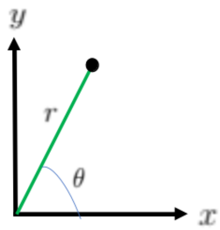
# From flat spacetime to Einstein equation: Hands-on session

## Part I: Flat space to Einstein's equations

**(1.1)** In the metric $ds^2 = g_{\mu\nu}x^\mu x^\nu$, once can also use the polar coordinates $(x'^1, x'^2) = (r, \theta)$:
Show that: $ds^2 = dr^2 + r^2 d\theta^2 = g'_{\mu\nu}dx'^\mu dx'^\nu$ where $g' = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$




**(1.2)** Consider beings living in a 2-dimensional surface with metric

$$ds^2 = R^2 d\theta^2 + R^2 \sinh^2\theta \, d\varphi^2, \tag{1}$$

where $R$ is a constant. (Note the hyperbolic sine in the metric, which makes it different from the metric on a spherical surface.) On this surface, consider a circle determined by $\theta = \alpha$, where $\alpha$ is some fixed number. What is the radius $\rho$ of the circle, as determined by how far the beings have to walk in the direction of constant $\varphi$ from $\theta = 0$ to $\theta = \alpha$? What is the circumference $C$ of the circle, as determined by how far the beings have to walk around the circle to get back to their starting point? Show that $C/\rho > 2\pi$, but if $\alpha$ is small, one has $C/\rho \simeq 2\pi$.
Use

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots, \tag{2}$$

**(1.3)** Coordinate transformations change the metric

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} \tag{3}$$

Explicitly show when going from $(x^1, x^2) = (x, y)$ to $(x'^1, x'^2) = (r, \theta)$, this changes the metric from $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $g' = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

**(1.4)** Defining tangent vector to a curve $x^\mu(\tau)$ as $V^\mu = \frac{dx^\mu}{d\tau}$, show that: $V_\mu V^\mu = -c^2$, where $c$ is the velocity of light.

# Part II: Einstein's equations to wave equation

**(2.1)** Show that to leading order in $h_{\mu\nu}$, the inverse of the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is given by

$$g^{\mu\nu} \equiv \eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}, \tag{4}$$

i.e. show that

$$g^{\mu\rho}g_{\rho\nu} = \delta^\mu{}_\nu + \mathcal{O}(h^2) \tag{5}$$

with $g^{\mu\nu}$ defined as above.

**(2.2)** Also show that to leading order, raising indices on $h_{\mu\nu}$ using $g^{\mu\nu}$ is the same as raising indices using $\eta^{\mu\nu}$; for example,

$$h^\mu{}_\nu = g^{\mu\rho}h_{\rho\nu} = \eta^{\mu\rho}h_{\rho\nu} + \mathcal{O}(h^2). \tag{6}$$

**(2.3)** Under small coordinate transformations, show:

$$h'_{\mu\nu} = h_{\mu\nu} - (\partial_\nu\xi_\mu + \partial_\mu\xi_\nu) \tag{7}$$

We defined

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{8}$$

with $h = h^\alpha{}_\alpha$. Show that under small coordinate transformations, or gauge transformations, $\bar{h}_{\mu\nu}$ transforms as

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho). \tag{9}$$

**(2.4)** It was stated in class that whatever $\bar{h}_{\mu\nu}$ is, there will always be a gauge transformation such that in the new coordinate system the Lorenz gauge holds, i.e.

$$\partial^\mu\bar{h}'_{\mu\nu} = 0 \tag{10}$$

Prove that such a gauge transformation indeed always exists.

*Hint*: You can show that

$$\partial^\mu(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho) = \Box\xi_\nu, \tag{11}$$

so that the required gauge transformation is a solution to the equation

$$\Box \xi_\nu = \partial^\mu \bar{h}_{\mu\nu}. \tag{12}$$

From this, obtain an explicit expression for $\xi_\nu$ in terms of the Green's function $\mathcal{G}(t, \mathbf{x})$ of the d'Alembertian, and explain why that expression is indeed the solution to the above equation.

**(2.5)** As we have seen, the Lorentz gauge reduces the Einstein equations to a particularly simple form. In vacuum ($T_{\mu\nu} = 0$) one has $\Box \bar{h}_{\mu\nu} = 0$, or

$$\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \bar{h}_{\mu\nu} = 0. \tag{13}$$

Consider a plane gravitational wave of the form

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \cos(\omega t - \mathbf{k} \cdot \mathbf{x}) \tag{14}$$

for some constant tensor $A_{\mu\nu}$. We have seen that this is a solution to (13) on condition that $\omega = c|\mathbf{k}|$. Show that the wavefronts are perpendicular to $\mathbf{k}$, and that $\omega = c|\mathbf{k}|$ implies that the wave propagates at the speed of light.