Five parton scattering in massless QCD and its high energy limit

Based on: Phys.Rev.D 109 (2024) 9, 094025, arXiv:2311.09870 with B. Agarwal, F. Buccioni, G. Gambuti, A. Von Manteuffel, L. Tancredi
+ ongoing work with F. Buccioni, F. Caola, G. Gambuti

Federica Devoto

Nikhef Theory Seminar, 11/07/2024

Outline

- Review of calculation of full color two-loop five-point scattering amplitudes in massless QCD

Why

How

- Five parton scattering in the high energy limit

Regge and Multi-Regge kinematics

Regge poles vs Regge cuts

Factorization beyond NLL

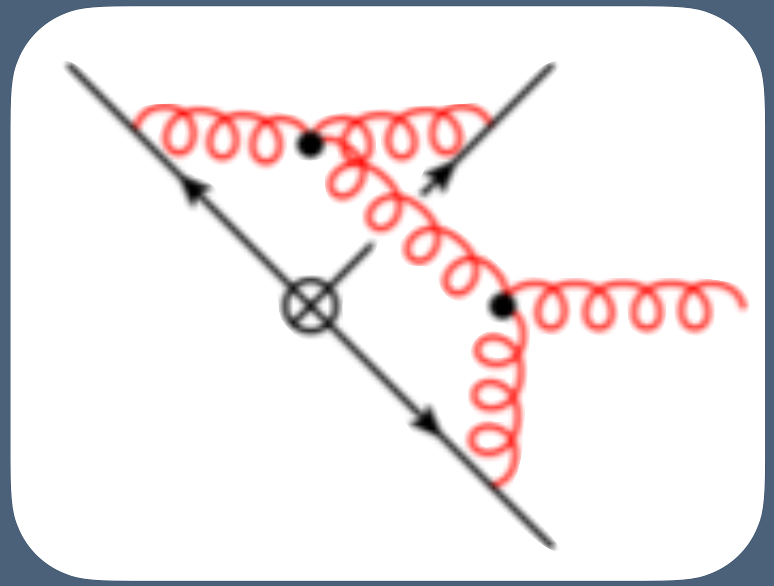
NEW!

"Cut contamination"

2-loop central emission vertex

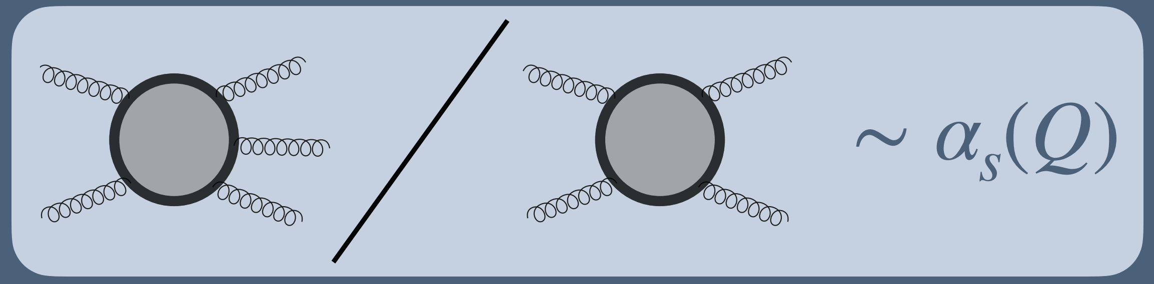
Why “Pain is inevitable, suffering is ~~optional~~” Worth it!

Simplest playground to study non planar sectors of QCD

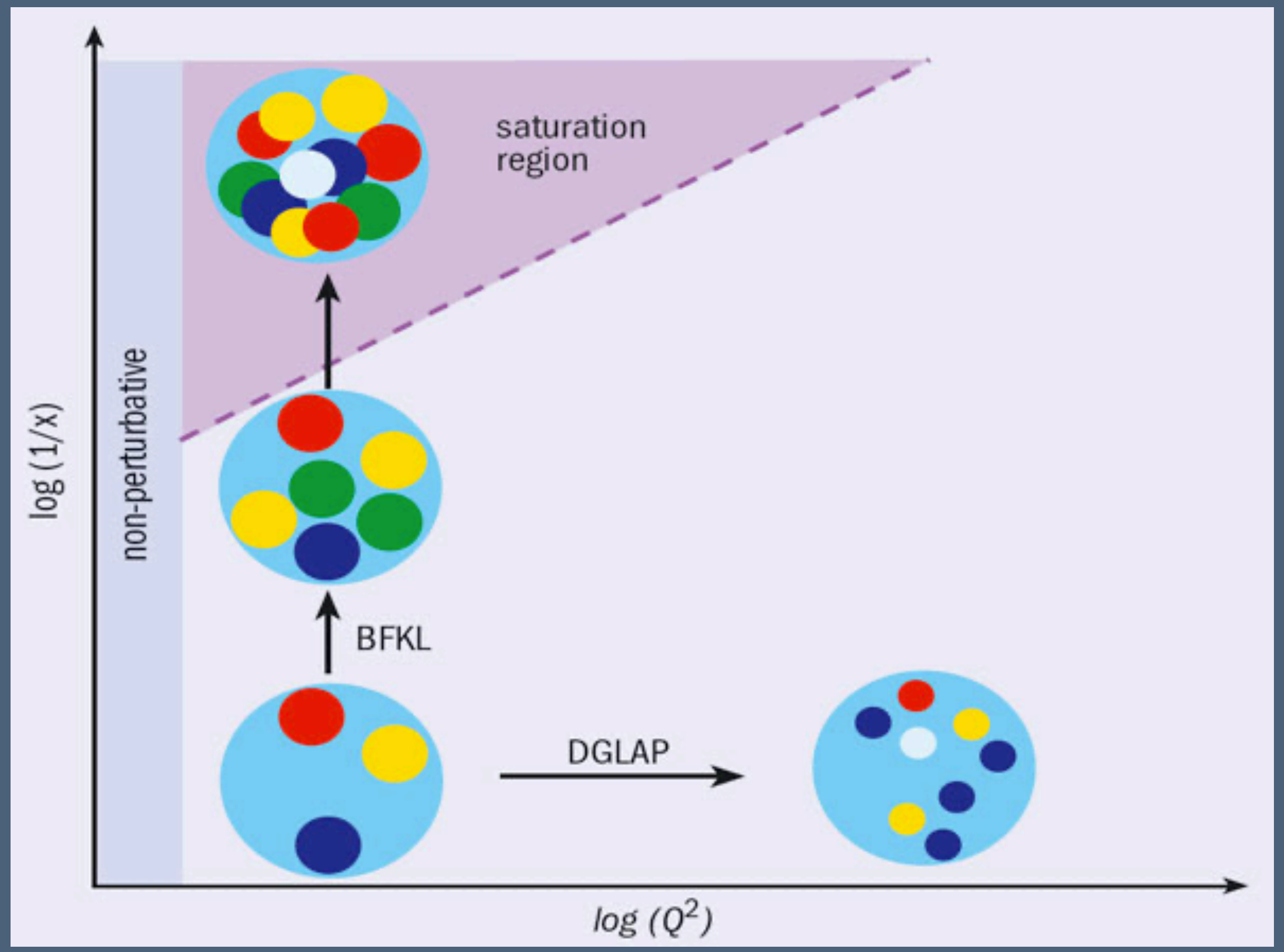


Infrared

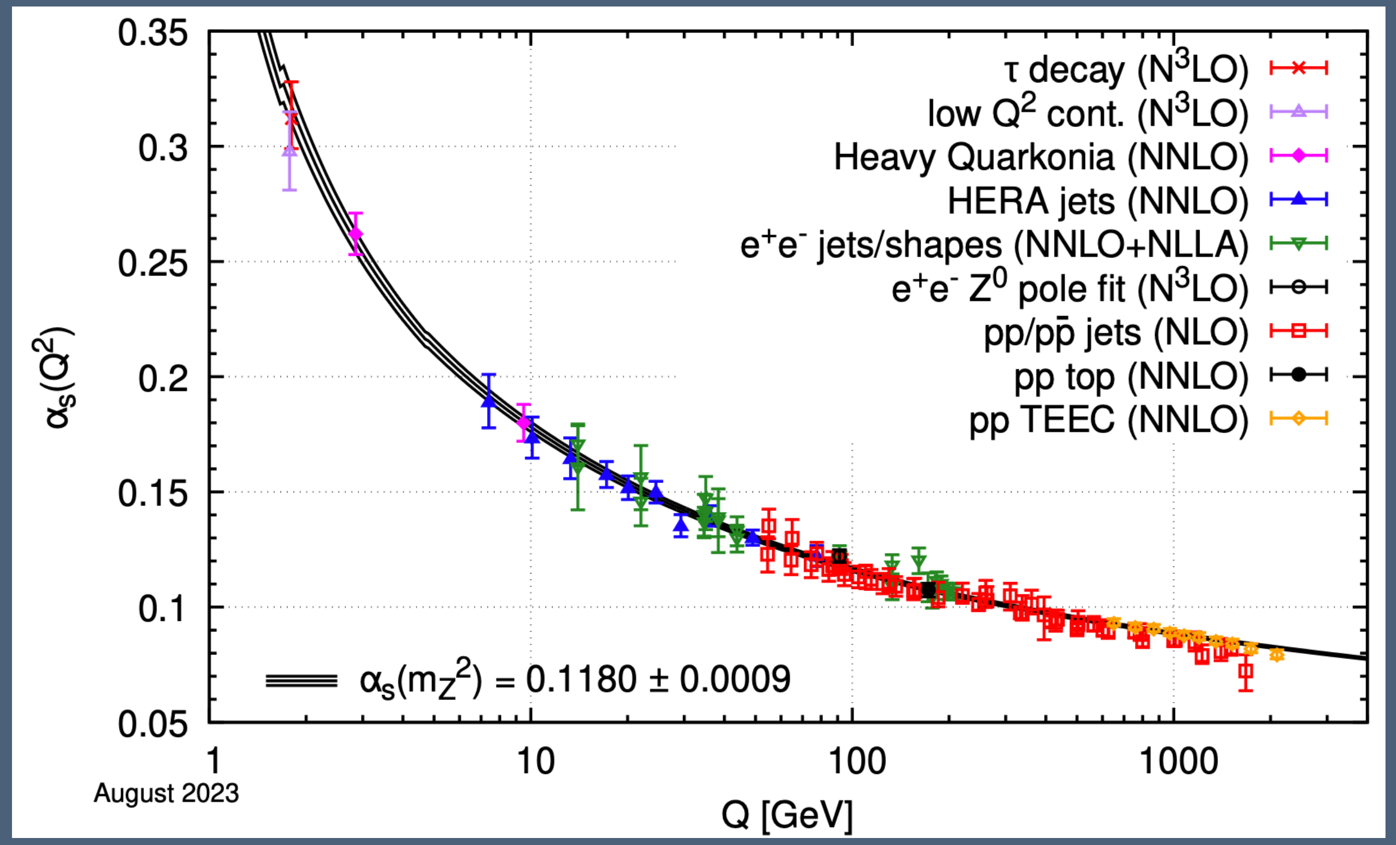
[Dixon et al 1912.09370]



High energy: Regge limit and factorization, BFKL evolution, PDFs@small x etc..



Testing running of strong coupling at TeV scale

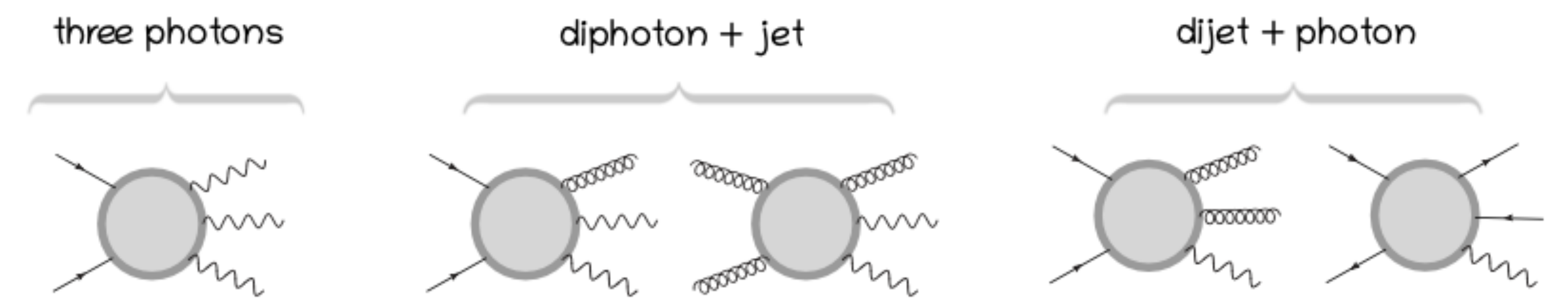


[PDG]

Input for higher order jet cross sections

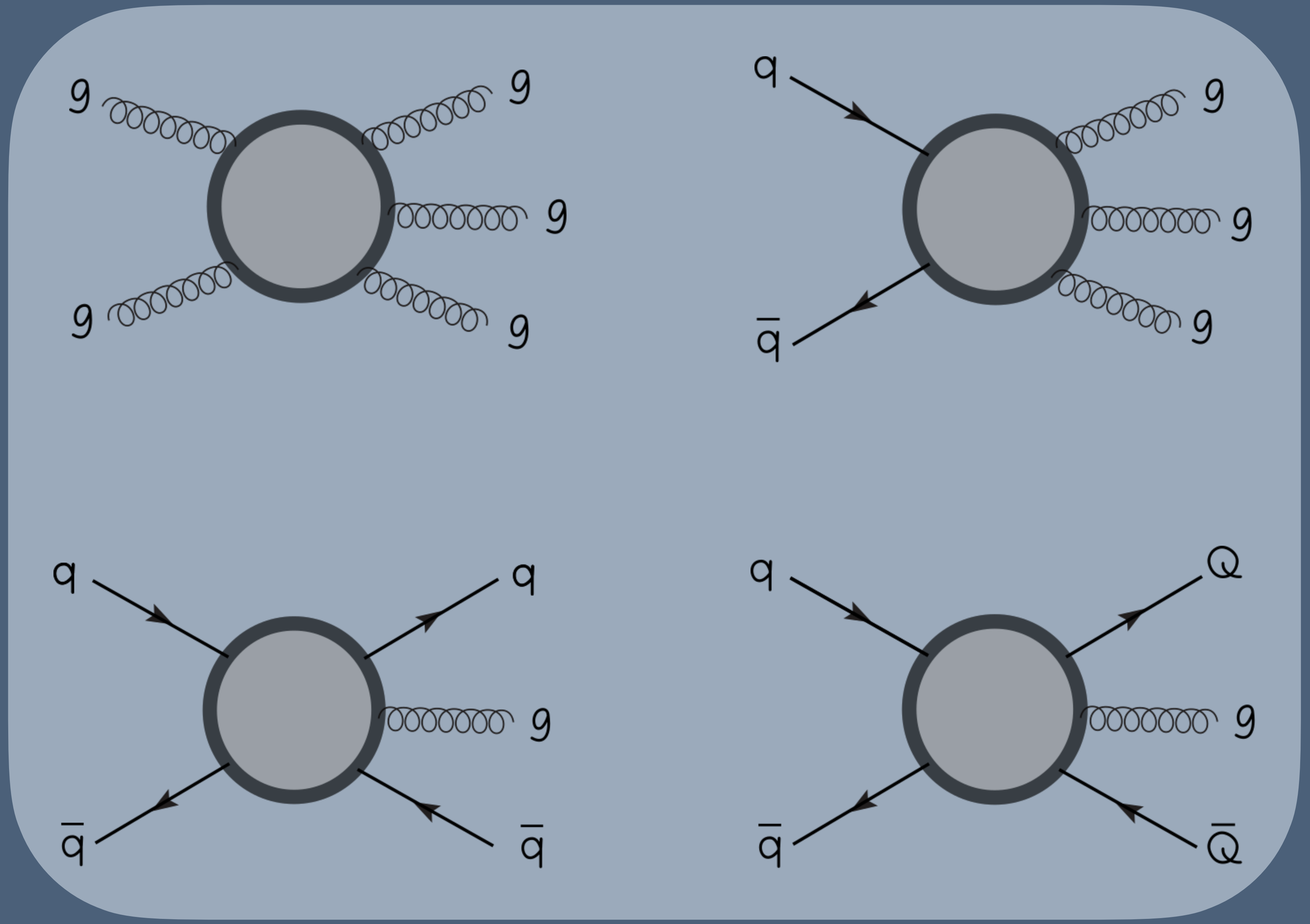
e.g. 3-jet XS [Czakon, Mitov, Poncelet 2106.05331]

Brief history of 5-point 2-loop QCD amplitudes



Contributors: [Abreu, Agarwal, Badger, Buccioni, Chawhdry, Chicherin, Czakon, Cordero Febres, De Laurentis, Gehrmann, Brønnum-Hansen, Hartanto, Henn, Ita, Klinkert, Kryś, Marcoli, Mitov, Moodie, Page, Pascual, Peraro, Poncelet, Sotnikov, Tancredi, Manteuffel von, Zoia, ...]

This talk



Leading color by Abreu et al in [2102.13609]

Full color: A. Agarwal, F. Buccioni, FD, G. Gambuti, A. von Manteuffel and L. Tancredi [2311.09870] + G. De Laurentis, H. Ita, M. Klinkert, V. Sotnikov in [2311.10086] & [2311.18752]

How

Old but gold

$$\mathscr{A} = \sum \text{Feynman diagrams}$$

> 40k

Helicity projection

Tancredi, Peraro:
1906.03298 & 2012.00820

$$\mathscr{A} = \sum H_h T_h$$

5 gluons: $$\mathscr{A} = A^{\mu_1\cdots\mu_5}\epsilon_{\mu_1}^{h_1}\ldots\epsilon_{\mu_5}^{h_5} = \sum_j H_j T_j^{\mu_1\cdots\mu_5}\epsilon_{\mu_1}^{h_1}\ldots\epsilon_{\mu_5}^{h_5}$$

$$T^{\mu_1\cdots\mu_5} \supset \begin{cases} p_i^{\mu_1}p_j^{\mu_2}\ldots p_k^{\mu_5} & i,j,k = 1,\ldots,5 \\ g^{\mu_1\mu_2}p_i^{\mu_3}p_j^{\mu_4}p_k^{\mu_5} & i,j,k = 1,\ldots,5 \\ g^{\mu_1\mu_2}g^{\mu_3\mu_4}p_k^{\mu_5} & k = 1,\ldots,5 \end{cases}$$

1724 structures in d dimensions!

$$\Big\downarrow \; \begin{matrix} \epsilon_i \cdot p_i = 0 \\ \epsilon_i \cdot p_{i+1} = 0 \end{matrix}$$

142 structures

In the end we care about 4-dimensional helicity amplitudes (tHV scheme): these structures are not all independent in 4 dimensions

$$g^{\mu\nu} = g^{\mu\nu}_{d=4} + g^{\mu\nu}_{d=-2\epsilon}$$

$\propto p_i^{\mu_1}p_j^{\mu_2}\ldots p_k^{\mu_5}$

Does not contribute to 4-dimensional helicity amplitudes, can be neglected

Left with $2^5 = 32$ combinations = number of helicity configurations!

4 quarks (1 gluon): even worse! Dirac algebra doesn't close in d dimensions

$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1)\,\bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$
$$T_2 = \bar{u}(p_2)\not{p}_3\,u(p_1)\,\bar{u}(p_4)\not{p}_1\,u(p_3)$$
$$T_3 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_1)\,\bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}u(p_3)$$
$$T_4 = \bar{u}(p_2)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3}u(p_1)\,\bar{u}(p_4)\gamma^{\mu_1}\not{p}_1\gamma^{\mu_3}u(p_3)$$

⋮ grows w/ # loops!

# of independent structures = # helicity configurations !

Key idea: rotate away the -2ϵ subspace

[Peraro, Tancredi 1906.03298, 2012.00820]

Only T_1, T_2 are left in d=4!

$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1)\,\bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$
$$T_2 = \bar{u}(p_2)\not{p}_3\,u(p_1)\,\bar{u}(p_4)\not{p}_1\,u(p_3)$$
$$T_3 = \cancel{T_3^{4}} + (d-4)\,T_3^{-2\varepsilon}$$
$$T_4 = \cancel{T_4^{4}} + (d-4)\,T_4^{-2\varepsilon}$$

Old but gold

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> 40k

Helicity projection

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$$\mathscr{A} = \sum H_h T_h$$

$$H = \sum_c H_c C_c$$

Color decomposition

Old but gold

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> 40k

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$$\mathscr{A} = \sum H_h T_h$$

$$H = \sum_c \boxed{H_c} C_c$$

Polynomial in N_c, n_f

Color decomposition

C_c	$ggggg$	$q\bar{q}ggg$	$q\bar{q}Q\bar{Q}g$
Tree level	$\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}) - \mathrm{Tr}(T^{a_5}T^{a_4}T^{a_3}T^{a_2}T^{a_1})$ + permutations	$(T^{a_1}T^{a_2}T^{a_3})_{ji}$ + permutations	$T^a_{ij}\delta_{kl}$ $T^a_{ik}\delta_{jl}$
Beyond tree	$\mathrm{Tr}(T^{a_1}T^{a_2}) \times (\mathrm{Tr}(T^{a_3}T^{a_4}T^{a_5}) - \mathrm{Tr}(T^{a_5}T^{a_4}T^{a_3}))$ + permutations	$\mathrm{Tr}(T^{a_1}T^{a_2})T^{a_3}_{ij}$ $(\mathrm{Tr}(T^{a_3}T^{a_4}T^{a_5}) - \mathrm{Tr}(T^{a_5}T^{a_4}T^{a_3}))\delta_{ij}$ $(\mathrm{Tr}(T^{a_3}T^{a_4}T^{a_5}) + \mathrm{Tr}(T^{a_5}T^{a_4}T^{a_3}))\delta_{ij}$	Same as tree

Optimal choice of color basis depends on problem at hand (see later in MRK)

$$H^{c_1\ldots c_5} = \sum_i^n \boxed{H_i} | c_1 \ldots c_5 >_i \equiv \sum_i^n H_i | \boxed{C_i} >$$

Partial amplitudes

Color basis

$$H_i = \boxed{b_i^{(2,0)} N_c^2} + \boxed{b_i^{(0,0)}\, 1} + \boxed{b_i^{(-2,0)} N_c^{-2}} + \boxed{b_i^{(1,1)} N_c^2 \frac{n_f}{N_c}} + \ldots$$

Leading color

Subleading color

Leading vs subleading?!

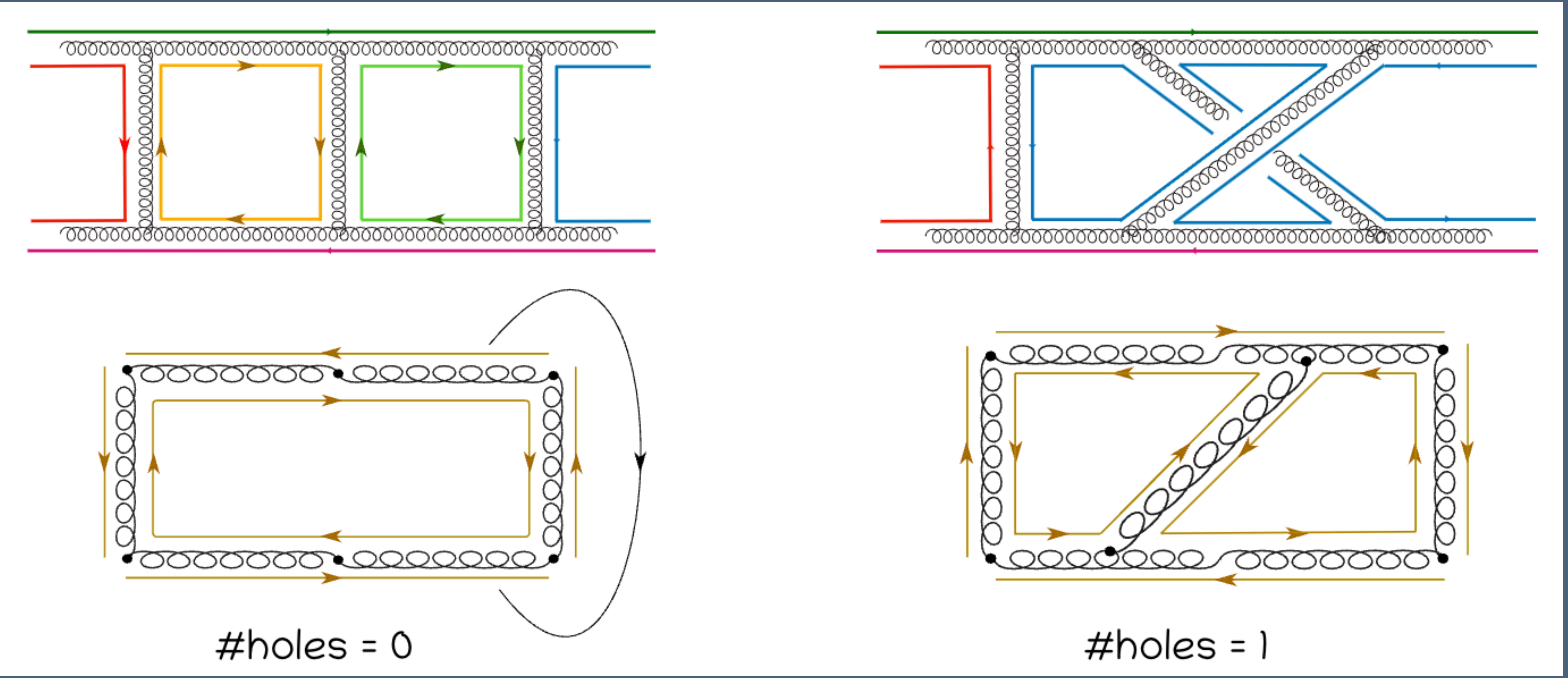
Leading color and planar diagrams

Consider U(N) (or SU(N)) gauge theory in the 't Hooft limit: $N \to \infty$ at $\lambda = g^2 N$ fixed ['t Hooft '73]

Diagram $\sim g^{V_3+2V_4} N^I \sim \lambda^{\frac{1}{2}V_3+V_4} N^{\chi}$

$\chi = 2 - 2H$, $H =$ # of "holes"

Sphere: $\chi = 2$ Torus: $\chi = 0$



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Color decomposition

$$H = \sum_c H_c C_c$$

Integration by parts identities

$$H = \sum_{m,c} R^{mc} M_m C_c$$

Master integrals & Pentagon functions

Chicherin, Sotnikov:
2009.07803

$$\frac{s_{23}s_{51}s_{45} + (d-4)s_{12}s_{13}s_{25}}{s_{12}s_{23}s_{34}}$$

Integration by parts: infinitesimal Feynman integral shift symmetry

Preserved by dimreg!

$$I(k) = \int \frac{d^dk}{(2\pi)^d} f(k+\alpha\, q) = I(k) + \alpha \int \frac{d^dk}{(2\pi)^d} \frac{\partial}{\partial k^\mu} \left(q^\mu f(k)\right) + \mathcal{O}(\alpha^2)$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{\partial}{\partial k^\mu} \left[q^\mu f(k)\right] = 0$$

$\mathcal{O}(10^6)$ Feynman integrals

$\mathcal{O}(10^3)$ master integrals

(Only ~500 indipendent functions)

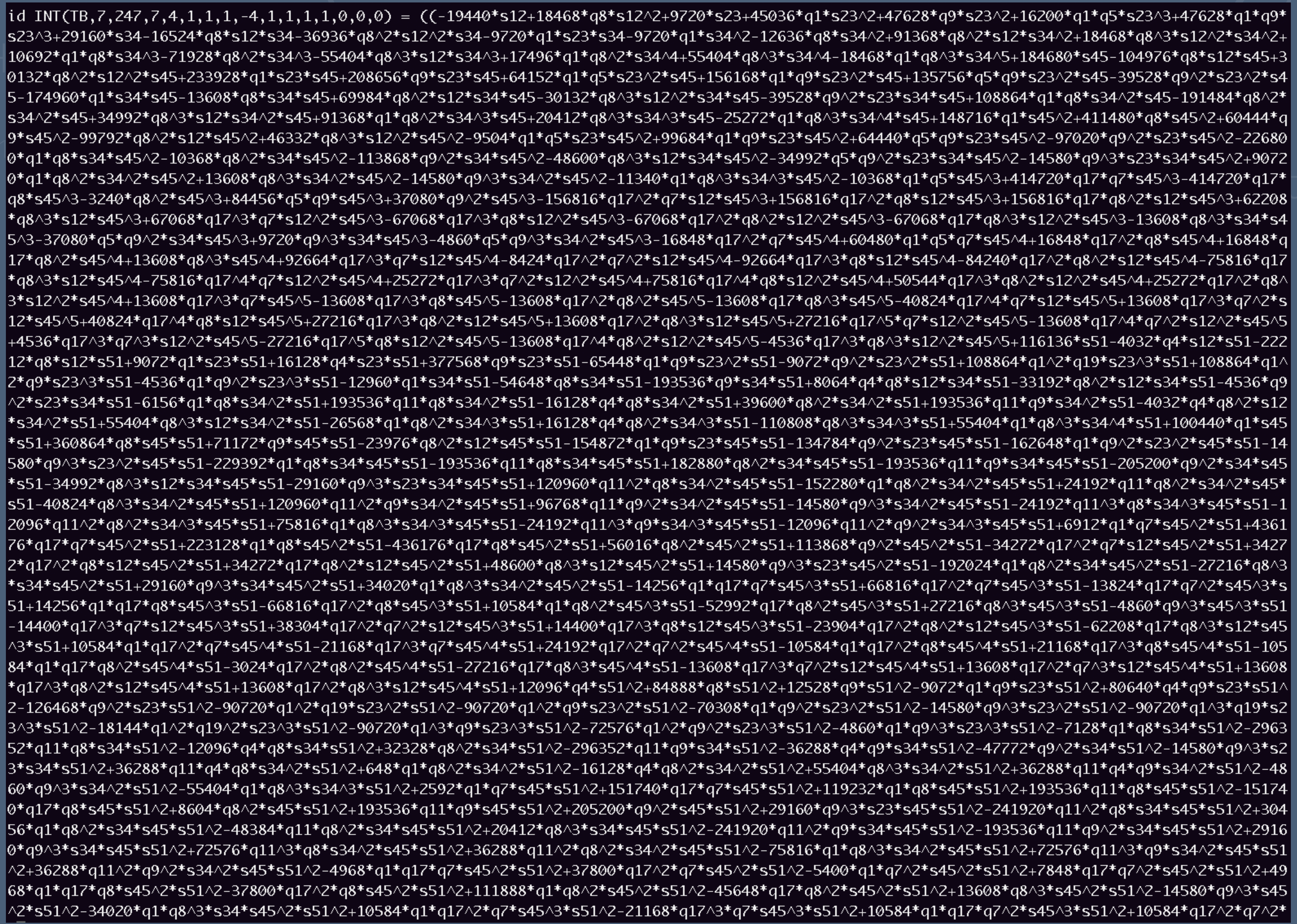
Finite fields reconstruction
Syzygy techniques

FinRed
(von Manteuffel)

Price to pay: deal with rational functions resulting from IBP reduction procedure...

$$\frac{s_{23}s_{51}s_{45} + (d-4)s_{12}s_{13}s_{25}}{s_{12}s_{23}s_{34}}$$

Instagram



$$H = \sum_{m,c} \boxed{R^{mc}} M_m C_c$$

Reality

Master integrals and differential equations

Canonical form, I_j vector of UT master integrals

$$dI_i(\mathbf{s}) = \epsilon \, dA_{ij}(\mathbf{s}) \, I_j(\mathbf{s}) \qquad dA_{ij}(\mathbf{s}) = \sum_{n=1} a_{ij}^n \, d\log(\boxed{W_n})$$

"Letters" of alphabet: algebraic functions of invariants

Need boundary conditions

Solution can be obtained systematically: iterated integrals

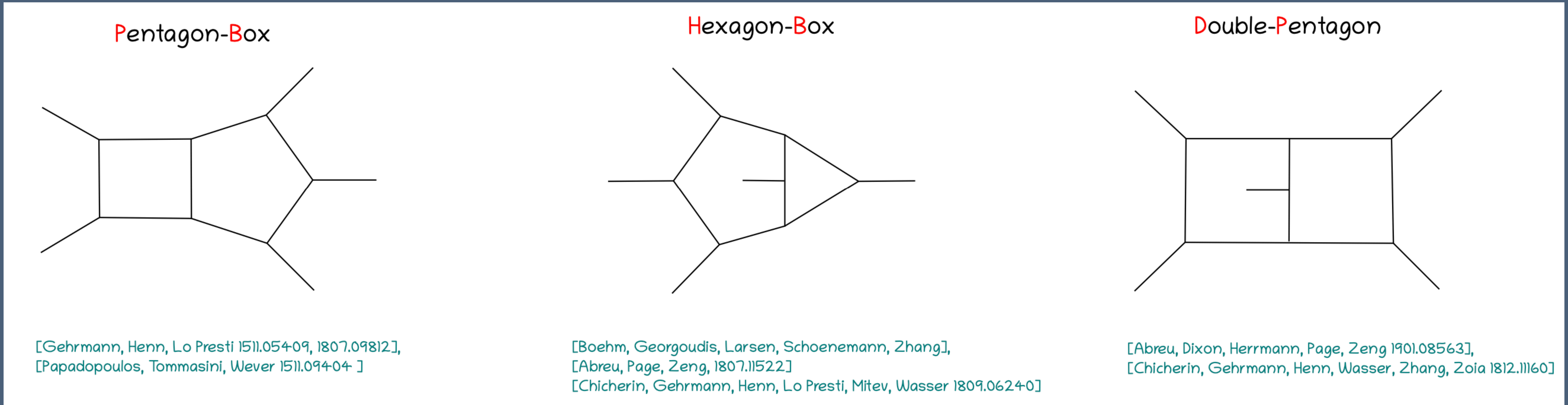
$$I(\vec{s}) = \mathrm{P}\exp\left[\epsilon \int^{\vec{s}} A(\vec{x})\mathrm{d}\vec{x}\right] I(\vec{s}_0)$$

$$I^{(\omega)}(\vec{s}) = \int_\gamma \underbrace{\mathrm{d}\log W_{i_1} \ldots \mathrm{d}\log W_{i_n}}_{\omega \text{ integrations}}$$

Typical difficulties:

- Find appropriate canonical basis
- Solution might be dependent on kinematic region
- Boundary conditions can be tricky to compute
- Non planar integrals much more complicated (cut structure,alphabet, larger # of independent functions)

Master integrals for $2 \to 3$ scattering and Pentagon Functions



Expressed as Chen iterated integrals, full set available. Results in the whole physical region
[Chicherin, Sotnikov 2009.07803]

$$f^{(\omega)}(\mathbf{x}) = \int_\gamma \mathrm{d}\log W_{i_1} \ldots \mathrm{d}\log W_{i_n}$$

Evaluation time: $\sim 1s$

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> 40k

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Color decomposition

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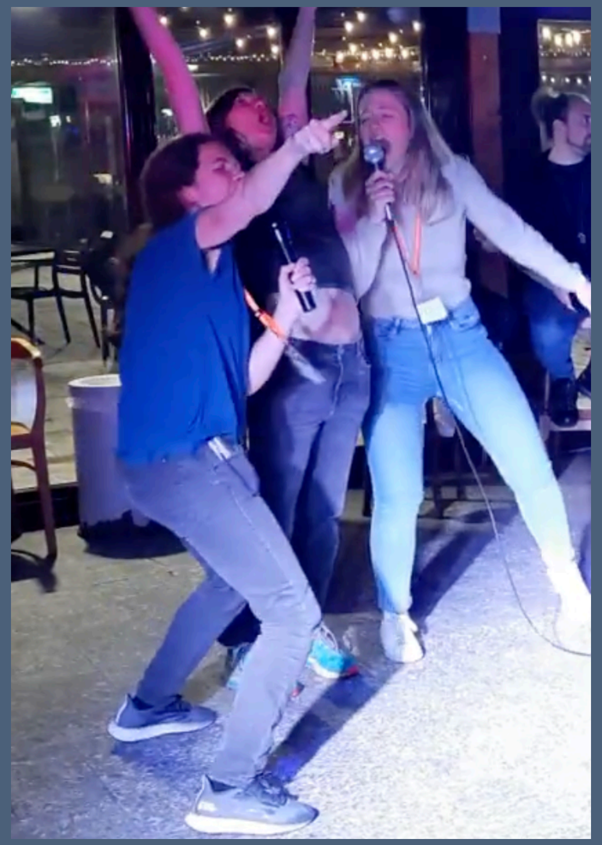
Integration by parts identities

$$H = \sum_{m,c} R^{mc} M_m C_c$$

This is where the magic happens!

Final result

It's a kind of magic



We employ MultivariateApart for multivariate partial fractioning:

[Heller, von Manteuffel 2101.08283]

- Avoids spurious denominators
- Produces unique results when applied to terms of a sum separately

$$I(s_{ij}, d) = \sum_{k=1}^{M} a_k(s_{ij}, d) \mathcal{I}(s_{ij}, d)$$

Common denominator

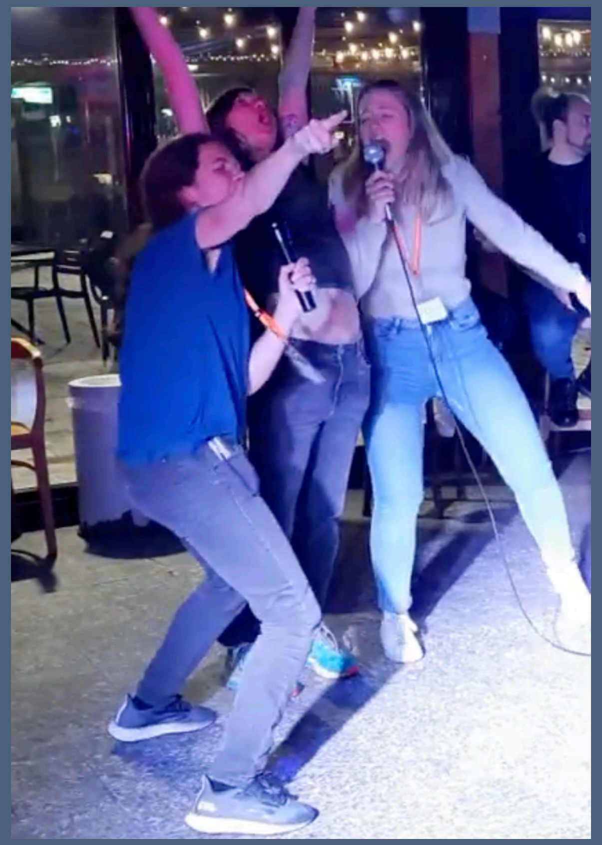
$$a_k(s_{ij}, d) = \frac{N(s_{ij}, d)}{Q(d) D(s_{ij})}$$

Partial-fractioned

$$a_k(s_{ij}, d) = \sum_{l} g_l(d) R_l(s_{ij})$$

Crucial step: drastic reduction in size!

It's a kind of magic



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Drastic simplifications occur:

	CD		PF
PB: INT[TA,8,255,8,5,{1,1,1,1,1,1,1,1,-5,0,0}]	162 MB	$\longrightarrow$	3.9 MB
HB: INT[TB,8,255,8,5,{1,1,1,1,1,1,1,1,-4,0,-1}]	513 MB	$\longrightarrow$	9.9 MB
DP: INT[TB,8,510,8,5,{0,1,1,1,1,1,1,1,1,-5,0}]	2.9 GB	$\longrightarrow$	24 MB

$\sim \mathcal{O}(100)$!!

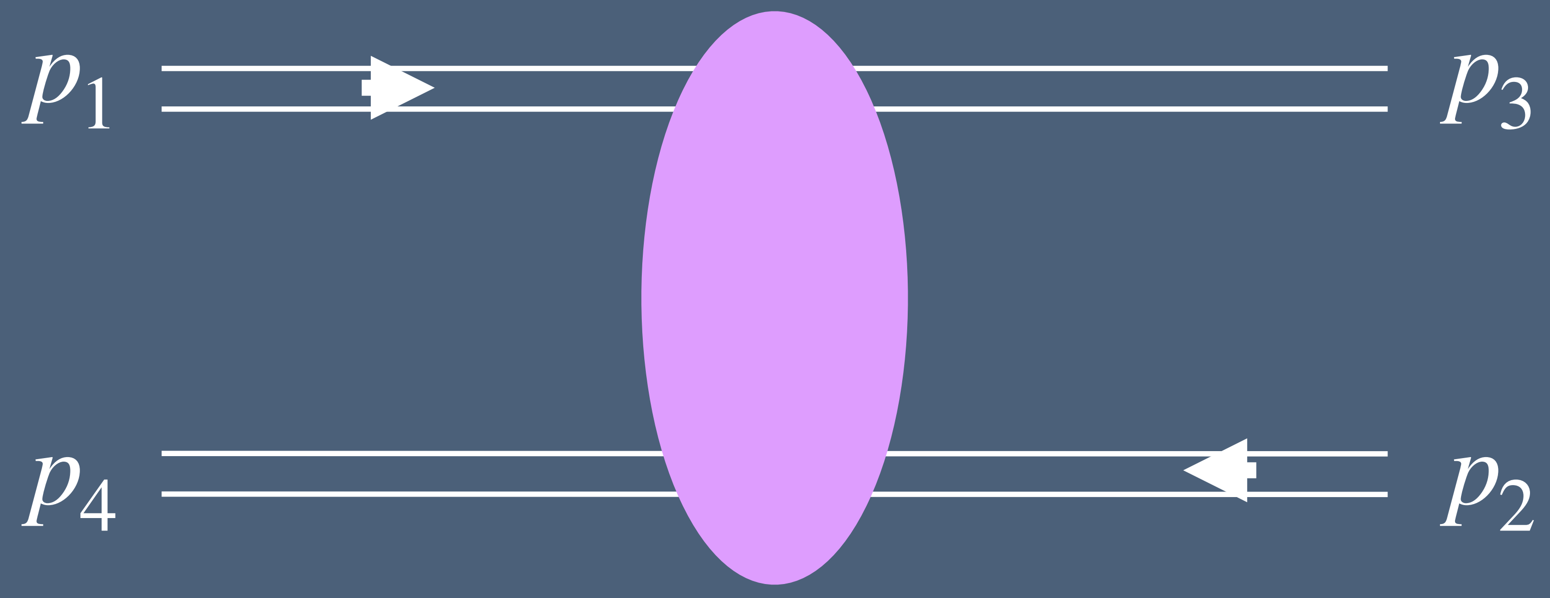
Final results available at https://zenodo.org/records/10227683

Go have a look and play with it :)

High energy limit

In collaboration with

Warm-up: $2 \to 2$ amplitudes in Regge kinematics



$$s \sim |u| \gg -t$$

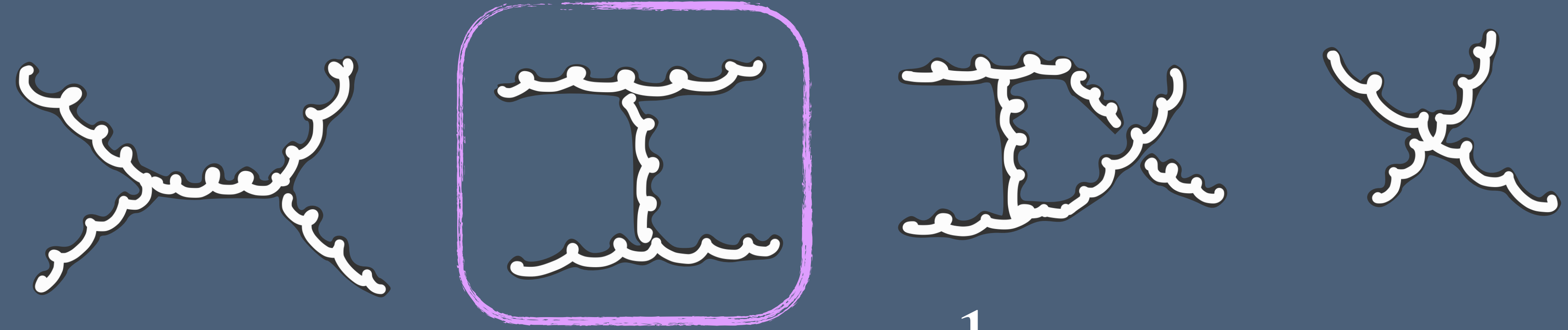
+ Large rapidity gap

$$x = \frac{-t}{s}$$

Light-cone components

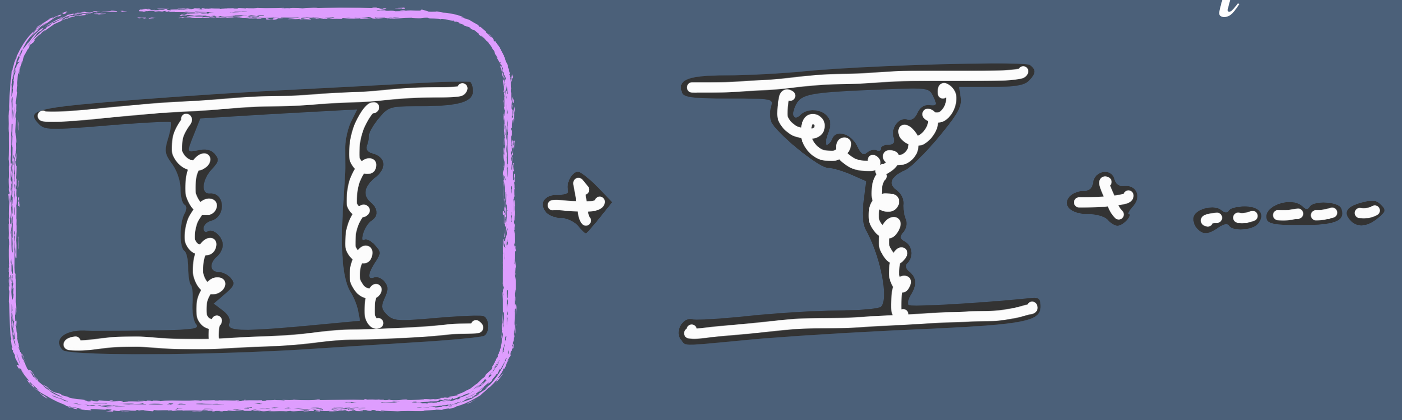
$$p_1^\mu \sim p_1^+ \quad p_2^\mu \sim p_2^-$$
$$p_3^+ \gg |p_3^\perp| \quad p_4^- \gg |p_4^\perp|$$
$$s_{12} = p_1^+ p_2^- \quad t \sim -k_\perp^2$$

Tree level:

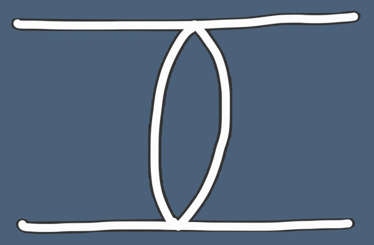


$$\frac{1}{t}$$

One loop:



Small x

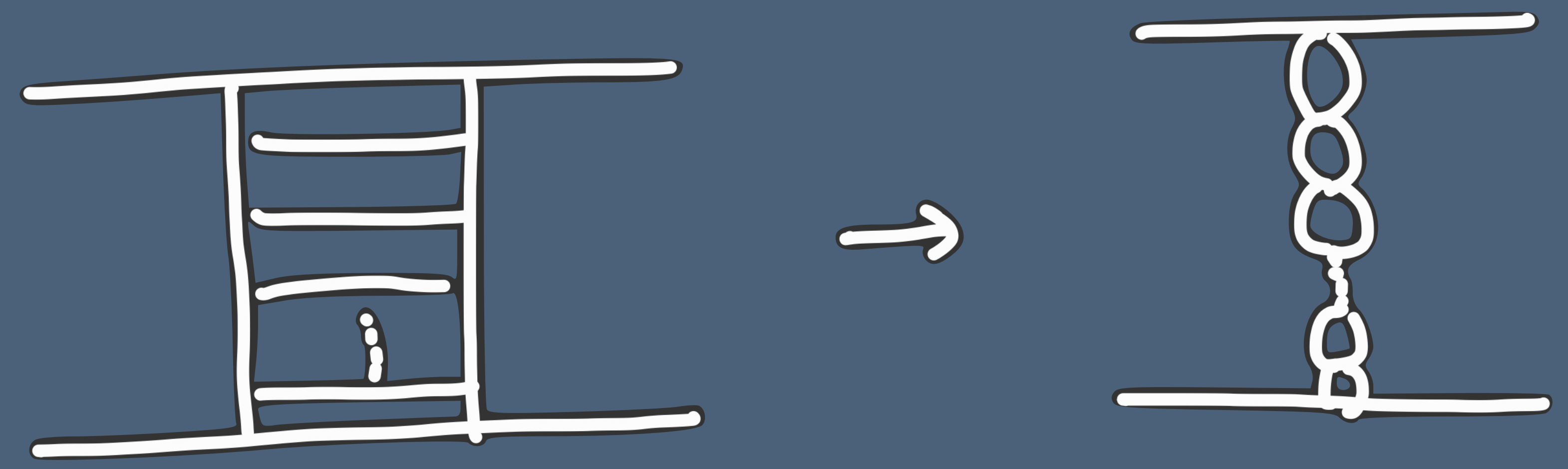


$$\times \frac{\log x}{x}$$

Dominant contribution (LL): $\mathscr{A} \sim \tau_g(t) \dfrac{\log x}{x}$

Warm-up: $2 \to 2$ amplitudes in Regge kinematics

Structure repeated to all orders: generalized ladder topologies



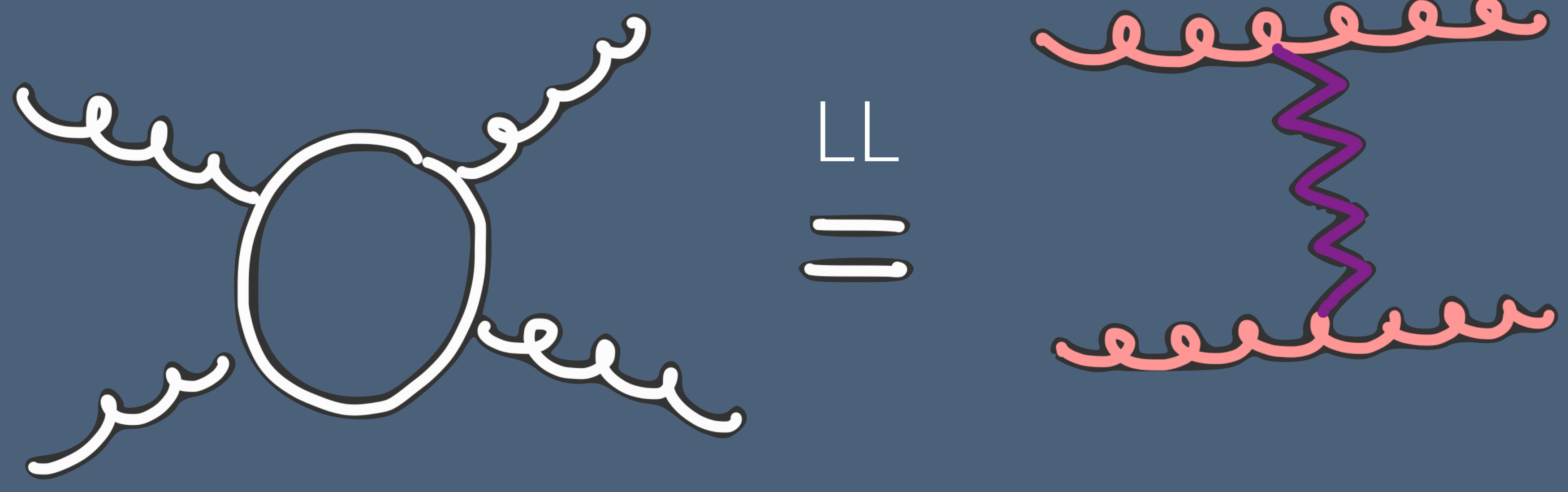
NB: this picture is schematic, in QCD things are more complicated

Factorization at LL

$$\mathscr{A}(s,t) \simeq \mathscr{A}^{(0)}(s,t)\left(\frac{s}{-t}\right)^{C_A\,\alpha_s\,\tau_g(t)}$$

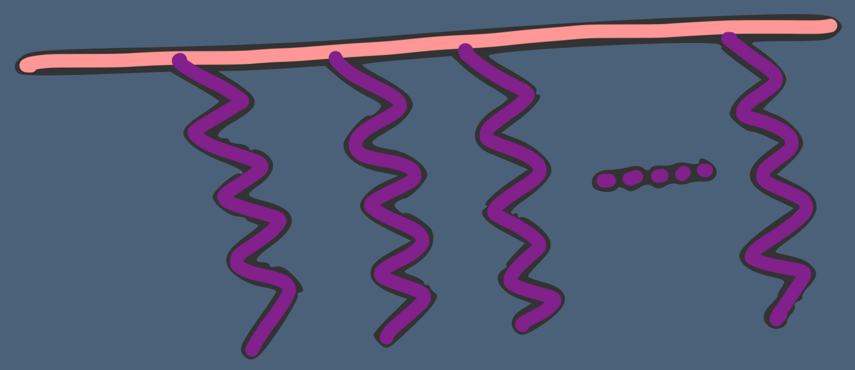
τ_g gluon Regge trajectory

Gluon “reggeization” at LL



Universal, does not depend on partonic nature of projectiles

Beyond LL: effective theory of “reggeons”



Regge pole is in the "odd amplitude": define signature eigenstates

$$\mathscr{A}^{\pm} = \frac{1}{2}\left(\mathscr{A}(s,t) \pm \mathscr{A}(-s-t,t)\right)$$

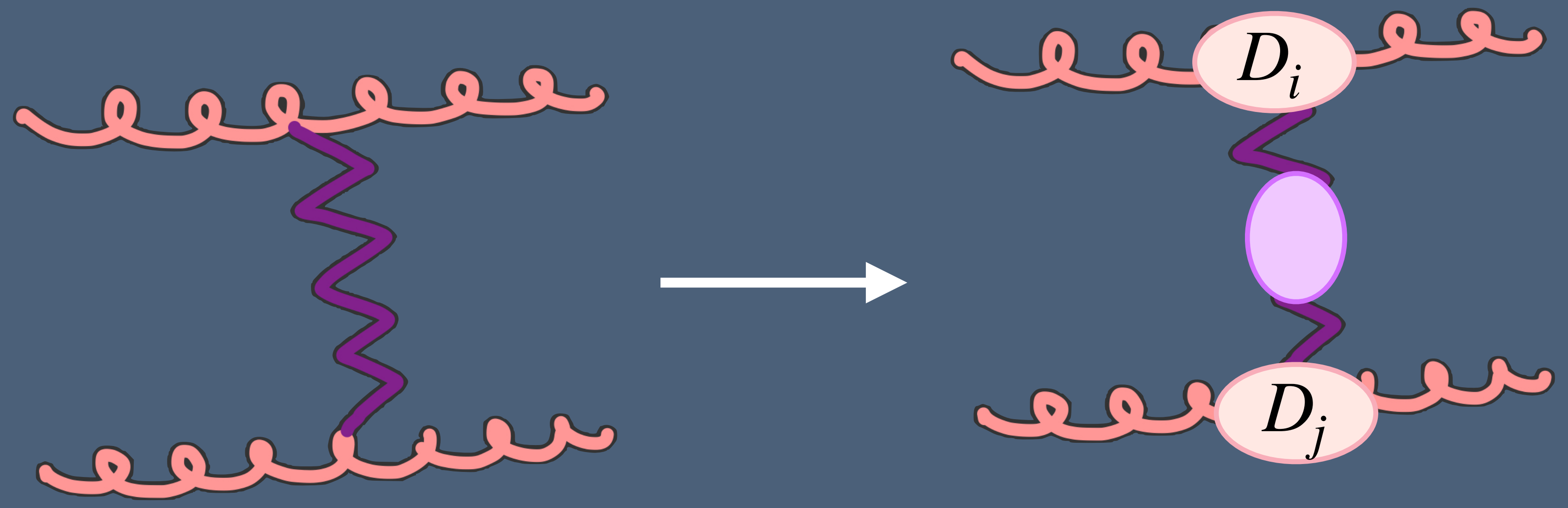
$\mathscr{A}^{+}$: "even" $\longrightarrow$ Starts with cut @1-loop

$\mathscr{A}^{-}$: "odd" $\longrightarrow$ Contains Regge pole

NLL factorization [Fadin, Lipatov hep-ph/9802290]

LL $\sim \left(\frac{\alpha_s}{2\pi}\right)^n \log^n x$

NLL $\sim \left(\frac{\alpha_s}{2\pi}\right)^n \log^{n-1} x$



Impact factors

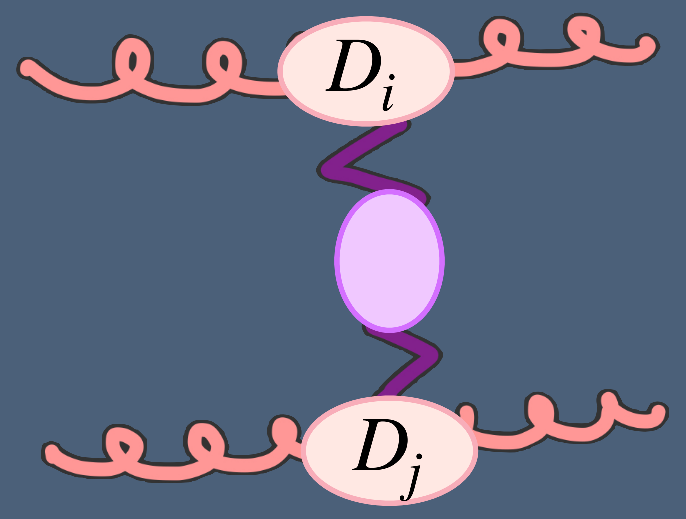
$$\mathscr{A}(s,t) \simeq \mathscr{A}^{(0)}(s,t) D_i(t) D_j(t) \left(\frac{s}{-t}\right)^{C_A \alpha_s \tau_g(t,\alpha_s)}$$

Corrections to Regge trajectory

Factorization still holds (in $\mathscr{A}^{-}$)

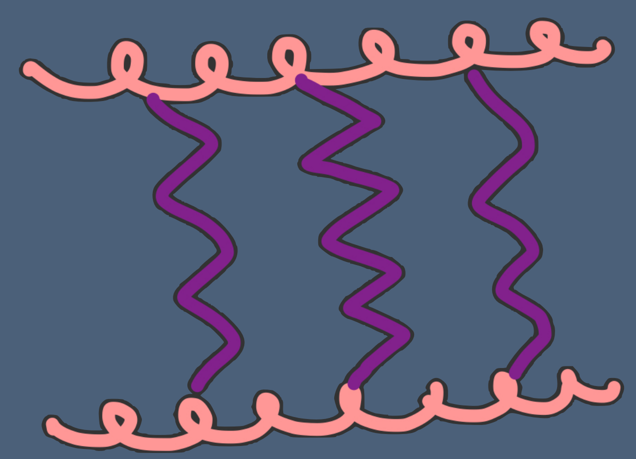
NNLL factorization

$$\text{NNLL} \sim \left(\frac{\alpha_s}{2\pi}\right)^n \log^{n-2} x$$



"Pole"

+


"Cut"

Regge cuts responsible for violation of factorization at NNLL

Factorization is restored once "cut contamination" is removed

How?

Wilson line approach

[Balitsky/JIMWLK + Caron-Huot 1309.6521, Caron-Huot, Gardi, Vernazza 1701.05241]

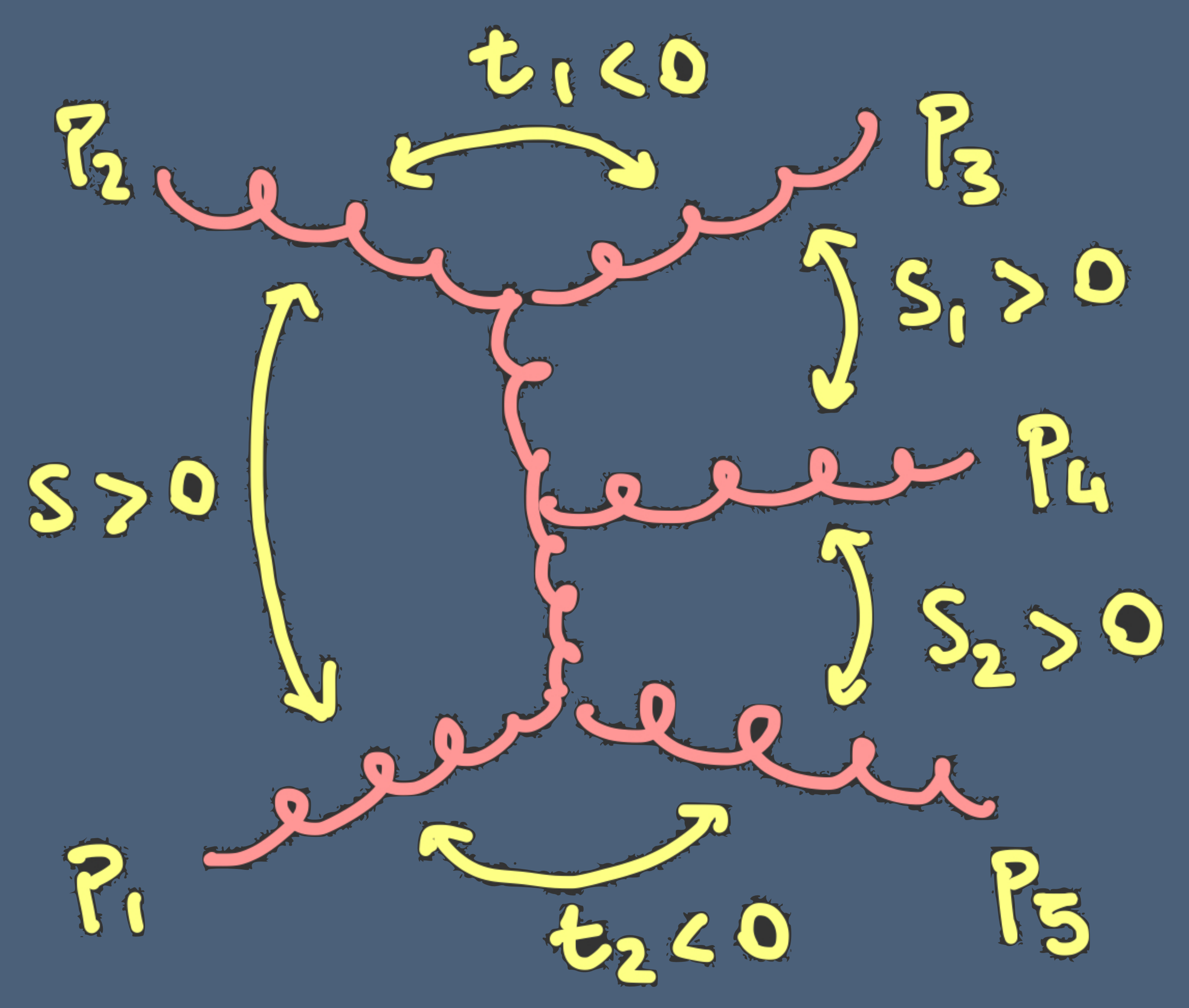
$$U_r(z) \equiv \mathscr{P} \exp\left(ig \int_{-\infty}^{+\infty} dx^+ A_+^a(x^+, x^- = 0, z) T_r^a \right)$$

$2 \to 3$ amplitudes: Multi-Regge kinematics

$$A^{(h_A)}(p_1)\, B^{(h_B)}(p_2) \to B'^{(h_{B'})}(p_3)\, g^{(h_g)}(p_4)\, A'^{(h_{A'})}(p_5)$$

Strong rapidity ordering, no ordering in transverse components, gluon centrally emitted

$$p_5^+ \gg p_4^+ \gg p_3^+\,, \quad p_3^- \gg p_4^- \gg p_5^-\,, \quad p_4^\pm \sim |p_{3,\perp}| \sim |p_{4,\perp}| \sim |p_{5,\perp}|$$



Longitudinal and transverse dynamics completely factorized

MRK limit: $x \to 0$

$$p_1^+, p_5^+, p_2^-, p_3^- \sim 1/x\,, \quad p_4^+, p_4^- \sim 1\,, \quad p_2^+, p_3^+, p_1^-, p_5^- \sim x$$

$$s_{12} = \frac{s}{x^2}, \quad s_{23} = -\frac{s_1 s_2}{s} z\bar{z}, \quad s_{34} = \frac{s_1}{x}$$

$$s_{45} = \frac{s_2}{x}, \quad s_{51} = -\frac{s_1 s_2}{s}(1-z)(1-\bar{z})$$

[Caron-Huot, Chicherin, Henn, Zhang, Zoia 2003.03120]

Signature and color decomposition

$$\mathscr{A}^{(\sigma_a,\sigma_b)}(p_1,p_2,p_3,p_4,p_5) \sim \mathscr{A}(p_1,p_2,p_3,p_4,p_5) + \sigma_a\mathscr{A}(p_5,p_2,p_3,p_4,p_1)$$
$$+\sigma_b\mathscr{A}(p_1,p_3,p_2,p_4,p_5) + \sigma_a\sigma_b\mathscr{A}(p_5,p_3,p_2,p_4,p_1)$$

$\mathscr{A}^{(-,-)}$ odd, odd

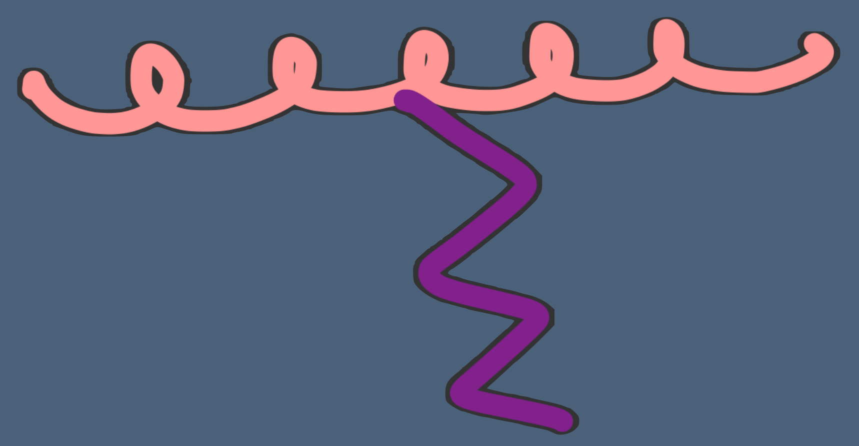
$\mathscr{A}^{(+,+)}$ even, even

$\mathscr{A}^{(+,-)}$ even, odd

$\mathscr{A}^{(-,+)}$ odd, even

Contains pole contributions, relevant part for factorization Beyond NLL, receives cut contamination

Color: t-channel in irreducible representation



symmetric

$$8 \otimes 8 = 0 + 1 + 8_a + 8_s + 10 + \overline{10} + 27$$

antisymmetric

Only $(8_a,8_a)$ amplitude receives pole contribution

Color basis

Elements of color basis denoted by (r_1, r_2), representations of the two t-channels

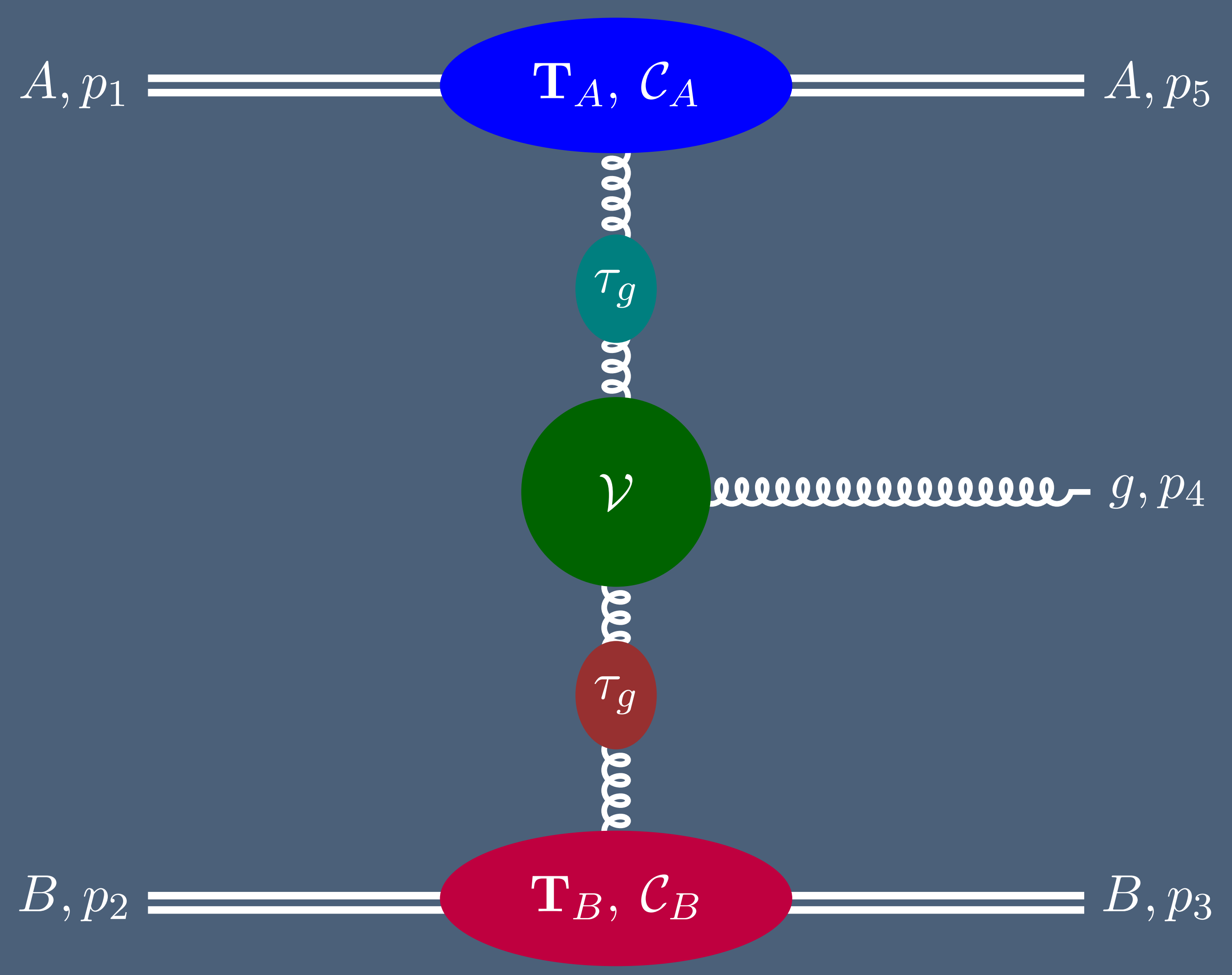
Example: $gg \to ggg$

$$8 \otimes 8 = 0 + 1 + 8_a + 8_s + 10 + \overline{10} + 27$$

1. $(8_a, 8_a)$
2. $(8_a, 8_s)$
3. $(8_s, 8_a)$
4. $(8_s, 8_s)_a$
5. $(1, 8_a)$
6. $(8_a, 1)$
7. $(8_a, 0)$
8. $(0, 8_a)$
9. $(8_a, 27)$
10. $(27, 8_a)$
11. $(8_a, 10 + \overline{10})$
12. $(10 + \overline{10}, 8_a)$
13. $(8_s, 10 - \overline{10})$
14. $(10 - \overline{10}, 8_s)$
15. $(27, 27)$
16. $(0, 0)$
17. $(10 + \overline{10}, 0)$
18. $(0, 10 + \overline{10})$
19. $(10 + \overline{10}, 27)$
20. $(27, 10 + \overline{10})$
21. $(10 + \overline{10}, 10 + \overline{10})$
22. $(10 - \overline{10}, 10 + \overline{10})$

Choice of color basis convenient to "separate" multi vs single reggeon exchanges

$2 \to 3$ amplitudes: factorization at NLL and beyond



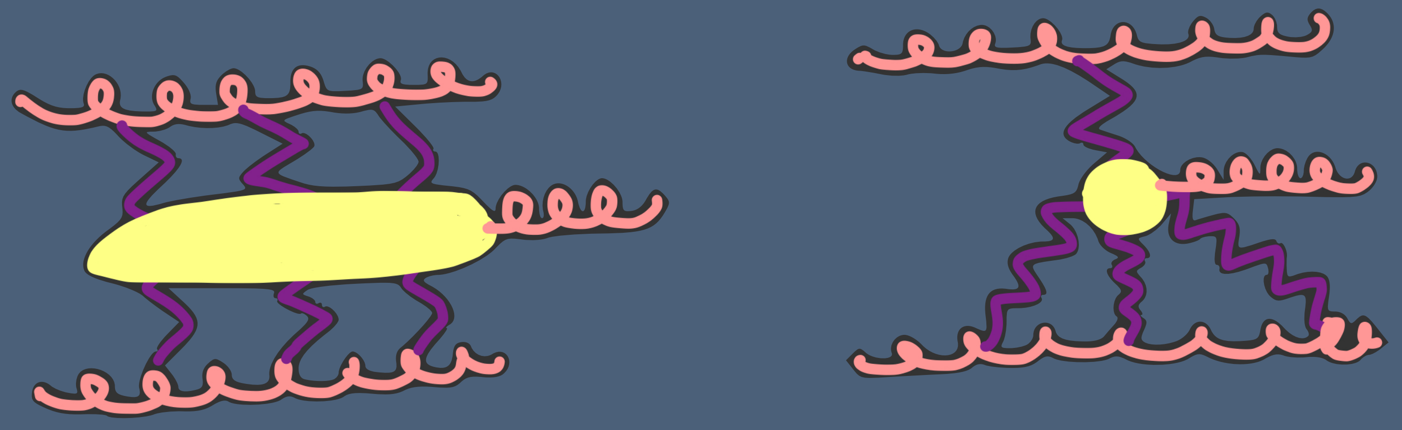
$\mathscr{A}^{(-,-)}$: Factorization at NLL works as expected, only pole contributions

$$\mathscr{A}^{\lambda}(s) = s_{12}\,[\mathbf{T}^a_A \mathscr{C}^{\lambda_A}_A(s_{51})]\,\frac{\mathscr{R}(s_{45}, s_{51})}{s_{51}}\,[f^{aba_4}\mathscr{V}^{h_g}]\,\frac{\mathscr{R}(s_{34}, s_{23})}{s_{23}}\,[\mathbf{T}^b_B \mathscr{C}^{\lambda_B}_B(s_{23})]$$

$$\mathscr{R}(s,t) = \frac{1}{2}\left[\left(\frac{s}{t}\right)^{\tau_g(t)} + \left(\frac{-s}{t}\right)^{\tau_g(t)}\right]$$

New ingredient: central emission vertex (Lipatov vertex)

At NNLL: cut contamination in $\mathscr{A}^{(-,-)}$, they enter $(8_a, 8_a)$



Strategy: compute the cut contributions from Wilson-line approach and subtract it to expansion of 5-point amplitudes to get universal structures

Need expansion of amplitude: rational + trascendental

Expansion of the amplitude in MRK

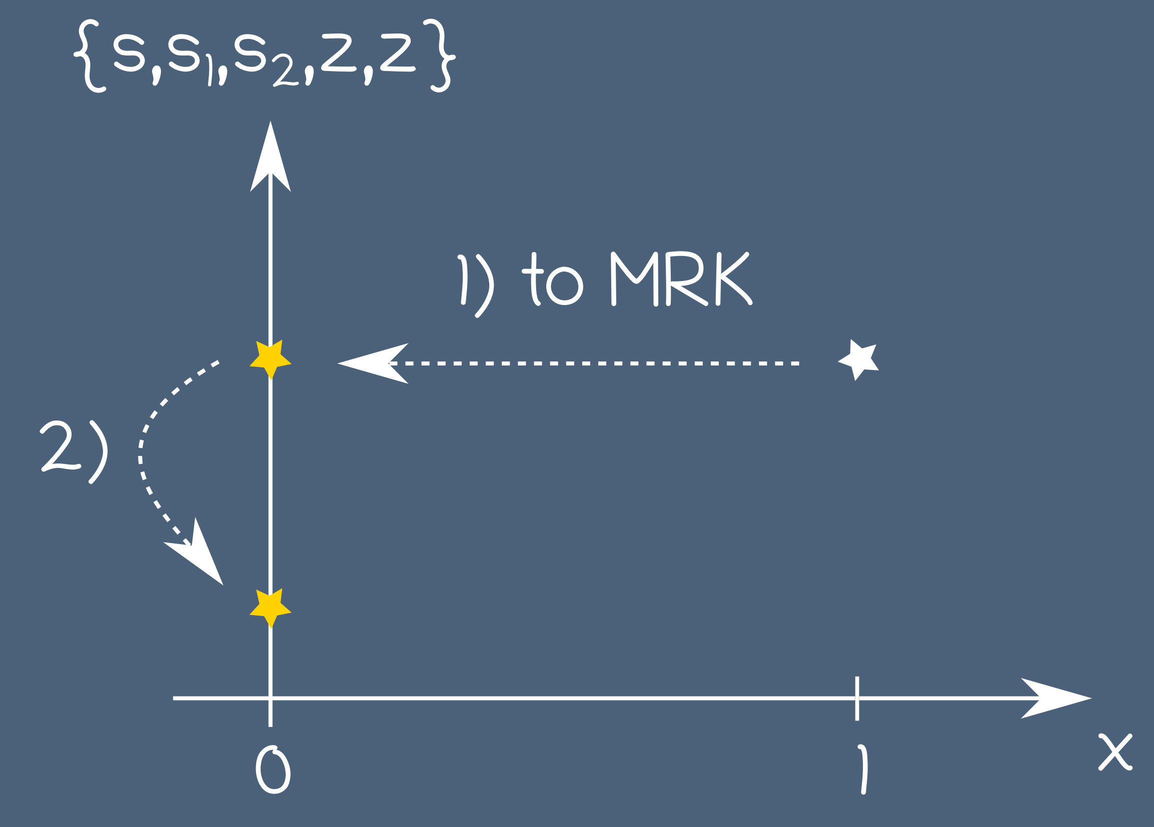
Rational

Transcendental

Scaling parameter
$x = 1$: "physical point",
$x = 0$: MRK

Set of PF and BC from [Chicherin, Sotnikov: 2009.07803]

$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\} \to \{s, s_1, s_2, z, \bar{z}\} + x$$



$$dI_i(\mathbf{s}) = \epsilon\, dA_{ij}(\mathbf{s})\, I_j(\mathbf{s}) \qquad dA_{ij}(\mathbf{s}) = \sum_{n=1} a^n_{ij}\, d\log(W_n)$$

$$\begin{cases} \dfrac{\partial}{\partial x}\mathbf{f}(x,y,\epsilon) = \epsilon\, A_x(x,y)\, \mathbf{f}(x,y,\epsilon) \\ \dfrac{\partial}{\partial y}\mathbf{f}(x,y,\epsilon) = \epsilon\, A_y(x,y)\, \mathbf{f}(x,y,\epsilon) \end{cases}$$

$$W_n \to W_n(x)$$

[Caron-Huot, Chicherin, Henn, Zhang, Zoia 2003.03120]

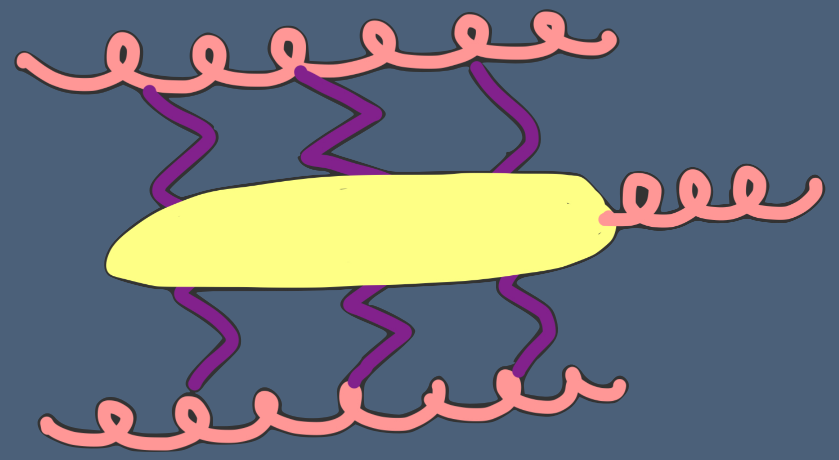
$$\{x\},\ \{\frac{s_1\, s_2}{s}\},\ \{s_1, s_2, s_1 - s_2, s_1 + s_2\},\ \{z, \bar{z}, 1 - z, 1 - \bar{z}, z - \bar{z}, 1 - z - \bar{z}\}$$

$$f^{(w)}(\mathbf{s}; x) = \sum_{n=0} \sum_{m=0}^{w} f^{(w)}_{mn}(\mathbf{s})\, x^n \log^m x$$

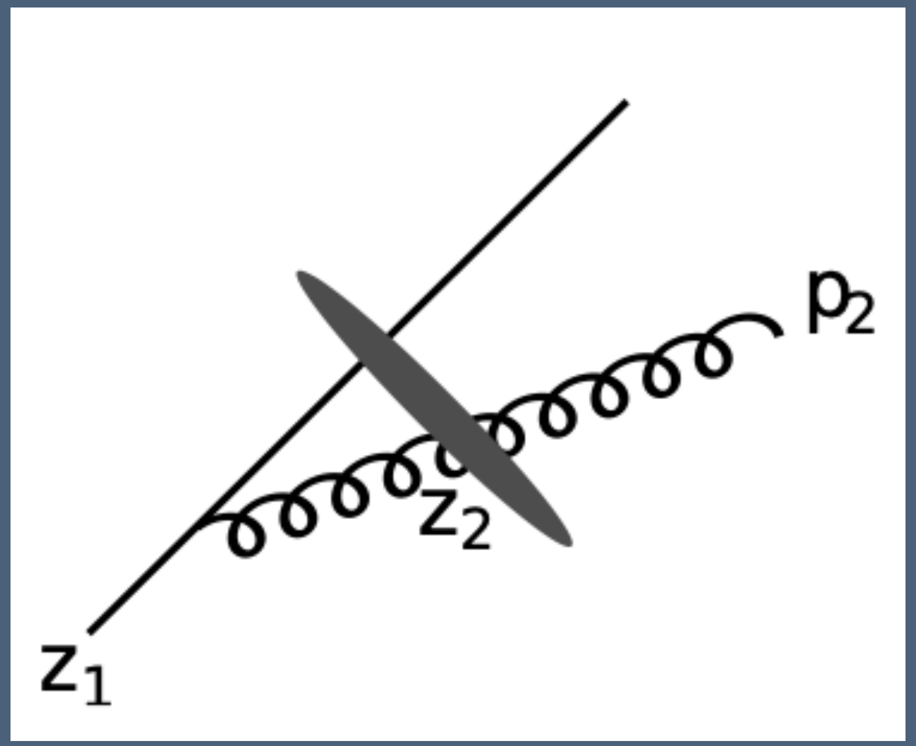
Cut contributions: OPE in the Wilson line approach

$$U = e^{i g_s T\cdot W}$$

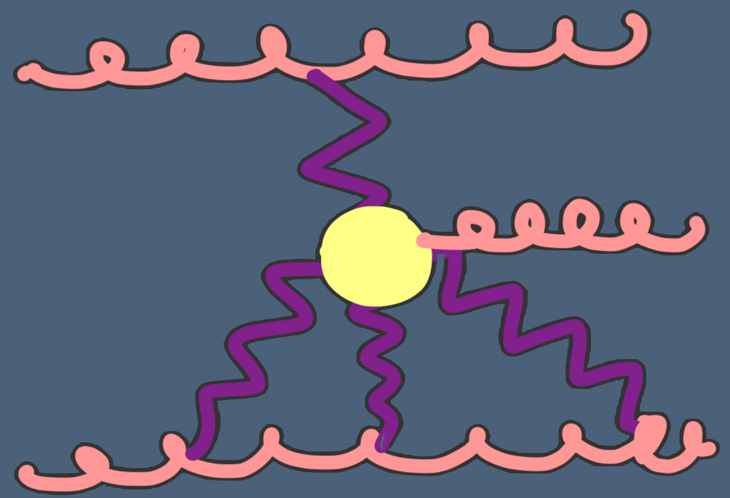
$$U(p_1)a^a(p_2) \sim -2g \int [dz_1][dz_2]\left[U^{ab}_{\mathrm{adj}}(z_2)\hat{T}^b_{R,1} - \hat{T}^a_{L,1}\right]U(z_1)e^{-ip_1\cdot z_1 - ip_2\cdot z_2}\int [dq]\frac{\epsilon\cdot q}{q^2}e^{iq\cdot(z_2-z_1)}$$



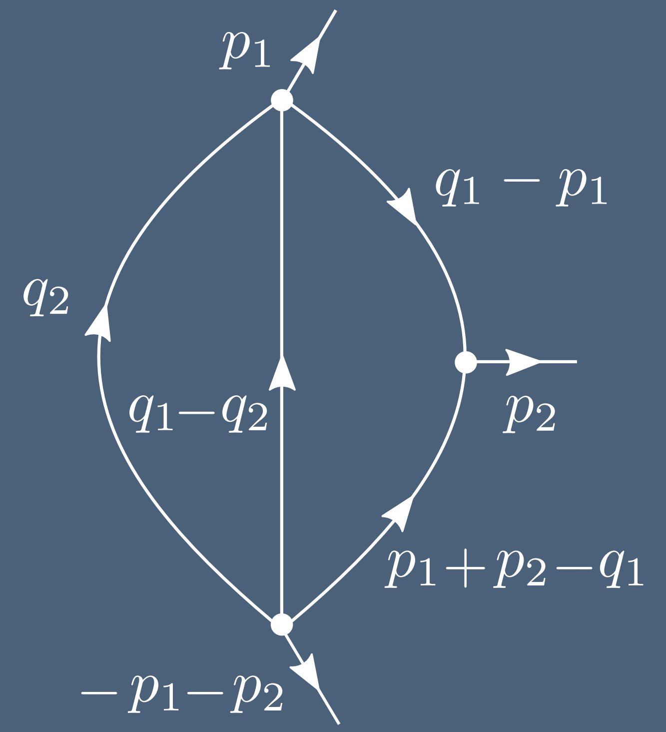
$$W_1^a W_1^b W_1^c a^d \sim 2g_s[F^{a_1}]^{da}\int [dq_1][dq_2]W^{a_1}(p_1+p_2-q_1)W^b(q_1-q_2)W^c(q_2)\times$$
$$\times\left[\frac{\epsilon\cdot p_2}{p_2^2} - \frac{\epsilon\cdot(q_1-p_1)}{(q_1-p_1)^2}\right] + (a\leftrightarrow b) + (a\leftrightarrow c) + \mathcal{O}(g_s^2)$$

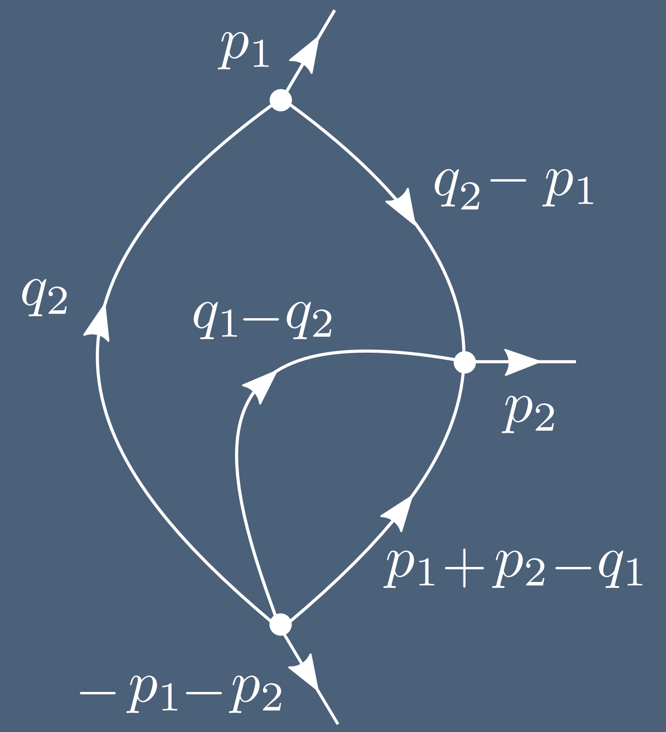


Caron-Huot [1309.6521]



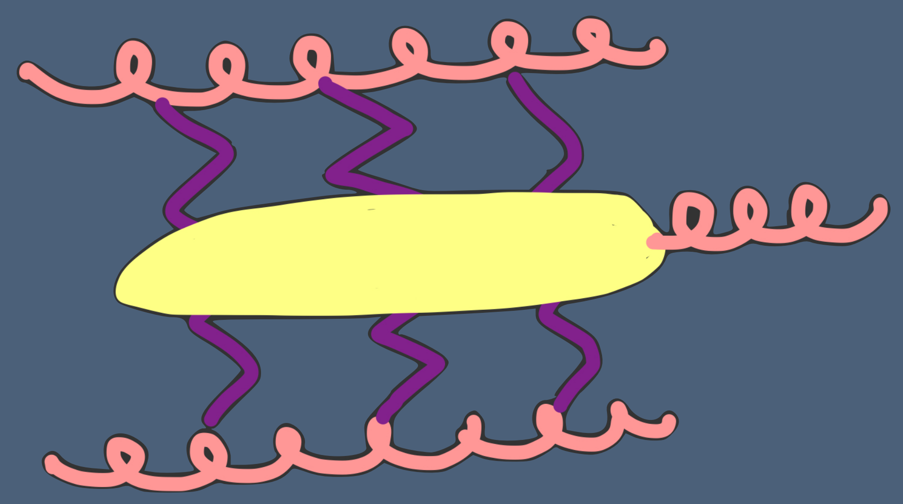
$$W_1^a a^b \sim -2g[\dots F\dots]^{ba} = 2g_s[F^{a_1}]^{ba}W^{a_1}(p_1+p_2)\left[\frac{\epsilon\cdot p_1}{p_1^2} + \frac{\epsilon\cdot p_2}{p_2^2}\right]$$
$$+ig_s^2[F^{a_1}F^{a_2}]^{ba}\int [dq_1]W^{a_1}(p_1+p_2-q_1)W^{a_2}(q_1)\left[\frac{\epsilon\cdot p_1}{p_1^2} + \frac{\epsilon\cdot(q_1-p_1)}{(q_1-p_1)^2}\right]+$$
$$+g_s^3[F^{a_1}F^{a_2}F^{a_3}]^{ba}\int [dq_1][dq_2]W^{a_1}(p_1+p_2-q_1)W^{a_2}(q_1-q_2)W^{a_3}(q_2)\times$$
$$\times\left[\frac{1}{6}\left(\frac{\epsilon\cdot(q_1-p_1)}{(q_1-p_1)^2}\right) - \frac{1}{2}\left(\frac{\epsilon\cdot(q_2-p_1)}{(q_2-p_1)^2}\right) - \frac{1}{3}\left(\frac{\epsilon\cdot p_1}{p_1^2}\right)\right] + \mathcal{O}(g_s^4)$$

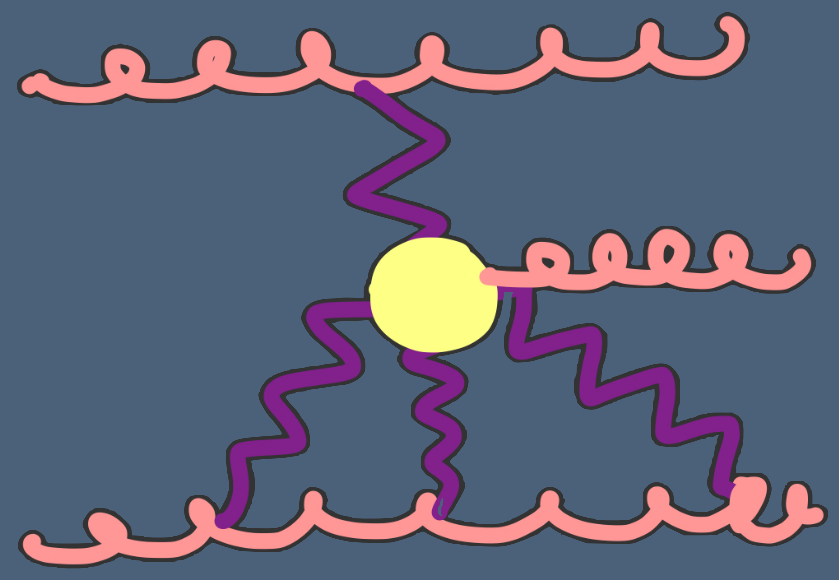




Our focus is (8,8)

Cut contributions in (8,8) amplitude: Wilson-line EFT approach





$$\mathscr{A}^{2,\mathrm{cut}}_{ab,[8,8]} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{i\pi}{3}\right)^2 \left(\frac{\mu^2}{|\mathbf{p}_4|^2}\right)^{2\epsilon} \left[\mathscr{F}_{\mathrm{LC}}(z,\bar{z},\epsilon) + \mathscr{F}_{ab}(z,\bar{z},\epsilon)\right]$$

$$\mathscr{F}_{\mathrm{LC}}(z,\bar{z},\epsilon) = N_c^2\left(\frac{2}{\epsilon^2} - \frac{\log(z\bar{z}) + \log((1-z)(1-\bar{z}))}{\epsilon} + 6iD_2(z,\bar{z}) - 2\zeta_2 \right.$$
$$\left. + \frac{5}{2}\log^2(z\bar{z}) + \frac{5}{2}\log^2((1-z)(1-\bar{z})) - \log(z\bar{z})\log((1-z)(1-\bar{z}))\right)$$

LC is universal

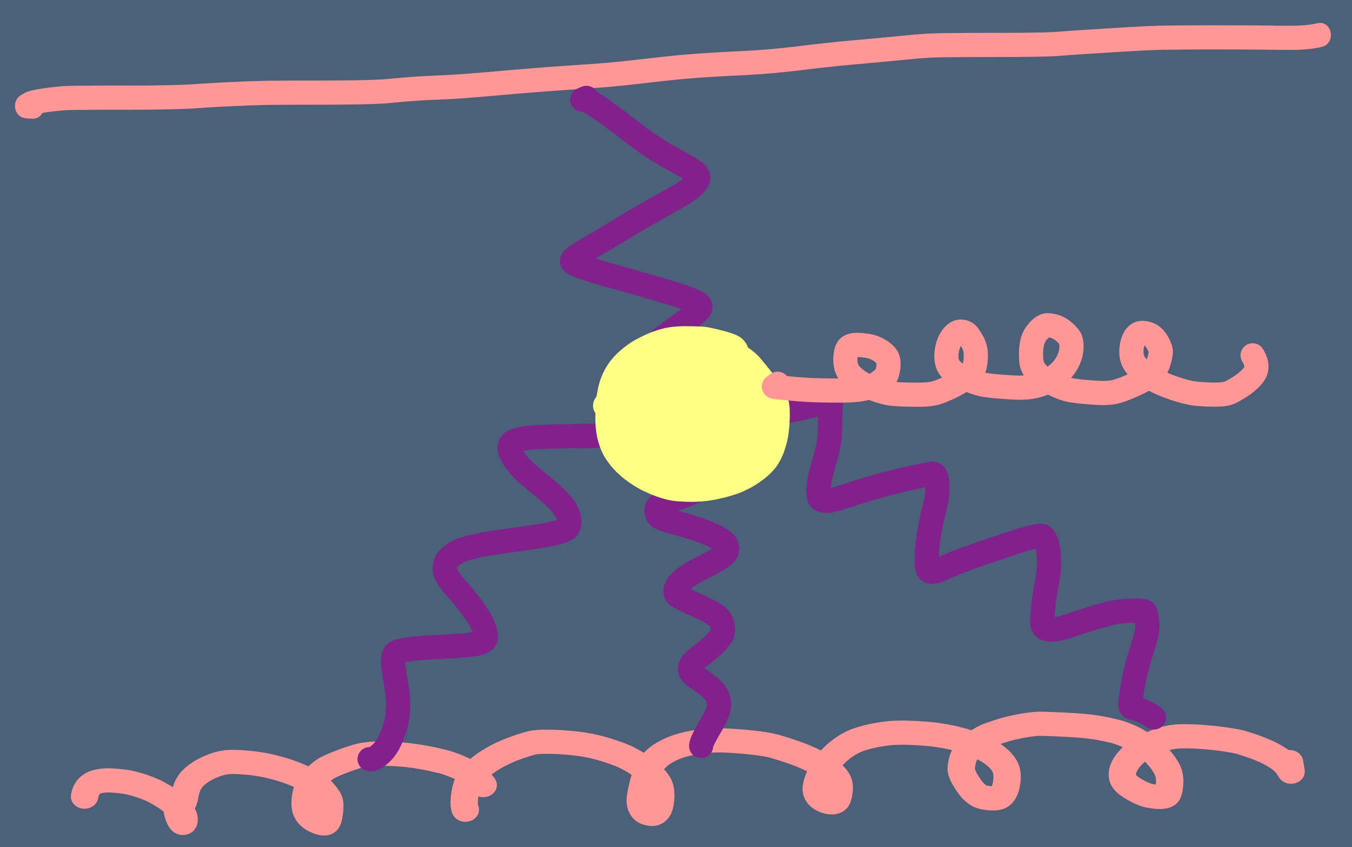
$$\mathscr{F}_{qq}(z,\bar{z},\epsilon) = \frac{18}{\epsilon}\big(\log((1-z)(1-\bar{z})) + \log(z\bar{z})\big) - 9\big(\log((1-z)(1-\bar{z})) + \log(z\bar{z})\big)^2$$
$$+108iD_2(z,\bar{z}) + \frac{1}{N_c^2}\left[\frac{18}{\epsilon^2} - \frac{36}{\epsilon}\big(\log((1-z)(1-\bar{z})) + \log(z\bar{z})\big) - 108iD_2(z,\bar{z})\right.$$
$$\left. -18\zeta_2 + 36\log^2((1-z)(1-\bar{z})) + 36\log^2(z\bar{z}) + 18\log((1-z)(1-\bar{z}))\log(z\bar{z})\right]$$

SLC is non-universal

$$\mathscr{F}_{qg}(z,\bar{z},\epsilon) = \frac{27}{\epsilon^2} + \frac{1}{\epsilon}\big(54\log((1-z)(1-\bar{z})) - 36\log(z\bar{z})\big) + 216iD_2(z,\bar{z}) - 27\zeta_2 + 45\log^2(z\bar{z}) - 36\log(z\bar{z})\log((1-z)(1-\bar{z}))$$

$$D_2(z,\bar{z}) = -i\left(\frac{\log(z\bar{z})}{2}\big(\log(1-z) - \log(1-\bar{z})\big) + \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\right)$$

In qg $1 \to 3$ reggeon interaction also contributes to $(8_a, 10+\overline{10})$: we can check it against the MRK expansion of the amplitude!



We find agreement

Odd-Odd: [3→1] in (8,10+10b)

```
(* result from expansion of two-loop amplitudes *)
Do[
  TwoLoopExp["+", "2q3g", "oo", col] =
   A2["2q3g", "MRK", "ppmpm", col] /. MRKpar /. sgn[_] → 1 /. G :→ tG /. {tG[0, x_] :→ Log[x], tG[1, x_] :→ Log[1 - x]} /. tG → G /. ss → s /.
    OddOdd;
  TwoLoopExp["+", "2q3g", "oo", col] = Collect[TwoLoopExp["+", "2q3g", "oo", col], {ep, Nc, π, _G, _Log}, Expand],
  {col, 1, 11}];

In[51]:= (* Prediction for 3→1 *)
PredictionCut2L["oo", 9] = (48 Nc π^4) Cut2L["oo", "3->1"];

In[52]:= (* check *)
Collect[(TwoLoopExp["+", "2q3g", "oo", 9] - PredictionCut2L["oo", 9] // PowerExpand // GToHPL // HPLConvertToKnownFunctions),
   ep, Factor] /. LisToGs // Expand // ShuffleG

Out[52]= 0
```

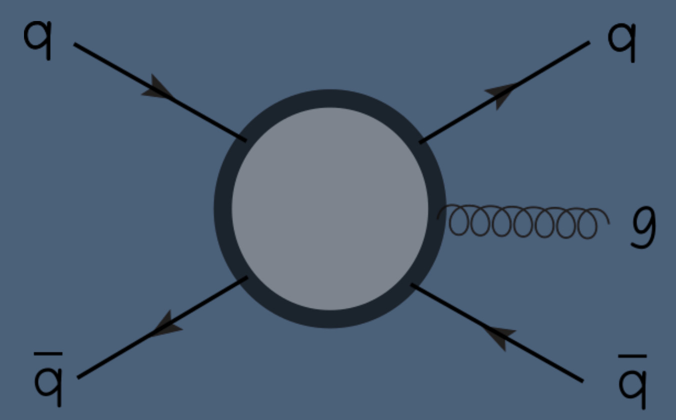
$$\mathscr{A}^{2,\mathrm{cut}}_{qg,[8,10+\overline{10}]} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{i\pi}{3}\right)^2 \left(\frac{\mu^2}{|\mathbf{p}_4|^2}\right)^{2\epsilon} \Bigg[\frac{9}{2\epsilon^2} + \frac{18\log((1-z)(1-\bar{z})) - 9\log(z\bar{z})}{\epsilon}$$

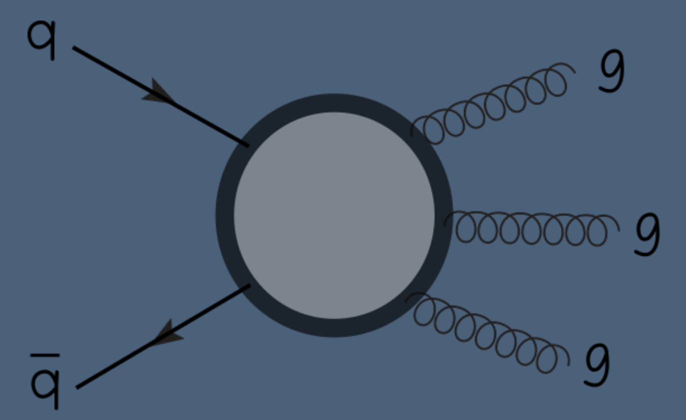
$$+54iD_2(z,\bar{z}) - \frac{9}{2}\log^2((1-z)(1-\bar{z})) + 9\log^2(z\bar{z}) - 9\log(z\bar{z})\log((1-z)(1-\bar{z})) - \frac{3\zeta_2}{4}\Bigg]$$

2-loop central emission vertex

NEW!!!

- Three-loop Regge trajectory
- Two-loop impact factors





2-loop Lipatov vertex

Universality at NNLL!!!

Final formula not instagrammable yet BUT

- Weight drop in finite remainder (weight 4 $\sim$ prod of lower weights)
- Spurious $z - \bar{z}$ cancels in finite remainder (seen at symbol level for now)

Summary and outlook

- Review of the computation of full color 5-point 2-loop QCD massless amplitudes
- Exploration of high energy regime: MRK kinematics
- Beyond NLL factorization in the Regge pole is broken by multi-reggeon exchanges
- We computed the cuts with EFT approach and subtracted from expansion of the amplitude
- Preliminary results for Lipatov vertex @2-loops from qqggg & qqqqg & ggggg amplitudes

TO DO:

- What about other color structures?, check with N=4?, ...

Thank you!

Backup slides

Old but gold

$$\mathscr{A} = \sum \text{Feynman diagrams}$$

> 40k

Helicity projection

Tancredi, Peraro:
1906.03298 & 2012.00820

$$\mathscr{A} = \sum H_h T_h$$

Color decomposition

$$H = \sum_c H_c C_c$$

Integration by parts identities

$$H = \sum_{m,c} R^{mc} M_m C_c$$

Key step for simplification

$$R^{mc} M_m = \sum_{k=1}^{M} r_k^c(s_{ij}, d) M(s_{ij}, d)$$

$$a_k(s_{ij}, d) = \frac{N(s_{ij}, d)}{Q(d) D(s_{ij})} \longrightarrow a_k(s_{ij}, d) = \sum_l g_l(d)\, R_l(s_{ij})$$

Uni (d) +multivariate PF

MultivariateApart [Heller, von Manteuffel 2101.08283]

Computation of master integrals + algebraic simplification

Final result

$$R_k = \sum_{m_1+\ldots+m_{30}\leq p} a_{k,m_1\ldots m_{30}} M_{m_1\ldots m_{30}}$$

$$M_{m_1\ldots m_{30}} = q_1^{m_1}\ldots q_{25}^{m_{25}} s_{12}^{m_{26}}\ldots s_{51}^{m_{30}}$$

Look for linear relations amongst rational functions

$$\sum_k R_k b_k = 0 \qquad \sum_k a_{k,m_1\ldots m_{30}} b_k = 0$$

$$H = \sum_{m,c} R^{mc} M_m C_c$$

Key step for simplification

$$R^{mc}M_m = \sum_{k=1}^{M} r_k^c(s_{ij},d)M(s_{ij},d)$$

$$r_k^c(s_{ij},d) = \frac{N(s_{ij},d)}{Q(d)D(s_{ij})} \longrightarrow r_k^c(s_{ij},d) = \sum_l g_l(d)\, R_l^c(s_{ij})$$

Uni (d) +multivariate PF

MultivariateApart [Heller, von Manteuffel 2101.08283]

Not independent!

Absolutely crucial step (see later)

Computation of master integrals + algebraic simplification

Final result

2-loop central emission vertex

NEW!!!

With cuts under control we extract the 2-loop Lipatov vertex from the qqggg amplitude and check against the 4q1g channel

Universality at NNLL!!!

Other ingredients are known:

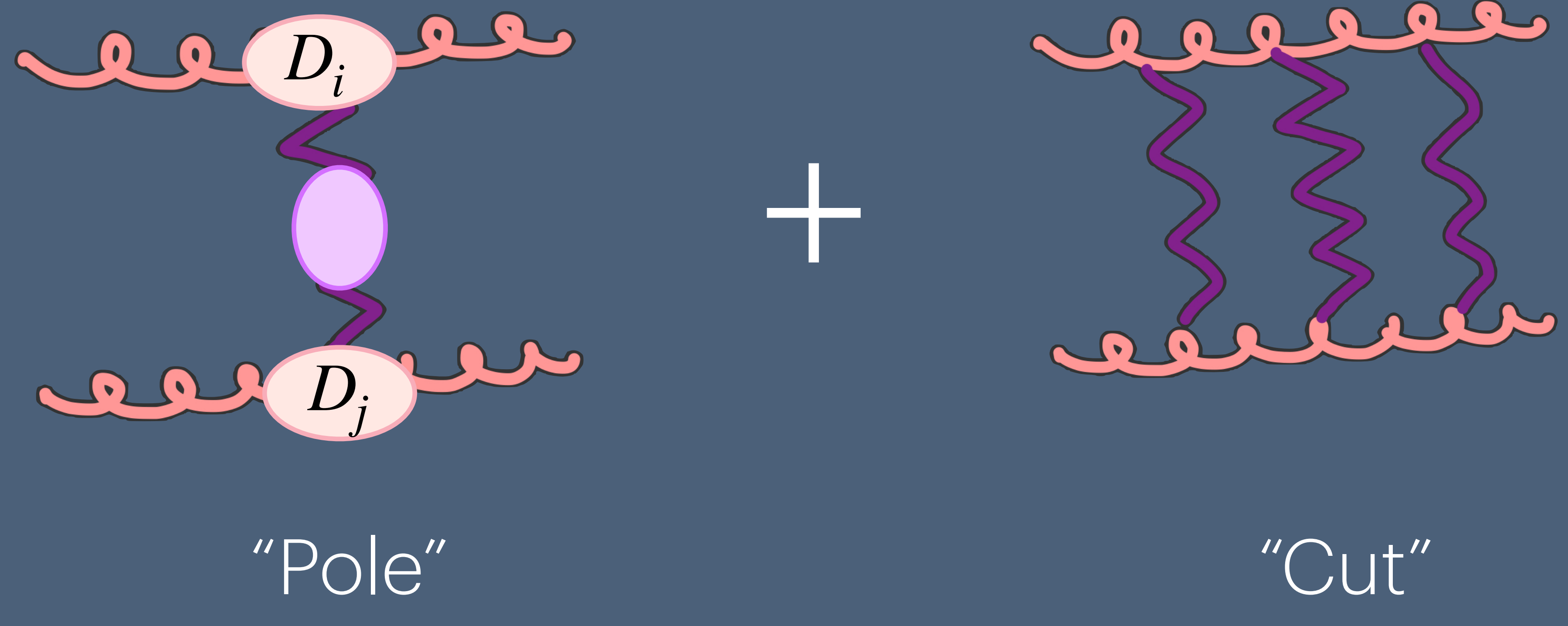
- Three-loop Regge trajectory
- Two-loop impact factors

Final formula not instagrammable yet BUT

- Weight drop in finite remainder (weight 4 $\sim$ prod of lower weights)
- Spurious $z - \bar{z}$ cancels in finite remainder (seen at symbol level for now)

NNLL factorization

NNLL $\sim \left(\frac{\alpha_s}{2\pi}\right)^n \log^{n-2} x$



Regge cuts responsible for violation of factorization at NNLL

Factorization is restored once "cut contamination" is removed

How?

Matched to amplitudes in a given theory

Leading color: universal

Sub leading color: non-universal

[Falcioni et al 2112.11098]

"LO" in the Wilson line correlators

Wilson line approach

[Balitsky/JIMWLK + Caron-Huot 1309.6521, Caron-Huot, Gardi, Vernazza 1701.05241]

Expansion of the amplitude in MRK

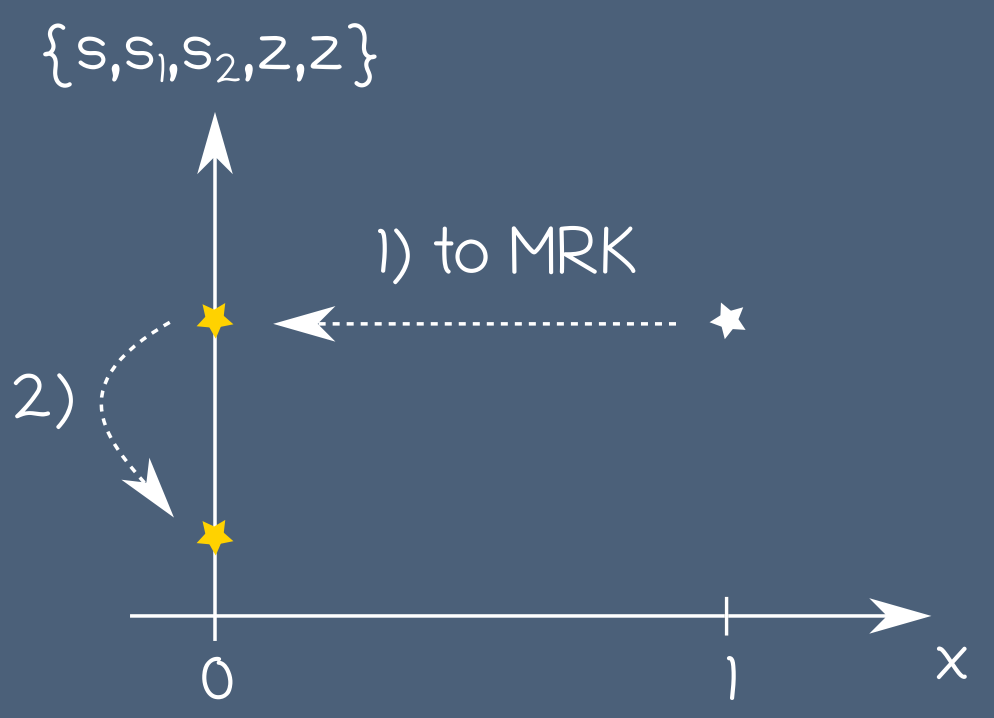
Rational

Transcendental

Scaling parameter

$x=1$: "physical point",

$x=0$:MRK



$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\} \to \{s, s_1, s_2, z, \bar{z}\} + x$$

$$W_n \to W_n(x)$$

[Caron-Huot, Chicherin, Henn, Zhang, Zoia 2003.03120]

$$dI_i(\mathbf{s}) = \epsilon\, dA_{ij}(\mathbf{s})\, I_j(\mathbf{s}) \qquad dA_{ij}(\mathbf{s}) = \sum_{n=1} a^n_{ij}\, d\log(W_n)$$

$$\{x\},\ \{\frac{s_1\, s_2}{s}\},\ \{s_1, s_2, s_1 - s_2, s_1 + s_2\},$$
$$\{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, 1-z-\bar{z}\}$$

$$\begin{cases} \dfrac{\partial}{\partial x}\mathbf{f}(x,y,\epsilon) = \epsilon\, A_x(x,y)\, \mathbf{f}(x,y,\epsilon) \\ \dfrac{\partial}{\partial y}\mathbf{f}(x,y,\epsilon) = \epsilon\, A_y(x,y)\, \mathbf{f}(x,y,\epsilon) \end{cases}$$

$$\mathbf{f}(x,y,z) = x^{\epsilon A_0} P \exp\left[\epsilon \int_{y_0}^{y} A_y(0,y')dy'\right] \mathbf{g}_0(\epsilon)$$

We need NNLP

$$A_x(x,y) = \frac{A_0}{x} + \sum_{k\geq 0} x^k A_{k+1}(y)$$

Solution at LP (x^0)

$$f^{(w)}(\mathbf{s};x) = \sum_{n=0} \sum_{m=0}^{w} f^{(w)}_{mn}(\mathbf{s})\, x^n \log^m x$$

Balitsky/JMWLK equation

$$-\frac{d}{d\eta}U(z_1)...U(z_n) = HU(z_1)...U(z_n),$$

$$\boxed{\begin{aligned}-\frac{d}{d\eta} = H = & \frac{\alpha_{s,b}}{2\pi^2}\int [dz_0][dz_i][dz_j]K_{ij,0}\times \\ & \left\{\left[\hat{T}^a_{i,L}\hat{T}^a_{j,L} + (L \leftrightarrow R)\right] - U^{ab}_{\text{adj}}(z_0)\left[\hat{T}^a_{i,L}\hat{T}^b_{j,R} + (i \leftrightarrow j)\right]\right\} + \mathcal{O}(\alpha^2_{s,b}),\end{aligned}}$$

$$\begin{aligned}H = & \int [dz_0][dz_i]K_{ii,0}\left\{-\frac{\alpha_{s,b}}{2\pi^2}C_A W^a_{0i}\frac{\delta}{\delta W^a_i} + \frac{\alpha^2_{s,b}}{3\pi}\text{Tr}\left[W_{0i}W_0W_{0i}\frac{\delta}{\delta W_i}\right] + ...\right\} + \\ & + \int [dz_0][dz_i][dz_j]K_{ij,0}\left\{-\frac{\alpha_{s,b}}{2\pi^2}[W_{0i}W_{0j}]^{xy} + \right. \\ & \left. + \frac{\alpha^2_{s,b}}{6\pi}[W_{0i}W_0W_0W_{0j} - W_{0i}W_0W_{0j}W_j - W_iW_{0i}W_0W_{0j}]^{xy} + ...\right\}\frac{\delta^2}{\delta W^x_i \delta W^y_j}.\end{aligned}$$

Poles and cuts

$$\mathcal{A}^{(-)} = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \sin\left(\frac{\pi j}{2}\right) a_j^{(-)}(t) e^{jL}$$

$$a_j^{(-)}(t) = \frac{1}{j-1-\alpha(t)}$$

$\alpha(t)$ → "Regge trajectory"

$$\mathcal{A}^{(-)}(s,t)|_{\text{Regge pole}} = \frac{\pi}{\sin\left(\frac{\pi\alpha(t)}{2}\right)} \frac{s}{t} e^{L\alpha(t)} + \text{sub-leading}$$

--"Regge Pole"

$$a_j^{(-)}(t) = \frac{1}{[j-1-\alpha(t)]^{1+\beta(t)}}$$

$$\mathcal{A}^{(-)}(s,t)|_{\text{Regge cut}} = \frac{\pi}{\sin\left(\frac{\pi\alpha(t)}{2}\right)} \frac{s}{t} \frac{1}{\Gamma(1+\beta(t))} L^{\beta(t)} e^{L\alpha(t)} + \text{sub-leading}$$

--"Regge cut"

Credits to Fabrizio :)

Some jargon…

$$\mathcal{A}(s,t) = \frac{1}{\pi}\int_0^\infty \frac{d\hat{s}}{\hat{s}-s-i\epsilon}\Delta_s(\hat{s},t) + \frac{1}{\pi}\int_0^\infty \frac{d\hat{u}}{\hat{u}+s+t-i\epsilon}\Delta_u(\hat{u},t)$$

Mellin moments

$$a_j^s(t) = \frac{1}{\pi}\int_0^\infty \frac{d\hat{s}}{\hat{s}}\Delta_s(\hat{s},t)\left(\frac{\hat{s}}{-t}\right)^{-j} \qquad \Delta_s(s,t) = \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} dj\, a_j^s(t)\left(\frac{s}{-t}\right)^j$$

Signature eigenstates

$$\mathcal{A}^{(\pm)}(s,t) = \frac{1}{2}\left(\mathcal{A}(s,t) \pm \mathcal{A}(-s-t,t)\right) \qquad a_j^{(\pm)}(t) = \frac{1}{2}\left(a_j^s(t) \pm a_j^u(t)\right)$$

$$\mathcal{A}^{(+)} = i\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \cos\left(\frac{\pi j}{2}\right) a_j^{(+)}(t) e^{jL} \qquad L = \frac{1}{2}\left(\ln\frac{-s-i\epsilon}{-t} + \ln\frac{-u-i\epsilon}{-t}\right)$$

$$\mathcal{A}^{(-)} = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \sin\left(\frac{\pi j}{2}\right) a_j^{(-)}(t) e^{jL} \qquad = \ln|s/t| - i\frac{\pi}{2}$$

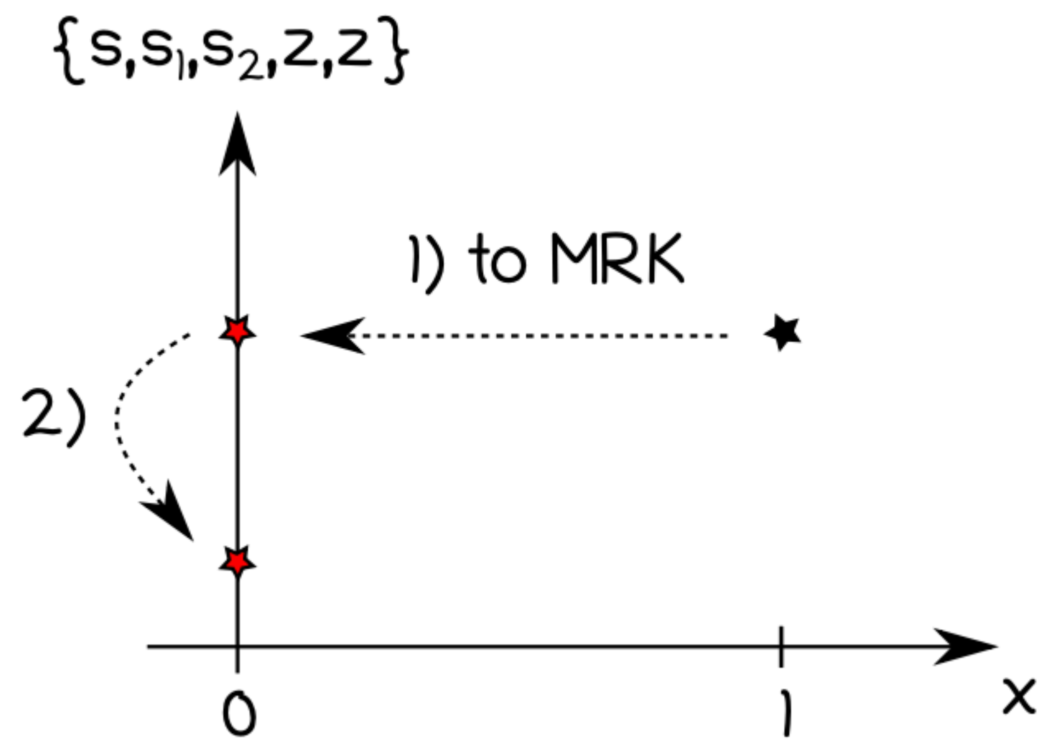
Expansion of pentagon functions in MRK

$$\mathrm{d}I_i(\vec{s}) = \epsilon\, \mathrm{d}A_{ij}(\vec{s}) I_j(\vec{s}) \qquad \mathrm{d}A_{ij}(\vec{s}) = \sum_{n=1} a_{ij}^n \mathrm{d}\log(W_n)$$

$W_n \to W_n(x)$ [Caron-Huot, Chicherin, Henn, Zhang, Zoia 2003.03120]

For fixed $\{s_{ij}\} \sim y$, one gets a 1-d differential equation in x

$$\begin{cases} \frac{\partial}{\partial x}\vec{f}(x,y,\epsilon) = \epsilon A_x(x,y)\vec{f}(x,y,\epsilon) \\ \frac{\partial}{\partial y}\vec{f}(x,y,\epsilon) = \epsilon A_y(x,y)\vec{f}(x,y,\epsilon) \end{cases} \qquad A_x(x,y) = \frac{A_0}{x} + \sum_{k\geq 0} x^k A_{k+1}(y)$$



$$\vec{f}(x,y,\epsilon) = x^{\epsilon A_0}\, \mathbb{P}\exp\left[\epsilon \int_{y_0}^{y} A_y(0,y')dy'\right] \vec{g}_0(\epsilon)$$

pentagon functions at LP

Nice property of MRK at LP:
Gram-determinant a perfect square
all square-roots in letters rationalized

$$\Delta = \epsilon_5^2 \underset{x\to 0}{\sim} \frac{s_1^2 s_2^2 (z-\bar{z})^2}{x^4} + \mathcal{O}\left(\frac{1}{x^3}\right)$$

$$\{x\}\,, \quad \left\{\frac{s_1 s_2}{s}\right\}\,, \quad \{s_1, s_2, s_1 - s_2, s_1 + s_2\}\,, \quad \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, 1-z-\bar{z}\}$$

Alphabet in MRK much simpler than in full kinematics (35→12 letters)

Federico Buccioni   Freiburg 04/06/2024   34

Credits to Federico :)