Experimental Higgs Quantum Observables in ATLAS

Performing a Bell test in Higgs to WW decays

ATLAS Nikhef brainstorm in May 2023

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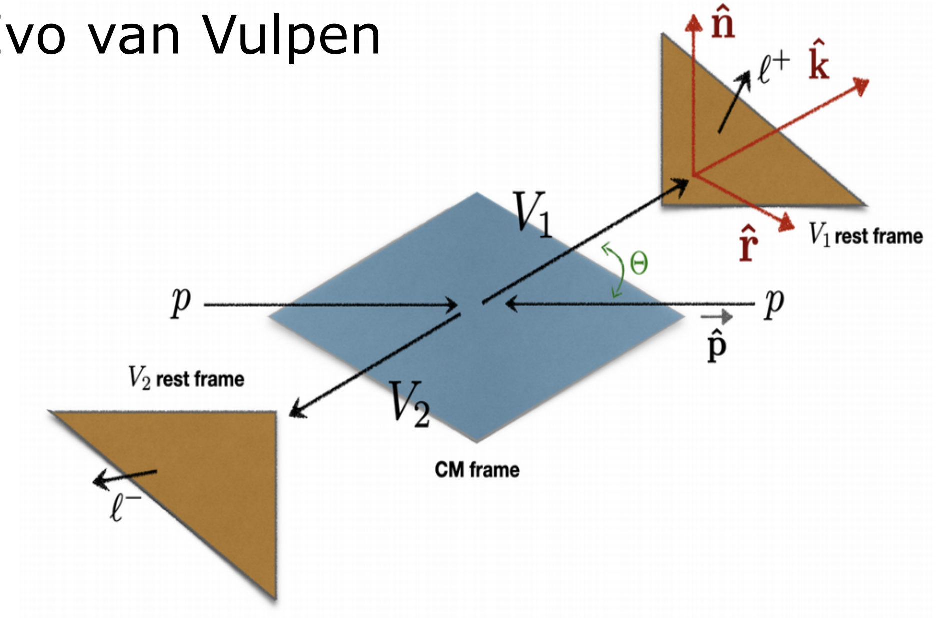
Triggered by a paper by Barr et al

'Testing Bell inequalities in Higgs boson decays'



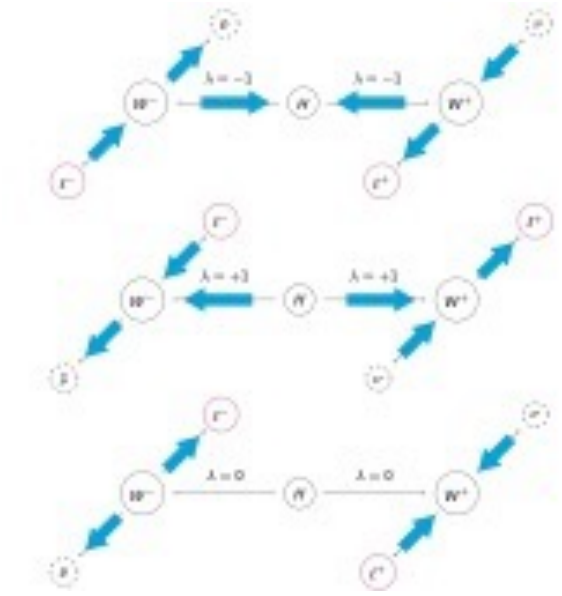
Nikhef

A Bell test in Higgs to WW decays

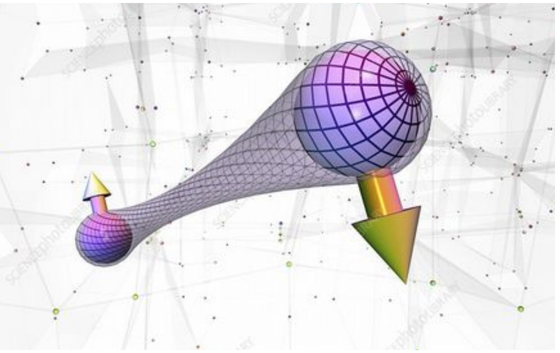
Barr argued: The Higgs decaying (S=0) into a pair of W bosons (S=1) is ideal for measuring spin if the W decays to a charged lepton and a neutrino. The lepton from the W is fully polarized.

A pair of bosons can form three states and are therefore called qutrits (see figure from Karsten Burgard).

To exploit the physics one has to reconstruct the lepton directions in the Higgs rest frame. Due to the spins the charged leptons are going in “similar” directions.



A Bell test in Higgs decays



Barr showed: by measuring the angular distribution of the two charged leptons in the Higgs rest frame, one can perform an angular analysis and a sensitive test of the Bell inequality, in this case the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality.

The rest frame definition is a bit tricky: a) do we mean the rest frame of the Higgs or the W and W^* rest frames? b) what is the precise orientation of the frame (next slide)? c) there is some freedom of choice of the orientation - like in the classical experiment - where the Bell or CGLMP theorem is maximally violated.

NB: In ATLAS all is measured, no need to rotate the detector (polarimeter)

A Bell test in Higgs to WW decays

Polarization density matrix of two spin-1 $V_1 V_2$

$$\bar{q}(p_1)\, q(p_2) \to V_1(k_1, \lambda_1)\, V_2(k_2, \lambda_2)$$

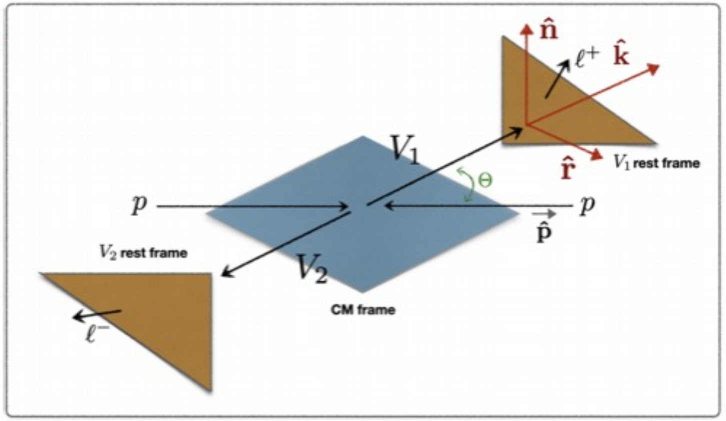
Spin eigenstates embedded in covariant polarization vectors

in the center of mass frame

$\{\hat{\mathbf{n}}, \hat{\mathbf{r}}, \hat{\mathbf{k}}\}$ right-handed basis

same procedure as for Qubits
but for spin-1 → 3 polarizations
→ Qutrits

more complicated !



$n^\mu_{\rm m}(1)$

$n^\mu_{\rm m}(2)$

$\rm m = 1, 2, 3$

$$\hat{\mathbf{r}} = \frac{1}{\sin\Theta}\left(\hat{\mathbf{p}} - \cos\Theta\hat{\mathbf{k}}\right), \quad \hat{\mathbf{n}} = \frac{1}{\sin\Theta}\left(\hat{\mathbf{p}} \times \hat{\mathbf{k}}\right)$$

scattering angle

quarks assumed massless

Boosted (n,r,k) basis in c.m. frame → n(1) and n(2) stands for particle V_1 and V_2

$$n_1^\mu(1) = n_1^\mu(2) = (0,\, \hat{\mathbf{n}}), \quad n_2^\mu(1) = n_2^\mu(2) = (0,\, \hat{\mathbf{r}})$$
$$n_3^\mu(1) = \gamma\,(\beta,\, \hat{\mathbf{k}}), \quad n_3^\mu(2) = \gamma\,(-\beta,\, \hat{\mathbf{k}}),$$

β = velocity in c.m. frame

7

A definition of the rest frame

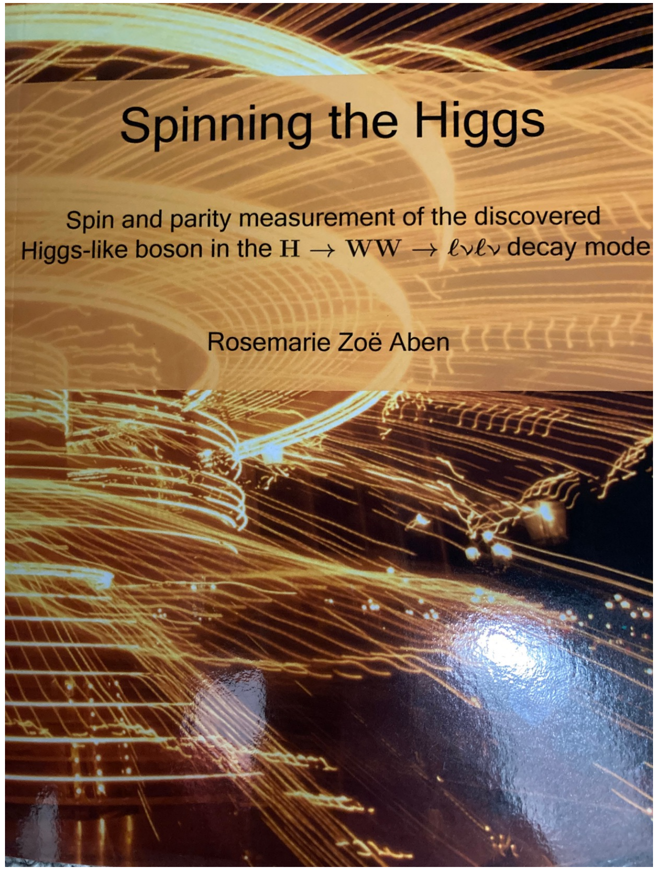
slide from E Gabrieli

arXiv: 2302.00683 [hep-ph]; EPJC 83 (2023) 2, 162, arXiv: 2208.11723 [hep-ph]

Can one reconstruct the Higgs rest frame in WW decays?

Rosemarie Aben wrote her thesis "Spinning the Higgs" (2015) exactly about this topic. And the answer is yes we can do that pretty well.

In ATLAS it was thought of little interest to reconstruct the Higgs rest frame. Contrary we were of the opinion that the rest frame variables are the key to Higgs physics, e.g. to determine the spin and parity of Higgs.
For a Bell test it is required to measure the lepton angles in the Higgs rest frame.



A Bell test in Higgs to WW decays

What is the expression for the differential cross section?

from E Gabrieli

Extract the density matrix for Two-Qutrits from data

Ashby-Pickering, Barr, Wierzchucka, 2209.13990 [quant-ph]

W W $\quad p\,p \to V_1 + V_2 + X \to \ell^+\ell^- + \text{jets} + E_T^{\text{miss}}$

Differential cross section

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega^+\, d\Omega^-} = \left(\frac{3}{4\pi}\right)^2 \text{Tr}\left[\rho_{V_1V_2}\,(\Gamma_+ \otimes \Gamma_-)\right]$$

depend on the invariant mass m_{VV} (or velocity β) and scattering angle Θ in the V_1V_2 cm frame

Rahaman, Singh, NPB 984 (2022), 2109.09345 [hep-ph]

$$d\Omega^\pm = \sin\theta^\pm d\theta^\pm\, d\phi^\pm$$

polar angle $\ell^\pm$ — azimuthal angle $\ell^\pm$

→ phase space written in terms of the spherical coordinates (with independent polar axis) for the momenta of the final charged leptons in the respective rest frames of the decaying spin-1 particles

$\rho_{V_1V_2}$ = density matrix of V_1V_2

$\Gamma_\pm$ → Density matrices that describe the polarization of the two decaying W into final leptons (the charged ones assumed to be massless)

these are projectors in the case of the W-bosons because of their chiral couplings to leptons

11

can be computed by rotating to an arbitrary polar axis the spin states of gauge bosons from the ones given in the k-direction quantization axis

$$\Gamma_\pm = \frac{1}{3}\mathbb{1} + \sum_{i=1}^{8} \mathfrak{q}^a_\pm T^a$$

→ Density matrices for W-bosons

$\mathfrak{q}^a_\pm$ (Winer q-symbols) are functions of the corresponding spherical coordinates

set of polynomials of spherical coordindates (see backup slide)

$$h_{ab} = \frac{1}{\sigma}\int\int \frac{d\sigma}{d\Omega^+\, d\Omega^-}\,\mathfrak{p}^a_+\,\mathfrak{p}^b_-\, d\Omega^+ d\Omega^-$$

$$f_a = \frac{1}{\sigma}\int \frac{d\sigma}{d\Omega^+}\,\mathfrak{p}^a_+\, d\Omega^+$$

$$g_a = \frac{1}{\sigma}\int \frac{d\sigma}{d\Omega^-}\,\mathfrak{p}^a_-\, d\Omega^-$$

$\mathfrak{p}^n_\pm$ a particular set of orthogonal functions (see next slide) ⇒ $\left(\frac{3}{4\pi}\right)\int \mathfrak{p}^n_\pm\,\mathfrak{q}^m_\pm\, d\Omega^\pm = \delta^{nm}$

For ZZ case, the set of functions are linear combinations of $\mathfrak{q}^a_\pm$ → see backup slides

12

A Bell test in Higgs to WW decays

from E Gabrieli

Inserting the **f,g** and **h** coefficients into the Gell-Mann basis for 2-qutrits

$$\rho_H = 2\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&h_{44}&0&h_{16}&0&h_{44}&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&h_{16}&0&2h_{33}&0&h_{16}&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&h_{44}&0&h_{16}&0&h_{44}&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\mathrm{Tr}\left[\rho_H\right] = 1$$

density matrix is idempotent

$$\rho_H^2 = \rho_H$$

Signaling that H → VV* is a **pure state**

$$\rho_H = |\Psi_H\rangle\langle\Psi_H|$$

Aguilar-Saavedra, Bernal, Casas, Moreno
2209.13441 [hep-ph]

using the basis

$$|\lambda\,\lambda'\rangle = |\lambda\rangle \otimes |\lambda'\rangle \text{ with } \lambda, \lambda' \in \{+, 0, -\}$$

Spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&1&0\\ 1&0&1\\ 0&1&0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&-i&0\\ i&0&-i\\ 0&i&0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1&0&0\\ 0&0&0\\ 0&0&-1 \end{pmatrix}$$

Expressed as a function of **Gell-Mann matrices**

$$S_1 = \frac{1}{\sqrt{2}}\left(T^1 + T^6\right), \quad S_2 = \frac{1}{\sqrt{2}}\left(T^2 + T^7\right), \quad S_3 = \frac{1}{2}T^3 + \frac{\sqrt{3}}{2}T^8$$

$$\begin{aligned} S_{31} = S_{13} &= \frac{1}{\sqrt{2}}\left(T^1 - T^6\right), \\ S_{12} = S_{21} &= T^5, \\ S_{23} = S_{32} &= \frac{1}{\sqrt{2}}\left(T^2 - T^7\right) \\ S_{11} &= \frac{1}{2\sqrt{3}}T^8 + T^4 - \frac{1}{2}T^3, \\ S_{22} &= \frac{1}{2\sqrt{3}}T^8 - T^4 - \frac{1}{2}T^3, \\ S_{33} &= T^3 - \frac{1}{\sqrt{3}}T^8, \end{aligned}$$

Gell-Mann basis

$$T^1 = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0&-i&0\\ i&0&0\\ 0&0&0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1&0&0\\ 0&-1&0\\ 0&0&0 \end{pmatrix}$$

$$T^4 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix}, \quad T^5 = \begin{pmatrix} 0&0&-i\\ 0&0&0\\ i&0&0 \end{pmatrix}, \quad T^6 = \begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix},$$

$$T^7 = \begin{pmatrix} 0&0&0\\ 0&0&-i\\ 0&i&0 \end{pmatrix}, \quad T^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&-2 \end{pmatrix}.$$

$\mathbb{1}$ being the 3×3 unit matrix

38

Wigner's *Q symbols*

$$\begin{aligned} \mathfrak{q}^1_\pm &= \frac{1}{\sqrt{2}}\sin\theta^\pm\left(\cos\theta^\pm \pm 1\right)\cos\phi^\pm, \\ \mathfrak{q}^2_\pm &= \frac{1}{\sqrt{2}}\sin\theta^\pm\left(\cos\theta^\pm \pm 1\right)\sin\phi^\pm, \\ \mathfrak{q}^3_\pm &= \frac{1}{8}\left(1 \pm 4\cos\theta^\pm + 3\cos 2\theta^\pm\right), \\ \mathfrak{q}^4_\pm &= \frac{1}{2}\sin^2\theta^\pm\cos 2\phi^\pm, \\ \mathfrak{q}^5_\pm &= \frac{1}{2}\sin^2\theta^\pm\sin 2\phi^\pm, \\ \mathfrak{q}^6_\pm &= \frac{1}{\sqrt{2}}\sin\theta^\pm\left(-\cos\theta^\pm \pm 1\right)\cos\phi^\pm, \\ \mathfrak{q}^7_\pm &= \frac{1}{\sqrt{2}}\sin\theta^\pm\left(-\cos\theta^\pm \pm 1\right)\sin\phi^\pm, \\ \mathfrak{q}^8_\pm &= \frac{1}{8\sqrt{3}}\left(-1 \pm 12\cos\theta^\pm - 3\cos 2\theta^\pm\right), \end{aligned}$$

$$\begin{aligned} \mathfrak{p}^1_\pm &= \sqrt{2}\sin\theta^\pm\left(5\cos\theta^\pm \pm 1\right)\cos\phi^\pm, \\ \mathfrak{p}^2_\pm &= \sqrt{2}\sin\theta^\pm\left(5\cos\theta^\pm \pm 1\right)\sin\phi^\pm, \\ \mathfrak{p}^3_\pm &= \frac{1}{4}\left(5 \pm 4\cos\theta^\pm + 15\cos 2\theta^\pm\right), \\ \mathfrak{p}^4_\pm &= 5\sin^2\theta^\pm\cos 2\phi^\pm, \\ \mathfrak{p}^5_\pm &= 5\sin^2\theta^\pm\sin 2\phi^\pm, \\ \mathfrak{p}^6_\pm &= \sqrt{2}\sin\theta^\pm\left(-5\cos\theta^\pm \pm 1\right)\cos\phi^\pm, \\ \mathfrak{p}^7_\pm &= \sqrt{2}\sin\theta^\pm\left(-5\cos\theta^\pm \pm 1\right)\sin\phi^\pm, \\ \mathfrak{p}^8_\pm &= \frac{1}{4\sqrt{3}}\left(-5 \pm 12\cos\theta^\pm - 15\cos 2\theta^\pm\right). \end{aligned}$$

An experimental point of view

The differential distribution is described by 8 fundamental distributions (Wigner $q^n_\pm$ in previous slide or $\phi_n^{Q\pm}$ from Barr). Experimentally one can measure the full 4D distribution – or 8x8 Matrix:

$$\frac{N\,(cos\vartheta^+, \varphi^+, cos\vartheta^-, \varphi^-)}{N_{total}} = \sum_{n,m=1}^{n,m=8} 9 < c_{nm} > \phi_n^{Q+}\,\phi_m^{Q-} + 3\sum_{n=1}^{n=8} < q_n^+ > \phi_n^{Q+} + 3\sum_{n=1}^{n=8} < q_n^- > \phi_n^{Q-}$$

where the off -diagonal $<c_{nm}>$ coefficients are the correlation factors, that are relevant for Bell tests. The last two terms give the uncorrelated distributions.

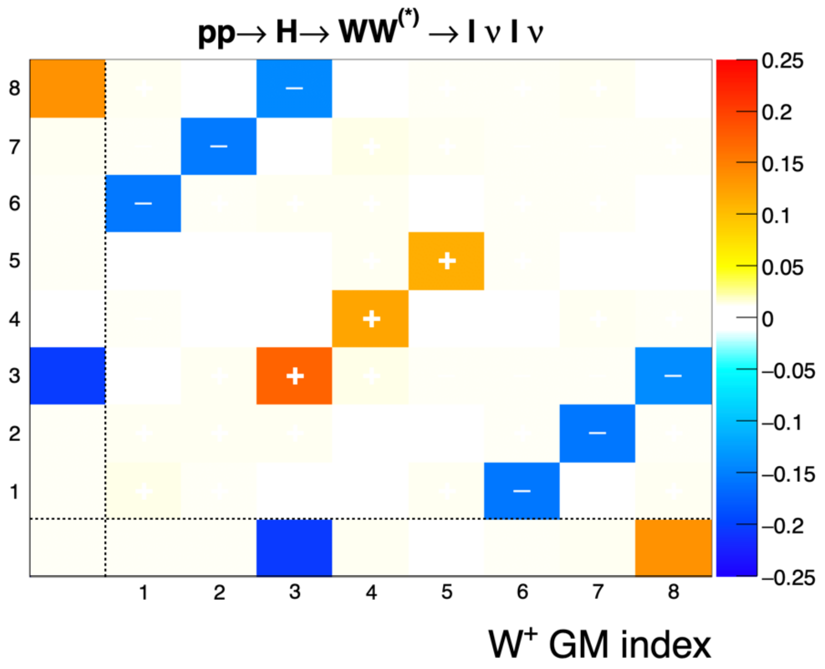
In general, large values for $|<c_{nm}>|$ - so large quantum correlations – mean large Bell violations or big entanglement and “spooky actions at a distance”.

As experimentalist, we cannot use the Wigner $p^n_\pm$ projectors (that assume 100% flat efficiency in phase space). We will have to fit the 4D angular distribution.

An experimental point of view

Recent review paper by Barr and Gabrieli al

Zoom slide 7 by Gabrieli



W^{+} GM index

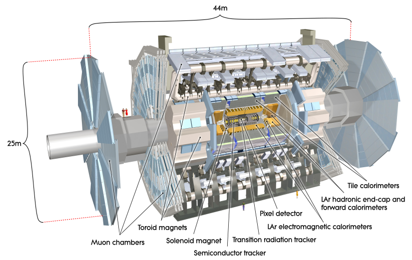
$$\rho_H = 2\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&h_{44}&0&h_{16}&0&h_{44}&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&h_{16}&0&2\,h_{33}&0&h_{16}&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&h_{44}&0&h_{16}&0&h_{44}&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

Ehh is there something I missed?

Shouldn't this be the same? It looks pretty different
Left plot is similar to the expression on the previous slide …

Reconstructing the Higgs rest frame

What is measured by ATLAS in H->WW* ->$\mu\ \nu$ e ν ?



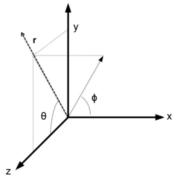


Figure 2.4: *Definition of the right-handed coordinate systems used to define positions and directions in the ATLAS detector. The z-axis points along the beam pipe in the anti-clockwise direction and the x-axis points towards the centre of the LHC. The azimuthal angle, ϕ, is defined in the x-y plane and, and the polar angle, θ, in the r-z plane.*

- The momentum vector of the two charged leptons (in ATLAS electron and muon). And the missing transverse momentum.
- In the xy frame the momentum of the Higgs is known.
- We don’t know the z momentum component (along beam) of the neutrino’s. The trick is to apply a Higgs mass constraint (125 GeV) and solve the z momentum component of the neutrino’s.
- There are some details like the imposed $M_{\nu\nu}$ mass and the choice of the solution as explained in the thesis chapter 5 and summarized on the next slide

Reconstructing the Higgs rest frame

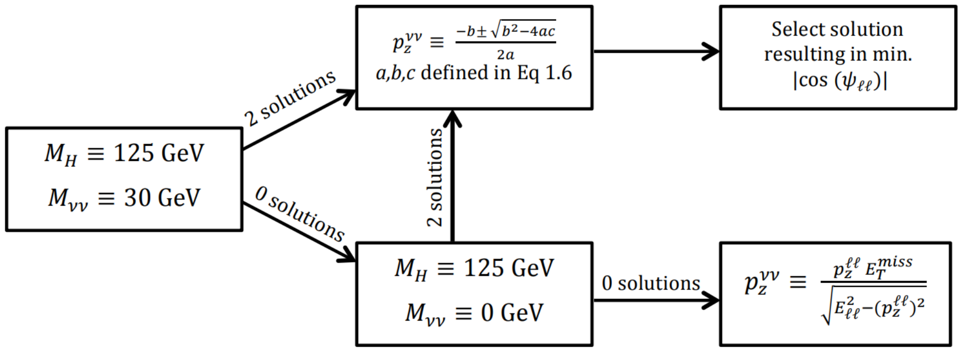
$\not{W}$ H rest frame reconstruction

- calculate $p_z^{\nu\nu}$ via

$$M_{\text{fix}}^2 = M_H^2 - M_{\ell\ell} - M_{\nu\nu}^2 + 2p_x^{\ell\ell}E_x^{\text{miss}} + 2p_y^{\ell\ell}E_y^{\text{miss}} \quad (1.5)$$

$$0 = \underbrace{\left(\left(p_z^{\ell\ell}\right)^2 - E_{\ell\ell}^2\right)}_{a}(p_z^{\nu\nu})^2 + \underbrace{M_{\text{fix}}^2 p_z^{\ell\ell}}_{b}\, p_z^{\nu\nu} + \frac{1}{4}\underbrace{M_{\text{fix}}^4 - E_{\ell\ell}^2\left(\left(E_T^{\text{miss}}\right)^2 + M_{\nu\nu}^2\right)}_{c} \quad (1.6)$$

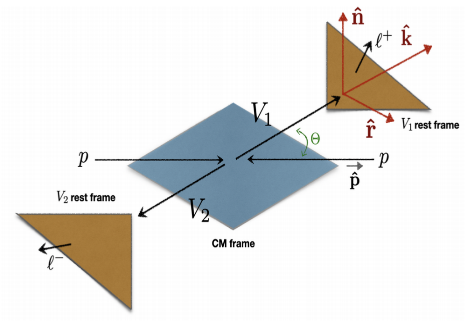
- $M_{\nu\nu}$ is a free parameter, set to the mean of the distribution at 30 GeV to avoid bias
- if system cannot be solved, instead try $M_{\nu\nu} = 0$ GeV
- in case of no solution, abandon M_H constraint and use $p_z^{\nu\nu}$ that allow for largest M_H
- prefer solutions with min $|\cos\psi_{\ell\ell}^*|$ to push the leptons close to the transverse plane
  - $\psi_{\ell\ell}^*$: the angle of the dilepton system in the $(r-z)$ plane of the H rest frame)
  - need to study bias introduced by this choice



Summary from Karsten Burgard

16

Reconstructing the Higgs rest frame



r,n along ATLAS x,y axis

What is measured and reconstructed in the experiment?
The momentum vector of the Higgs and the momenta of electron, muon and the νν system are measured.

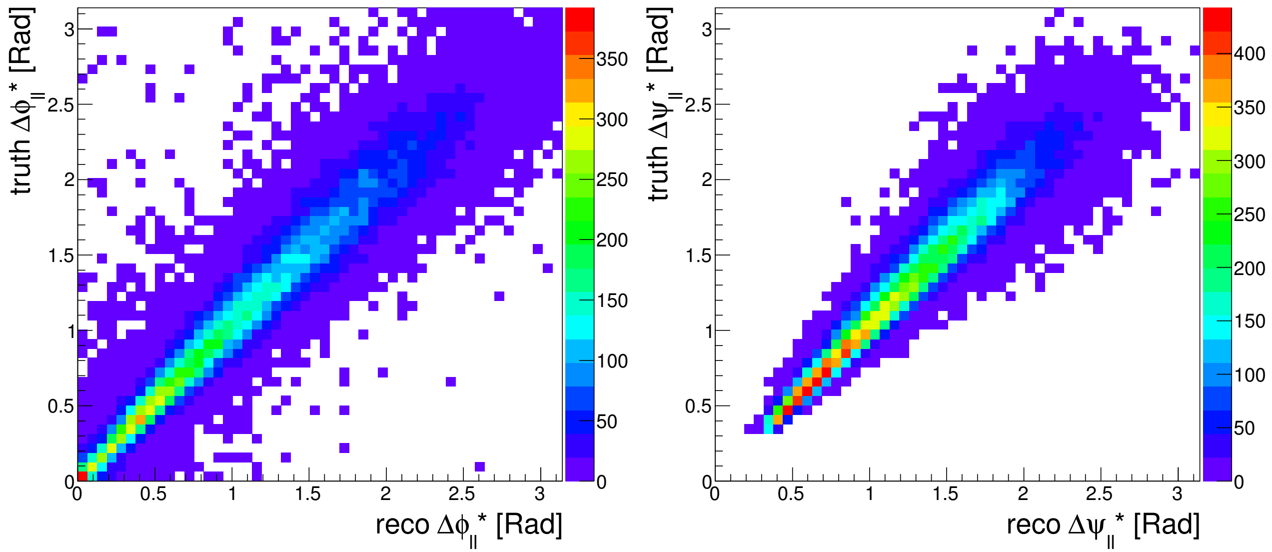
Because the Higgs momentum vector is know, one can go to the rest frame of the Higgs, and Lorentz boost the electron and muon. This gives the angles and momenta in the rest frame.

That is in principle enough for the proposed Bell measurements.
NB we can use any well-defined frame of axes.

Reconstructing the Higgs rest frame

From the thesis opening angle in rest frame in xy (ϕ) and 3D opening angle (ψ) vs truth

Pretty nice reconstruction of the di-lepton opening angles



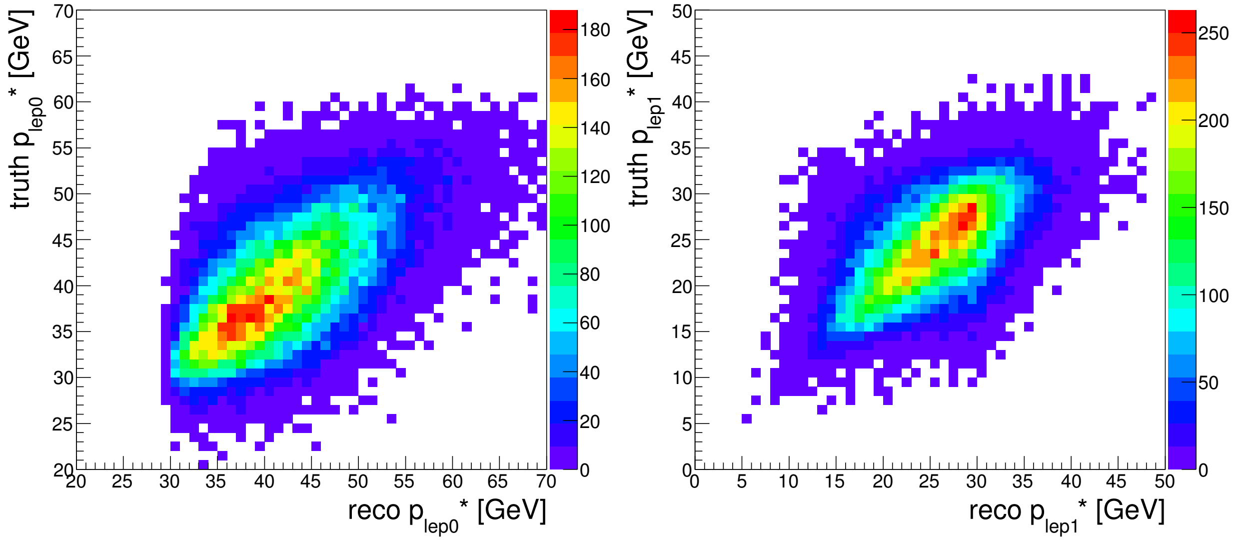
Experimental resolutions are best in the xy frame (ϕ) compared to rz (θ)

Reconstructing the Higgs rest frame

From the thesis momenta of the leptons in rest frame in vs truth

Mind: different horizontal scale

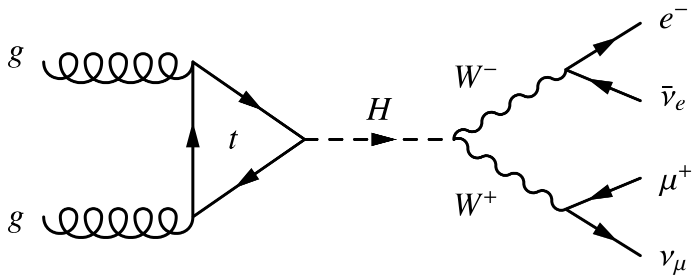
Reflects off-shell ness of W*



Off the road: theorist propose a Bell test without angles but using the W* (Z*) mass. Can we define an observable like $m_{W*}/m_W = \min(p_{l0},p_{l1}) / \max(p_{l0}, p_{l1})$?

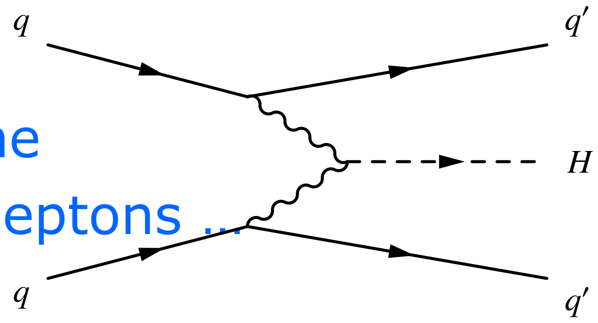
More Bell tests in Higgs to WW decays?

What about Higgs production modes?



The Higgs to WW Bell test only takes into account the decay part of the process. For gluon-gluon we cannot measure the initial state.

Off the road



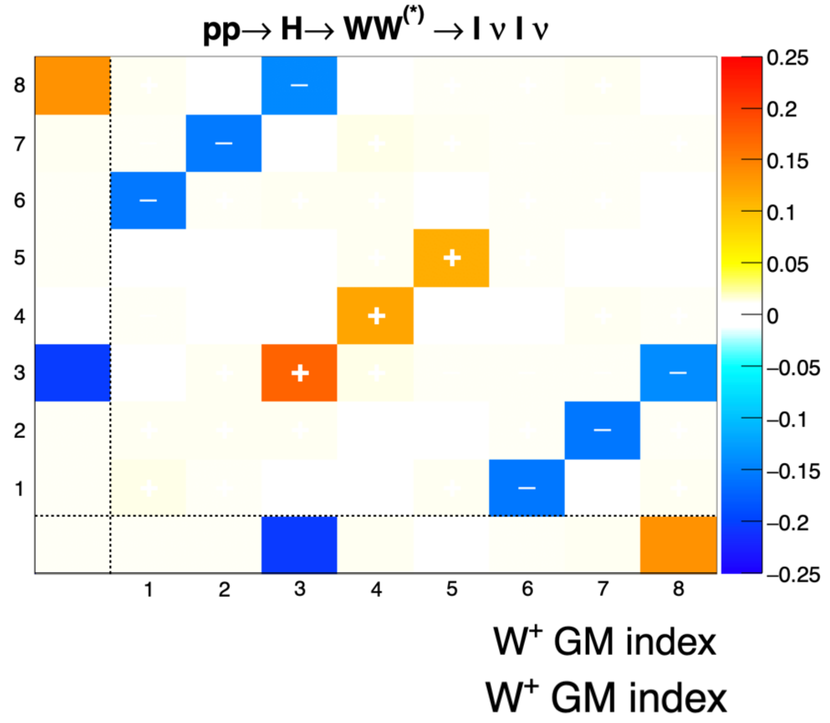
Higgs production in the Vector Boson process?
Here one measures the jets and one can imagine
a Bell test by correlating the jet directions and leptons ...
What is a suitable reference (rest) frame here?
What are important Bell observables?

Perspective of a Bell test in Higgs to WW decays

- The idea of performing a Bell test in the Higgs to WW decays, reconstructing the angles in the Higgs rest frame looks both challenging, interesting and feasible to me.
- Need to works out better the mathematical framework to apply to the ATLAS data (reference frame, the 8 Wigner/Gell-Mann Matrix, and Bell sensitive variables).
- Experimentally, quite a lot of work to do – but e.g. a H to WW event selection is available. Backgrounds need to be studied in the 4D differential cross section ($\cos\theta^+, \phi^+, \cos\theta^-, \phi^-$) in particular for the Bell sensitive observables (off-diagonal elements).

An experimental point of view

Recent review paper by Barr and Gabrieli al



arXiv.2302.00683 by Gabrieli

The non-vanishing f_a elements are

$$f_3 = \frac{1}{6}\frac{-m_H^4 + 2(1+f^2)m_H^2M_V^2 - (1-f^2)^2M_V^4}{m_H^4 - 2(1+f^2)m_H^2M_V^2 + (1+10f^2+f^4)M_V^4}\,,$$

$$f_8 = -\frac{1}{\sqrt{3}}f_3\,, \qquad (3.5)$$

and we find $g_a = f_a$ for $a \in \{1,\dots,8\}$. The non-vanishing h_{ab} elements are

$$h_{16} = h_{61} = h_{27} = h_{72} = \frac{fM_V^2\left[-m_H^2 + (1+f^2)M_V^2\right]}{m_H^4 - 2(1+f^2)m_H^2M_V^2 + (1+10f^2+f^4)M_V^4}\,,$$

$$h_{33} = \frac{1}{4}\frac{\left[m_H^2 - (1+f^2)M_V^2\right]^2}{m_H^4 - 2(1+f^2)m_H^2M_V^2 + (1+10f^2+f^4)M_V^4}\,,$$

$$h_{38} = h_{83} = -\frac{1}{4\sqrt{3}}$$

$$h_{44} = h_{55} = \frac{2f^2M_V^4}{m_H^4 - 2(1+f^2)m_H^2M_V^2 + (1+10f^2+f^4)M_V^4}\,,$$

$$h_{88} = \frac{1}{12}\frac{m_H^4 - 2(1+f^2)m_H^2M_V^2 + (1-14f^2+f^4)M_V^4}{m_H^4 - 2(1+f^2)m_H^2M_V^2 + (1+10f^2+f^4)M_V^4}\,, \qquad (3.6)$$

Coefficients look the same f3, f8, h16 etc.

An experimental point of view

Recent review paper by Barr and Gabrieli al

Similarly to the case (2.39) for qubits, the combination of probabilities in $\mathcal{I}_3$ can be expressed in quantum mechanics as an expectation value of a suitable Bell operator $\mathscr{B}$ as

$$\mathcal{I}_3 = \mathrm{Tr}\big[\rho\,\mathscr{B}\big]\ , \tag{2.59}$$

where ρ is the 9×9 density matrix representing the state of the two qutrits. Following the current convention[5], we denote

$$f_i = \frac{1}{9}\,\mathcal{A}_i^{(3)}\ ,\quad g_j = \frac{1}{9}\,\mathcal{B}_j^{(3)}\quad \text{and}\quad h_{ij} = \frac{1}{9}\,\mathcal{C}_{ij}^{(3)}\ . \tag{2.60}$$

The density operator in Eq. (2.50) can thus be written

$$\rho = \frac{1}{9}\,[\mathbb{1}\otimes\mathbb{1}] + \sum_{a=1}^{8} f_a\,[T^a\otimes\mathbb{1}] + \sum_{a=1}^{8} g_a\,[\mathbb{1}\otimes T^a] + \sum_{a,b=1}^{8} h_{ab}\,\big[T^a\otimes T^b\big]\ , \tag{2.61}$$

in the form of (2.50), specialised to $d=3$, where now the generators are the the standard Gell-Mann matrices T^a.

The explicit form of $\mathscr{B}$ depends on the choice of the four measured operators $\hat{A}_i$ and $\hat{B}_i$. For the case of the maximally correlated qutrit state, analogous to the qubit state in (2.47), the problem of finding an optimal choice of measurements has been solved [103], and the Bell operator takes a particular simple form [105]:

$$\mathscr{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 2 & 0 & 0 \\ 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & 2 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\ . \tag{2.62}$$

The observable $\mathcal{I}_3$ defined in Eq. (2.59), which parametrizes the violations of Bell inequalities for two qutrits systems, then can be written in terms of the coefficients h_{ab} as

$$\mathcal{I}_3 = 4\Big(h_{44}+h_{55}\Big) - \frac{4\sqrt{3}}{3}\Big[h_{61}+h_{66}+h_{72}+h_{77}+h_{11}+h_{16}+h_{22}+h_{27}\Big]\ . \tag{2.63}$$

Within the choice of measurements leading to the Bell operator (2.62), there is still the freedom of modifying the measured observables through local unitary transformations, which effectively corresponds to local changes of basis, separately at Alice and Bob's sites. Correspondingly, the Bell operator undergoes the change:

$$\mathscr{B} \to (U\otimes V)^\dagger\cdot\mathscr{B}\cdot(U\otimes V)\ , \tag{2.64}$$

where U and V are independent 3×3 unitary matrices. One can use this additional freedom in order to maximize the value of $\mathcal{I}_3$ for any given qutrit state ρ.

I_3 Bell = 4 ($h_{44}+h_{55} > 0$) – $4/\sqrt{3}$ ($2\,h_{16} + 2\,h_{17} < 0$)

$h_{44} > 0$ $h_{55} > 0$ $h_{61}=h_{16} < 0$ $h_{27}=h_{72} < 0$

h_{11}, h_{22}, h_{66} and $h_{77} = 0$