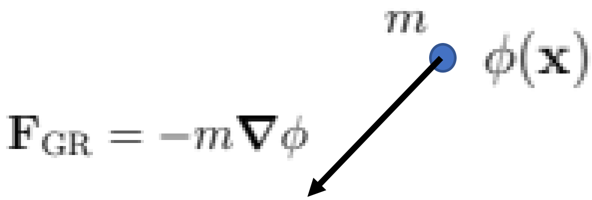
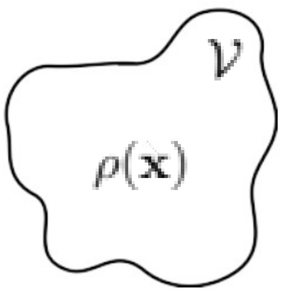
Topic 2: Lecture 2

Anuradha Samajdar
Utrecht University
a.samajdar@uu.nl

From Einstein Equations to Wave equation

Newton's theory of gravity





- Force and potential

$$\mathbf{F}_{\mathrm{GR}} = -m\boldsymbol{\nabla}\phi \qquad \nabla^2\phi(\mathbf{x}) = 4\pi G\rho(\mathbf{x})$$

- Given $\rho(\mathbf{x})$, how do we solve for $\phi(\mathbf{x})$?
  - Suppose we had a function $G(\mathbf{x})$ such that $\nabla^2 G(\mathbf{x}) = \delta^3(\mathbf{x})$
  - Then solution to Poisson equation:

$$\phi(\mathbf{x}) = 4\pi G \int_{\mathcal{V}} d^3\mathbf{x}'\,\rho(\mathbf{x}')\,G(\mathbf{x}-\mathbf{x}')$$

because

$$\begin{aligned}
\nabla^2\phi(\mathbf{x}) &= 4\pi G \nabla^2_{\mathbf{x}} \int_{\mathcal{V}} d^3\mathbf{x}'\,\rho(\mathbf{x}')\,G(\mathbf{x}-\mathbf{x}') \\
&= 4\pi G \int_{\mathcal{V}} d^3\mathbf{x}'\,\rho(\mathbf{x}')\,\nabla^2_{\mathbf{x}} G(\mathbf{x}-\mathbf{x}') \\
&= 4\pi G \int_{\mathcal{V}} d^3\mathbf{x}'\,\rho(\mathbf{x}')\,\delta^3(\mathbf{x}-\mathbf{x}') \\
&= 4\pi G\rho(\mathbf{x})
\end{aligned}$$

✓

Newton's theory of gravity

- Want to find a function $G(\mathbf{x})$ such that

$$\nabla^2 G(\mathbf{x}) = \delta^3(\mathbf{x}) \qquad (*)$$

This is called the **Green's function** of the Laplacian operator

- To find $G(\mathbf{x})$, take as ansatz that $G(\mathbf{x}) = g(r)$ where $r = |\mathbf{x}|$
  - Integrate both sides of $(*)$ over sphere of radius R centered on origin:

$$\int_{V_R} \nabla^2 G(\mathbf{x})\, d^3\mathbf{x} = 1 \quad \text{because} \quad \int_{V_R} \delta^3(\mathbf{x})\, d^3\mathbf{x} = 1$$

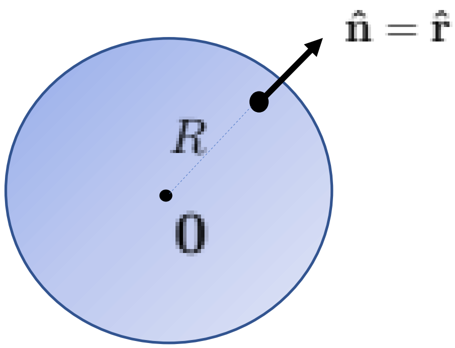
$$\Longrightarrow \int_{V_R} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} G(\mathbf{x})\, d^3\mathbf{x} = 1$$

$$\Longrightarrow \int_{S_R} \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} G(\mathbf{x})\, dA = 1$$

$$\Longrightarrow \int_{S_R} \frac{d}{dr} g(r)\, dA = 1$$

$$\Longrightarrow 4\pi R^2 \frac{d}{dr} g(R) = 1$$

$$\Longrightarrow \frac{d}{dr} g(R) = \frac{1}{4\pi R^2} \Longrightarrow g(r) = -\frac{1}{4\pi r} \Longrightarrow \boxed{G(\mathbf{x}) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}}$$



Newton's theory of gravity

- Solution to the Poisson equation thanks to Green's function:

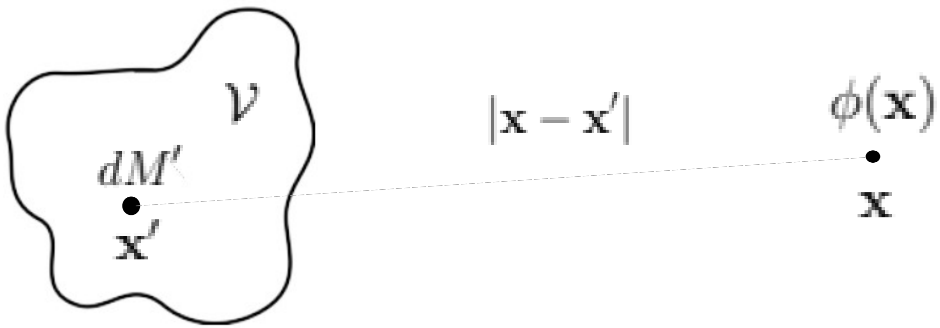
$$\phi(\mathbf{x}) = 4\pi G \int_{\mathcal{V}} d^3x'\, \rho(\mathbf{x}')\, G(\mathbf{x} - \mathbf{x}') \implies \boxed{\phi(\mathbf{x}) = -\int_{\mathcal{V}} d^3x' \frac{G\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}}$$

This makes sense!

  - Contribution to $\phi(\mathbf{x})$ from infinitesimal mass element dM' at $\mathbf{x}'$:

$$-\frac{G\,dM'}{|\mathbf{x} - \mathbf{x}'|} = -\frac{G\,\rho(\mathbf{x}')\,d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$

  - Total potential $\phi(\mathbf{x})$ at $\mathbf{x}$ obtained by "summing" over all contributions



Newton's theory of gravity

- Solution to the Poisson equation thanks to Green's function:

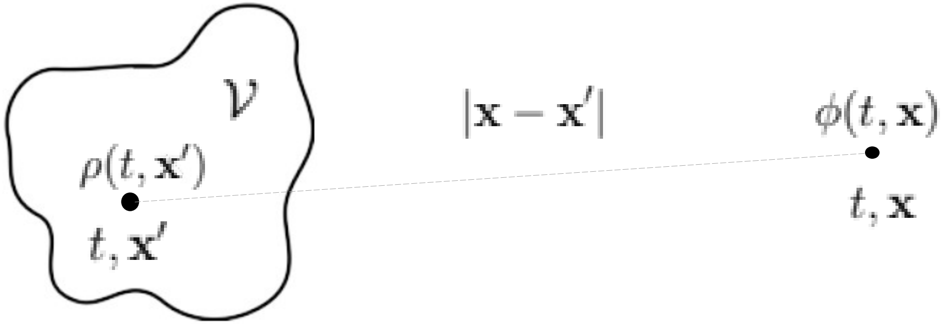
$$\phi(\mathbf{x}) = 4\pi G \int_{\mathcal{V}} d^3\mathbf{x}'\, \rho(\mathbf{x}')\, G(\mathbf{x}-\mathbf{x}') \implies \boxed{\phi(\mathbf{x}) = -\int_{\mathcal{V}} d^3\mathbf{x}'\, \frac{G\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}}$$

- Note that the density distribution is allowed to be time-dependent!
  - Poisson equation:

    $$\nabla^2\phi(t,\mathbf{x}) = 4\pi G\, \rho(t,\mathbf{x})$$
  - Green's function of the Laplacian:

    $$\nabla^2 G(\mathbf{x}) = \delta^3(\mathbf{x})$$
  - Solution to the Poisson equation remains the same:

    $$\phi(t,\mathbf{x}) = \int_{\mathcal{V}} d^3\mathbf{x}'\, \rho(t,\mathbf{x}')\, G(\mathbf{x}-\mathbf{x}') \implies \boxed{\phi(t,\mathbf{x}) = -\int_{\mathcal{V}} d^3\mathbf{x}'\, \frac{G\,\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}}$$



Any change in the density causes an immediate change in potential, no matter how far away!

Gravity and action at a distance

- Newtonian gravity: **instantaneous action at a distance**
- Not true in Maxwell's theory of electromagnetism!
  - Fields **E**, **B** at distance *D* from a charge/current distribution: What happens at time *t* depends on what charge/current distribution was at earlier time, $t - D/c$
  - **Wave equation** for **E**, **B** with propagation speed *c*
- Special relativity**: speed of light c is "speed limit"** for any kind of information transfer
- General relativity is a dynamical theory (involves time derivatives!)
  - Does it imply finite propagation speed for gravity?
  - Is there a wave equation for the gravitational field?

General relativity for weak gravitational fields

- Einstein equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Left hand side involves derivatives of the **metric** $g_{\mu\nu}$ w.r.t. time and space

- Equations will simplify when gravitational fields are weak:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

- General relativity allows for general coordinate transformations $x'^{\mu}(x)$
  - Many of those will make $h_{\mu\nu}$ large!
  - Will restrict ourselves to "small" coordinate transformations

$$x'^{\mu} = x^{\mu} + \xi^{\mu}$$

- Coordinate transformations acting on the metric: $g'_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}$

Here $\xi^{\mu}(x)$ is allowed to be different at different points, but has to have small effect!

General relativity for weak gravitational fields

- With the notation $\partial_\alpha \equiv \dfrac{\partial}{\partial x^\alpha}$:

$$h'_{\mu\nu} = h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

“gauge transformations”

- For all intents and purposes, we can view $h_{\mu\nu}$ as a tensor that lives on a flat spacetime

  - To linear order, raising and lowering of indices happens with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$:

$$h^\mu{}_\nu = g^{\mu\rho} h_{\rho\nu} = \eta^{\mu\rho} h_{\rho\nu} + \mathcal{O}(|h|^2)$$

Exercise

- Convenient **definition** before we continue:

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

with $h = \eta^{\alpha\beta} h_{\alpha\beta}$

which transforms under gauge transformations as

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho)$$

Exercise

“Linearized” general relativity

- Start from the full Einstein equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Substitute

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

and keep only terms linear in (derivatives of) $h_{\mu\nu}$ or equivalently $\bar{h}_{\mu\nu}$

See lecture notes for more details

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \partial^\rho \partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

where $\partial^\rho = \eta^{\rho\alpha}\partial_\alpha$

and $\Box$ is the **d’Alembertian operator**

$$\begin{aligned}
\Box &\equiv \partial_\mu \partial^\mu \\
&= \eta^{\mu\nu}\partial_\mu \partial_\nu \\
&= -\frac{\partial^2}{(\partial x^0)^2} + \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2} \\
&= -\frac{\partial^2}{c^2 \partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\
&= -\frac{\partial^2}{c^2 \partial t^2} + \nabla^2
\end{aligned}$$

“Linearized” general relativity

- Linearized Einstein equations:

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu \bar{h}_{\mu\rho} - \partial^\rho\partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu}$$

- Some terms can be gotten rid of by using gauge transformations!

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho)$$

  - Whatever $\bar{h}_{\mu\nu}$ is, there always exists a gauge transformation such that

$$\boxed{\partial^\mu \bar{h}'_{\mu\nu} = 0}$$

“**Lorentz gauge**” **Exercise**

  - In that case the linearized Einstein equations become a lot simpler!

$$\Box \bar{h}'_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}'_{\rho\sigma} - \partial^\rho\partial_\nu \bar{h}'_{\mu\rho} - \partial^\rho\partial_\mu \bar{h}'_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu}$$

  - Result:

$$\boxed{\Box \bar{h}'_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}}$$

“Linearized” general relativity

- Linearized Einstein equations (dropping the prime):

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Lorentz gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$

- Given an energy-momentum tensor $T_{\mu\nu}$, how do we solve for $\bar{h}_{\mu\nu}$?
  - Suppose we had a Green's function $\mathcal{G}(t, \mathbf{x})$ for the d'Alembertian $\Box$:

    $$\Box \mathcal{G}(t, \mathbf{x}) = \delta^4(t, \mathbf{x})$$ where $\delta^4(t, \mathbf{x}) = \delta(t)\, \delta^3(\mathbf{x})$

  - Then the solution to the linearized Einstein equations would be

    $$\bar{h}_{\mu\nu}(t, \mathbf{x}) = -\frac{16\pi G}{c^4} \int dt'\, d^3\mathbf{x}'\, T_{\mu\nu}(t', \mathbf{x}')\, \mathcal{G}(t - t', \mathbf{x} - \mathbf{x}')$$

    because

    $$\begin{aligned} \Box \bar{h}_{\mu\nu}(t, \mathbf{x}) &= -\frac{16\pi G}{c^4} \int dt'\, d\mathbf{x}'\, T_{\mu\nu}(t', \mathbf{x}')\, \Box \mathcal{G}(t - t', \mathbf{x} - \mathbf{x}') \\ &= -\frac{16\pi G}{c^4} \int dt'\, d\mathbf{x}'\, T_{\mu\nu}(t', \mathbf{x}')\, \delta^4(t - t', \mathbf{x} - \mathbf{x}') \\ &= -\frac{16\pi G}{c^4} T_{\mu\nu}(t, \mathbf{x}) \end{aligned}$$

Green's function of the d'Alembertian

- Need the Green's function of the d'Alembertian:

$$\Box \mathcal{G}(t,\mathbf{x}) = \delta^4(t,\mathbf{x})$$ where $$\delta^4(t,\mathbf{x}) = \delta(t)\,\delta^3(\mathbf{x})$$

In full:

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathcal{G}(t,\mathbf{x}) + \nabla^2\mathcal{G}(t,\mathbf{x}) = \delta^4(t,\mathbf{x})$$

- Away from the origin ($\delta^4(t,\mathbf{x}) = 0$) this looks like the equation for a wave!

  - Wave equation in **one spatial dimension**:

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}g(t,x) + \frac{\partial^2}{\partial x^2}g(t,x) = 0$$

Solutions:
$$\begin{cases} g(t,x) = f\left(t - \frac{x}{c}\right) & \text{wave traveling in positive x direction} \\ g(t,x) = f\left(t + \frac{x}{c}\right) & \text{wave traveling in negative x direction} \end{cases}$$

  - Ansatz in **three spatial dimensions**:

$$\mathcal{G}(t,r) = \frac{1}{r}F\left(t - \frac{r}{c}\right)$$

wave traveling away from the origin $r = 0$ and decreasing in strength as it does so

Green's function of the d'Alembertian

- Need the Green's function of the d'Alembertian:

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathcal{G}(t,\mathbf{x})+\nabla^2\mathcal{G}(t,\mathbf{x})=\delta^4(t,\mathbf{x})$$

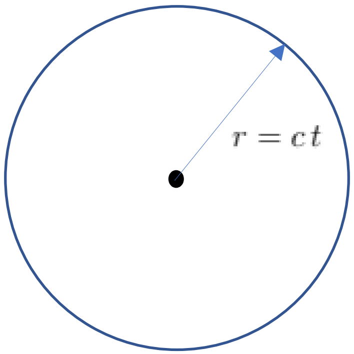
- Ansatz: $\mathcal{G}(t,r)=\frac{1}{r}F\left(t-\frac{r}{c}\right)$, or $\mathcal{G}(t,\mathbf{x})=\frac{1}{|\mathbf{x}|}F\left(t-\frac{|\mathbf{x}|}{c}\right)$

- Substituting into the equation leads to

$$-4\pi F(t)=\delta(t)$$

Hence

$$\boxed{\mathcal{G}(t,\mathbf{x})=-\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}\delta\left(t-\frac{|\mathbf{x}|}{c}\right)}$$



  - A single wavefront spreading from the origin at speed of light: $r=|\mathbf{x}|=ct$

- Side note:
  - $\mathcal{G}(t,\mathbf{x})=-\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}\delta\left(t-\frac{|\mathbf{x}|}{c}\right)$ is the *retarded* Green's function
  - $\mathcal{G}(t,\mathbf{x})=-\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}\delta\left(t+\frac{|\mathbf{x}|}{c}\right)$ is the *advanced* Green's function (time-reversed!)

Green's function of the d'Alembertian

- Linearized Einstein equations:

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- General solution in terms of Green's function:

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = -\frac{16\pi G}{c^4}\int dt'\, d^3\mathbf{x}'\, T_{\mu\nu}(t',\mathbf{x}')\,\mathcal{G}(t-t',\mathbf{x}-\mathbf{x}')$$

- We have found

$$\mathcal{G}(t,\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}\,\delta\left(t-\frac{\mathbf{x}}{c}\right)$$

- Substituting:

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4}\int dt'\, d^3\mathbf{x}'\, T_{\mu\nu}(t',\mathbf{x}')\,\delta\left(t-t'-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

- Integrating over time:

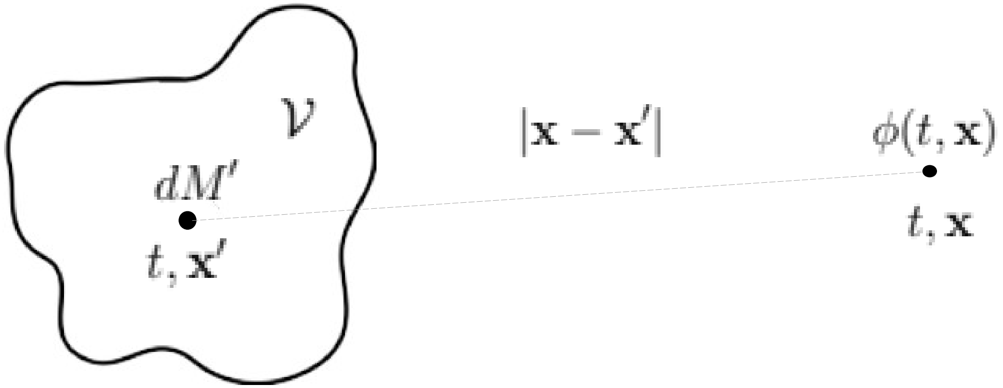
$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4}\int_{\mathcal{V}} d^3\mathbf{x}'\,\frac{T_{\mu\nu}\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}$$

No instantaneous action at a distance!

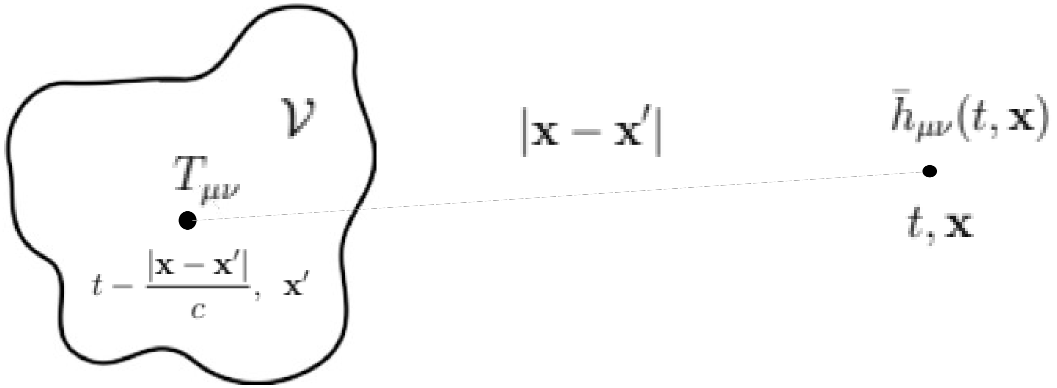
- Gravitational field due to arbitrary energy-momentum distribution:

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4}\int_{\mathcal{V}} d^3\mathbf{x}'\,\frac{T_{\mu\nu}\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}$$

- Previously, in **Newtonian gravity**:



- The picture in **Einsteinian gravity**:



No instantaneous action at a distance! ✓

Gravitational waves

- Linearized Einstein equations:

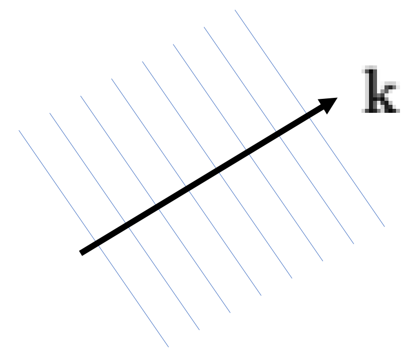
$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- Away from matter/energy distributions, in vacuum ($T_{\mu\nu} = 0$):

$$\Box \bar{h}_{\mu\nu} = 0$$

- This is the familiar wave equation with propagation speed c :

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}_{\mu\nu} = 0$$



- Example: **plane wave**

$$\bar{h}_{\mu\nu} = A_{\mu\nu}\cos(\omega t - \mathbf{k}\cdot\mathbf{x})$$

  - For this to be a solution: $\frac{\omega^2}{c^2} - \mathbf{k}\cdot\mathbf{k} = 0 \implies \omega = c|\mathbf{k}|$
  - If traveling in z direction: $\bar{h}_{\mu\nu} = A_{\mu\nu}\cos\left[\omega(t - z/c)\right]$
  - Usually one has **superpositions** of plane waves traveling in different directions

Summary

- Newtonian gravity has **instantaneous action at a distance**:

$$\nabla^2\phi(t,\mathbf{x}) = 4\pi G\,\rho(t,\mathbf{x}) \quad \Longrightarrow \quad \phi(t,\mathbf{x}) = -\int_{\mathcal{V}} d^3\mathbf{x}'\,\frac{G\,\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

- For **weak gravitational fields** the Einstein equations become

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}$$ in the Lorentz gauge $$\partial^\mu\bar{h}_{\mu\nu} = 0$$

- In Einsteinian gravity, time dependence in the source is communicated at the **speed of light**:

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4}\int_{\mathcal{V}} d^3\mathbf{x}'\,\frac{T_{\mu\nu}\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}$$

- In vacuum ($T_{\mu\nu} = 0$) the linearized Einstein equations become a **wave equation**

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)\bar{h}_{\mu\nu} = 0$$

gravitational waves