Topical lectures:

(1.4) $$c\,d\tau = \sqrt{|ds^2|} = \sqrt{-g_{\mu\nu}\,dx^\mu dx^\nu}$$

$$= \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\;d\lambda$$

Proper time elapsed b/w 2 pts A & B

$$c\,\Delta\tau_{AB} = \int_A^B \sqrt{|ds^2|} = \int_A^B \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\;d\lambda$$

Using $\lambda = \tau$,

$$c\,\cancel{d\tau} = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\;\cancel{d\tau}$$

$$\Rightarrow\; g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2$$

$$V^\mu = \frac{dx^\mu}{d\tau} \qquad \Rightarrow\; V^\mu V_\mu = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2$$

Topical lectures

(2.3) $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \; ; \; h = h^{\alpha}_{\alpha}$

Show under gauge transformation,

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho\right)$$

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

Weak-fld limit : $g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}$,

$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$.

Gauge trans: $x'^\alpha = x^\alpha + \xi^\alpha$

$\Rightarrow \frac{\partial x'^\alpha}{\partial x^\beta} = \delta^\alpha_\beta + \partial_\beta \xi^\alpha \; ; \; \frac{\partial x^\beta}{\partial x'^\alpha} = \frac{\partial}{\partial x'^\alpha}\left(x'^\beta - \xi^\beta\right)$

$= \delta^\beta_\alpha - \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial \xi^\beta}{\partial x^\mu}$

$\Rightarrow \frac{\partial x^\beta}{\partial x'^\alpha} = \delta^\beta_\alpha - \left(\delta^\mu_\alpha - \partial'_\alpha \xi^\mu\right)\partial_\mu \xi^\beta \simeq \delta^\beta_\alpha - \delta^\mu_\alpha \partial_\mu \xi^\beta$

[note the recursive use]

$= \delta^\beta_\alpha - \partial_\alpha \xi^\beta$

$g'_{\mu\nu} = \left(\delta^\alpha_\mu - \partial_\mu \xi^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu \xi^\beta\right)\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right)$

$\cancel{= \delta^\alpha_\mu \delta^\beta_\nu \eta_{\alpha\beta} + \delta^\alpha_\mu \delta^\beta_\nu h_{\alpha\beta}}$

$\simeq \left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\mu \partial_\nu \xi^\beta - \delta^\beta_\nu \partial_\mu \xi^\alpha\right)\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right)$

$\simeq \delta^\alpha_\mu \delta^\beta_\nu \eta_{\alpha\beta} + \delta^\alpha_\mu \delta^\beta_\nu h_{\alpha\beta} - \delta^\alpha_\mu \partial_\nu \xi^\beta \eta_{\alpha\beta} - \delta^\beta_\nu \partial_\mu \xi^\alpha \eta_{\alpha\beta}$

$\cancel{= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi}$

(2.3) Continued.

$\xi_\mu = \eta_{\mu\alpha}\xi^\alpha$

$\Rightarrow \delta^\alpha_\mu \partial_\nu \xi^\beta \eta_{\alpha\beta} = \delta^\alpha_\mu \partial_\nu \xi_\alpha$ (note $\eta_{\alpha\beta}$ is const; can take inside diff)

$= \partial_\nu \xi_\mu$

Similarly,

$\delta^\beta_\nu \partial_\mu \xi^\alpha \eta_{\alpha\beta} = \delta^\beta_\nu \partial_\mu \xi_\beta$

$= \partial_\mu \xi_\nu$

$\Rightarrow g'_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu$

$\Rightarrow \boxed{h'_{\mu\nu} = h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu}$

Now

$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h'$

~~Substituting~~,

~~$\bar{h}'_{\mu\nu} = (h_{\mu\nu} - \partial_\nu\xi_\mu - \partial_\mu\xi_\nu) - \frac{1}{2}\eta_{\mu\nu} h$~~

~~$\Rightarrow \bar{h}'_{\mu\nu} = h_{\mu\nu}$~~

$h' = \eta^{\mu\nu} h'_{\mu\nu}$

$= \eta^{\mu\nu}(h_{\mu\nu} - \partial_\nu\xi_\mu - \partial_\mu\xi_\nu)$

$= h - 2\partial_\sigma \xi^\sigma$

Now substituting,

$\bar{h}'_{\mu\nu} = (h_{\mu\nu} - \partial_\nu\xi_\mu - \partial_\mu\xi_\nu) - \frac{1}{2}\eta_{\mu\nu}(h - 2\partial_\sigma\xi^\sigma)$

$= \bar{h}_{\mu\nu} - \partial_\nu\xi_\mu - \partial_\mu\xi_\nu - \eta_{\mu\nu}\partial_s\xi^s$