Quark Hadron Duality Violation

And its implications in inclusive semi-leptonic B decays

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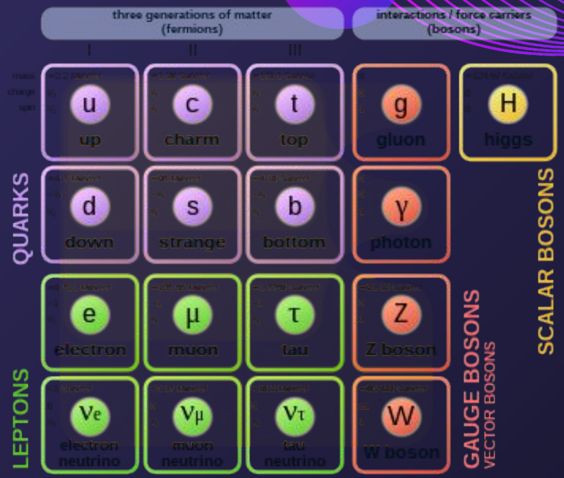
arXiv:2407.01473

Introduction

- Standard Model of particles



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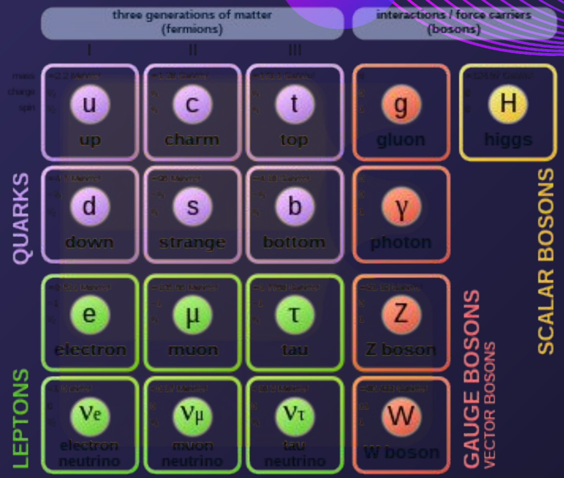

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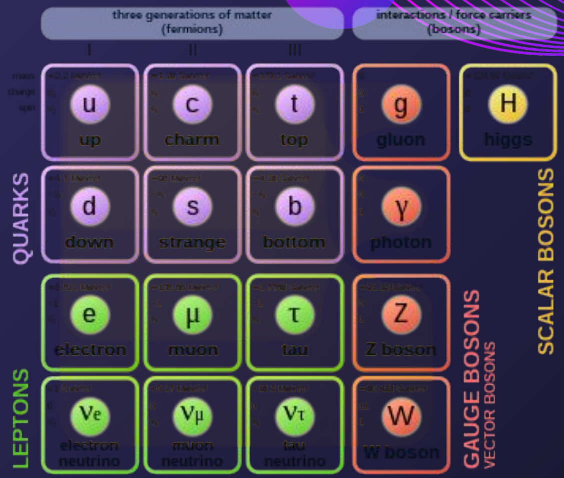

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- Extensions for Dark Matter,
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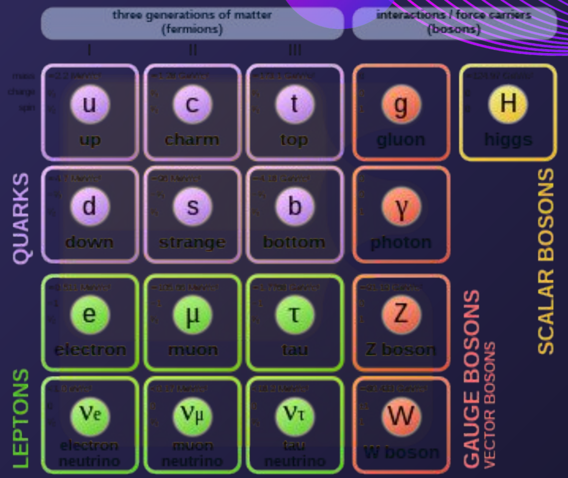
- Puzzles in the fundamental parameters of the SM
- CKM matrix elements essential for quark interactions



$$\frac{-g}{\sqrt{2}}\left(\overline{u_L}, \overline{c_L}, \overline{t_L}\right)\gamma^\mu W_\mu^+ V_{\mathrm{CKM}}\begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}$$

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

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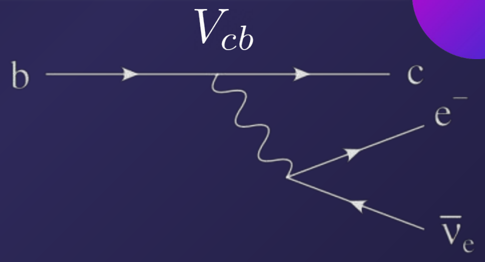
- Extraction of CKM element V_{cb} from weak decays



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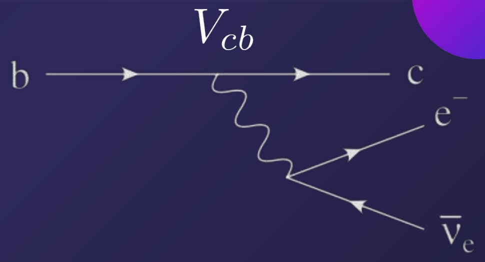

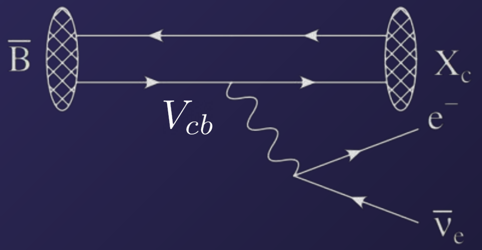

Manohar and Wise, 2000

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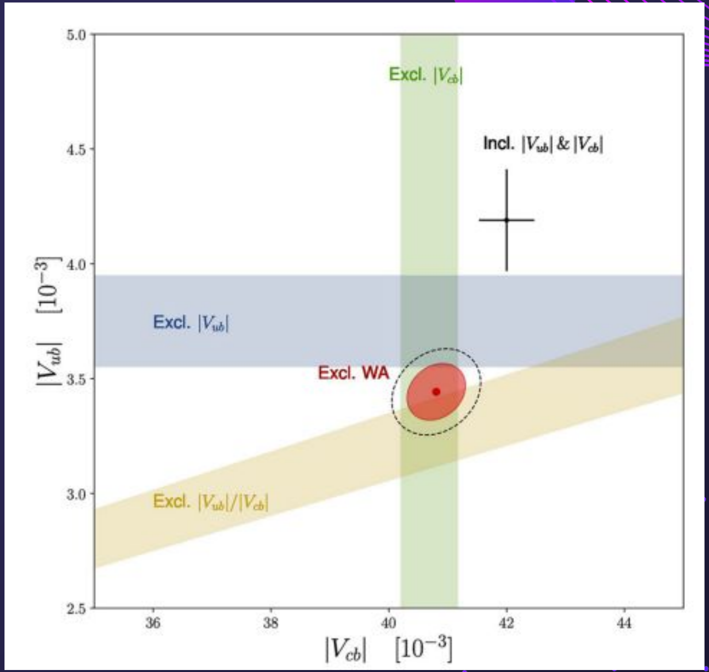

Bernlochner, Prim, Vos, 2024, EPJ. Spec. Top.

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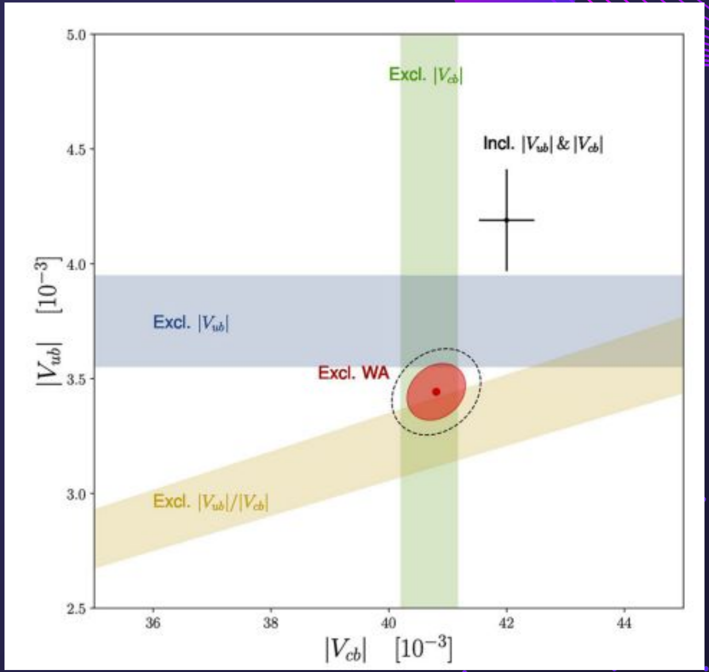
- We focus on inclusive decays calculable using Heavy Quark Expansion (HQE)



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Introducing Heavy Quark Expansion

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$$\Gamma(\overline{B} \to X_c \ell \overline{\nu}) = \Gamma^{(0)} + \frac{\Gamma^{(2)}}{m_b^2} + \frac{\Gamma^{(3)}}{m_b^3} + \ldots$$

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- Each term has α_s corrections and depends on so called HQE parameters that need to be extracted from data

$$\Gamma^{(3)}(\mu_\pi^2, \mu_G^2, \rho_D^3, \ldots)$$

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$$\Gamma^{(3)}(\mu_\pi^2, \mu_G^2, \rho_D^3, \ldots)$$

- Extract V_{cb} at percent precision

$$V_{cb} = (42.00 \pm 0.47) \times 10^{-3}$$

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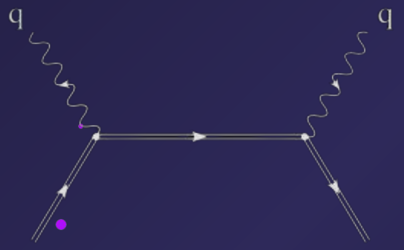
- Quark Hadron Duality (QHD) allows translation
  of predictions at the quark level to observables at the hadron level

Quark



Duality

Hadron

D

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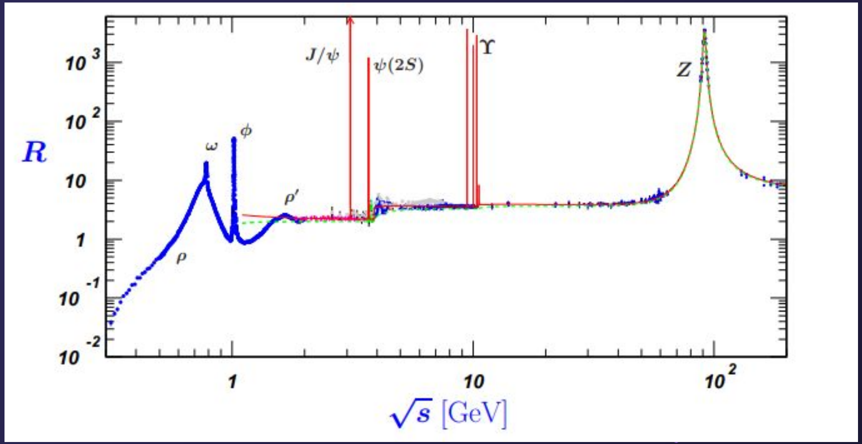
$$R = \frac{\sigma(\mathrm{e}^+\mathrm{e}^- \to \mathrm{hadrons})}{\sigma(\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-)} = 3\sum_q e_q^2$$



Ratio of cross section e+ e- as a function of centre of mass energy,

M. Tanabashi et al. (PDG), 2019

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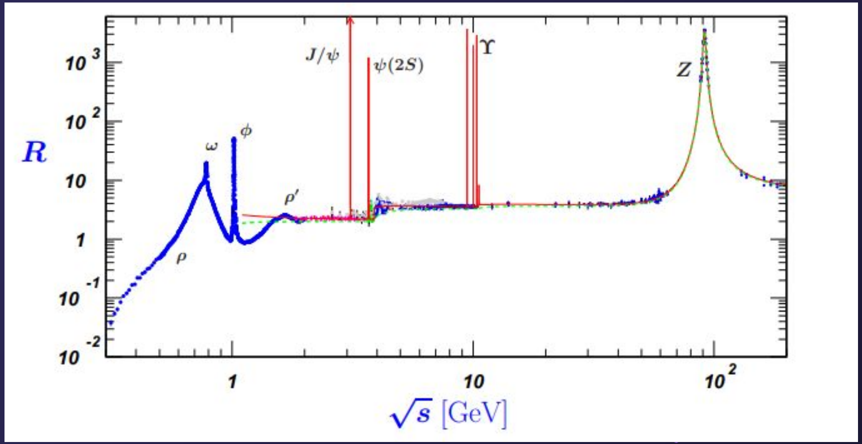
- Integrating away resonances similar to looking at the inclusive decay



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- Develop a model of QHDV in the context of the Heavy Quark Expansion (HQE)

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- In the quest of sub-percent $|V_{cb}|$ precision QHD Violation (QHDV) might become the limiting factor
- Develop a model of QHDV in the context of the Heavy Quark Expansion (HQE)
- Apply our model to observables of Semi-leptonic inclusive B decays

$Q = m_b\, v - q$

Modelling Duality Violation

- Asymptotic behaviour of the OPE expansion in $\frac{\Lambda_{QCD}}{Q}$ resulting in a non-converging series (like the perturbative case)

Chibisov et al. 1996, hep-ph/9605465

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Modelling Duality Violation

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- For example, an exponential function contributing (e.g. instanton)

$$\Pi(\tilde{Q}^2) \sim \exp\left(-\frac{1}{\omega}\sqrt{\tilde{Q}^2}\right)$$

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- Would induce factorial growth of HQE coefficients

$$C_l^{HQE} \sim (2l)!$$

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- Develop a model that estimates the uncertainty in the expansion using a Borel transform

- How does this model enter into the theory expression we need?

$$Q = m_b\, v - q$$

$$v = p/M_B$$

Modelling Duality Violation

- Differential decay rate from leptonic tensor hadronic tensor

$$\frac{d\Gamma}{dq^2 dE_\nu dE_\ell} \propto L^{\mu\nu} W_{\mu\nu}$$

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- Differential decay rate from leptonic tensor hadronic tensor

$$\frac{d\Gamma}{dq^2 dE_\nu dE_\ell} \propto L^{\mu\nu} W_{\mu\nu}$$

- Decompose in 5 scalar functions

$$W_{\mu\nu}(vQ, Q^2) = X_{\mu\nu}(W_1, W_2, W_3, W_4, W_5, vQ, Q^2)$$

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- Modelled Duality Violation contribution for the 1st and 4th scalar correlation function

$$\hat{\Delta}_{\mathrm{DV}} W_{1,4}(vQ; Q^2) = \frac{1}{\Lambda_{HQE} - vQ} \quad \frac{vQ}{\sqrt{Q^2}} \left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}} \right) - \sqrt{\frac{vQ}{\Lambda_{HQE}}} \sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}} \sqrt{\frac{Q^2}{vQ}} \right) \right)$$

- Our model estimates the shape but not the size of the DV contribution

Duality Violation model

- OPE + DV model $$W_i \to W_i^{(\mathrm{OPE})} + 0.25\,\mathcal{C}_{\mathrm{DV}}\hat{\Delta}_{\mathrm{DV}} W_i(s, \hat{q}^2, \Lambda_{HQE})$$

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- Default scale choice $$\Lambda_{HQE} = 0.5\,\mathrm{GeV}$$

$m_c = 1.092\,\mathrm{GeV}$ $m_b = 4.573\,\mathrm{GeV}$

LLSA: arXiv:1407.4384

$$\Gamma_0 = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3}$$

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- Normalised so that partonic and DV contributions are equal for C_{DV}=1 (= 100% Duality Violation)

$$\frac{\Gamma}{\Gamma_0} = 0.657 + 0.657\,\mathcal{C}_{\mathrm{DV}} - 0.025|_{m_b^2} - 0.026|_{m_b^3} + 0.0003|_{m_b^4} + 0.007|_{m_b^5}$$

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Observables

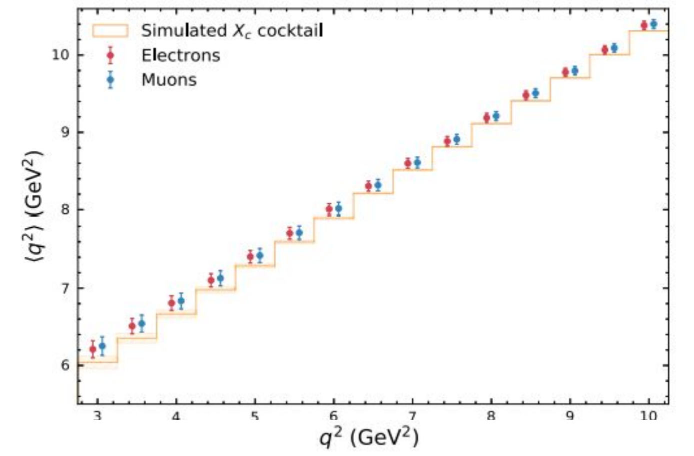
- Moments of the spectrum integrated with different lower bound (q^2_{cut})

$$Q_n(q^2_{\rm cut}) \equiv \frac{1}{\Gamma_0}\int_{q^2_{\rm cut}} \mathrm{d}q^2\,(q^2)^n \frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}$$

- Normalised to the rate

$$\bar{q}_n \equiv \langle (q^2)^n \rangle_{q^2 \geq q^2_{\rm cut}} \equiv \frac{Q_n(q^2_{\rm cut})}{Q_0(q^2_{\rm cut})}$$

- Measured in Belle and Belle II



Belle II: F. Abudinén et al. 2023, Phys. Rev. D 107, 072002

R. van Tonder et al. (Belle Collaboration) ,2021, Phys. Rev. D 104, 11201

DV sensitive observables

- q^2 moment decomposition

$$\bar{q}_i = C_i^{(0)} + \frac{\mu_G^2}{m_b^2} C_i^{(2)} + \frac{\tilde{\rho}_D^3}{m_b^3} C_i^{(3)} + R_i$$

$$R_i = R_{DV} + \sum_{n=0}^{\infty} R_{m_b^{4+n}}$$

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- Construct observables depending only on R_i by cancelling lower order contributions

$$R_i = R_{DV} + \sum_{n=0}^{\infty} R_{m_b^{4+n}}$$

$$O_{\mathrm{DV}}^{(3)} = \xi_1 \frac{\bar{q}_1}{m_b^2} + \xi_2 \frac{\bar{q}_2}{m_b^4} + \xi_3 \frac{\bar{q}_3}{m_b^6} + \xi_4 \frac{\bar{q}_4}{m_b^8}$$

$$\xi_{(2..4)}(q_{cut}^2, \xi_1)$$

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$$\xi_{(2..4)}(q^2_{cut}, \xi_1)$$

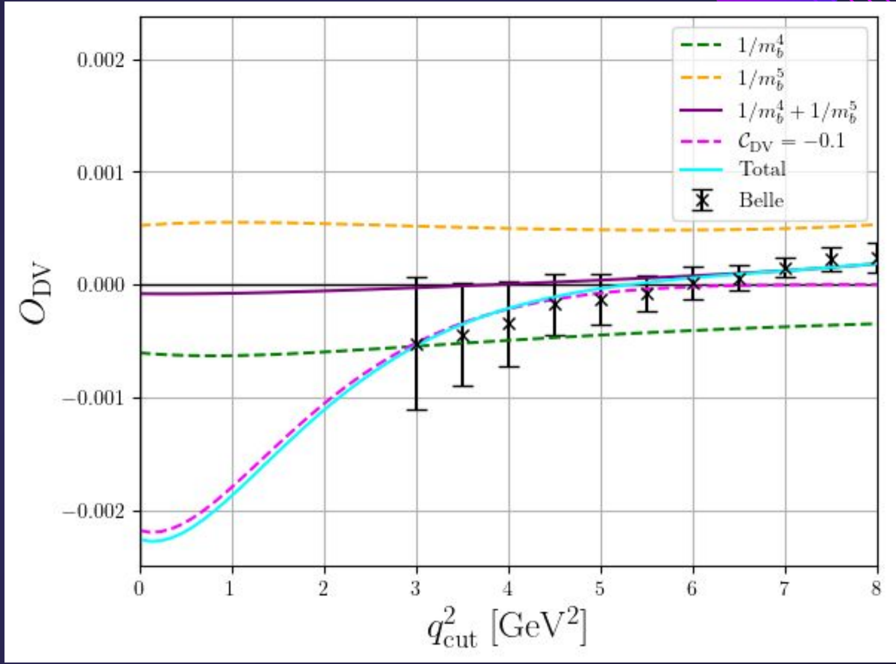
- No contribution from lower orders HQE

$$O_{\mathrm{DV}}^{(k)} \sim \Lambda_{HQE}^{k+1} / m_n^{k+1}$$

Data used from Belle collaboration, 2021
arxiv 2109.01685

QHDV from Belle data

- q^2 moment data from Belle electron channel (2021)



O_{DV} (q^2_{cut}) obtained from Belle data compared to theory and model predictions

QHDV from Belle data

- Comparison with theory (LLSA)

$$O^{(3)}_{\mathrm{DV}} = (5.182\,\mathcal{C}_{\mathrm{DV}} - 0.546|_{m_b^4} + 0.519|_{m_b^5}) \times 10^{-3} \qquad (q^2_{\mathrm{cut}} = 3.0\ \mathrm{GeV}^2)$$

LLSA: arXiv:1407.4384

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- Determine C_{DV} from data using LLSA values

$$\mathcal{C}_{\mathrm{DV}} = -0.10 \pm 0.11 \qquad (q^2_{\mathrm{cut}} = 3.0\ \mathrm{GeV}^2)$$
$$\mathcal{C}_{\mathrm{DV}} = -0.16 \pm 0.17 \qquad (q^2_{\mathrm{cut}} = 4.0\ \mathrm{GeV}^2)$$
$$\mathcal{C}_{\mathrm{DV}} = -0.30 \pm 0.30 \qquad (q^2_{\mathrm{cut}} = 5.0\ \mathrm{GeV}^2)$$

- Strongest constraint at low cuts

- Results consistent with $C_{DV} = 0$

LLSA: arXiv:1407.4384

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  - Procedure could also constraint higher order terms
- Combining different cuts and E_ℓ moments could further constrain Duality Violation

References presentation

- B. Chibisov, R. D. Dikeman, M. A. Shifman and N. Uraltsev, *Operator product expansion, heavy quarks, QCD duality and its violations*, Int. J. Mod. Phys. A 12 (1997) 2075–2133, [hep-ph/9605465]
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- F. Abudinén et al. (Belle II Collaboration), 2023, F. Abudinén et al. (Belle II Collaboration)Phys. Rev. D 107, 072002

BACKUP

$$\Gamma_\mu = \frac{\gamma_\mu(1-\gamma_5)}{2} \qquad Q = m_b\, v - q$$

$$v = p/M_B$$

Setting up the HQE

- Differential decay rate from leptonic tensor hadronic tensor

$$\frac{d\Gamma}{dq^2 dE_\nu dE_\ell} \propto L^{\mu\nu} W_{\mu\nu}$$

- Decompose in 5 scalar functions

$$W_{\mu\nu}\left(vQ, Q^2\right) = W_1\left(g_{\mu\nu} + \frac{Q_\mu v_\nu + Q_\nu v_\mu - i\epsilon_{\mu\nu\alpha\beta}Q^\alpha v^\beta}{vQ}\right) - W_2 g_{\mu\nu} + W_3 v_\mu v_\nu + W_4 \frac{(Q_\mu v_\nu + Q_\nu v_\mu)}{vQ} + W_5 \frac{Q_\mu Q_\nu}{(vQ)^2}$$

- Functions of the HQE parameters

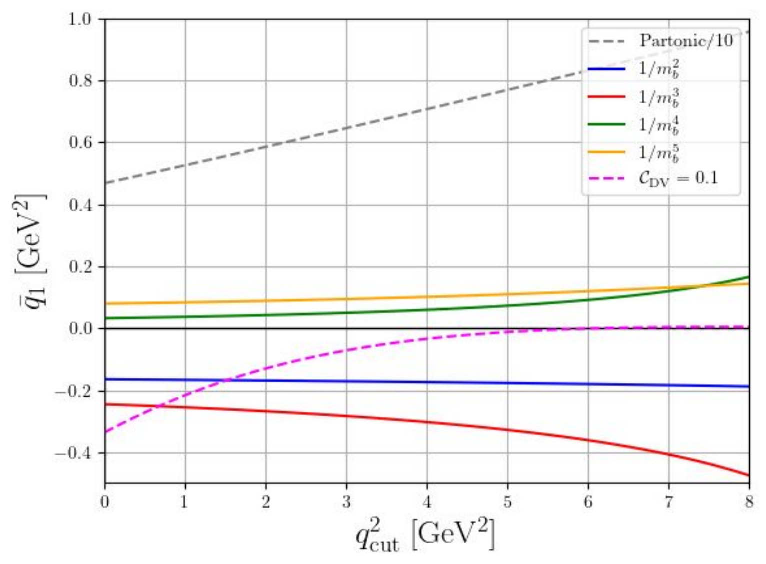
$$W_i = W_i(vQ, Q^2, \mu_\pi^2, \mu_G^2, \rho_D^3, \dots)$$

q^2 moments

$$Q_n(q^2_{\mathrm{cut}}) \equiv \frac{1}{\Gamma_0}\int_{q^2_{\mathrm{cut}}} \mathrm{d}q^2\,(q^2)^n \frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}$$

$$\bar{q}_n \equiv \langle (q^2)^n \rangle_{q^2 \geq q^2_{\mathrm{cut}}} \equiv \frac{Q_n(q^2_{\mathrm{cut}})}{Q_0(q^2_{\mathrm{cut}})}$$

- Non centralised q^2 moments
- DV most pronounced at low cut
- DV cut dependance differs slightly from power corrections
- Higher moments show a similar picture (see backup slides)
- DV may be difficult to disentangle from power corrections



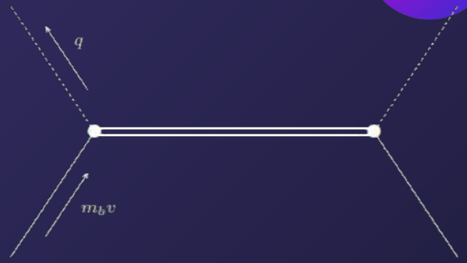
cut q^2 moments using LLSA values with DV contribution for Λ_{HQE} =0.5 GeV and C_{DV} = 0.1

$$Q = m_b v - q$$

Setting up the HQE

- Background field propagator

$$iS_{BGF} = \frac{1}{\not{Q} + i\not{D} - m_c}$$



- Expand in orders of the covariant derivative

$$\frac{1}{\not{Q} - m_c + i\not{D}} = \frac{1}{\not{Q} - m_c} - \frac{1}{\not{Q} - m_c}(i\not{D})\frac{1}{\not{Q} - m_c} + \frac{1}{\not{Q} - m_c}(i\not{D})\frac{1}{\not{Q} - m_c}(i\not{D})\frac{1}{\not{Q} - m_c} + \dots$$

- Choose $m_c = 0$

$$\frac{1}{\not{Q} + i\not{D}} = \sum_{k=0}^{\infty}\left(\frac{1}{Q^2}\right)^{k+1} \not{Q}[-(i\not{D})\,\not{Q}]^k$$

Dassigner et al. 2006
ArXiV: 0611168

$$Q = m_b v - q$$

Setting up the HQE

- Obtained by taking the forward matrix element

$$T_{\mu\nu}(Q) = \sum_{k=0}^{\infty} \left(\frac{1}{Q^2}\right)^{k+1} \langle B(v)|\bar{b}_v \Gamma_\mu \not{Q}[-(i\not{D}) \not{Q}]^k \bar{\Gamma}_\nu b_v(0)|B(v)\rangle$$

$$2m_B \mu_\pi^2 = -\langle B|\bar{b}_v (iD)^2 b_v|B\rangle$$

- Scalar Hadronic Structure functions

$$T_i(t, Q^2) = \frac{1}{\Lambda_{HQE}} \sum_{l=0}^{\infty} \left(\frac{\Lambda_{HQE}^2}{Q^2}\right)^{l+1} P_l^{(i)}(t)$$

- Polynomial containing non-perturbative HQE parameters

$$t \equiv \frac{vQ}{\Lambda_{HQE}}$$

Modelling Duality Violation

- Function with factorial growth
  Only converges if a_{2n} suppresses the factorial

$$F(\lambda) = \sum_{n=0}^{\infty} a_{2n} (2n)! (\lambda^2)^n$$

- Borel Transform to kill the factorial

$$B[F](M) = \sum_{n=0}^{\infty} a_{2n} M^{2n}$$

- Inverse Borel to re-obtain the function

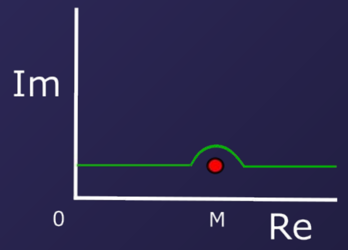
$$F(\lambda) = \int_0^{\infty} \mathrm{d}M\, e^{-M} B[F](\lambda M)$$

Modelling Duality Violation

- Asymptotic function! $\tilde{B}[F](M) = \sum_{n=0}^{\infty} M^{2n} = \frac{1}{1-M^2} = \frac{1}{1+M}\frac{1}{1-M}$

  $a_{2n} = 1$

- One has to deal with the poles



- Choice introduces an ambiguity

- We identify this ambiguity with QHDV $$\frac{1}{1-M+i\epsilon} - \frac{1}{1-M-i\epsilon} = 2i\pi\delta(1-M)$$

- Why does this identification make sense?

Illustrative example

- Expand at $x^2 = 0$ to form a kind of "OPE"

$$f(Q) = \int_0^\infty \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{r^{2k+2}} e^{iQx} dx$$

- Clearly missing the exponential term

$$f(Q) = \frac{i}{r} \sum_{k=0}^{\infty} \frac{(2k)!}{(Qr)^{2k+1}}$$

- Symmetric combination captures the uncertainty of the expansion coming from the singularity

$$\frac{f(Q) + f(-Q)}{2} = \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{1}{x^2 + \rho^2} e^{iQx}$$

$$= \frac{\pi}{2} \frac{e^{-Q\rho}}{\rho}$$

- We found the lost exponential!

Chibisov et al. 1996
hep-ph/9605465

Duality Violation model

- Anzats model polynomials based on the HQE parameters

$$p_l^{(1,4)}(t) = \sum_{m=1}^{l+1} t^m = \frac{t-t^{l+2}}{1-t}$$

- Identifying the ambiguity through Borel transform

- Use optical theorem to obtain DV contribution to hadronic tensor

$$\hat{\Delta}_{\mathrm{DV}} W_{1,4}(vQ,Q^2) = -\frac{1}{\pi}\hat{\Delta}_{\mathrm{DV}} \mathrm{Im}\left[T_{1,4}(vQ,Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE}-vQ} \quad \frac{vQ}{\sqrt{Q^2}}\left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}}\right) - \sqrt{\frac{vQ}{\Lambda_{HQE}}}\sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}}\sqrt{\frac{Q^2}{vQ}}\right)\right)$$

Duality Violation model

- Choose polynomials based on calculating the parameters
  up to l=5 and $1/mb^5$

$$p_l^{(1,4)}(t) = \sum_{m=1}^{l+1} t^m = \frac{t-t^{l+2}}{1-t}$$

$$p_l^{(2,3)}(t) = \sum_{m=0}^{l} t^m = \frac{1-t^{l+1}}{1-t}$$

$$p_0^{(5)}(t) = 0 \qquad p_{l\geq 1}^{(5)}(t) = \sum_{m=2}^{l+1} t^m$$

$$\Rightarrow p_{l\geq 0}^{(5)}(t) = \frac{t^2-t^{l+2}}{1-t}$$

Duality Violation model

- Model ansatz polynomials based on the HQE parameters assuming factorial growth*

$$P_l^{(1,4)}(t) = (2l)! \sum_{m=1}^{l+1} t^m = (2l)! \frac{t - t^{l+2}}{1-t}$$

* Similar models for P 2,3 and 5

Duality Violation model

- Model ansatz polynomials based on the HQE parameters assuming factorial growth*

$$P_l^{(1,4)}(t) = (2l)! \sum_{m=1}^{l+1} t^m = (2l)! \frac{t - t^{l+2}}{1-t}$$

- Model scalar hadronic structure functions

$$T_{1,4}(t, \lambda^2) \quad = \frac{1}{\Lambda_{HQE}} \frac{t\lambda^2}{1-t} \left(F_1(\lambda) - tF_2(\lambda)\right)$$

$$\lambda \equiv \frac{\Lambda_{HQE}}{\sqrt{Q^2}}$$

$$F_1(\lambda) \quad = \sum_{l=0}^{\infty} (2l)!(\lambda^2)^l$$

$$F_2(\lambda) \quad = \sum_{l=0}^{\infty} (2l)!(t\lambda^2)^l$$

* Similar models for P 2,3 and 5

Duality Violation model

$$\hat{\Delta}_{\rm DV} W_{1,4}(vQ,Q^2) = -\frac{1}{\pi}\hat{\Delta}_{\rm DV}\mathrm{Im}\left[T_{1,4}(vQ,Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE}-vQ}\quad\frac{vQ}{\sqrt{Q^2}}\left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}}\right)-\sqrt{\frac{vQ}{\Lambda_{HQE}}}\sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}}\sqrt{\frac{Q^2}{vQ}}\right)\right)$$

$$\hat{\Delta}_{\rm DV} W_{2,3}(vQ,Q^2) = -\frac{1}{\pi}\hat{\Delta}_{\rm DV}\mathrm{Im}\left[T_{2,3}(vQ,Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE}-vQ}\quad\frac{\Lambda_{HQE}}{\sqrt{Q^2}}\left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}}\right)-\sqrt{\frac{vQ}{\Lambda_{HQE}}}\sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}}\sqrt{\frac{Q^2}{vQ}}\right)\right)$$

$$\hat{\Delta}_{\rm DV} W_{5}(vQ,Q^2) = -\frac{1}{\pi}\hat{\Delta}_{\rm DV}\mathrm{Im}\left[T_{5}(vQ,Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE}-vQ}\quad\frac{(vQ)^2}{\Lambda_{HQE}\sqrt{Q^2}}\left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}}\right)-\sqrt{\frac{\Lambda_{HQE}}{vQ}}\sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}}\sqrt{\frac{Q^2}{vQ}}\right)\right)$$

Differential rate

$$\frac{\mathrm{d}^3\Gamma}{\mathrm{d}\hat{q}^2\mathrm{d}s\mathrm{d}y} = 48 m_b \Gamma_0 \left[\frac{2ys - y^2 - 2\hat{q}^2 + y\hat{q}^2}{1-s} W_1 + \hat{q}^2 W_2 + \frac{1}{2}\left(2ys - y^2 - \hat{q}^2\right) W_3 \right.$$

$$\left. + \frac{2ys - y^2 - \hat{q}^2}{1-s} W_4 + \frac{2ys - y^2 - \hat{q}^2}{2(1-s)^2} W_5 \right] \theta\left(\hat{q}^2\right) \theta\left(2ys - y^2 - \hat{q}^2\right)$$

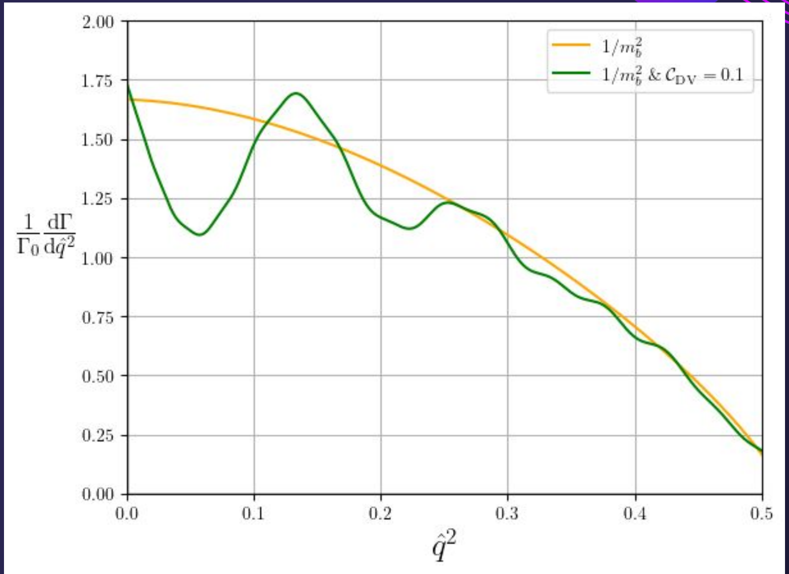
$$\Gamma_0 = \frac{G_F^2 |V_{cb}|^2 m_b^5}{192\pi^3} \qquad \hat{q}^2 = \frac{q^2}{m_b^2} \qquad s = \frac{v \cdot q}{m_b} \qquad y = \frac{2E_\ell}{m_b}$$

Instanton-like contribution

- Comparison with instanton terms motivates to keep the scale as a free fit parameter

- Choosing a small scale produces the expected 'wiggle' around the OPE

- For larger scale the period increases beyond the q^2 interval



Differential spectrum up to $1/mb^2$ with DV for Λ DV = 10^{-4} GeV using N = 0.2508 C_{DV}

Kinematic moments

- q^2 moments

$$Q_n(q^2_{\rm cut}) \equiv \frac{1}{\Gamma_0}\int_{q^2_{\rm cut}} {\rm d}q^2\,(q^2)^n\,\frac{{\rm d}\Gamma}{{\rm d}q^2}$$

- Lepton Energy moments

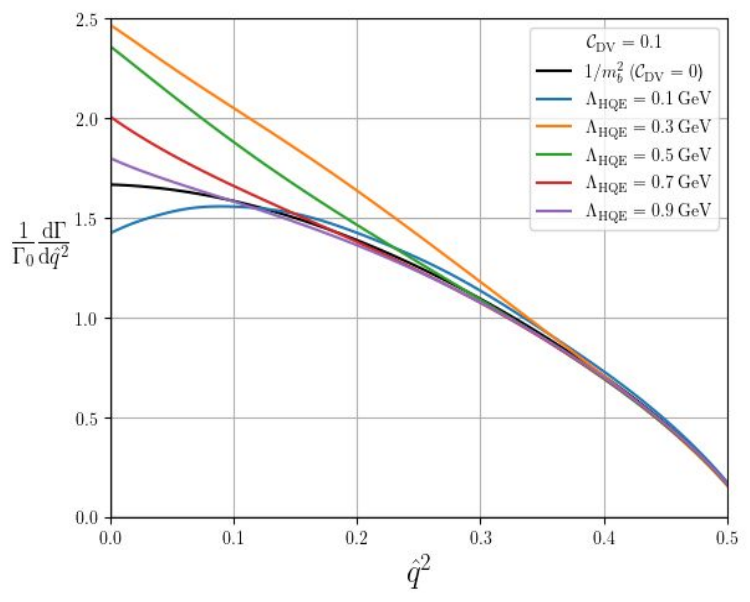
$$L_n(E_\ell^{\rm cut}) \equiv \frac{1}{\Gamma_0}\int_{E_\ell^{\rm cut}} {\rm d}E_\ell\,E_\ell^n\,\frac{{\rm d}\Gamma}{{\rm d}E_\ell}$$

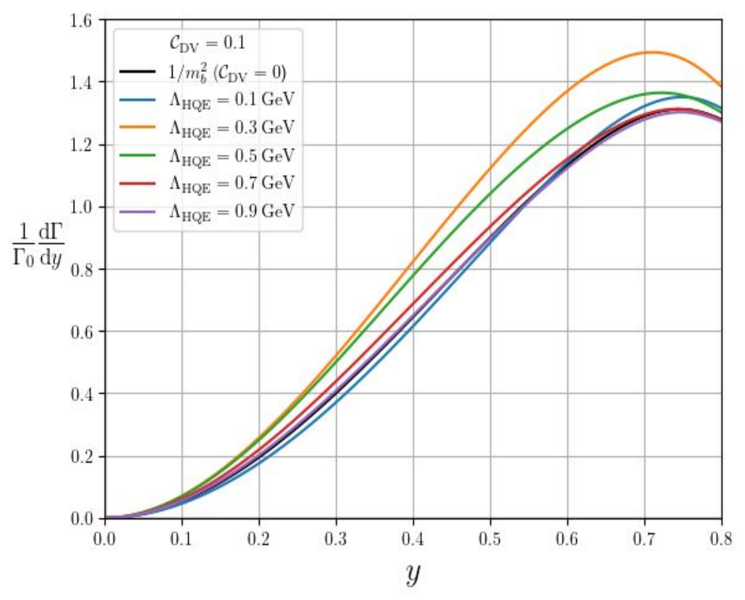
- Normalised and re-expanded in 1/mb and C_{DV} neglecting C_{DV}/mb terms

$$\bar{q}_n \equiv \langle (q^2)^n \rangle_{q^2 \geq q^2_{\rm cut}} \equiv \frac{Q_n(q^2_{\rm cut})}{Q_0(q^2_{\rm cut})}$$

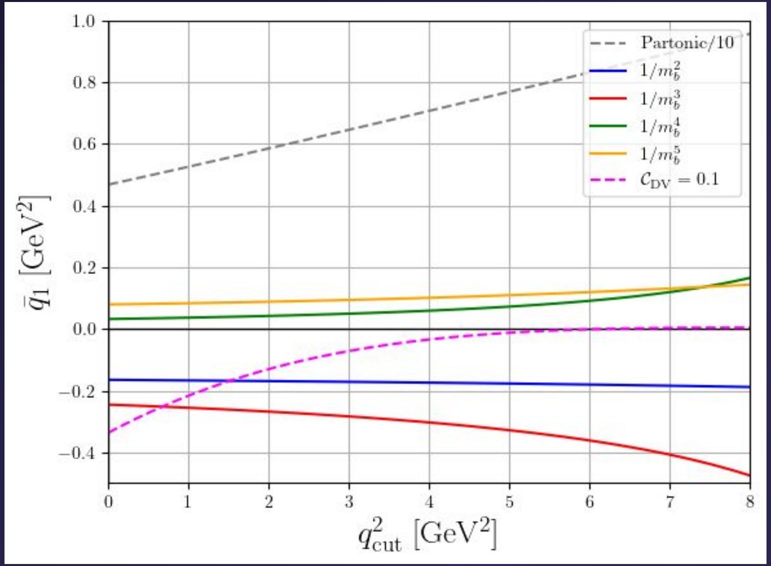
$$\bar{\ell}_n \equiv \langle E_\ell^n \rangle_{E_\ell \geq E_\ell^{\rm cut}} \equiv \frac{L_n(E_\ell^{\rm cut})}{L_0(E_\ell^{\rm cut})}$$

Effect of the scale parameter

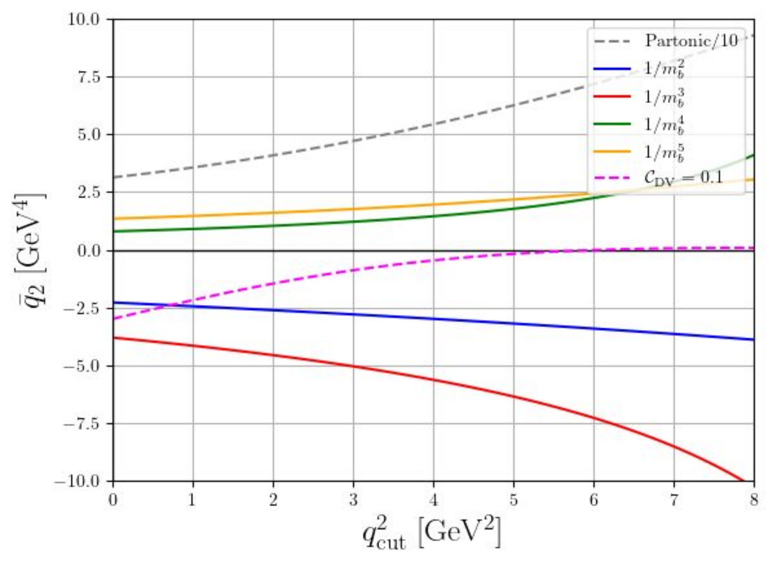




q^2 moments

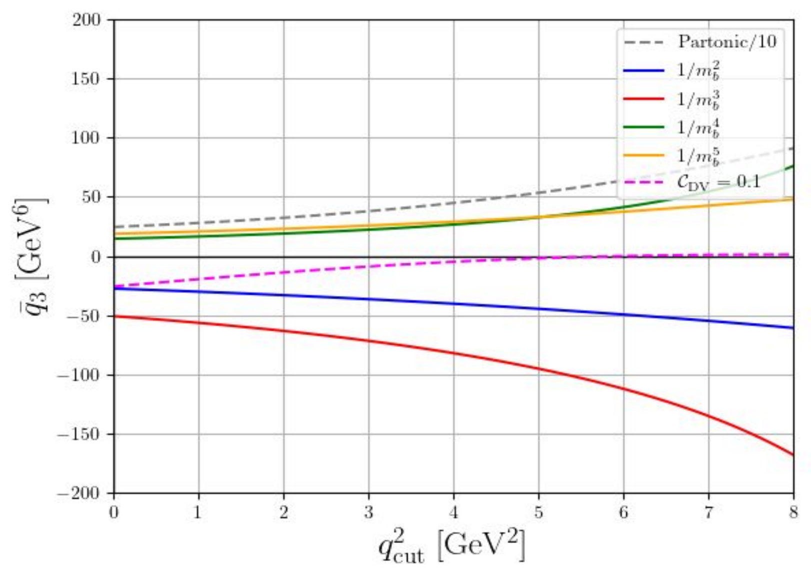


Partonic/10
$1/m_b^2$
$1/m_b^3$
$1/m_b^4$
$1/m_b^5$
$\mathcal{C}_{\rm DV} = 0.1$
$\bar{q}_1$ [GeV^2]
$q^2_{\rm cut}$ [GeV^2]
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0
1
2
3
4
5
6
7
8

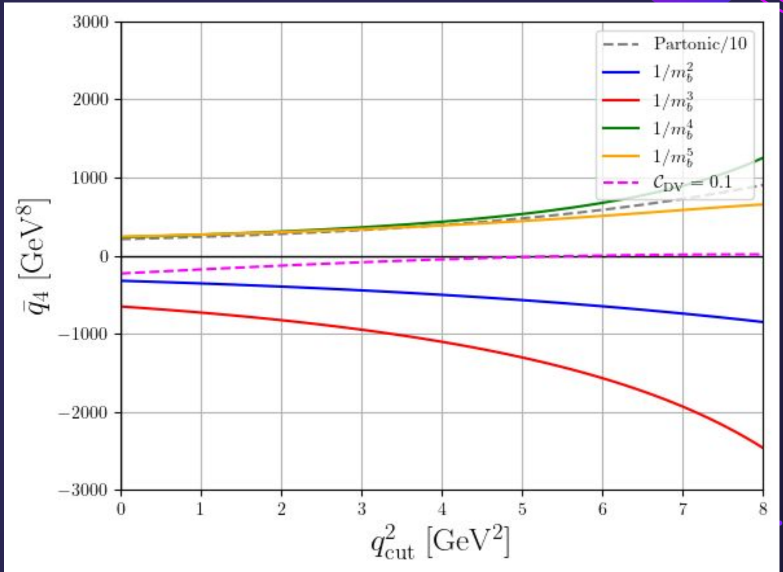


Partonic/10
$1/m_b^2$
$1/m_b^3$
$1/m_b^4$
$1/m_b^5$
$\mathcal{C}_{\rm DV} = 0.1$
$\bar{q}_2$ [GeV^4]
$q^2_{\rm cut}$ [GeV^2]
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
0
1
2
3
4
5
6
7
8

q^2 moments

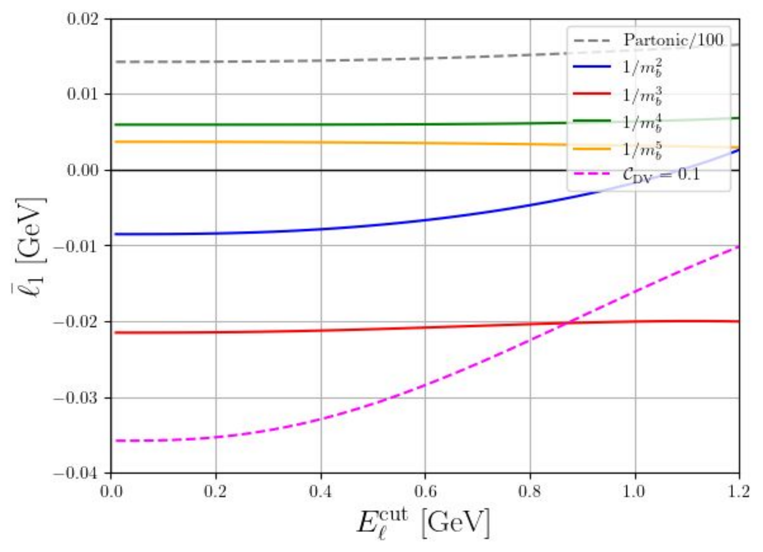


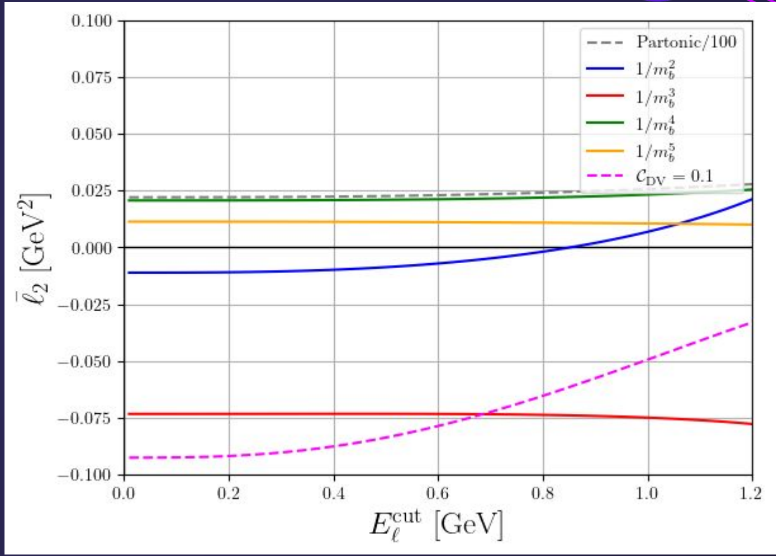
Partonic/10
$1/m_b^2$
$1/m_b^3$
$1/m_b^4$
$1/m_b^5$
$\mathcal{C}_{\rm DV} = 0.1$
$\bar{q}_3$ [GeV^6]
$q^2_{\rm cut}$ [GeV^2]



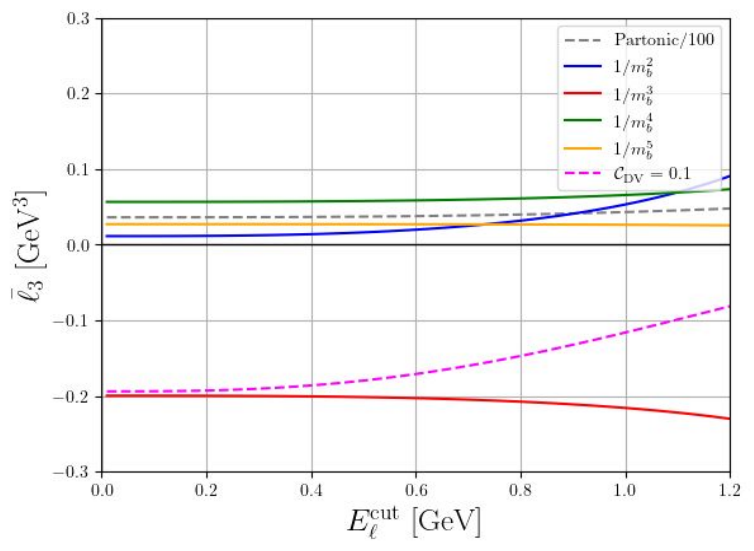
Partonic/10
$1/m_b^2$
$1/m_b^3$
$1/m_b^4$
$1/m_b^5$
$\mathcal{C}_{\rm DV} = 0.1$
$\bar{q}_4$ [GeV^8]
$q^2_{\rm cut}$ [GeV^2]

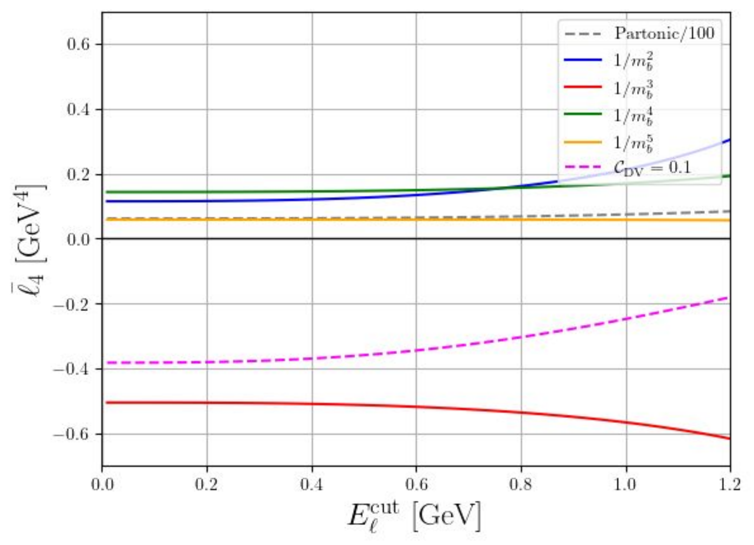
Lepton energy moments





Lepton energy moments





$\boldsymbol{T_i}$				
$\boldsymbol{l=0}$	$b_0^{(i,0)}$	$b_1^{i,1}$	-	-
$\boldsymbol{l=1}$	$b_0^{(i,1)}$	$b_1^{i,2}$	$b_2^{i,3}$	-
$\boldsymbol{l=2}$	$b_0^{(i,2)}$	$b_1^{i,3}$	$b_2^{i,4}$	$b_3^{i,5}$
$\boldsymbol{l=3}$	$b_0^{(i,3)}$	$b_1^{i,4}$	$b_2^{i,5}$	$\mathcal{O}(1/m_b^6)$
$\boldsymbol{l=4}$	$b_0^{(i,4)}$	$b_1^{i,5}$	$\mathcal{O}(1/m_b^6)$	$\mathcal{O}(1/m_b^6)$
$\boldsymbol{l=5}$	$b_0^{(i,5)}$	$\mathcal{O}(1/m_b^6)$	$\mathcal{O}(1/m_b^6)$	$\mathcal{O}(1/m_b^6)$

$\boldsymbol{T_1}$				
$\boldsymbol{l=0}$	-0.5	0	-	-
$\boldsymbol{l=1}$	0.032	-0.265	0	-
$\boldsymbol{l=2}$	-0.052	0.050	0.002	0
$\boldsymbol{l=3}$	-0.003	0.001	-0.0005	$\mathcal{O}$
$\boldsymbol{l=4}$	-0.0002	0.0004	$\mathcal{O}$	$\mathcal{O}$
$\boldsymbol{l=5}$	-0.000007	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$

T_2				
$l = 0$	0	0.032	-	-
$l = 1$	0	-0.310	0.570	-
$l = 2$	0	-0.043	0.049	0.031
$l = 3$	0	-0.005	0.017	$\mathcal{O}$
$l = 4$	0	-0.0003	$\mathcal{O}$	$\mathcal{O}$
$l = 5$	0	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$

T_3				
$l = 0$	0	0.064	-	-
$l = 1$	0	-0.620	1.119	-
$l = 2$	0	-0.086	0.154	0.015
$l = 3$	0	-0.010	0.036	$\mathcal{O}$
$l = 4$	0	-0.0006	$\mathcal{O}$	$\mathcal{O}$
$l = 5$	0	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$

T_4				
$l = 0$	1	0	-	-
$l = 1$	-0.064	0.317	0	-
$l = 2$	0.103	-0.136	-0.004	0
$l = 3$	0.006	-0.007	0.001	$\mathcal{O}$
$l = 4$	0.0003	-0.001	$\mathcal{O}$	$\mathcal{O}$
$l = 5$	0.00001	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$

T_5				
$l = 0$	0	0	-	-
$l = 1$	0.026	0	0	-
$l = 2$	0.003	0.035	0	0
$l = 3$	0.0003	0.001	0.001	$\mathcal{O}$
$l = 4$	0.00002	0.0002	$\mathcal{O}$	$\mathcal{O}$
$l = 5$	0	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$

$$T_{1,4}(t,\lambda^2) = \frac{1}{\Lambda_{HQE}} \frac{t\lambda^2}{1-t} \left(F_1(\lambda) - tF_2(\lambda)\right) ,$$

$$T_{2,3}(t,\lambda^2) = \frac{1}{\Lambda_{HQE}} \frac{\lambda^2}{1-t} \left(F_1(\lambda) - tF_2(\lambda)\right) ,$$

$$T_5(t,\lambda^2) = \frac{1}{\Lambda_{HQE}} \frac{t^2\lambda^2}{1-t} \left(F_1(\lambda) - F_2(\lambda)\right) ,$$

$$F_1(\lambda) = \sum_{l=0}^{\infty} (2l)!(\lambda^2)^l$$

$$F_2(\lambda) = \sum_{l=0}^{\infty} (2l)!(t\lambda^2)^l$$

Input values

Input values

m_b^{kin}	4.573 GeV	[20]
$\overline{m}_c(2\text{ GeV})$	1.092 GeV	[20]
m_B	5.279 GeV	[30]
$\epsilon_{1/2}$	0.390 GeV	[23]
$\epsilon_{3/2}$	0.476 GeV	[23]
$(\mu_\pi^2)^\perp$	0.477 GeV^2	[20]
$(\mu_G^2)^\perp$	0.306 GeV^2	[20]

LSSA

LLSA approximation Historical basis

$(\rho_D^3)^\perp$	0.232 GeV^3
$(\rho_{LS}^3)^\perp$	-0.161 GeV^3
m_1	0.126 GeV^4
m_2	-0.112 GeV^4
m_3	-0.062 GeV^4
m_4	0.397 GeV^4
m_5	0.081 GeV^4
m_6	0.062 GeV^4
m_7	-0.039 GeV^4
m_8	-1.17 GeV^4
m_9	-0.393 GeV^4

LLSA approximation Historical basis

r_1	0.049 GeV^5
r_2	-0.106 GeV^5
r_3	-0.027 GeV^5
r_4	-0.043 GeV^5
r_5	0.00 GeV^5
r_6	0.00 GeV^5
r_7	0.00 GeV^5
r_8	-0.039 GeV^5
r_9	0.074 GeV^5
r_{10}	0.068 GeV^5
r_{11}	0.0059 GeV^5
r_{12}	0.010 GeV^5
r_{13}	-0.055 GeV^5
r_{14}	0.039 GeV^5
r_{15}	0.00 GeV^5
r_{16}	0.00 GeV^5
r_{17}	0.00 GeV^5
r_{18}	0.00 GeV^5

LLSA approximation RPI-basis

μ_π^2	0.477 GeV^2
μ_G^2	0.290 GeV^2
$\tilde{\rho}_D^3$	0.205 GeV^3
$\tilde{r}_E^4$	0.098 GeV^4
r_G^4	0.16 GeV^4
$\tilde{s}_E^4$	-0.074 GeV^4
s_B^4	-0.14 GeV^4
s_{qB}^4	-1.00 GeV^4
X_1^5	0.049 GeV^5
X_2^5	0.00 GeV^5
X_3^5	0.094 GeV^5
X_4^5	-0.41 GeV^5
X_5^5	-0.039 GeV^5
X_6^5	0.00 GeV^5
X_7^5	0.091 GeV^5
X_8^5	-0.0030 GeV^5
X_9^5	0.27 GeV^5
X_{10}^5	0.025 GeV^5