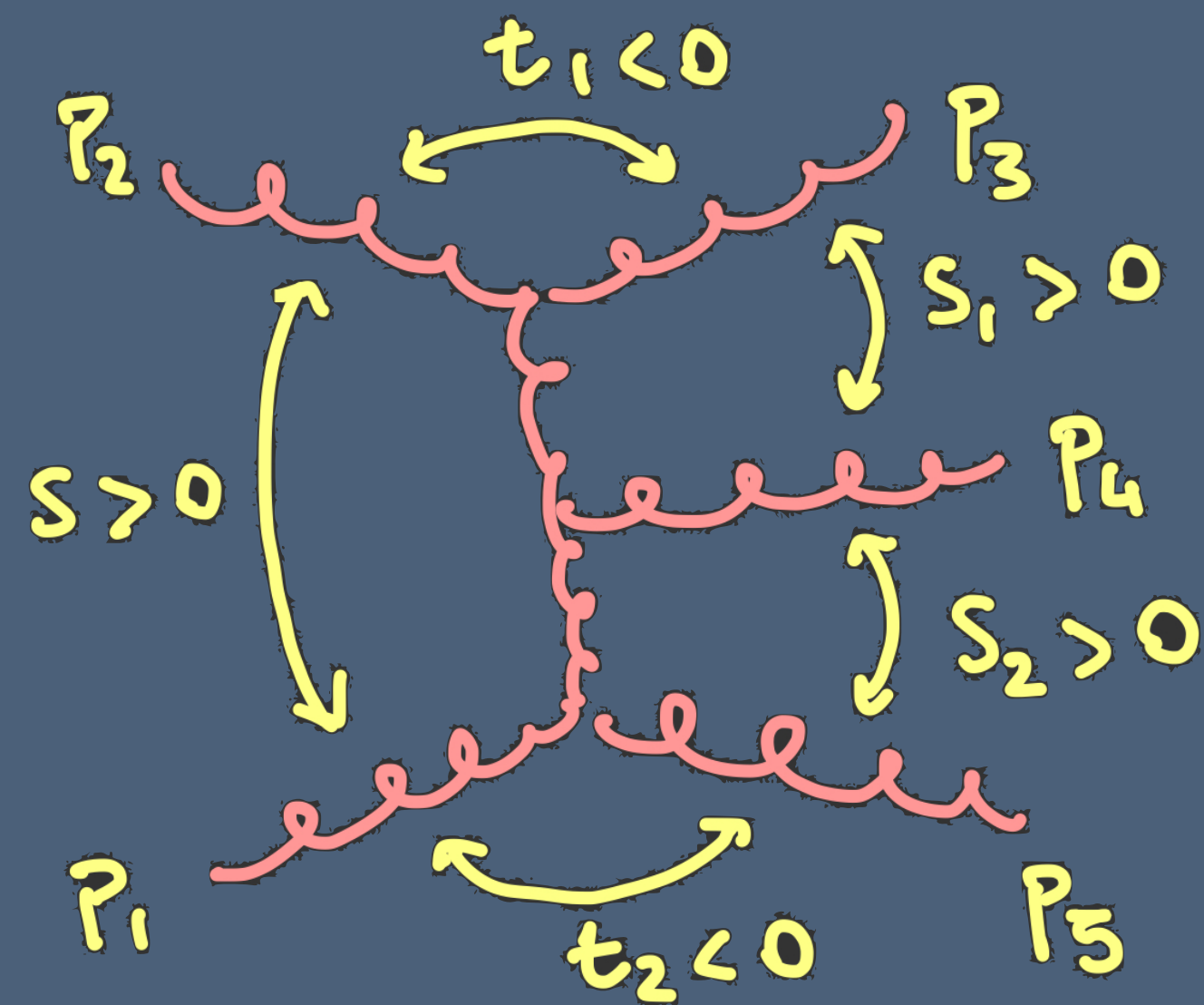
[Balitsky/JIMWLK + Caron-Huot 1309.6521, Caron-Huot, Gardi, Vernazza 1701.05241]

$$U_r(z) \equiv \mathcal{P} \exp \left( ig \int_{-\infty}^{+\infty} dx^+ A_+^a(x^+, x^- = 0, z) T_r^a \right)$$

## 2 → 3 amplitudes: Multi-Regge kinematics

$$A^{(h_A)}(p_1) B^{(h_B)}(p_2) \rightarrow B'^{(h_{B'})}(p_3) g^{(h_g)}(p_4) A'^{(h_{A'})}(p_5)$$

Strong rapidity ordering, no ordering in transverse components, gluon centrally emitted



$$p_5^+ \gg p_4^+ \gg p_3^+, \quad p_3^- \gg p_4^- \gg p_5^-, \quad p_4^\pm \sim |p_{3,\perp}| \sim |p_{4,\perp}| \sim |p_{5,\perp}|$$

Longitudinal and transverse dynamics completely factorized

$$p_1^+, p_5^+, p_2^-, p_3^- \sim 1/x, \quad p_4^+, p_4^- \sim 1, \quad p_2^+, p_3^+, p_1^-, p_5^- \sim x$$

MRK limit:  $x \rightarrow 0$

$$s_{12} = \frac{s}{x^2}, \quad s_{23} = -\frac{s_1 s_2}{s} z \bar{z}, \quad s_{34} = \frac{s_1}{x}$$

$$s_{45} = \frac{s_2}{x}, \quad s_{51} = -\frac{s_1 s_2}{s} (1-z)(1-\bar{z})$$

[Caron-Huot, Chicherin, Henn, Zhang, Zoia 2003.03120]

# Signature and color decomposition

$$\mathcal{A}^{(\sigma_a, \sigma_b)}(p_1, p_2, p_3, p_4, p_5) \sim \mathcal{A}(p_1, p_2, p_3, p_4, p_5) + \sigma_a \mathcal{A}(p_5, p_2, p_3, p_4, p_1) \\ + \sigma_b \mathcal{A}(p_1, p_3, p_2, p_4, p_5) + \sigma_a \sigma_b \mathcal{A}(p_5, p_3, p_2, p_4, p_1)$$

$\mathcal{A}^{(-,-)}$  odd, odd

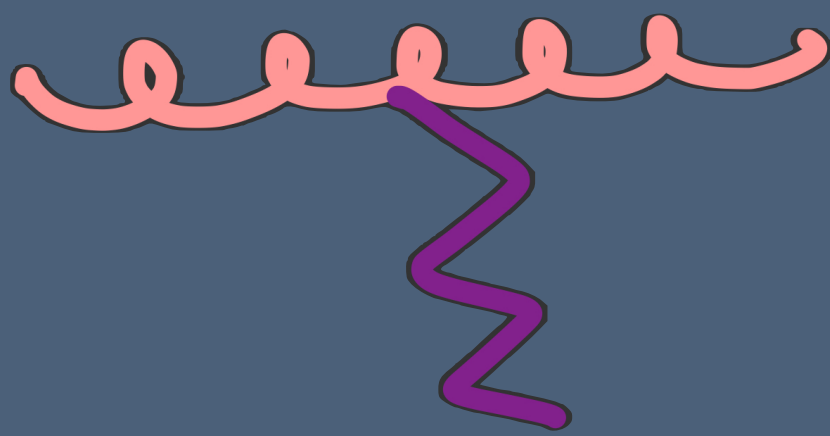
$\mathcal{A}^{(+,+)}$  even, even

$\mathcal{A}^{(+,-)}$  even, odd

$\mathcal{A}^{(-,+)}$  odd, even

Contains pole contributions,  
relevant part for factorization  
Beyond NLL, receives cut  
contamination

Color: t-channel in irreducible representation



$$8 \otimes 8 = 0 + 1 + 8_a + 8_s + 10 + \overline{10} + 27$$

symmetric

antisymmetric

Only  $(8_a, 8_a)$  amplitude  
receives pole  
contribution

# Color basis

Elements of color basis denoted by  $(r_1, r_2)$ , representations of the two t-channels

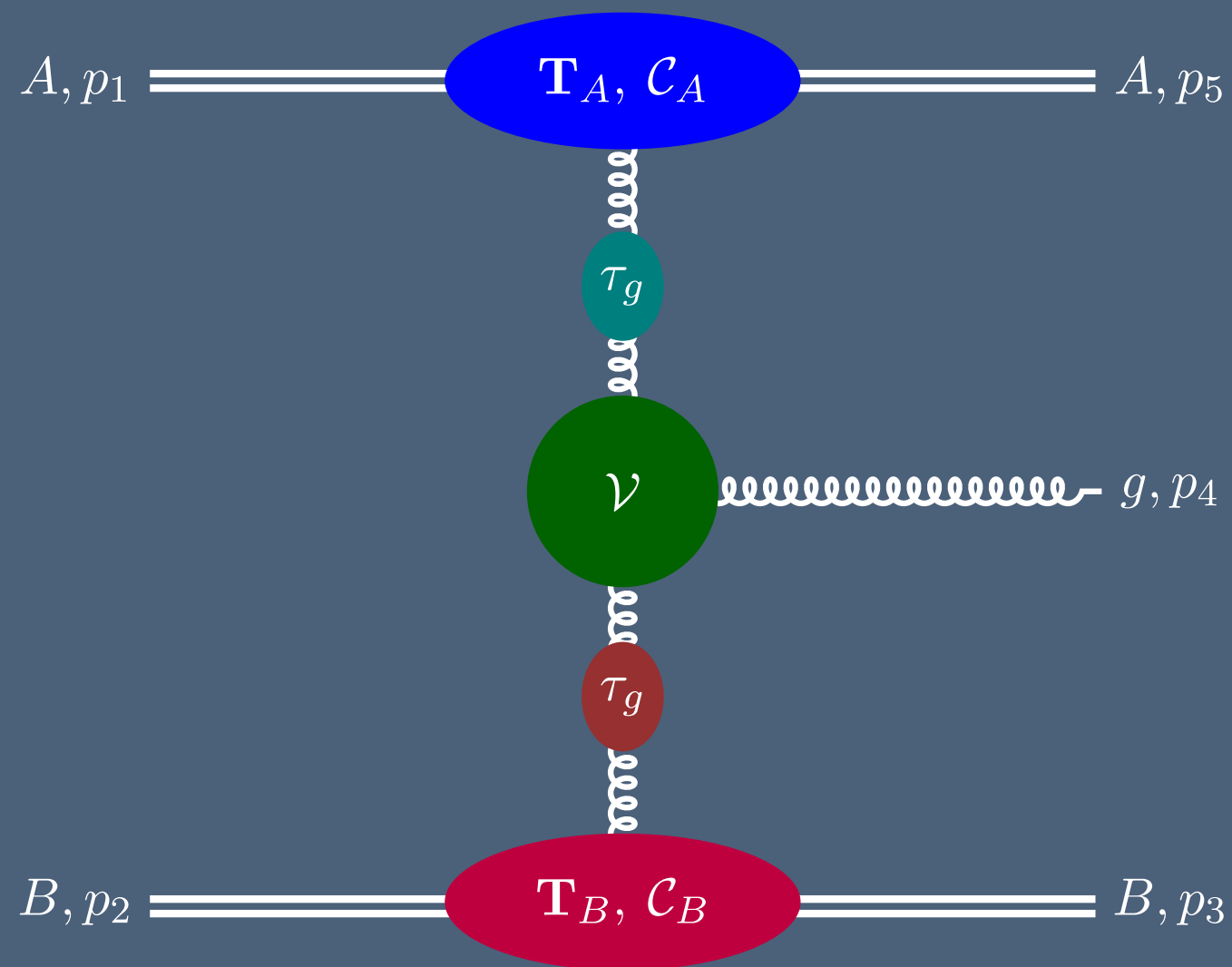
Example:  $gg \rightarrow ggg$

$$8 \otimes 8 = 0 + 1 + 8_a + 8_s + 10 + \overline{10} + 27$$

- |                   |                                 |  |
|-------------------|---------------------------------|--|
| 1. $(8_a, 8_a)$   | 9. $(8_a, 27)$                  |  |
| 2. $(8_a, 8_s)$   | 10. $(27, 8_a)$                 |  |
| 3. $(8_s, 8_a)$   | 11. $(8_a, 10 + \overline{10})$ | 18. $(0, 10 + \overline{10})$                  |
| 4. $(8_s, 8_s)_a$ | 12. $(10 + \overline{10}, 8_a)$ | 19. $(10 + \overline{10}, 27)$                 |
| 5. $(1, 8_a)$     | 13. $(8_s, 10 - \overline{10})$ | 20. $(27, 10 + \overline{10})$                 |
| 6. $(8_a, 1)$     | 14. $(10 - \overline{10}, 8_s)$ | 21. $(10 + \overline{10}, 10 + \overline{10})$ |
| 7. $(8_a, 0)$     | 15. $(27, 27)$                  | 22. $(10 - \overline{10}, 10 + \overline{10})$ |
| 8. $(0, 8_a)$     | 16. $(0, 0)$                    |  |
|                   | 17. $(10 + \overline{10}, 0)$   |  |

Choice of color basis convenient to “separate” multi vs single reggeon exchanges

# 2 → 3 amplitudes: factorization at NLL and beyond



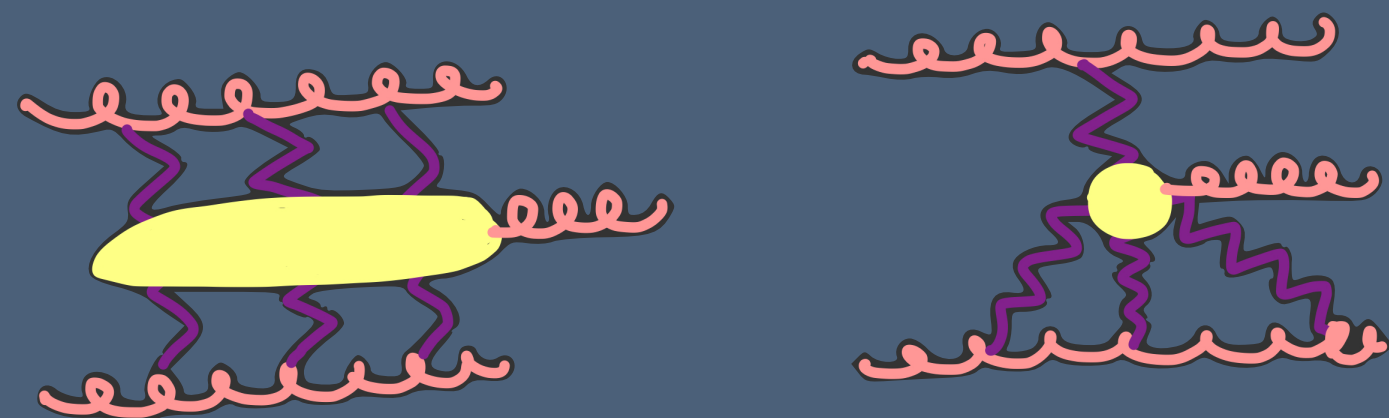
$\mathcal{A}^{(-,-)}$ : Factorization at NLL works as expected, only pole contributions

$$\mathcal{A}^\lambda(s) = s_{12} [\mathbf{T}_A^a \mathcal{C}_A^\lambda(s_{51})] \frac{\mathcal{R}(s_{45}, s_{51})}{s_{51}} [f_{aba_4} \mathcal{V}^{h_g}] \frac{\mathcal{R}(s_{34}, s_{23})}{s_{23}} [\mathbf{T}_B^b \mathcal{C}_B^\lambda(s_{23})]$$

$$\mathcal{R}(s, t) = \frac{1}{2} \left[ \left( \frac{s}{t} \right)^{\tau_g(t)} + \left( \frac{-s}{t} \right)^{\tau_g(t)} \right]$$

New ingredient: central emission vertex (Lipatov vertex)

At NNLL: cut contamination in  $\mathcal{A}^{(-,-)}$ , they enter  $(\delta_a, \delta_a)$



Strategy: compute the cut contributions from Wilson-line approach and subtract it to expansion of 5-point amplitudes to get universal structures

Need expansion of amplitude: rational + transcendental

# Expansion of the amplitude in MRK

Rational

Transcendental

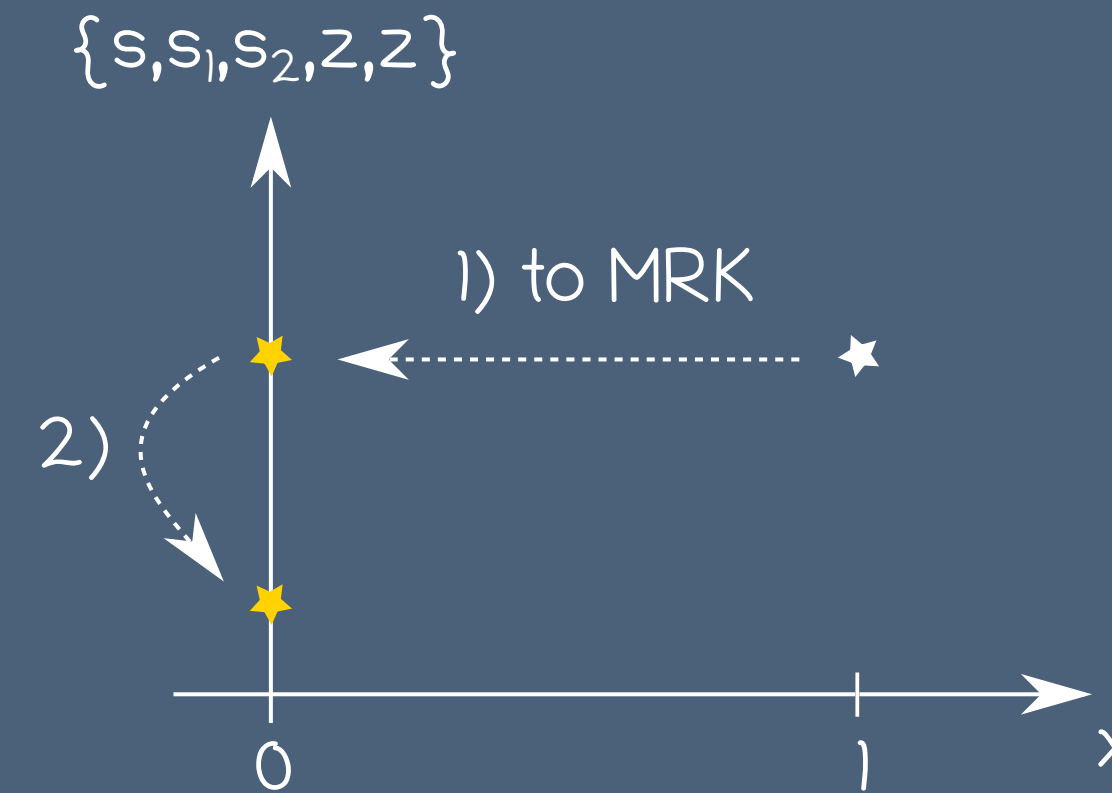
Scaling parameter  
 $x = 1$ : "physical point",  
 $x = 0$ :MRK

Set of PF and BC from  
 [Chicherin, Sotnikov:  
 2009.07803]

$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\} \rightarrow \{s, s_1, s_2, z, \bar{z}\} + x$$

$$dI_i(\mathbf{s}) = \epsilon dA_{ij}(\mathbf{s}) I_j(\mathbf{s}) \quad dA_{ij}(\mathbf{s}) = \sum_{n=1} a_{ij}^n d \log(W_n)$$

$$\begin{cases} \frac{\partial}{\partial x} \mathbf{f}(x, y, \epsilon) = \epsilon A_x(x, y) \mathbf{f}(x, y, \epsilon) \\ \frac{\partial}{\partial y} \mathbf{f}(x, y, \epsilon) = \epsilon A_y(x, y) \mathbf{f}(x, y, \epsilon) \end{cases}$$



$$W_n \rightarrow W_n(x)$$

[Caron-Huot, Chicherin, Henn, Zhang, Zoia 2003.03120]

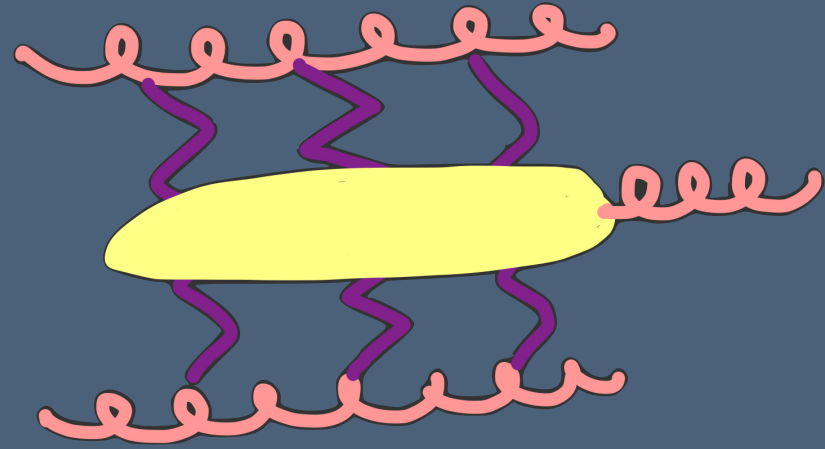
$$f^{(w)}(\mathbf{s}; x) = \sum_{n=0} \sum_{m=0}^w f_{mn}^{(w)}(\mathbf{s}) x^n \log^m x$$

$$\{x\}, \left\{ \frac{s_1 s_2}{s} \right\}, \{s_1, s_2, s_1 - s_2, s_1 + s_2\}, \\ \{z, \bar{z}, 1 - z, 1 - \bar{z}, z - \bar{z}, 1 - z - \bar{z}\}$$

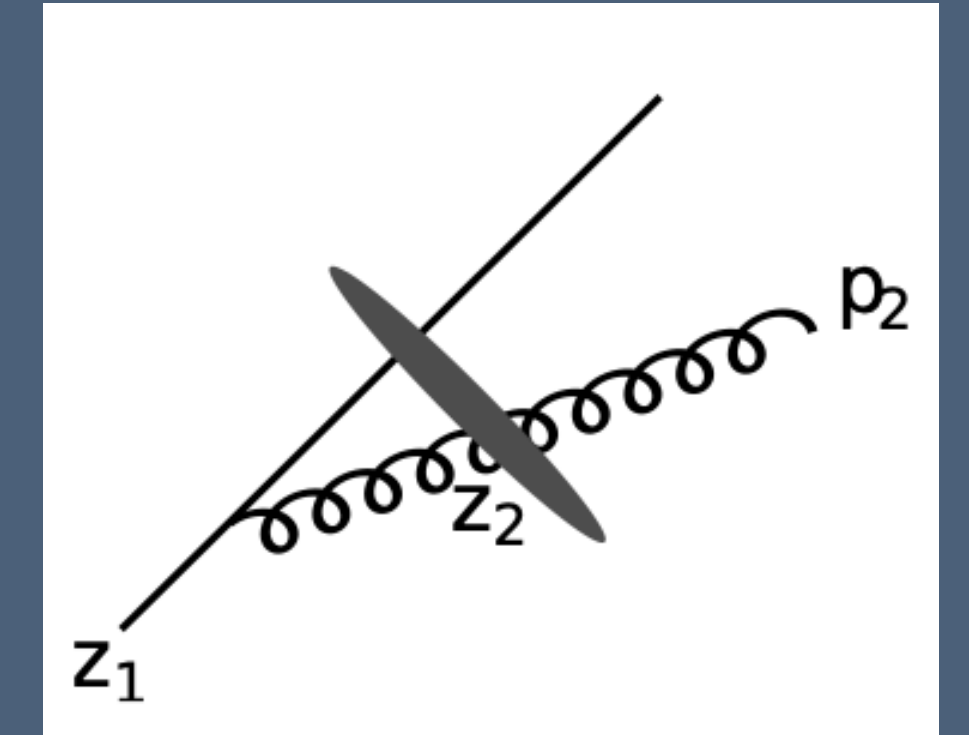
# Cut contributions: OPE in the Wilson line approach

$$U = e^{ig_s T \cdot W}$$

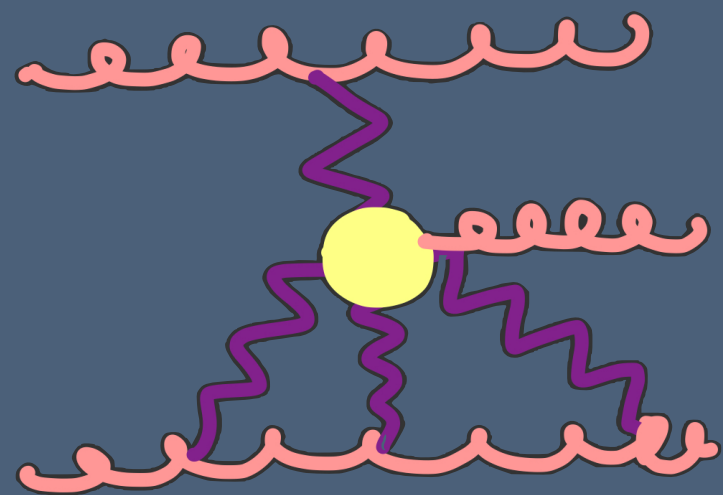
$$U(p_1)a^a(p_2) \sim -2g \int [dz_1][dz_2] [U_{\text{adj}}^{ab}(z_2)\hat{T}_{R,1}^b - \hat{T}_{L,1}^a] U(z_1)e^{-ip_1 \cdot z_1 - ip_2 \cdot z_2} \int [dq] \frac{\epsilon \cdot q}{q^2} e^{iq \cdot (z_2 - z_1)}$$



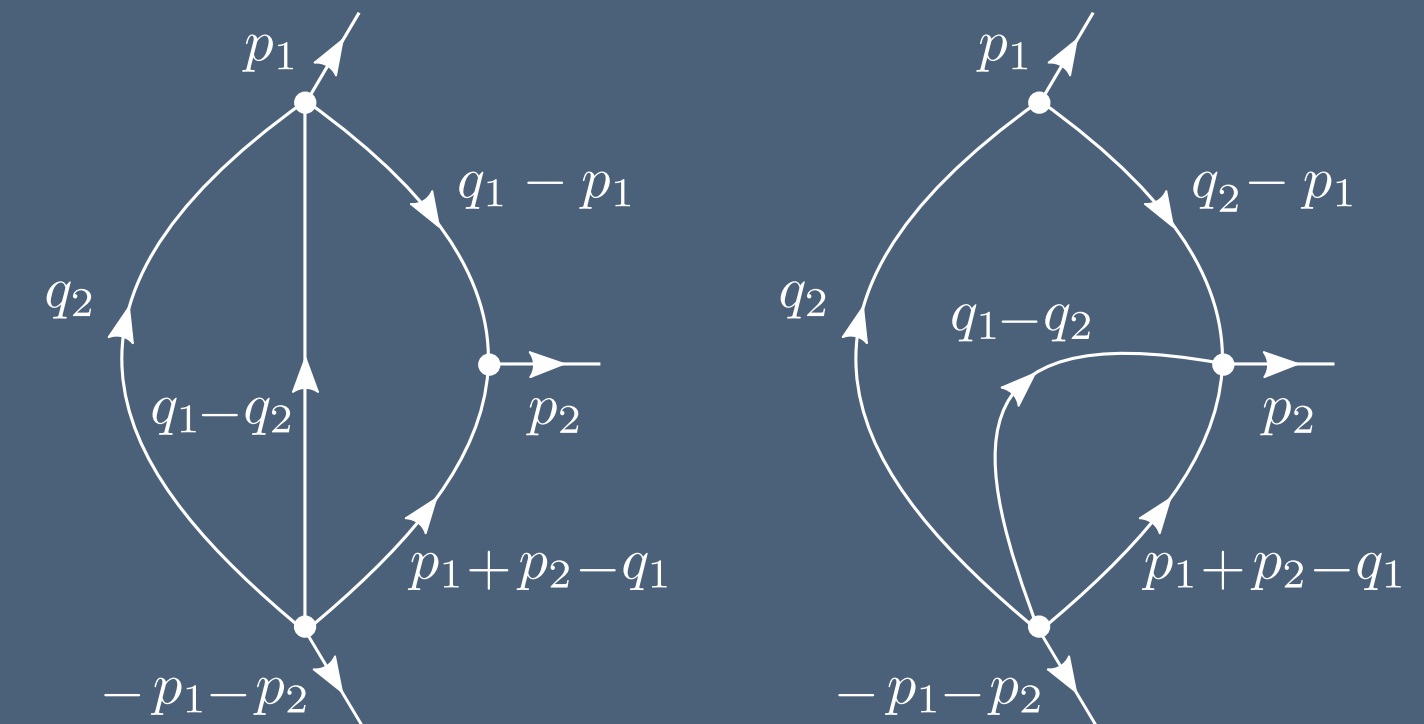
$$W_1^a W_1^b W_1^c a^d \sim 2g_s [F^{a_1}]^{da} \int [dq_1][dq_2] W^{a_1}(p_1 + p_2 - q_1) W^b(q_1 - q_2) W^c(q_2) \times \left[ \frac{\epsilon \cdot p_2}{p_2^2} - \frac{\epsilon \cdot (q_1 - p_1)}{(q_1 - p_1)^2} \right] + (a \leftrightarrow b) + (a \leftrightarrow c) + \mathcal{O}(g_s^2)$$



Caron-Huot [1309.6521]

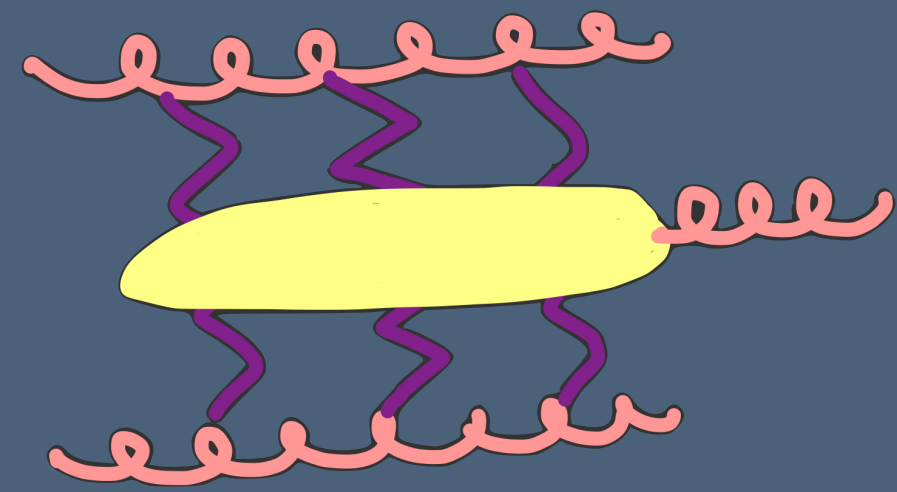


$$W_1^a a^b \sim -2g [\dots F \dots]^{ba} = 2g_s [F^{a_1}]^{ba} W^{a_1}(p_1 + p_2) \left[ \frac{\epsilon \cdot p_1}{p_1^2} + \frac{\epsilon \cdot p_2}{p_2^2} \right] + ig_s^2 [F^{a_1} F^{a_2}]^{ba} \int [dq_1] W^{a_1}(p_1 + p_2 - q_1) W^{a_2}(q_1) \left[ \frac{\epsilon \cdot p_1}{p_1^2} + \frac{\epsilon \cdot (q_1 - p_1)}{(q_1 - p_1)^2} \right] + g_s^3 [F^{a_1} F^{a_2} F^{a_3}]^{ba} \int [dq_1][dq_2] W^{a_1}(p_1 + p_2 - q_1) W^{a_2}(q_1 - q_2) W^{a_3}(q_2) \times \left[ \frac{1}{6} \left( \frac{\epsilon \cdot (q_1 - p_1)}{(q_1 - p_1)^2} \right) - \frac{1}{2} \left( \frac{\epsilon \cdot (q_2 - p_1)}{(q_2 - p_1)^2} \right) - \frac{1}{3} \left( \frac{\epsilon \cdot p_1}{p_1^2} \right) \right] + \mathcal{O}(g_s^4)$$



Our focus is (8,8)

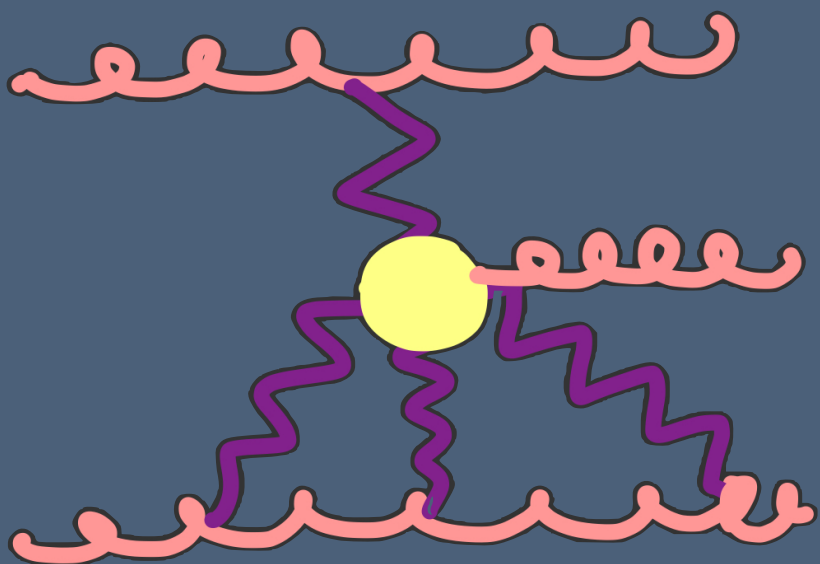
Cut contributions in (8,8) amplitude: Wilson-line EFT approach



$$\mathcal{A}_{ab,[8,8]}^{2,\text{cut}} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{i\pi}{3}\right)^2 \left(\frac{\mu^2}{|\mathbf{p}_4|^2}\right)^{2\epsilon} \left[ \mathcal{F}_{\text{LC}}(z, \bar{z}, \epsilon) + \mathcal{F}_{ab}(z, \bar{z}, \epsilon) \right]$$

$$\mathcal{F}_{\text{LC}}(z, \bar{z}, \epsilon) = N_c^2 \left( \frac{2}{\epsilon^2} - \frac{\log(z\bar{z}) + \log((1-z)(1-\bar{z}))}{\epsilon} + 6iD_2(z, \bar{z}) - 2\zeta_2 \right. \\ \left. + \frac{5}{2} \log^2(z\bar{z}) + \frac{5}{2} \log^2((1-z)(1-\bar{z})) - \log(z\bar{z})\log((1-z)(1-\bar{z})) \right)$$

LC is universal



$$\mathcal{F}_{qq}(z, \bar{z}, \epsilon) = \frac{18}{\epsilon} \left( \log((1-z)(1-\bar{z})) + \log(z\bar{z}) \right) - 9 \left( \log((1-z)(1-\bar{z})) + \log(z\bar{z}) \right)^2 \\ + 108iD_2(z, \bar{z}) + \frac{1}{N_c^2} \left[ \frac{18}{\epsilon^2} - \frac{36}{\epsilon} \left( \log((1-z)(1-\bar{z})) + \log(z\bar{z}) \right) - 108iD_2(z, \bar{z}) \right. \\ \left. - 18\zeta_2 + 36 \log^2((1-z)(1-\bar{z})) + 36 \log^2(z\bar{z}) + 18 \log((1-z)(1-\bar{z}))\log(z\bar{z}) \right]$$

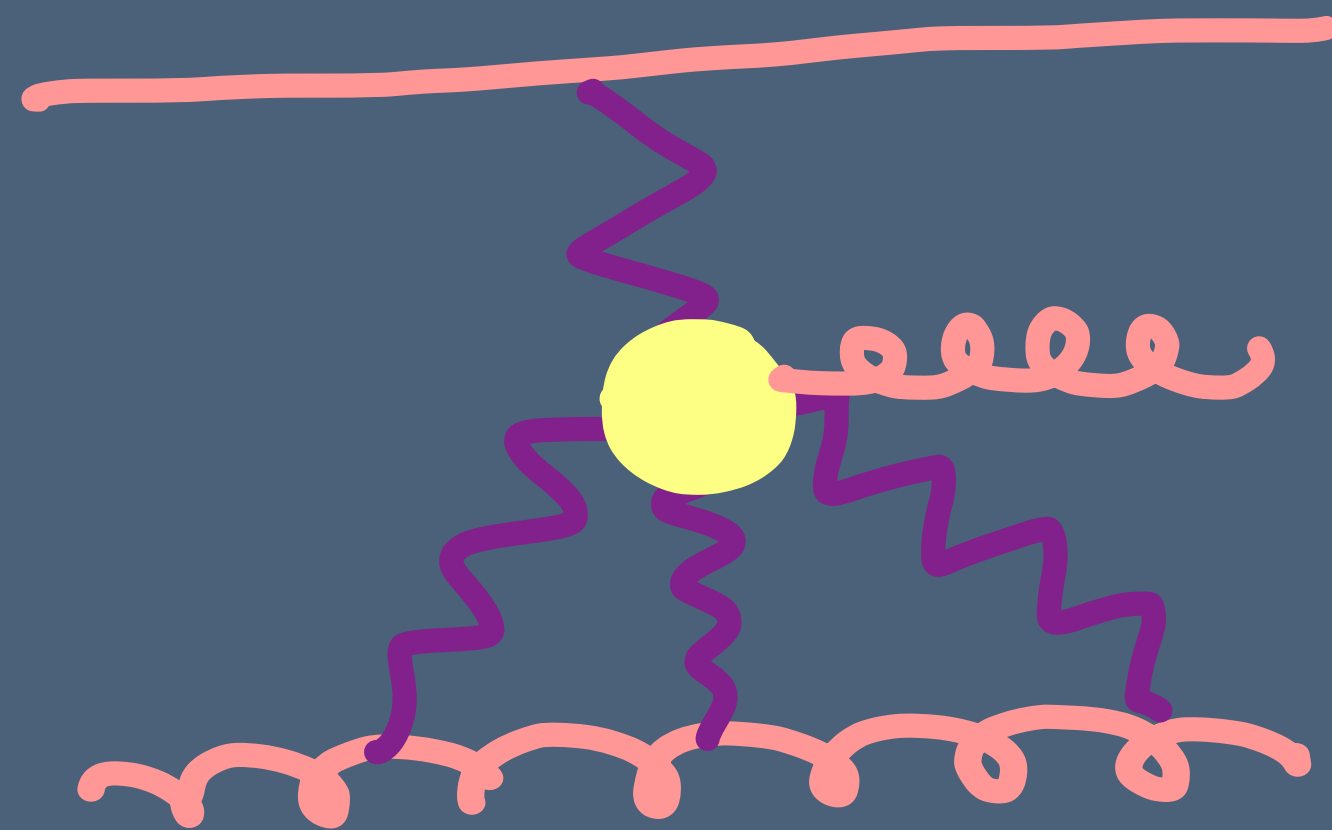
SLC is non-universal

$$\mathcal{F}_{qg}(z, \bar{z}, \epsilon) = \frac{27}{\epsilon^2} + \frac{1}{\epsilon} \left( 54 \log((1-z)(1-\bar{z})) - 36 \log(z\bar{z}) \right) + 216iD_2(z, \bar{z}) - 27\zeta_2 + 45 \log^2(z\bar{z}) - 36 \log(z\bar{z})\log((1-z)(1-\bar{z}))$$

$$D_2(z, \bar{z}) = -i \left( \frac{\log(z\bar{z})}{2} \left( \log(1-z) - \log(1-\bar{z}) \right) + \text{Li}_2(z) - \text{Li}_2(\bar{z}) \right)$$



In  $qg\ 1 \rightarrow 3$  reggeon interaction also contributes to  $(8_a, 10 + \overline{10})$ : we can check it against the MRK expansion of the amplitude!



We find agreement

```

Odd-Odd: [3→1] in (8,10+10b)

(* result from expansion of two-loop amplitudes *)
Do[
  TwoLoopExp["+", "2q3g", "oo", col] =
  A2["2q3g", "MRK", "ppmpm", col] /. MRKpar /. sgn[_] → 1 /. G → tG /. {tG[0, x_] → Log[x], tG[1, x_] → Log[1 - x]} /. tG → G /. ss → s /.
  OddOdd;
  TwoLoopExp["+", "2q3g", "oo", col] = Collect[TwoLoopExp["+", "2q3g", "oo", col], {ep, Nc, π, _G, _Log}, Expand],
  {col, 1, 11}];

In[51]:= (* Prediction for 3→1 *)
PredictionCut2L["oo", 9] = (48 Nc π^4) Cut2L["oo", "3→1"];

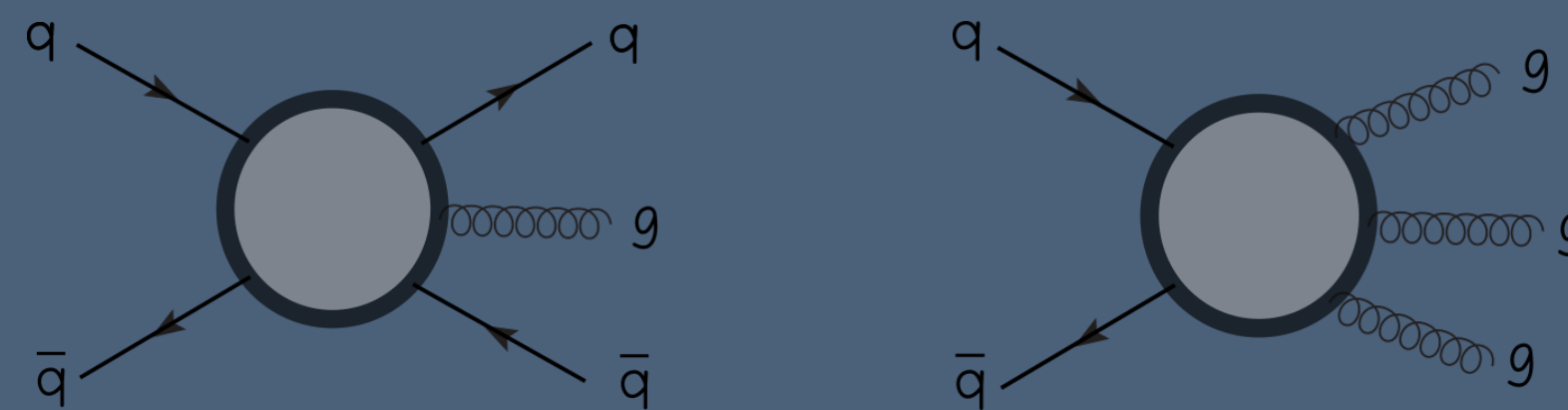
In[52]:= (* check *)
Collect[(TwoLoopExp["+", "2q3g", "oo", 9] - PredictionCut2L["oo", 9]) // PowerExpand // GToHPL // HPLConvertToKnownFunctions],
  ep, Factor] /. LisToGs // Expand // ShuffleG

Out[52]= 0
  
```

$$\begin{aligned}
\mathcal{A}_{qg,[8,10+\overline{10}]}^{2,\text{cut}} = & \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{i\pi}{3}\right)^2 \left(\frac{\mu^2}{|\mathbf{p}_4|^2}\right)^{2\epsilon} \left[ \frac{9}{2\epsilon^2} + \frac{18 \log((1-z)(1-\bar{z})) - 9 \log(z\bar{z})}{\epsilon} \right. \\
& \left. + 54iD_2(z, \bar{z}) - \frac{9}{2} \log^2((1-z)(1-\bar{z})) + 9 \log^2(z\bar{z}) - 9 \log(z\bar{z}) \log((1-z)(1-\bar{z})) - \frac{3\zeta_2}{4} \right]
\end{aligned}$$

# 2-loop central emission vertex

- Three-loop Regge trajectory
- Two-loop impact factors



2-loop Lipatov vertex



Universality at NNLL!!!

Final formula not instagrammable yet **BUT**

- Weight drop in finite remainder (weight 4  $\sim$  prod of lower weights)
- Spurious  $z - \bar{z}$  cancels in finite remainder (seen at symbol level for now)

# Summary and outlook

- Review of the computation of full color 5-point 2-loop QCD massless amplitudes
- Exploration of high energy regime: MRK kinematics
- Beyond NLL factorization in the Regge pole is broken by multi-reggeon exchanges
- We computed the cuts with EFT approach and subtracted from expansion of the amplitude
- **Preliminary** results for Lipatov vertex @2-loops from  $qqggg$  &  $qqqqg$  &  $ggggg$  amplitudes

TO DO:

- What about other color structures?, check with  $N=4$ ?, ....

Thank you!

Backup slides

Old but gold

$\mathcal{A} = \sum$  Feynman diagrams  
 > 40k

Helicity projection  
 Tancredi, Peraro  
 1906.03296 & 2012.00820

$$\mathcal{A} = \sum H_h T_h$$

Color decomposition

$$H = \sum_c H_c C_c$$

Integration by parts identities

Key step for simplification

$$R^{mc} M_m = \sum_{k=1}^M r_k^c(s_{ij}, d) M(s_{ij}, d)$$

$$a_k(s_{ij}, d) = \frac{N(s_{ij}, d)}{Q(d)D(s_{ij})} \longrightarrow a_k(s_{ij}, d) = \sum_l g_l(d) R_l(s_{ij})$$

Uni (d) + multivariate PF

MultivariateApart [Heller, von Manteuffel 2101.08283]

$$H = \sum_{m,c} R^{mc} M_m C_c$$

Computation of master integrals + algebraic simplification

Final result

Old but gold

$$R_k = \sum_{m_1 + \dots + m_{30} \leq p} a_{k,m_1 \dots m_{30}} M_{m_1 \dots m_{30}}$$

$$M_{m_1 \dots m_{30}} = q_1^{m_1} \dots q_{25}^{m_{25}} s_{12}^{m_{26}} \dots s_{51}^{m_{30}}$$

Look for linear relations amongst rational functions

$$\sum_k R_k b_k = 0 \quad \sum_k a_{k,m_1 \dots m_{30}} b_k = 0$$

$$H = \sum_{m,c} R^{mc} M_m C_c$$

Key step for simplification

$$R^{mc} M_m = \sum_{k=1}^M r_k^c(s_{ij}, d) M(s_{ij}, d)$$

$$r_k^c(s_{ij}, d) = \frac{N(s_{ij}, d)}{Q(d)D(s_{ij})} \longrightarrow r_k^c(s_{ij}, d) = \sum_l g_l(d) R_l^c(s_{ij})$$

Uni (d) + multivariate PF

MultivariateApart [Heller, von Manteuffel 2101.08283]

Not independent!

Final result

Computation of master integrals + algebraic simplification

Absolutely crucial step (see later)

# 2-loop central emission vertex

With cuts under control we extract the 2-loop Lipatov vertex from the qqggg amplitude and check against the 4q1g channel ✓

Other ingredients are known:

- Three-loop Regge trajectory
- Two-loop impact factors

Final formula not instagrammable yet **BUT**

- Weight drop in finite remainder (weight 4  $\sim$  prod of lower weights)
- Spurious  $z - \bar{z}$  cancels in finite remainder (seen at symbol level for now)

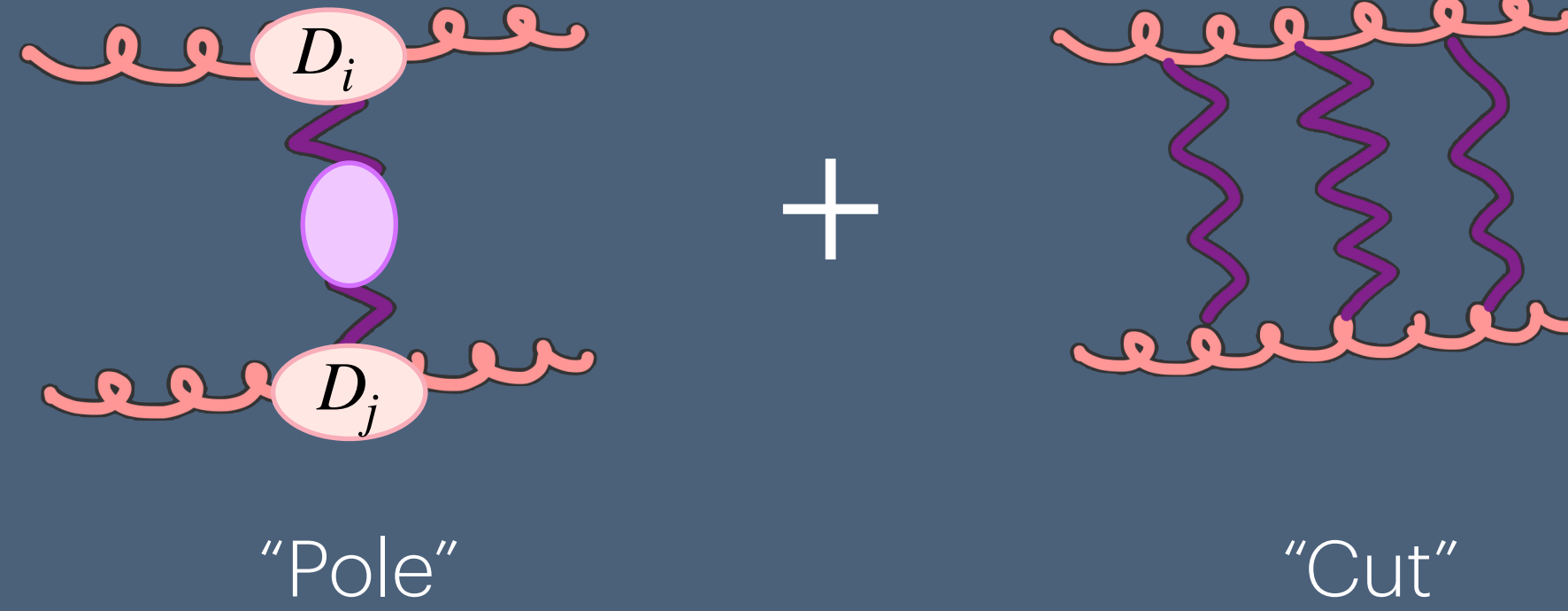


Universality at NNLL!!!



# NNLL factorization

$$\text{NNLL} \sim \left(\frac{\alpha_s}{2\pi}\right)^n \log^{n-2} x$$



Regge cuts responsible for violation of factorization at NNLL

Factorization is restored once "cut contamination" is removed

How?

Leading color: universal  
 Sub leading color: non-universal

[Falcioni et al 2112.11098]

Matched to amplitudes in a given theory

Wilson line approach

[Balitsky/JIMWLK + Caron-Huot 1309.6521, Caron-Huot, Gardi, Vernazza 1701.05241]

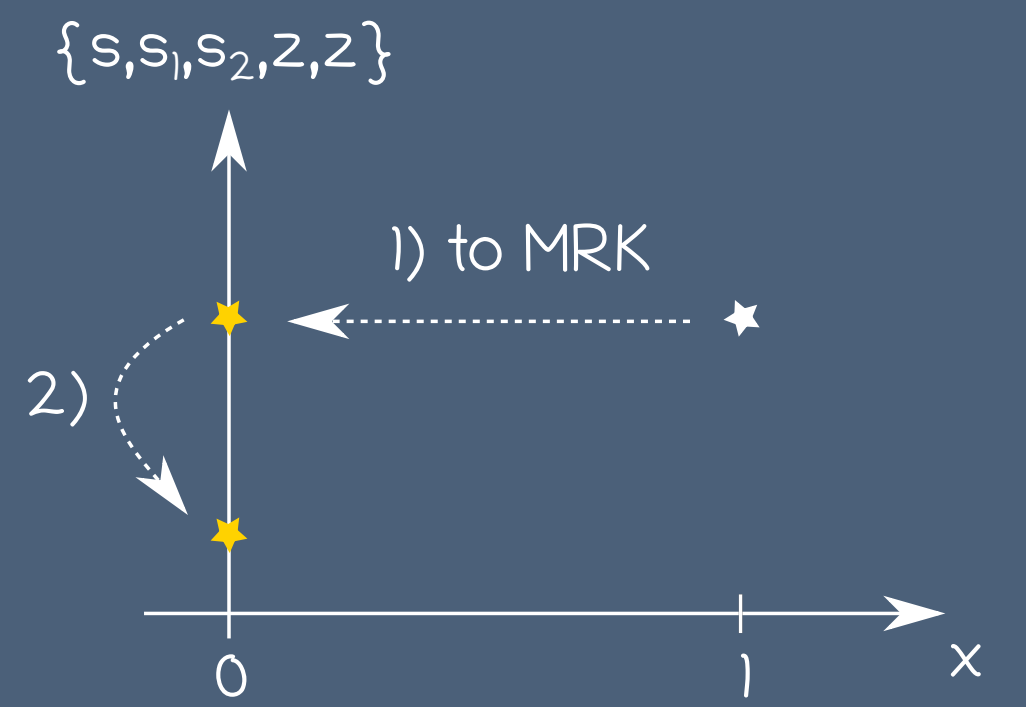
"LO" in the Wilson line correlators

# Expansion of the amplitude in MRK

Rational

Transcendental

Scaling parameter  
 $x = 1$ : "physical point",  
 $x = 0$ : MRK



$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\} \rightarrow \{s, s_1, s_2, z, \bar{z}\} + x$$

$$W_n \rightarrow W_n(x)$$

[Caron-Huot, Chicherin, Henn, Zhang, Zoia 2003.03120]

$$dI_i(\mathbf{s}) = \epsilon dA_{ij}(\mathbf{s}) I_j(\mathbf{s}) \quad dA_{ij}(\mathbf{s}) = \sum_{n=1} a_{ij}^n d \log(W_n)$$

$$\{x\}, \left\{\frac{s_1 s_2}{s}\right\}, \{s_1, s_2, s_1 - s_2, s_1 + s_2\}, \\ \{z, \bar{z}, 1 - z, 1 - \bar{z}, z - \bar{z}, 1 - z - \bar{z}\}$$

$$\begin{cases} \frac{\partial}{\partial x} \mathbf{f}(x, y, \epsilon) = \epsilon A_x(x, y) \mathbf{f}(x, y, \epsilon) \\ \frac{\partial}{\partial y} \mathbf{f}(x, y, \epsilon) = \epsilon A_y(x, y) \mathbf{f}(x, y, \epsilon) \end{cases}$$

$$\mathbf{f}(x, y, z) = x^{\epsilon A_0} P \exp \left[ \epsilon \int_{y_0}^y A_y(0, y') dy' \right] \mathbf{g}_0(\epsilon)$$

We need NNLP

$$A_x(x, y) = \frac{A_0}{x} + \sum_{k \geq 0} x^k A_{k+1}(y)$$

Solution at LP ( $x^0$ )

$$f^{(w)}(\mathbf{s}; x) = \sum_{n=0} \sum_{m=0}^w f_{mn}^{(w)}(\mathbf{s}) x^n \log^m x$$

# Balitsky/JMWLK equation

$$-\frac{d}{d\eta}U(z_1)\dots U(z_n) = HU(z_1)\dots U(z_n),$$

$$-\frac{d}{d\eta} = H = \frac{\alpha_{s,b}}{2\pi^2} \int [dz_0][dz_i][dz_j] K_{ij,0} \times \left\{ [\hat{T}_{i,L}^a \hat{T}_{j,L}^a + (L \leftrightarrow R)] - U_{\text{adj}}^{ab}(z_0) [\hat{T}_{i,L}^a \hat{T}_{j,R}^b + (i \leftrightarrow j)] \right\} + \mathcal{O}(\alpha_{s,b}^2),$$

$$H = \int [dz_0][dz_i] K_{ii,0} \left\{ -\frac{\alpha_{s,b}}{2\pi^2} C_A W_{0i}^a \frac{\delta}{\delta W_i^a} + \frac{\alpha_{s,b}^2}{3\pi} \text{Tr} \left[ W_{0i} W_0 W_{0i} \frac{\delta}{\delta W_i} \right] + \dots \right\} +$$

$$+ \int [dz_0][dz_i][dz_j] K_{ij,0} \left\{ -\frac{\alpha_{s,b}}{2\pi^2} [W_{0i} W_{0j}]^{xy} + \right.$$

$$\left. + \frac{\alpha_{s,b}^2}{6\pi} [W_{0i} W_0 W_0 W_{0j} - W_{0i} W_0 W_{0j} W_j - W_i W_{0i} W_0 W_{0j}]^{xy} + \dots \right\} \frac{\delta^2}{\delta W_i^x \delta W_j^y}.$$

## Poles and cuts

$$\mathcal{A}^{(-)} = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \sin\left(\frac{\pi j}{2}\right) a_j^{(-)}(t) e^{jL}$$

$$a_j^{(-)}(t) = \frac{1}{j-1-\alpha(t)} \rightarrow \text{"Regge trajectory"}$$

--Regge Pole--

$$\mathcal{A}^{(-)}(s, t)|_{\text{Regge pole}} = \frac{\pi}{\sin\left(\frac{\pi\alpha(t)}{2}\right)} \frac{s}{t} e^{L\alpha(t)} + \text{sub-leading}$$

$$a_j^{(-)}(t) = \frac{1}{[j-1-\alpha(t)]^{1+\beta(t)}}$$

--Regge cut--

$$\mathcal{A}^{(-)}(s, t)|_{\text{Regge cut}} = \frac{\pi}{\sin\left(\frac{\pi\alpha(t)}{2}\right)} \frac{s}{t} \frac{1}{\Gamma(1+\beta(t))} L^{\beta(t)} e^{L\alpha(t)} + \text{sub-leading}$$

Credits to Fabrizio :)

## Some jargon...

$$\mathcal{A}(s, t) = \frac{1}{\pi} \int_0^\infty \frac{d\hat{s}}{\hat{s}-s-i\epsilon} \Delta_s(\hat{s}, t) + \frac{1}{\pi} \int_0^\infty \frac{d\hat{u}}{\hat{u}+s+t-i\epsilon} \Delta_u(\hat{u}, t)$$

### Mellin moments

$$a_j^s(t) = \frac{1}{\pi} \int_0^\infty \frac{d\hat{s}}{\hat{s}} \Delta_s(\hat{s}, t) \left(\frac{\hat{s}}{-t}\right)^{-j} \quad \Delta_s(s, t) = \frac{1}{2i} \int_{\gamma-i\infty}^{\gamma+i\infty} dj a_j^s(t) \left(\frac{s}{-t}\right)^j$$

### Signature eigenstates

$$\mathcal{A}^{(\pm)}(s, t) = \frac{1}{2} (\mathcal{A}(s, t) \pm \mathcal{A}(-s-t, t)) \quad a_j^{(\pm)}(t) = \frac{1}{2} (a_j^s(t) \pm a_j^u(t))$$

$$\mathcal{A}^{(+)} = i \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \cos\left(\frac{\pi j}{2}\right) a_j^{(+)}(t) e^{jL} \quad L = \frac{1}{2} \left( \ln \frac{-s-i\epsilon}{-t} + \ln \frac{-u-i\epsilon}{-t} \right)$$

$$\mathcal{A}^{(-)} = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \sin\left(\frac{\pi j}{2}\right) a_j^{(-)}(t) e^{jL} \quad = \ln |s/t| - i\frac{\pi}{2}$$

# Expansion of pentagon functions in MRK

$$dI_i(\vec{s}) = \epsilon dA_{ij}(\vec{s}) I_j(\vec{s}) \quad dA_{ij}(\vec{s}) = \sum_{n=1} a_{ij}^n d \log(W_n)$$

$$W_n \rightarrow W_n(x) \quad \text{[Caron-Huot, Chicherin, Henn, Zhang, Zola 2003.03120]}$$

For fixed  $\{s_{ij}\} \sim y$ , one gets a 1-d differential equation in  $x$

$$\begin{cases} \frac{\partial}{\partial x} \vec{f}(x, y, \epsilon) = \epsilon A_x(x, y) \vec{f}(x, y, \epsilon) \\ \frac{\partial}{\partial y} \vec{f}(x, y, \epsilon) = \epsilon A_y(x, y) \vec{f}(x, y, \epsilon) \end{cases} \quad A_x(x, y) = \frac{A_0}{x} + \sum_{k \geq 0} x^k A_{k+1}(y)$$

$$\vec{f}(x, y, \epsilon) = x^{\epsilon A_0} \mathbb{P} \exp \left[ \epsilon \int_{y_0}^y A_y(0, y') dy' \right] \vec{g}_0(\epsilon)$$

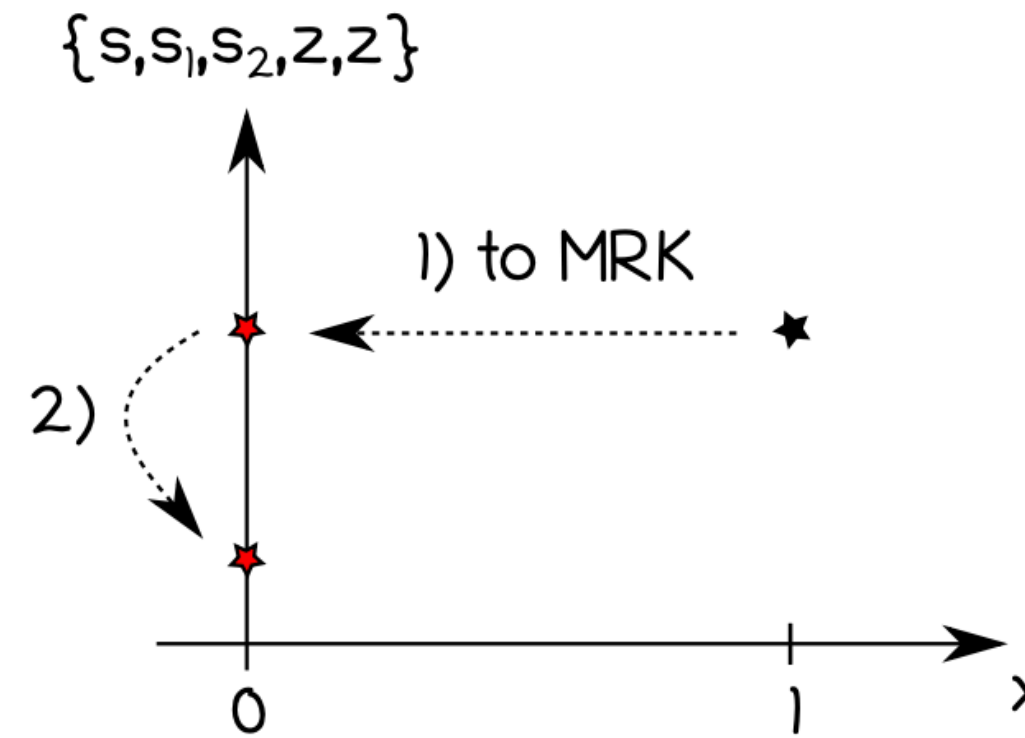
Nice property of MRK at LP:  
Gram-determinant a perfect square  
all square-roots in letters rationalized

$$\Delta = \epsilon_5^2 \underset{x \rightarrow 0}{\sim} \frac{s_1^2 s_2^2 (z - \bar{z})^2}{x^4} + \mathcal{O} \left( \frac{1}{x^3} \right)$$

Alphabet in MRK much simpler than in full kinematics (35 → 12 letters)

- $\{x\}$ ,
- $\left\{ \frac{s_1 s_2}{s} \right\}$ ,
- $\{s_1, s_2, s_1 - s_2, s_1 + s_2\}$ ,
- $\{z, \bar{z}, 1 - z, 1 - \bar{z}, z - \bar{z}, 1 - z - \bar{z}\}$

pentagon functions at LP



Credits to Federico :)