Recap of derivation of gravitational wave equation

Starting with the Einstein Field Equations (EFE)

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{curvature}} = \underbrace{8\pi}_{\substack{\text{very small number:}\\ \text{takes a lot of mass/energy}\\ \text{to curve spacetime.}}} \underbrace{T_{\mu\nu}}_{\text{energy momentum (density).}}$$

LHS is made up entirely of the metric tensor $g_{\mu\nu}$ ('gravity').

So: put in some $T_{\mu\nu}$, solve system for $g_{\mu\nu}$. From $g_{\mu\nu}$, gravitational interactions follow, as well as curved spacetime rules via the line-element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

i.e. deviations from a Pythagorean theorem. This measures curvature.

Important: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ consists of second derivatives of the metric tensor.

Reason: first derivatives can be coordinate transformed away, meaning that there exists some coordinate transformation that will make gravity disappear.

So, real (non-gaugable) gravity must be measured by second derivatives.

Finally, the metric tensor can also be seen as an energy: $\frac{d^2 x^\lambda}{d\tau^2} = -\Gamma^{\lambda}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$

consists of first derivatives of $g_{\mu\nu}$

force on mass

if $\Gamma^{\lambda}_{\alpha\beta} \propto \frac{dg}{dx} =$ force,

then $g_{\mu\nu}$ must be a measure of energy: Curvature carries its own energy

Derivation of GW equation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,,$$

assumption $h_{\mu\nu} \ll \eta_{\mu\nu}$ generally justified because of smallness of $\frac{8\pi G}{c^4}$.

so that:

$\mathcal{O}(h^2) = 0$, (GW small)

$\mathcal{O}(h \partial_\lambda h) = 0$. (GW stays small)

Then (skipping many steps!),

$$LHS = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$= \dots = \frac{1}{2} \partial_\alpha \partial^\alpha h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu$$

where $V_\mu = \partial_\gamma h^\gamma_\mu - \frac{1}{2} \partial_\mu h^\gamma_\gamma$

This is almost the wave equation, but for the final two terms.

Solution: use the freedom of General Relativity to go to a coordinate system in which $V_\mu \stackrel{!}{=} 0$

What remains is the wave equation.

Coordinate transformation:

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu \,, \quad \xi \ll 1$$

(small coordinate transformation).

This in turn will change the gravitational wave $h_{\mu\nu}$ as:

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu .$$

Putting into the definition of V_α results in

$$V_\alpha \to V'_\alpha = \partial_\gamma h'^\gamma{}_\alpha - \frac{1}{2}\partial_\alpha h'^\gamma{}_\gamma$$

$$= \partial_\gamma (h^\gamma{}_\alpha - \partial^\gamma \xi_\alpha - \partial_\alpha \xi^\gamma)$$

$$- \frac{1}{2}\partial_\alpha (h^\gamma{}_\gamma - \partial_\gamma \xi^\gamma - \partial^\gamma \xi_\gamma)$$

cancellation

$$= \partial_\gamma h^\gamma{}_\alpha - \Box^2 \xi_\alpha - \frac{1}{2}\partial_\alpha h^\gamma{}_\gamma$$

$$\stackrel{!}{=} 0$$

$$\Rightarrow \quad \Box^2 \xi_\alpha = S_\alpha(t, \vec{x})$$

$$S_\alpha(t, \vec{x}) \equiv \partial_\gamma h^\gamma{}_\alpha - \frac{1}{2}\partial_\alpha h^\gamma{}_\gamma$$

Now, this is a sourced Laplace equation $\Box^2 f(t,\vec{x}) = S(t,\vec{x})$, which is known to have the solution

$$f(t,\vec{x}) = -\frac{1}{4\pi}\int \frac{S(t_R,\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$$

$$t_R = t - |\vec{x}-\vec{x}'|.$$

The fact that $\Box^2 \xi_\alpha = S_\alpha$ has a solution, is enough to conclude that the coordinate transformation $x \to x + \xi$ exists that will make $V_\alpha = 0$, and hence turn the gravitational wave equation into the familiar wave equation

$$\Box^2 h_{\alpha\beta} = 0$$

This has solution $h_{\alpha\beta} = h_{\alpha\beta}(t - \vec{r})$

i.e. travelling through the vacuum with the speed of light, not changing shape (i.e. very clean signal).

If one adds a source $8\pi T_{\mu\nu}$ to the Einstein Field Equation and redoes the whole exercise (including the gauging), the resulting wave equation reads:

$$\frac{1}{2}\Box^2 \bar{h}_{\alpha\beta} = -8\pi T_{\alpha\beta}\,, \quad \bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h^\lambda_\lambda$$

which has as its solution:

$$\boxed{\bar{h}_{\alpha\beta} = 4\int \frac{T_{\alpha\beta}(t_R, x')}{|x-x'|}\, d^3x'}$$

So! Put in source of GW, perform integration so as to obtain $\bar{h}_{\alpha\beta}$. Then, solve for $h_{\alpha\beta}(t,\vec{x})$. This will determine the shape of the gravitational wave; once detached from the source, the unsourced (vacuum) wave equation takes over and states that the wave stays the same shape while travelling through the Universe with the speed of light.

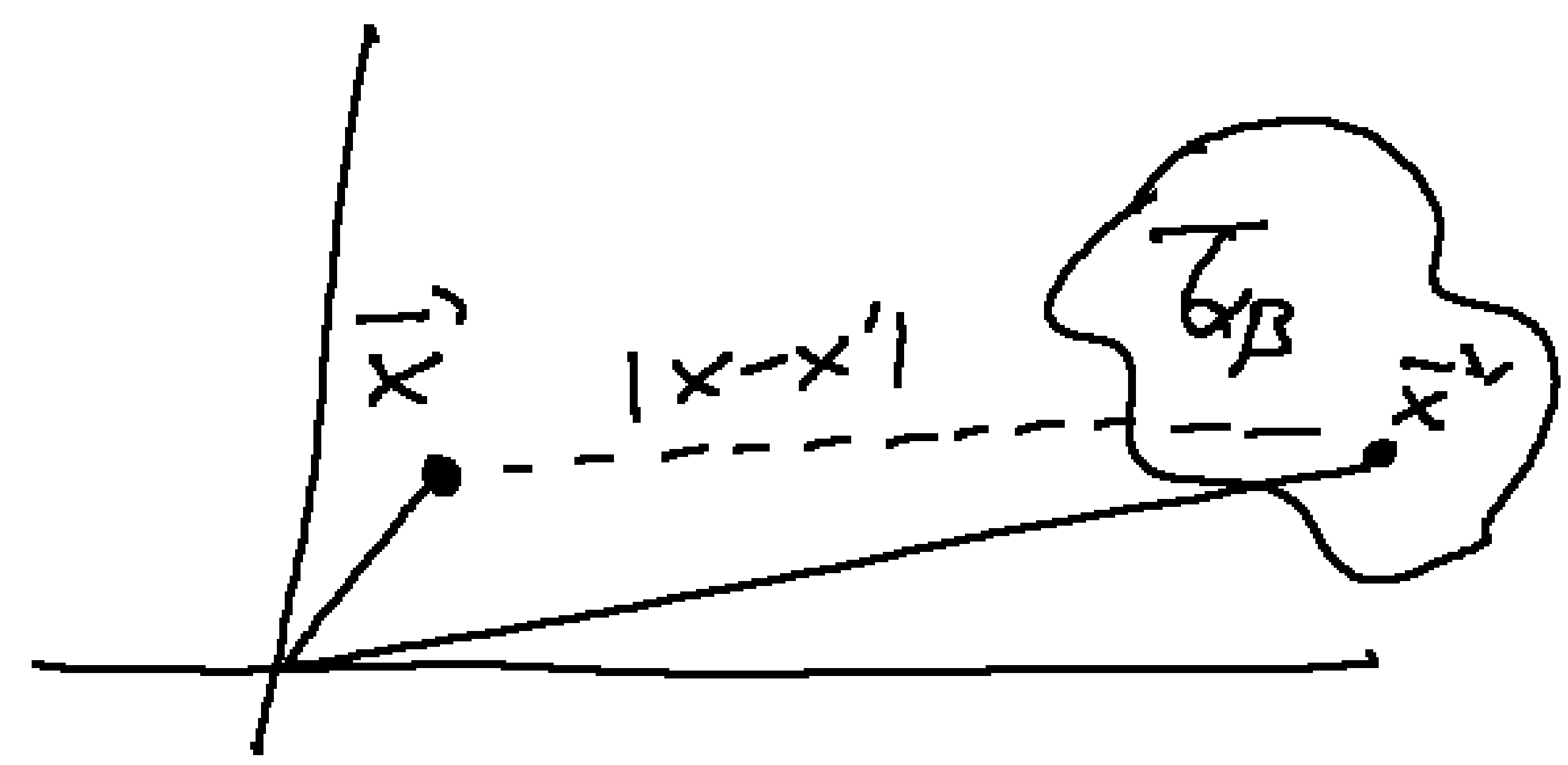
Derivation of $\bar{h}^{ij} = \frac{2}{r}\ddot{I}^{ij}(t-r)$

Derived before was the solution $\bar{h}_{\alpha\beta}$ of the sourced linearised gravitational wave equation:

$$\Box^2 \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}$$

$$\Rightarrow \bar{h}_{\alpha\beta} = 4\int \frac{T_{\alpha\beta}(t_R, \vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

with $t_R \equiv t - |\vec{x}-\vec{x}'|$.



This solution holds under assumption of $h_{\mu\nu} \ll \eta_{\mu\nu}$, and in the Lorentz-gauge.

If also the TT-gauge is applied, only the ij-components need to be considered:

$$\bar{h}^{ij} = 4\int \frac{T^{ij}(t_R, \vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'.$$

Also, if all points $\vec{x}'$ in the source are very far away from us at $\vec{x}$, then $\vec{x}-\vec{x}'$ by good approximation becomes a constant, so

$$\bar{h}^{ij} = \frac{4}{r}\int T^{ij}(t-r,\vec{x}')\,d^3x'.$$

Now, from conservation of energy momentum,

$$D_\mu T^{\mu\nu} = 0,$$

linearised version:

$$\partial_\mu T^{\mu\nu} = 0 \implies \boxed{\begin{aligned} &\partial_t T^{tt} + \partial_i T^{it} = 0 \qquad (i) \\ &\partial_t T^{tj} + \partial_i T^{ij} = 0. \qquad (ii) \end{aligned}}$$

These two conservation laws will turn

$$\bar{h}^{ij} = \frac{4}{r}\int T^{ij}(t-r,\vec{x}')\,d^3x'$$

into

$$\bar{h}^{ij} = \frac{2}{r}\ddot{I}^{ij}(t-r), \quad I^{ij} = \int \mu(t-r)\,x^i x^j\,d^3x.$$

As follows:

$$\partial_t T^{tt} + \partial_i T^{it} \overset{(i)}{=} 0$$

$$\Rightarrow \partial_t^2 T^{tt} = -\partial_i(\partial_t T^{it})$$

$$\overset{(ii)}{=} \partial_i \partial_j T^{ij}$$

Multiply both sides by $x^l x^k$, integrate by parts:

$$\int x^l x^k \partial_t^2 T^{tt} d^3x = \int x^l x^k \partial_i \partial_j T^{ij} d^3x$$

$$= \int x^l x^k (\partial_i \partial_j T^{ij}) dx^1 dx^2 dx^3$$

partial integration over x_i

$$= \underbrace{\int x^l x^k \partial_j T^{ij} \Big|_{x_i = \pm\infty} d^2x}_{=0,\ \text{because source does not extend to } \infty} - \int \partial_i (x^l x^k) \partial_j T^{ij} d^3x$$

$$= -\int \partial_i (x^l x^k) \partial_j T^{ij} d^3x$$

$$= -\int (\delta_i^l x^k + \delta_i^k x^l) \partial_j T^{ij} d^3x$$

$$= -\int (\partial_j T^{lj} x^k + \partial_j T^{kj} x^l) d^3x$$

That is:

$$\int x^l x^k \partial_t^2 T^{tt} d^3x = -\int \left(\partial_j T^{lj} x^k + \partial_j T^{kj} x^l\right) d^3x$$

This trick of partial integration can be done once more, to get

$$= \int \left(T^{lk} + T^{kl}\right) d^3x$$

$$\boxed{\int x^l x^k \ddot{T}^{tt} d^3x = 2\int T^{lk} d^3x}$$

Using this, the GW solution has become:

$$\bar{h}^{ij} = \frac{4}{r}\int T^{ij}(t-r, \vec{x}') d^3x'$$

$$= \frac{2}{r}\partial_t^2\left(\int x^i x^j T^{00} d^3x\right)$$

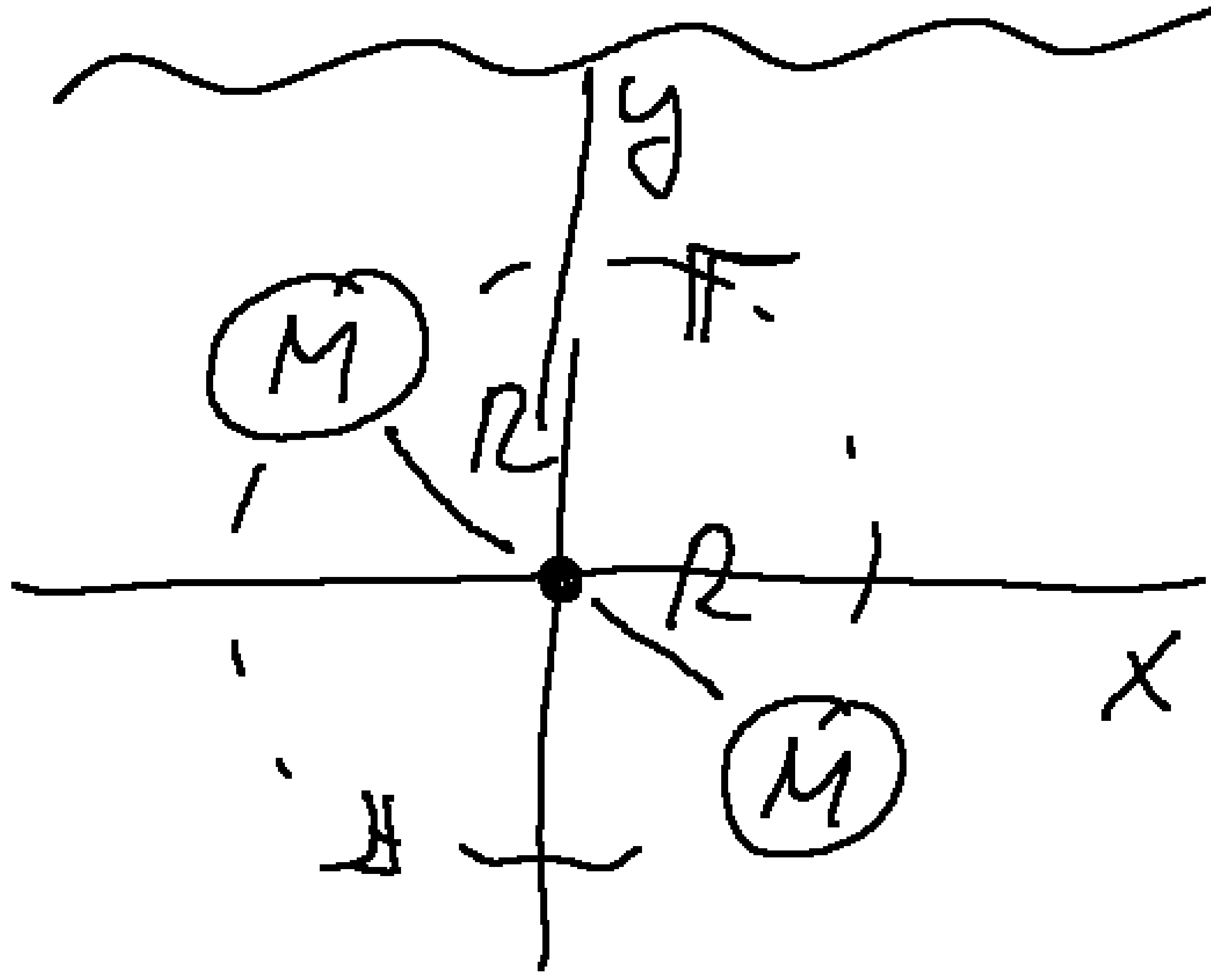
and since, for low velocities all T^{00} is mass density μ,

$$\boxed{\bar{h}^{ij} = \frac{2}{r}\ddot{I}^{ij}, \quad I^{ij} \equiv \int x^i x^j \mu(t-r) d^3x}$$

* Thus: gravitational waves require an acceleration (double time-derivative).
* Also, they require non-spherical symmetry: for spherical symmetry, I^{ij} is constant, so $\ddot{I} = 0 \Rightarrow$ no gravitational waves.
* Thus, in order to get GWs, a single black hole cannot do it due to its spherical shape, so one will require a binary system.
* If we are to use the formula derived, it must be for a situation in which two masses accelerate with respect to each other, but do not move too fast (to spoil approximation $T^{00} \approx \mu$), and are not too heavy (to spoil approximation of Minkowski background), and do not make gravitational waves that are too big (to spoil approximation $h_{\mu\nu} \ll 1$).

$$\boxed{\text{Two stars in Newtonian mechanics will fulfill these requirements}}$$

Worked example:
gravitational waves from a Newtonian binary system



Equal mass system (so centre of mass is exactly in the middle). Can easily be adapted to non-equal mass, via M and μ.

Via Newtonian mechanics, per star:

$$x(t) = R\cos(\omega t)$$
$$y(t) = R\sin(\omega t)$$

Needed: $I^{ij} \equiv \int x^i x^j \mu(t-r)\, d^3x$

E.g. $I^{xx} = \int R^2 \cos^2(\omega t)\, \mu(t-r)\, d^3x$

with $\cos^2 = \frac{1}{2} + \frac{1}{2}\cos(2\omega t)$

and $\mu(t-r) = M_1 \delta(t-R) + M_2 \delta(t-R)$
$= 2M\delta(t-R)$

$\Rightarrow I^{xx} = R^2 M\left(1 + \cos(2\omega t)\right)$

Likewise:

$$I^{xy} = MR^2 \sin(2\omega t),$$
$$I^{yy} = MR^2(1-\cos(2\omega t)),$$
$$I^{xx} = MR^2(1+\cos(2\omega t)).$$

So, in TT-gauge then:

$$\boxed{\bar{h}^{ij} = \frac{2}{r}\ddot{I}^{ij} = -\frac{8\omega^2}{r}MR^2\begin{pmatrix}\cos(2\omega t) & \sin(2\omega t)\\ \sin(2\omega t) & -\cos(2\omega t)\end{pmatrix}}$$

Key features:

* Gravitational waves go down with distance (the tidal forces, however, go down with $1/r^2$)
* The frequency of a gravitational wave is twice the orbital frequency: $\omega_{gw} = 2\omega$.
* The amplitude $\frac{8\omega^2}{r}MR^2 = \frac{2}{r}MR^2\omega_{gw}^2$ goes up with frequency. So, during inspiral phase when R goes down, by Kepler's Third Law, ω goes up. $\omega^2 \propto R^{-3}$, so net effect is amplitude going up.
* Source motion can be Fouriered into modes, each having twice GW-frequency.

Energy of a gravitational wave

Energy is a complicated matter in GR:

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{curvature}} = \underbrace{8\pi T_{\mu\nu}}_{\text{energy (density)}}.$$

Where exactly is the split between energy (RHS) and curvature (LHS)?
For example, from the geodesic equation it follows that $g_{\mu\nu}$ itself is a source of energy, and hence so is $h_{\mu\nu}$: a gravitational wave carries energy, simply from its existence alone!
Another example: Black holes! They are vacuum solutions to the EFE, yet they curve spacetime. We even talk about black holes having mass, despite we also say that they are vacuum solutions. This is because their own curvature is their energy!

* So, how do we know which part of 'the' energy of a curved spacetime 'should' be written at the LHS of the EFE, and which part 'should' be written as RHS? This ambiguity results in more than one definition of what constitutes 'mass', 'energy', and curvature in a spacetime, depending on where the split is set.

* For linearised gravity this ambiguity has a natural resolution!
By premisse, LHS will only consist of first order terms in $h_{\mu\nu}$. But, since EFE is non-linear, LHS will consist of higher-order terms. Ferrying these over to RHS thus creates an energy-momentum (pseudo-) tensor that therefore denotes the energy carried by the gravitational wave.

* Its leading term will be $O(h_{\mu\nu}^2)$, or, in terms of second mass moment, $O(I^{ij\,2})$.
Its the self coupling of curvature to itself (in particle physics, this would be the mass of a particle field).

The resulting expression, in leading term, is the gravitational wave power:

$$\boxed{L_{GW} = \frac{1}{5} \langle \dddot{\mathcal{I}}_{ij} \dddot{\mathcal{I}}^{ij} \rangle}$$

third derivative, as second der. gives energy, so third der. gives power

where $\mathcal{I}^{ij} \equiv \underbrace{I^{ij} - \frac{1}{3}\delta^{ij} I^k{}_k}$

this makes sure a spherically symm. mass distribution does not send out gravitational energy.

This is the Quadrupole formula.

Example: applied to the Newtonian binary system, where

$$I^{xx} = MR^2 \left(1 + \cos(2\omega t)\right)$$

$$I^{yy} = MR^2 \left(1 - \cos(2\omega t)\right)$$

$$I^{xy} = MR^2 \sin(2\omega t)$$

$$\Rightarrow I^k{}_k = I^{xx} + I^{yy}$$

$$= 2MR^2 .$$

So $$\dddot{I}^{ij} = MR^2 \begin{pmatrix} (1+\cos(2\omega t))-2 & \sin(2\omega t) \\ \sin(2\omega t) & (1-\cos(2\omega t))-2 \end{pmatrix}$$

and:
$$\dddot{I}^{ij} = MR^2 8\omega^3 \begin{pmatrix} -\sin(2\omega t) & \cos(2\omega t) \\ \cos(2\omega t) & \sin(2\omega t) \end{pmatrix}$$

So next:
$$\langle \dddot{I}^{ij} \dddot{I}_{ij} \rangle = 2M^2R^4 64\omega^6$$

and averaging over spacetime adds a factor of 2. Finally then:

$$\boxed{L_{GW} = \frac{128}{5} M^2 R^4 \omega^6 \frac{c^5}{G}}$$

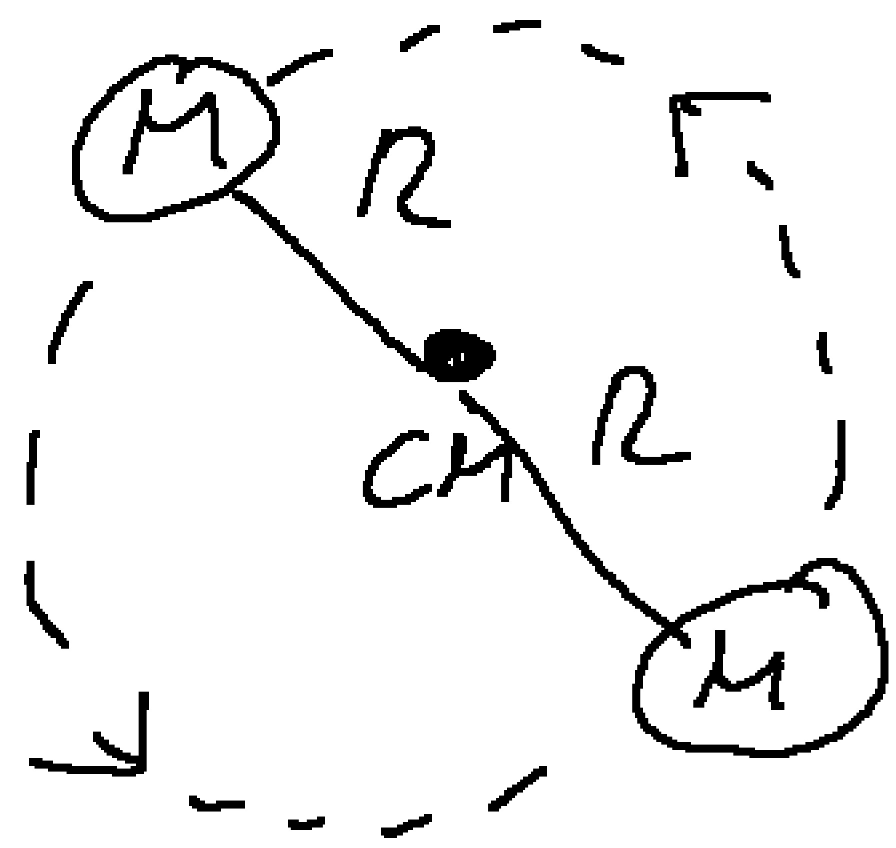
dimensional analysis: LHS $= \frac{kg\, m^2}{s^3}$, RHS is $kg\, m^4 \frac{1}{s^6}$, giving that c^5/G be added.

tends to be huge numbers!

Gravitational waves carry huge amounts of energy!

This is due to the fact that it cost enormous amounts to create them to begin with: the $\frac{8\pi G}{c^4}$ that had to be overcome, is now carried by the gravitational wave!

Effect of gravitational wave emission on the Newtonian binary system



Binary system from before, equal mass, each orbiting around the centre of mass CM, at equal distance R.

Argued before: formulas derived in Minkowski spacetime apply for this system, if M not too big and orbital velocity not too big, and small gravitational waves.

So, we can use Newtonian dynamics:

$$E = \frac{1}{2} M_1 v^2 + \frac{1}{2} M_2 v^2 - \frac{M_1 M_2}{2R}$$

$$= M v^2 - \frac{M^2}{2R}$$

with v the orbital velocity, and $2R$ the distance between the two stars.

Plan: write this total energy as a function of the orbital period P (Kepler's Third Law), then set its time derivative equal to the quadrupole formula: $\dot{E} = -L_{GW}$, solve for $P(t)$.

Now, Kepler's Third Law states that

$$T^2 = \frac{4\pi^2}{GM_{total}} R^3$$

for each star. Here, $T = P$, and $M_{total} = 2M$.

So: $$R = \sqrt[3]{\frac{MG}{2\pi^2} P^2}.$$

Also: $$v = \frac{2\pi R}{P} = 2\pi \frac{1}{P} \sqrt[3]{\frac{MG}{2\pi^2} P^2}$$

From these, the energy E of the binary system becomes

$$E = Mv^2 - \frac{M^2}{2R}$$
$$= M \frac{4\pi^2}{P^2} \left(\frac{MG}{2\pi^2} P^2 \right)^{2/3} - \frac{M^2}{2} \sqrt[3]{\frac{2\pi^2}{M} \frac{1}{P^2}}$$

Cleaned up:

$$\boxed{E = -\frac{1}{4} M \left(\frac{4\pi M}{P}\right)^{2/3}}$$

Energy of a Newtonian binary system, expressed in orbital period P

Using now that the system leaks energy via the Quadrupole formula:

$$\dot{E} = -L_{GW},$$

i.e.

$$\dot{E} = \frac{2}{12} M (4\pi M)^{2/3} P^{-5/2} \dot{P}$$

$$\stackrel{!}{=} L_{GW} = \frac{128}{5} M^2 R^4 \omega^6$$

$$= \frac{128}{5} M^2 \left(\frac{M}{2\pi^2} P^2\right)^{4/3} \left(\frac{2\pi}{P}\right)^6$$

$$\propto P^{8/3} P^{-6} = P^{-10/3}$$

$$\Rightarrow \quad \boxed{\dot{P} = -A P^{-5/3}, \quad A = \frac{96}{5} \pi \, 4^{-1/3} (2\pi M)^{5/3}}$$

which has as its solution:

$$\boxed{P(t) = \frac{2^{9/8}}{3^{3/8}} \left(A(t_0 - t)\right)^{3/8}}$$

$$\boxed{P(t) = \frac{2^{9/8}}{3^{3/8}}\left(A(t_0 - t)\right)^{3/8}}$$

From this expression, much physics follows:

* The period goes down as time progresses: orbital dynamics speeds up. (this is just Kepler's Third Law)

* GW frequency $\omega_{gw} = 2\omega_{orbital}$

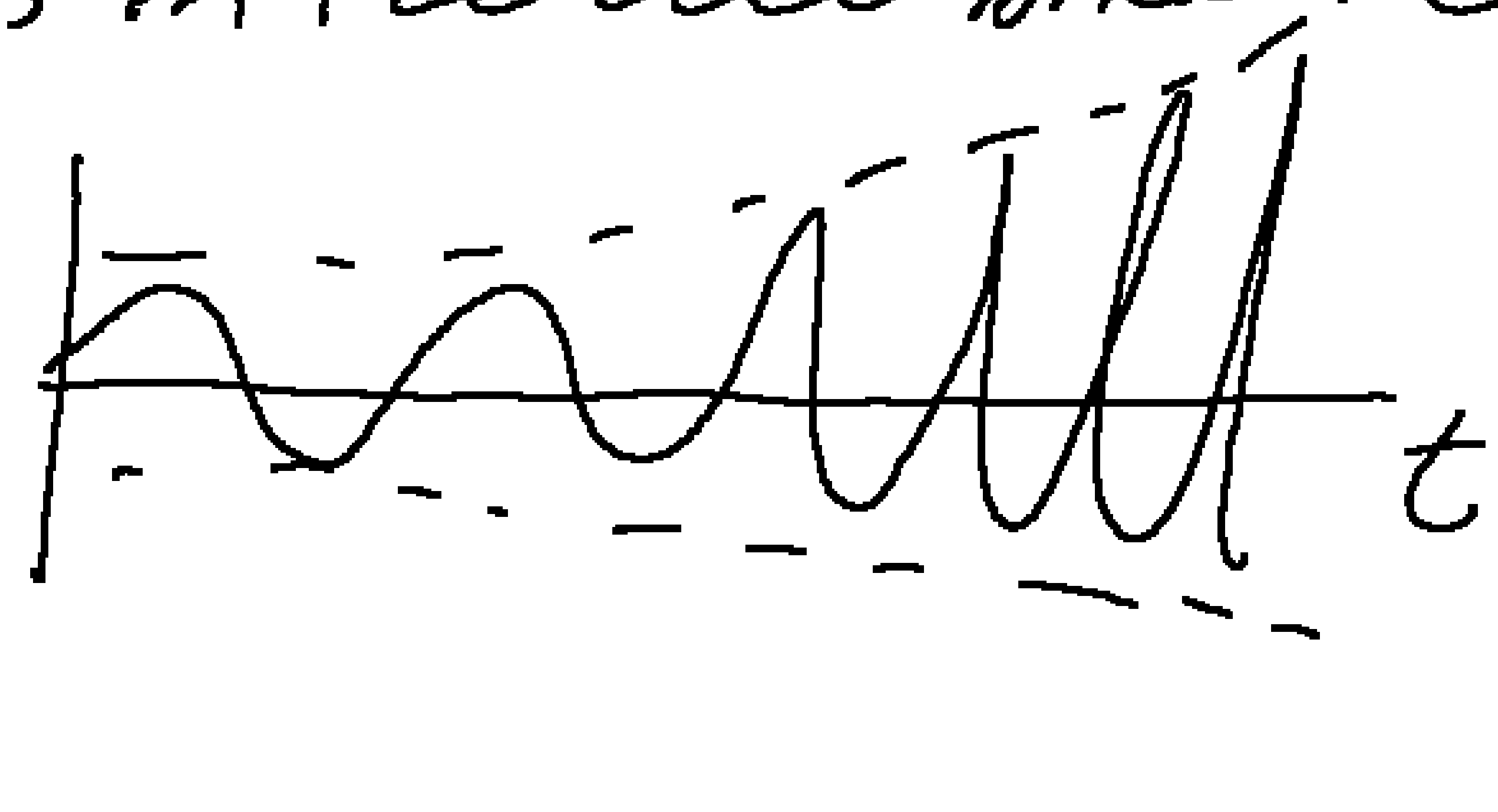
$$= 2 \cdot \frac{2\pi}{P}$$

goes up as time progresses.

* Plugged back into

$$\bar{h}_{ij} = \frac{8\omega^2}{2} MR \begin{pmatrix} \cos(2\omega t) & \sin(2\omega t) \\ \sin(2\omega t) & -\cos(2\omega t) \end{pmatrix}$$

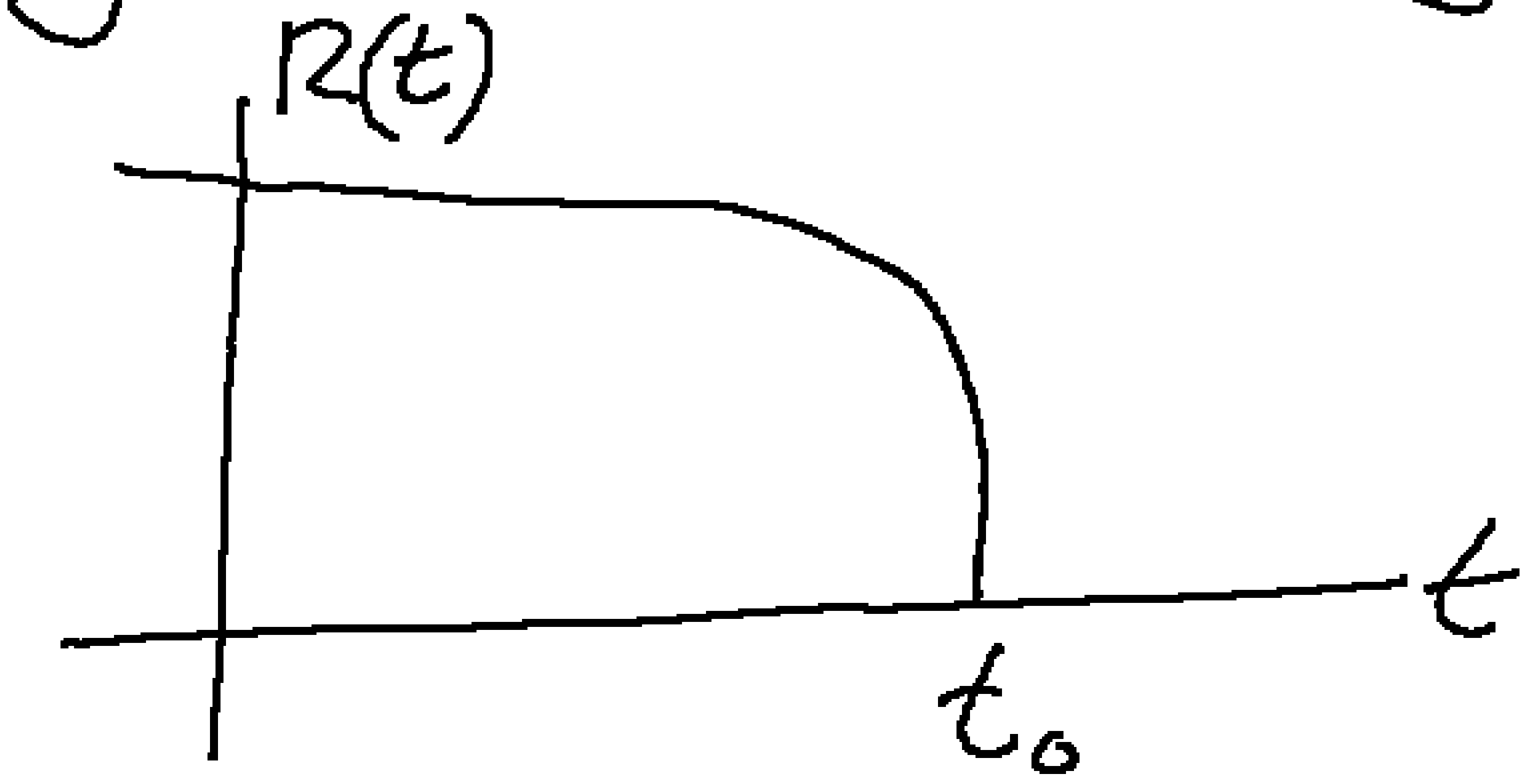
the amplitude of the GW goes up, too, as the stars orbit closer and closer. This results in the well-known chirp signal:



* The radius $R(t) = \sqrt[3]{\frac{M}{2\pi^2} P^2(t)}$

goes down : $\propto \sqrt[4]{A(t-t_0)}$

This is of course the leak of energy.



* The constant of integration to represent a final moment, when the solutions stops holding. It has to be put in by hand. Where? Well, the Newtonian dynamics does not know about horizons (or ISCO's, ICO's), and treats the two stars M as point particles. This is represented by the fact that $R(t=t_0)=0$, ie. M's get to reach each other infinitely closely. But, but for real-life stars, they are not point particles, and t_0 has to be set by their 'moment of meeting'.