# Modelling magnetically formed neutron star mountains

Amlan Nanda

RESCEU, The University of Tokyo
Coauthors: Prof. Kotaro Fujisawa, Prof. Shota Kisaka, and Prof. Yasufumi Kojima

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*"The first step to breach an impregnable fortress is to attack it where the enemy least expects it. That's right, from the inside!"*
*- Sanada Masayuki.*

# Contents

# Introduction

# Motivation

1. Highly-anticipated detection of Continuous Gravitational Waves (CGWs) is important in more ways than one
2. Non-axisymmetric deformation of neutron stars: *Mountains*
3. **O**3 run estimates of LVK: Ellipticity constraints of $\approx 10^{-8}$
4. Mountain-building papers: Ushomirsky et. al (0001136), Gittins et al. (2009.12794) and Morales & Horowitz (2209.03222)

# *Magnetic Mountains*

1. Mountains generated by internal magnetic fields: *Magnetic mountains*
2. Case I: Magnetic fields inside the star generating mountains atop the elastic crust of the star
3. Case II: Magnetic fields inside the star and Surface currents at the crustal boundaries generating mountains atop the elastic crust

# Background neutron star

1. Newtonian, non-rotating, barotropic fluid star with $n = 1$ polytropic equation of state; Euler equation and Poisson's equation
2. Lagrangian perturbation of background parameters: $\vec{\xi}$
3. Crustal elasticity and strain is treated perturbatively: $t_{ij} = \mu(\nabla_i \xi_j + \nabla_j \xi_i - \frac{2}{3} g_{ij} \nabla_k \xi^k)$
4. Quadrupolar moments $I_{22} = \int_0^R \delta\rho_{22}(r) r^4 dr$ and fiducial ellipticity estimates $\epsilon = \sqrt{(8\pi/15)} * (I_{22}/I_{zz})$

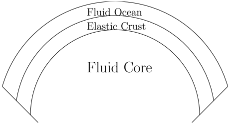


Figure 1: Interior of the neutron star

# Internal magnetic fields

1. Magnetic fields inside the star expanded using the multipolar decomposition: $\vec{B} = B_r Y_{lm}\hat{r} + \frac{r}{\beta} B_\perp \vec{\nabla} Y_{lm} + \frac{r}{\beta} B_\times (\hat{r} \times \vec{\nabla} Y_{lm})$
2. Maximum strength possible $\approx 10^{15}$G; treated perturbatively
3. Choice: dipolar ($(l, m) = (1, 0)$) magnetic fields generating quadrupolar ($(l, m) = (2, 2)$) Lorentz force
4. VSH normalization using Wigner symbols, Orientation of the fields, and split into Poloidal (radial and angular) and Toroidal (axial) parts
5. MHD equilibrium: $\vec{\nabla} \times (\vec{L}/\rho) = 0$

# Mountain-building schemes

# Ushomirsky scheme

1. Ushomirsky scheme: Based on the formation history of the star
2. Entire crust strained and elastic crust is maximally deformed

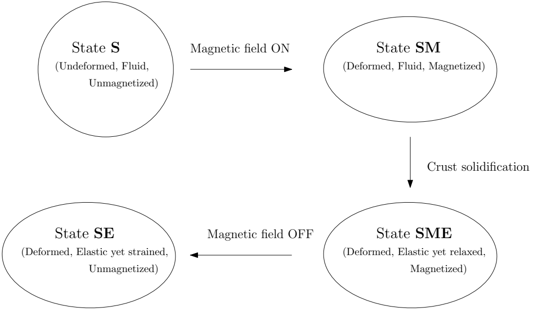


Figure 2: Ushomirsky scheme of mountain formation

# Gittins scheme

1. Gittins scheme: Force-based approach to mountain formation
2. Boundary conditions satisfied with one point in the crust strained
3. Differences with Ushomirsky scheme is of higher perturbative order

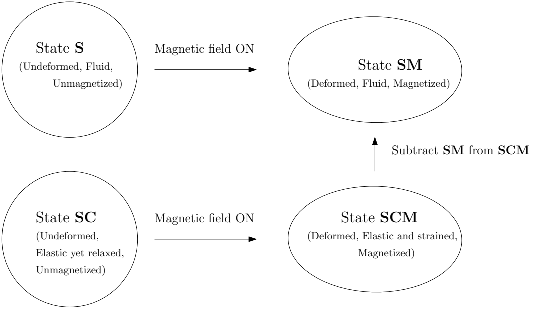


Figure 3: Gittins scheme of mountain formation

# Case I: Magnetic fields forming Mountains

# Magnetic field configuration

1. Poloidal magnetic fields: $B_r(r) = \left(\frac{4\pi}{3}\right)^{3/2}\frac{2F_0\rho_c}{\pi^5}\left[\left(\frac{3\pi^2R^3}{r}-\frac{6R^5}{r^3}\right)\sin(\pi r/R)+\frac{6\pi R^4}{r^2}\cos(\pi r/R)+\pi^3R^2\right]$ and $B_\perp(r) = \left(\frac{4\pi}{3}\right)^{3/2}\frac{\beta F_0\rho_c}{\pi^5}\left[\left(\frac{3\pi^2R^3}{r}+\frac{6R^5}{r^3}\right)\sin(\pi r/R)+\left(3\pi^2R^3-\frac{6R^5}{r^2}\right)\frac{\pi}{R}\cos(\pi r/R)-\frac{6\pi^2R^3}{r}\sin(\pi r/R)+2\pi^3R^2\right]$.
2. Toroidal magnetic fields: $B_\times(r) = \left(\frac{4\pi}{3}\right)^{1/2}\frac{\beta S_0\rho_c}{\pi^5}\sin\left(\pi r/R\right)$.

# Lorentz force components and perturbative equations

1. Poloidal case: $f_r = -\frac{1}{4\pi}\left(\frac{dB_\perp}{dr} + \frac{B_\perp}{r} - \frac{\beta B_r}{r}\right) B_\perp \frac{r^2}{\beta^2}\mathbf{A}$;
$f_\perp = \frac{1}{4\pi}\left(\frac{dB_\perp}{dr} + \frac{B_\perp}{r} - \frac{\beta B_r}{r}\right) B_r \frac{r}{\beta}\mathbf{B}$
2. Toroidal case: $f_r = -\frac{1}{4\pi}\left(\frac{dB_\times}{dr} + \frac{B_\times}{r}\right) B_\times \frac{r^2}{\beta^2}\mathbf{A}$; $f_\perp = \frac{1}{4\pi} B_\times^2 \mathbf{B}$
3. Perturbative equations: $\mathbf{a} d\vec{\xi}/dr = \mathbf{b}\vec{\xi} + \mathbf{c}\vec{T}$ and
$\mathbf{A} d\vec{T}/dr = \mathbf{B}\vec{\xi} + \mathbf{C}\vec{T} + \vec{f} + \vec{\nabla}\delta\Phi$
4. Boundary conditions: Usual conditions of the fluid star at the centre and the surface of the star. At the crustal boundaries; continuity of the perturbed traction vector $(\delta p - T_1;\ T_2)$. No change in the boundary conditions due to the continuity of the magnetic fields.

# Results: Plots of Traction: Poloidal case

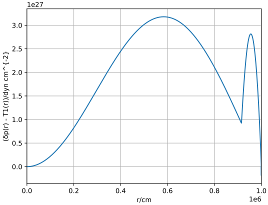


Figure 4: Continuity of radial perturbed traction

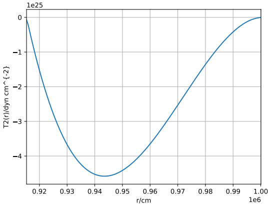


Figure 5: Continuity of angular perturbed traction

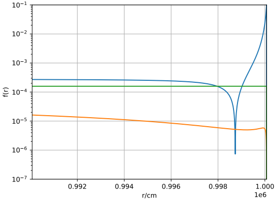


Figure 6: The strain components maximized over $(\theta, \phi)$ at the point of crust breaking

# Results: Plots of Traction: Toroidal case

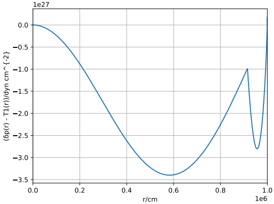


Figure 7: Continuity of radial perturbed traction

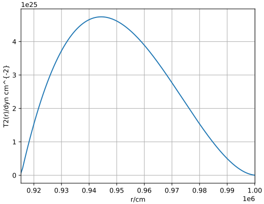


Figure 8: Continuity of angular perturbed traction

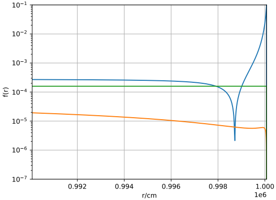


Figure 9: The strain components maximized over $(\theta, \phi)$ at the point of crust breaking

# Results: Ellipticity estimates

1. Poloidal case: Quadrupolar moment: $|Q_{strain} - Q_{relax}| = 2.69167 \times 10^{37}$
2. Poloidal case: Ellipticity estimate: $|\epsilon_{strain} - \epsilon_{relax}| = 3.48451 \times 10^{-8}$
3. Toroidal case: Quadrupolar moment: $|Q_{strain} - Q_{relax}| = 2.89424 \times 10^{37}$
4. Toroidal case: Ellipticity estimate: $|\epsilon_{strain} - \epsilon_{relax}| = 3.74636 \times 10^{-8}$

# Case II: Magnetic fields and Surface currents forming Mountains

# Surface currents Model

1. Crustal surface currents: $K_i(r) = K_0(\delta(r - r_1) - \delta(r - r_2))$
2. Choice: dipolar surface currents $((l, m) = (1, 0))$ with poloidal (for toroidal fields) and toroidal (for poloidal fields) components
3. Strength of surface currents: Determined by boundary conditions at the crustal interfaces: $K_0 \approx 10^{15}\mathtt{G}$
4. Confining strong magnetic fields inside the crust of the star

# Magnetic field configuration

1. Poloidal magnetic fields: $B_r(r) = \left(\frac{4\pi}{3}\right)^{3/2} \frac{2F_0\rho_c}{\pi^5}\left[\left(\frac{3\pi^2 R^3}{r} - \frac{6R^5}{r^3}\right)\sin(\pi r/R) + \frac{6\pi R^4}{r^2}\cos(\pi r/R) + \pi^3 R^2\right] + 2K_0\left(\frac{4\pi}{3}\right)^{3/2}\left(1 - \frac{1}{r^3}\right)$

   and $B_\perp(r) = \left(\frac{4\pi}{3}\right)^{3/2} \frac{\beta F_0\rho_c}{\pi^5}\left[\left(\frac{3\pi^2 R^3}{r} + \frac{6R^5}{r^3}\right)\sin(\pi r/R) + \left(3\pi^2 R^3 - \frac{6R^5}{r^2}\right)\frac{\pi}{R}\cos(\pi r/R) - \frac{6\pi^2 R^3}{r}\sin(\pi r/R) + 2\pi^3 R^2\right] + \beta K_0\left(\frac{4\pi}{3}\right)^{3/2}\left(2 + \frac{1}{r^3}\right)$

2. Toroidal magnetic fields:

   $B_\times(r) = \left(\frac{4\pi}{3}\right)^{1/2} \frac{\beta S_0\rho_c}{\pi^5}\sin(\pi r/R) + K_0\left(\frac{4\pi}{3}\right)^{3/2}\left(2(1-\beta) - \frac{2+\beta}{r^3}\right)$

# Lorentz force and perturbative equations

1. Poloidal case: $f_r = -\frac{1}{4\pi}\left(\frac{dB_\perp}{dr} + \frac{B_\perp}{r} - \frac{\beta B_r}{r}\right) B_\perp \frac{r^2}{\beta^2}\mathbf{A}$;
   $f_\perp = \frac{1}{4\pi}\left(\frac{dB_\perp}{dr} + \frac{B_\perp}{r} - \frac{\beta B_r}{r}\right) B_r \frac{r}{\beta}\mathbf{B}$
2. Toroidal case: $f_r = -\frac{1}{4\pi}\left(\frac{dB_\times}{dr} + \frac{B_\times}{r}\right) B_\times \frac{r^2}{\beta^2}\mathbf{A}$; $f_\perp = \frac{1}{4\pi} B_\times^2 \mathbf{B}$
3. Perturbative equations: $\mathbf{a} d\vec{\xi}/dr = \mathbf{b}\vec{\xi} + \mathbf{c}\vec{T}$ and
   $\mathbf{A} d\vec{T}/dr = \mathbf{B}\vec{\xi} + \mathbf{C}\vec{T} + \vec{f} + \vec{\nabla}\delta\Phi$
4. Boundary conditions: Usual conditions of the fluid star at the centre and the surface of the star. At the crustal boundaries; the continuity of the perturbed traction vector is modified by surface currents.
5. Poloidal case: continuity of $\delta p - T_1 + \frac{1}{8\pi}((B^\perp)^2 - (B^r)^2)$,
   $T_2 - \frac{1}{4\pi} B^r B^\perp$; Toroidal case: continuity of $\delta p - T_1 + \frac{1}{8\pi}(B^\times)^2$, $T_2$

# Results: Plots of Traction: Poloidal case

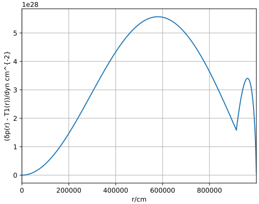


Figure 10: Continuity of radial perturbed traction

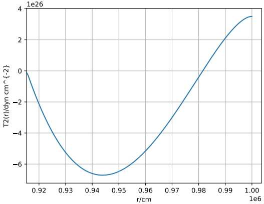


Figure 11: Continuity of angular perturbed traction

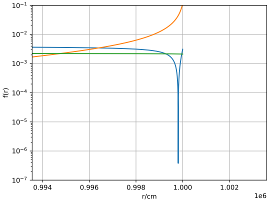


Figure 12: The strain components maximized over $(\theta, \phi)$ at the point of crust breaking

# Results: Plots of Traction: Toroidal case

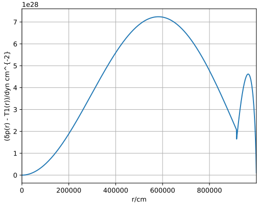


Figure 13: Continuity of radial perturbed traction

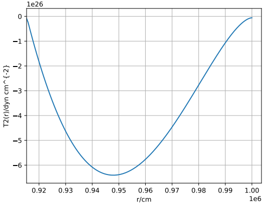


Figure 14: Continuity of angular perturbed traction

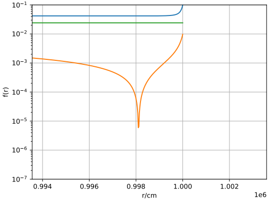


Figure 15: The strain components maximized over $(\theta, \phi)$ at the point of crust breaking

# Results: Ellipticity estimates

1. Poloidal case: Quadrupolar moment: $|Q_{strain} - Q_{relax}| = 8.36263 \times 10^{38}$
2. Poloidal case: Ellipticity estimate: $|\epsilon_{strain} - \epsilon_{relax}| = 2.43050 \times 10^{-6}$
3. Toroidal case: Quadrupolar moment: $|Q_{strain} - Q_{relax}| = 6.38721 \times 10^{38}$
4. Toroidal case: Ellipticity estimate: $|\epsilon_{strain} - \epsilon_{relax}| = 8.26771 \times 10^{-7}$

# Discussion

# Conclusion

1. Mountains generated/ellipticity estimated for the star of the range of $10^{-8}$ due to internal magnetic fields of the star; within previous theoretical and observational constraints.
2. Mountains generated/ellipticity estimated for the star of the range of $10^{-6}$ due to internal magnetic fields of the star with crustal surface currents; possible explanation of higher ellipticities if observed.
3. Radial dependence of the forces and boundary conditions across crustal boundaries are the key to higher ellipticities.
4. Special magnetic field configurations are the most natural explanation of such conditions inside a neutron star.

# The way ahead

1. Multiple other deformation strategies can be implemented
2. Mixing of poloidal and toroidal magnetic fields
3. Vacuum magnetic fields of extraordinary strength
4. Leaky Surface currents? From Dirac-delta to Gaussian or Block
5. Relativistic formulation of the problem

# Summary

1. What we did: Generation of *Mountains* atop neutron stars by internal magnetic fields of the star
2. How we did: Poloidal or Toroidal fields without/with surface currents deforming the elastic crust of the neutron star
3. Why we did: Expectations of high deformation of the neutron star due to magnetic fields' geometrical configuration
4. What we got: Normal estimates of $10^{-8}$ ellipticity without surface currents; increased estimates of $10^{-6}$ ellipticity with surface currents
5. What more can we do: Different configurations of magnetic fields or considering other phenomena.

# Thank you!

# Backup slides

# Tackling a subtlety: MHD vs MED

1. MED equilibrium is extremely complicated; coupling the magnetic fields' geometrical configuration with the crustal perturbations
2. Not a problem with the Ushomirsky scheme and is of higher perturbative order in the Gittins scheme

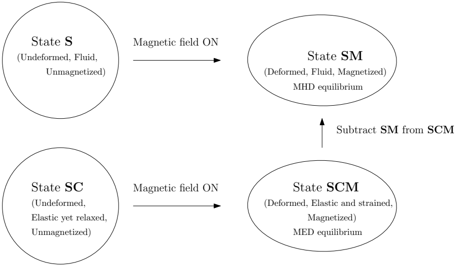


Figure 16: Gittins scheme with MED equilibrium

# Coding strategy

1. System of equations solved using `Vern6` solver which is part of `DifferentialEquations` package of `Julia`
2. Check the von Mises strain at the maxima point in the outer boundary of the elastic crust
3. Normalize that to $\sigma_{\text{max}} = 0.1$ and then calculate the fiducial ellipticity

# VSH integrals

1. Radial component:

$$\mathbf{A} \equiv \int Y^*_{l_3 m_3}(\vec{\nabla} Y_{l_1 m_1} \cdot \vec{\nabla} Y_{l_2 m_2}) d\Omega$$

2. Angular component:

$$\mathbf{B} \equiv \int \vec{\nabla} Y^*_{l_3 m_3} \cdot Y_{l_1 m_1} \vec{\nabla} Y_{l_2 m_2} d\Omega$$

3. Axial component (for MHD calculation):

$$\mathbf{C} \equiv \int (\hat{r} \times \vec{\nabla} Y^*_{l_3 m_3}) \cdot (Y_{l_1 m_1} \hat{r} \times \vec{\nabla} Y_{l_2 m_2}) d\Omega = \mathbf{B}$$

# Orientation of the field

1. Aligning the magnetic field:

$$Y_{lm}(\theta, \phi) = \sum_{m'} \mathcal{D}^{(l)}_{m'm}(\alpha, \beta) Y_{lm'}(\theta', \phi')$$

2. Wigner-$\mathcal{D}$ function:

$$\mathcal{D}^{(2)}_{0,2}(\alpha, \beta) = \frac{1}{4\sqrt{6}} (\sin \beta/2)^4 (\tan \beta/2)^2$$

# Equivalence of field decompositions

1. Poloidal-Toroidal decomposition $\equiv$ Multipolar decomposition
2. Poloidal-Toroidal decomposition: $\vec{B} = \vec{\nabla} \times (\hat{r}\Psi) + \vec{\nabla} \times (\vec{\nabla} \times (\hat{r}\Phi))$
3. Using a spherical harmonics basis: $\Psi = \Psi_{lm} Y_{lm}, \Phi = \Phi_{lm} Y_{lm}$
4. Insertion into the equation and then we re-label the parameters

# Future strategies

# Deformation strategies

1. MED equilibrium star consideration
2. Higher multipolar orders
3. Hot neutron star and effect of temperature on the magnetic fields
4. Superconducting core
5. Superfluidity of the star fluid
6. Plasticity of the crust
7. Realistic Equations of State

# Mixed magnetic fields

1. Mixed poloidal-toroidal magnetic fields:
   $\vec{B} = B_r Y_{lm}\hat{r} + \frac{r}{\beta}B_{\perp}\vec{\nabla}Y_{lm} + \frac{r}{\beta}B_{\times}(\hat{r}\times\vec{\nabla}Y_{lm})$
2. Lorentz force has axial component now $\implies$ Displacement vector and thus, the Traction vector will also have axial components now $\implies$ System of coupled ODEs goes from 5 equations to 7 equations
3. Modified von Mises strain: Has two extra parameters: $\xi_{\times}$ and $T_3$

# Vacuum magnetic fields

1. Vacuum magnetic fields: $\vec{B} = B_0(Y_{lm}\hat{r} + \frac{r}{\beta}\vec{\nabla} Y_{lm})$
2. Possibly of external origin; only interesting when it is strong enough
3. Perturbations induced by perturbed Lorentz force:
   $\delta\vec{f}_L = \frac{1}{4\pi}\left((\vec{\nabla} \times \vec{B}) \times \delta\vec{B} + (\vec{\nabla} \times \delta\vec{B}) \times \vec{B}\right)$
4. Which is generated by the perturbed Magnetic field:
   $\delta\vec{B} = \vec{\nabla} \times (\vec{\xi} \times \vec{B})$
5. Perturbed Lorentz force has an axial component again $\implies$ the system of coupled ODEs is again enlarged and the von Mises strain is again modified

# Leaky currents

1. Modification of geometrical configuration of surface currents
2. Possible Leaky currents?:

$$K_i(r) = K_0(\Theta(r - r_1) - \Theta(r - r_2))$$

3. Gaussian Leaky currents?:

$$K_i(r) = K_0(\mathcal{N}(r - r_1) - \mathcal{N}(r - r_2))$$

# Relativistic Extension

1. To solve: $\delta G_{\mu\nu} = 8\pi G[\delta T^M_{\mu\nu} + T^B_{\mu\nu}]$
2. Extra energy-momentum tensor components due to the magnetic field:

$$T^B_{\mu\nu} = \begin{bmatrix} \frac{B^2}{8\pi} & 0 & 0 & 0 \\ 0 & -\frac{B_x^2}{8\pi} & -\frac{B_x B_y}{4\pi} & -\frac{B_x B_z}{4\pi} \\ 0 & -\frac{B_x B_y}{4\pi} & -\frac{B_y^2}{8\pi} & -\frac{B_z B_y}{4\pi} \\ 0 & -\frac{B_x B_z}{4\pi} & -\frac{B_z B_y}{4\pi} & -\frac{B_z^2}{8\pi} \end{bmatrix}$$