From flat spacetime to Einstein equation: Hands-on session

Part I: Flat space to Einstein's equations

(1.2) When going along the radius of the circle, φ is constant, so $d\varphi = 0$, hence $ds^2 = R^2 d\theta^2$. Therefore the radius is

$$\rho = \int_0^\alpha \sqrt{R^2 d\theta^2} = R \int_0^\alpha d\theta = R\alpha. \tag{1}$$

When going along the circle itself one instead has $\theta = \alpha$, so this time $d\theta = 0$, and $ds^2 = R^2 \sinh^2 \alpha \, d\varphi^2$. The circumference of the circle is then

$$C = \int_0^{2\pi} \sqrt{R^2 \sinh^2 \alpha \, d\varphi^2} \, d\varphi = R \sinh \alpha \int_0^{2\pi} d\varphi = 2\pi R \sinh \alpha. \tag{2}$$

Hence the ratio of circumference to radius is

$$\frac{C}{\rho} = 2\pi \frac{\sinh \alpha}{\alpha}. \tag{3}$$

Using

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots, \tag{4}$$

clearly $\sinh \alpha / \alpha > 1$, hence $C/\rho > 2\pi$. For $\alpha \ll 1$ one has $\sinh \alpha \simeq \alpha$ and $C/\rho \simeq 2\pi$.

Part II: Einstein's equations to wave equation

(2.4) One must have

$$\partial^\mu \bar{h}'_{\mu\nu} = 0 = \partial^\mu \bar{h}_{\mu\nu} - \partial^\mu (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho), \tag{5}$$

from which it follows that

$$\partial^\mu (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho) = \partial^\mu \bar{h}_{\mu\nu}. \tag{6}$$

For the left hand side one has

$$\begin{aligned} &\partial^\mu (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho) \\ &\qquad = \Box \xi_\nu + \partial_\nu \partial^\mu \xi_\mu - \partial_\nu \partial_\rho \xi^\rho \\ &\qquad = \Box \xi_\nu + \partial_\nu \partial^\mu \xi_\mu - \partial_\nu \partial_\mu \xi^\mu \\ &\qquad = \Box \xi_\nu, \end{aligned} \tag{7}$$

where in the last term of the next-to-last line we renamed the dummy index ρ to μ. Thus, the required gauge transformation must satisfy

$$\Box\xi_\nu = \partial^\mu \bar{h}_{\mu\nu}. \tag{8}$$

This equation always has a solution, which we can explicitly construct using the Green's function of the d'Alembertian, $\mathcal{G}(t, \mathbf{x})$. Specifically,

$$\xi_\nu(t, \mathbf{x}) = \int dt' d\mathbf{x}' \partial^\mu \bar{h}_{\mu\nu}(t', \mathbf{x}')\, \mathcal{G}(t - t', \mathbf{x} - \mathbf{x}'). \tag{9}$$

This is indeed the solution, because

$$\begin{aligned}
\Box\xi_\nu(t, \mathbf{x}) &= \Box \int dt' d\mathbf{x}' \partial^\mu \bar{h}_{\mu\nu}(t', \mathbf{x}')\, \mathcal{G}(t - t', \mathbf{x} - \mathbf{x}') \\
&= \int dt' d\mathbf{x}' \partial^\mu \bar{h}_{\mu\nu}(t', \mathbf{x}')\, \Box\mathcal{G}(t - t', \mathbf{x} - \mathbf{x}') \\
&= \int dt' d\mathbf{x}' \partial^\mu \bar{h}_{\mu\nu}(t', \mathbf{x}')\, \delta(t - t', \mathbf{x} - \mathbf{x}') \\
&= \partial^\mu \bar{h}_{\mu\nu}(t, \mathbf{x}),
\end{aligned} \tag{10}$$

where in the last line we have used the defining equation of the Green's function.

(2.5) Consider a wavefront at a give time t defined by

$$\omega t - \mathbf{k} \cdot \mathbf{x} = K \tag{11}$$

for some constant K. Let $\mathbf{y}$ be another point on the same wavefront, so also

$$\omega t - \mathbf{k} \cdot \mathbf{y} = K. \tag{12}$$

Writing $\mathbf{\Delta x} = \mathbf{y} - \mathbf{x}$, one finds

$$\omega t - \mathbf{k} \cdot \mathbf{x} - \mathbf{k} \cdot \mathbf{\Delta x} = K. \tag{13}$$

Subtracting Eq. (11),

$$\mathbf{k} \cdot \mathbf{\Delta x} = 0. \tag{14}$$

Hence any two points on a wavefront are connected by a vector that is perpendicular to $\mathbf{k}$.

Now let time evolve. If at time t a point on the wavefront was at $\mathbf{x}$, then at a later time $t' = t + \Delta t$ it will have been moved along $\mathbf{k}$, so $\mathbf{x}' = \mathbf{x} + a\mathbf{k}$ for some a. Plugging this into Eq. (11) we get

$$\omega(t + \Delta t) - \mathbf{k} \cdot \mathbf{x} - a\mathbf{k} \cdot \mathbf{k} = K, \tag{15}$$

so that

$$\omega \Delta t = a|\mathbf{k}|^2. \tag{16}$$

Since $\omega = c|\mathbf{k}|$,

$$c\Delta t = a|\mathbf{k}|. \tag{17}$$

Hence, the wavefront traverses the distance $\Delta D = a|\mathbf{k}|$ in a time Δt such that $\Delta D/\Delta t = c$.