

Topics

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1. Symmetries & quantum effects

- Conformal symmetry
- The logs of Asymptotic Freedom
- Calculation of $\langle F^2(x) F^2(0) \rangle$

2. The running coupling

- $\lambda \phi^4$ theory to one loop
- Calculation of $\Lambda(\phi\phi \rightarrow \phi\phi)$

3. The Callan-Symanzik equation

- $\beta(g)$ and $\gamma_0(g)$
- Perturbative expansions
- Universality of coefficients
- $g(x)$ from $\beta(g)$
- $\beta(g)$ in $d = 4 - 2\varepsilon$ dimensions

- Wilson-Fisher fixed point in
 - QED₃
 - QCD₄ above the CW

- Solution of the CS equation
(m=2 case)

4. RG flow à la Wilson.

- Integrate "out" high-momentum modes
- Relate classification
relevant / marginal / irrelevant
to the classification of Lagrangians in
superrenormalizable / renormalizable /
nonrenormalizable.
- More on CS eq. versus Wilson flow
- More on dimensional analysis.

NB : The • indicates topics that are
further treated in these notes -

Solutions
Discussion

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NB Cite appropriately if this content is used.

• Conformal Ward Identities

A generic correlator respects conformal symmetry if it fulfills the Ward Identities below-

a) Dilatations Ward Identity

$$0 = \sum_{J=1}^m \left(\Delta_J + x_J^\mu \frac{\partial}{\partial x_J^\mu} \right) \langle O_1(x_1) \dots O_m(x_m) \rangle$$

$$\boxed{m=2}$$

$$\Delta_1 = \Delta_2 = \Delta \quad O_1 = O_2$$

$$0 = \left(2\Delta + x^\mu \frac{\partial}{\partial x^\mu} \right) \langle O(x) O(0) \rangle$$

The conformal correlator $\langle O(x) O(0) \rangle = \frac{C}{x^{2\Delta}}$ preserves dilatations. Indeed:

$$\begin{aligned} x^\mu \frac{\partial}{\partial x^\mu} \frac{1}{x^{2\Delta}} &= x^\mu \frac{\partial x^2}{\partial x^\mu} \frac{\partial}{\partial x^2} (x^2)^{-\Delta} = x^\mu 2x_\mu (-\Delta)(x^2)^{-\Delta-1} \\ &= \frac{-2\Delta}{x^{2\Delta}} \end{aligned}$$

Hence:

$$\begin{aligned} \left(2\Delta + x^\mu \frac{\partial}{\partial x^\mu} \right) \frac{1}{x^{2\Delta}} &= (2\Delta - 2\Delta) \frac{1}{x^{2\Delta}} \\ &= 0 \quad \checkmark \end{aligned}$$

b) Special conformal Ward Id. (inversion \times
translation)

$$O = \sum_{J=1}^m \left(2\Delta_J x_J^\mu + 2x_J^\mu x_J^\nu \frac{\partial}{\partial x_J^\nu} - x_J^2 \frac{\partial}{\partial x_{J\mu}} \right) <..>$$

$$\boxed{m=2} \quad \Delta_1 = \Delta_2 \equiv \Delta \quad O_1 = O_2 \equiv O$$

$$O = \left(2\Delta x^\mu + 2x^\mu x^\nu \frac{\partial}{\partial x^\nu} - x^2 \frac{\partial}{\partial x_\mu} \right) <O(x)O(0)>$$

The conformal correlator preserves special conformal symmetry:

$$\left(2\Delta x^\mu + 2x^\mu x^\nu \frac{\partial}{\partial x^\nu} - x^2 \frac{\partial}{\partial x_\mu} \right) \frac{1}{x^{2\Delta}}$$

$$= 2\Delta \frac{x^\mu}{x^{2\Delta}} + 2x^\mu (-2\Delta) \frac{1}{x^{2\Delta}} - x^2 \frac{(-2\Delta)x^\mu}{(x^2)^{\Delta+1}}$$

$$= 2\Delta \frac{x^\mu}{x^{2\Delta}} - 4\Delta \frac{x^\mu}{x^{2\Delta}} + 2\Delta \frac{x^\mu}{x^{2\Delta}}$$

$$= 0 \quad \checkmark$$

• Beta function $\beta(g)$

a) Solve for the running coupling in $d=4$, i.e. integrate the following identity:

$$\beta(g) = \frac{dg}{d \log \mu}$$

Integrate both sides:

$$\int_{g(\mu)}^{g(z)} \frac{dg}{\beta(g)} = \int_{\mu}^{z^{-1}} d \log \mu \quad \begin{matrix} (z \text{ is a coordinate}) \\ z \sim \text{energy}^{-1} \\ z = |z| \end{matrix}$$

where $g(z) = g(z\mu, g(\mu))$ with initial condition

$g(1, g(\mu)) = g(\mu)$. Then, insert $\beta(g) = -\beta_0 g^3 + \dots$ in the LHS to obtain:

$$\int_{g(\mu)}^{g(z)} \frac{dg}{-\beta_0 g^3 + \dots} = -\log(z\mu)$$

Solve at leading order (LO) in perturbation theory:

$$\int_{g(\mu)}^{g(z)} \frac{dg}{g^3} = \beta_0 \log z\mu$$

Hence:

$$-\frac{1}{2} \left(\frac{1}{g^2(z)} - \frac{1}{g^2(\mu)} \right) = \beta_0 \log z\mu$$

$$\frac{g^2(\mu)}{g^2(z)} = 1 - 2\beta_0 g^2(\mu) \log z\mu$$

Its inverse is :

$$g^2(z) = \frac{g^2(\mu)}{1 - 2\beta_0 g^2(\mu) \log z\mu}$$

whose perturbative expansion for $g^2(\mu) \rightarrow 0$
and logs of order one is:

$$g^2(z) = g^2(\mu) \left(1 + 2\beta_0 g^2(\mu) \log z\mu + \dots \right)$$

- Beta function in $d = 4 - 2\epsilon$ dimensions

It is interesting to see what happens when we move slightly away from $d=4$ space-time dimensions. We discover that the beta function changes.

In $d = 4 - 2\epsilon$ the beta function is:

$$\begin{aligned}\beta(g, \epsilon) &= -\epsilon g + \beta(g) \\ &= -\epsilon g - \beta_0 g^3 + \dots\end{aligned}$$

where $\beta(g)$ is the beta function in $d=4$. What are the zeroes of $\beta(g, \epsilon)$?

First, $g=0$ remains a zero.

Second, depending on the sign of β_0 a new non-trivial (i.e. at $g > 0$) zero may also exist.

Solving to \mathcal{O} in g the equation $\beta(g, \epsilon) = 0$ we obtain:

$$-\epsilon - \beta_0 g^2 = 0$$

Hence:

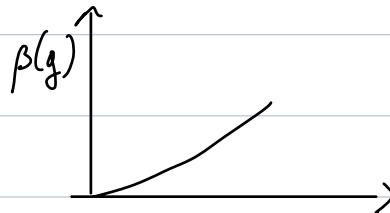
$$g^2 = -\frac{\epsilon}{\beta_0}$$

Hence, $\alpha^2 > 0$ if $\beta_0 < 0$!

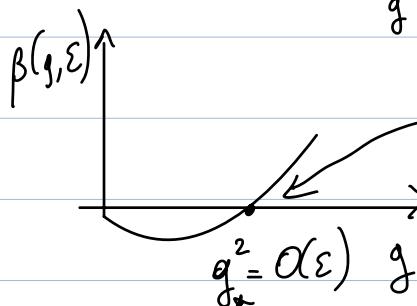
This happens in at least two relevant cases :

a) QED

$$d = \hbar \Rightarrow$$



$$d = \hbar - 2\varepsilon \Rightarrow$$



Wilson - Fisher
Fixed Point

(relevant in
condensed
matter)

b) Above the CW of QCD i.e. $N_f > N_f^{\text{AF}}$

CW = Conformal Window

N_f = Number of fermion "flavors"

In $SU(N)$ QCD :

$$\beta_0 = \frac{1}{16\pi^2} \left(\frac{11}{3} N - \frac{2}{3} N_f \right)$$

colors # flavors

It changes sign at :

$$\frac{11}{3} N - \frac{2}{3} N_f = 0$$

That is :

$$N_f^{\text{AF}} = \frac{11}{2} N$$

Hence:	$\beta_0 > 0$	for	$N_f < \frac{11}{2} N$
	$\beta_0 < 0$	for	$N_f > \frac{11}{2} N$

For $N=3$ $\overset{\text{AF}}{N_f} = \frac{11}{2} \cdot 3 = 16.5$ and

for $N_f > 16.5$ QCD is no longer asymptotically free!

- Universality of coefficients in $\beta(g)$, $\gamma(g)$

Consider $\beta(g)$ and a generic anomalous dimension $\gamma(g)$ in perturbation theory, i.e. for g close to zero. Then they admit the following expansions in g :

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 + \dots$$

$$\gamma(g) = -\gamma_0 g^2 - \gamma_1 g^4 + \dots$$

The coefficients β_0 , β_1 and γ_0 are special in the sense that they do not depend on the choice of "renormalization scheme". We say that they are universal. They carry crucial physical information!

β_0 and β_1 are the one-loop and two-loop coefficients of $\beta(g)$ respectively - They both are universal -

In $\gamma(g)$, only γ_0 is universal, γ_1 is not.

Moreover, if $\gamma_0 = 0$, one can make $\gamma(g)$ zero by a choice of renormalization scheme!

A renormalization (RG) scheme change implies:

$$\left\{ \begin{array}{l} g \rightarrow g' = g + a_1 g^3 + a_2 g^5 + \dots \\ \varphi \rightarrow \varphi' = \varphi (1 + c_1 g^2 + c_2 g^4 + \dots) \\ m^2 \rightarrow m'^2 = m^2 (1 + b_1 g^2 + b_2 g^4 + \dots) \end{array} \right.$$

Prove the universality of β_0, β_1 as follows :

$$\beta(g) = \mu \frac{dg}{d\mu} = -\beta_0 g^3 - \beta_1 g^5 + \dots$$

The beta function in the new scheme is :

$$\begin{aligned} \beta'(g') &= \mu \frac{dg'}{d\mu} = \mu \frac{dg}{d\mu} \frac{\partial g'(g)}{\partial g} \quad (\text{I}) \\ &= \beta(g) \frac{\partial g'(g)}{\partial g} \end{aligned}$$

that admits the general expansion :

$$\beta'(g') = -\beta'_0 g'^3 - \beta'_1 g'^5 + \dots \quad (\text{II})$$

with β'_0 and β'_1 to be determined -

Now, rewrite (I) and (II) in terms of

$$g^1 = g + a_1 g^3 + a_2 g^5 + \dots$$

and equate them to obtain :

$$\beta_0^1 = \beta_0 \quad \beta_1^1 = \beta_1$$

Hint :

$$\begin{aligned} (\text{I}) \quad \beta^1(g^1) &= \beta(g) \frac{\partial g^1(g)}{\partial g} \\ &= (-\beta_0 g^3 - \beta_1 g^5 + \dots) \frac{\partial}{\partial g} (g + a_1 g^3 + a_2 g^5 + \dots) \\ &= (-\beta_0 g^3 - \beta_1 g^5 + \dots) (1 + 3a_1 g^2 + 5a_2 g^4 + \dots) \end{aligned}$$

$$\begin{aligned} (\text{II}) \quad \beta^1(g^1) &= -\beta_0^1 g^{1^3} - \beta_1^1 g^{1^5} + \dots \\ &= -\beta_0^1 (g + a_1 g^3 + a_2 g^5 + \dots)^3 \\ &\quad - \beta_1^1 (g + a_1 g^3 + a_2 g^5 + \dots)^5 + \dots \end{aligned}$$

Equate (I) and (II) term by term in powers of g .

* Massless case

Solution of the CS eq. $N=2$

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + 2\gamma_0(g) \right) G_2(x, \mu, g(\mu)) = 0$$

The conformal ansatz

$$[O_0] = D$$

$$G_2|_{\text{conf.}} = \frac{C}{x^{2D}}$$

solves the free limit. Hence, we write the general solution as:

$$G_2(x, \mu, g(\mu)) = \frac{1}{x^{2D}} \underbrace{C(g(x))}_{\uparrow} \mathcal{Z}^2(x_\mu, g(\mu))$$

and solve CS for the adimensioned factor.

In this case it holds:

$$\mu \frac{\partial}{\partial \mu} F(x_\mu) = x \frac{\partial}{\partial x} F(x_\mu)$$

so that we can equivalently solve the eq.:

$$\left(x \frac{\partial}{\partial x} + \beta(g) \frac{\partial}{\partial g} + 2\gamma_0(g) \right) C(g(x)) \mathcal{Z}^2(x_\mu, g(\mu)) = 0$$

Setting for simplicity $C(g(x)) = 1$
 we can show that:

$$Z(x_\mu, g_\mu) = e^{2 \int_{g(\mu)}^{g(x)} \frac{r_O(g)}{\beta(g)} dg}$$

is solution of the CS equation. The factor Z contains the correction to the free-field solution and possibly breaks conformal invariance -
 It depends on the beta function of the theory, $\beta(g)$, and the anomalous dimension $r_O(g)$ of the given operator O that enters the correlator -

A factor $C(g(x))$ function of $g(x)$ is also allowed in general -

④ Proof that ζ^2 is solution of the CS eq.

$$x \frac{\partial}{\partial x} \zeta^2 = x \frac{\partial}{\partial x} e^{\int_{g(\mu)}^{g(x)} \frac{r_0(g)}{\beta(g)} dg}$$

$$= 2 \times \frac{\partial g(x)}{\partial x} \left. \frac{r_0(g)}{\beta(g)} \right|_{g=g(x)} \zeta^2$$

$$= -2 \beta(g(x)) \frac{r_0(g(x))}{\beta(g(x))} \zeta^2$$

$$= -2 r_0(g(x)) \zeta^2$$

$$\beta(g(\mu)) \frac{\partial}{\partial g(\mu)} \zeta^2 = \beta(g(\mu)) \frac{\partial}{\partial g(\mu)} e^{\int_{g(\mu)}^{g(x)} \frac{r_0(g)}{\beta(g)} dg}$$

$$= \beta(g(\mu)) \left\{ 2 \underbrace{\frac{\partial g(x)}{\partial g(\mu)}}_{\frac{r_0(g(x))}{\beta(g(x))}} - 2 \frac{r_0(g(\mu))}{\beta(g(\mu))} \right\} \zeta^2$$

$$= \frac{\beta(g(x))}{\beta(g(\mu))}$$

$$= \left\{ 2 r_0(g(x)) - 2 r_0(g(\mu)) \right\} \zeta^2$$

Hence the sum of all terms gives :

$$\left(x \frac{\partial}{\partial x} + \beta(g) \frac{\partial}{\partial g} + 2r_0(g) \right) Z^2(x, g(\mu))$$

$$= -2r_0(g(x)) Z^2$$

$$+ \left(2r_0(g(x)) - 2r_0(g(\mu)) \right) Z^2$$

$$+ 2r_0(g(\mu)) Z^2$$

$$= 0$$



- RG flow à la Wilson

(see Peskin & Schroeder QFT)

$$Z = \int \partial q_1 e^{-S} \frac{1}{2} (\partial_\mu q)^2 + \frac{1}{2} m^2 q^2 + \frac{\lambda}{4!} q^4$$

define $q = q_{\text{high}} + q_{\text{low}}$

or equivalently $q \rightarrow q + \hat{q}$

$$\begin{cases} q(k) = 0 & \text{for } |k| \geq b\Lambda \\ \hat{q}(k) = 0 & \text{for } |k| < b\Lambda \end{cases}$$

Integrate on $\hat{q}(k)$ i.e. on the shell $b\Lambda \leq |k| \leq \Lambda$

$$Z = \int \partial q_{b\Lambda} \int \partial \hat{q} e^{-S} \left\{ \frac{1}{2} (\partial_\mu q + \partial_\mu \hat{q})^2 + \right. \\ \left. + \frac{1}{2} m^2 (q + \hat{q})^2 + \frac{\lambda}{4!} (q + \hat{q})^4 \right\}$$

NB The terms $q \hat{q}$ give zero in the integral since Fourier components with different momentum are orthogonal. Hence:

$$-\int d^d x \mathcal{L}(q)$$

$$\begin{aligned} Z = & \int \partial \mathcal{L}_{b\Lambda} e^{-\int d^d x \left\{ \frac{1}{2} (\partial_\mu \hat{q})^2 + \frac{1}{2} m^2 \hat{q}^2 + \right.} \\ & \times \int \partial \hat{q} e^{-\int d^d x \left\{ \frac{1}{2} (\partial_\mu \hat{q})^2 + \frac{1}{2} m^2 \hat{q}^2 + \right.} \\ & \left. + \lambda \left(\frac{1}{6} \hat{q}^3 \hat{q}' + \frac{1}{4} \hat{q}^2 \hat{q}''^2 + \frac{1}{6} \hat{q} \hat{q}'''^3 + \frac{1}{4!} \hat{q}^4 \right) \right\} \end{aligned}$$

where $\hat{q} \neq 0$ only on $b\Lambda \leq |k| \leq \Lambda$

We treat all terms as perturbations to $\frac{1}{2} (\partial_\mu \hat{q})^2$

The theory with $\mathcal{L}_0 = \frac{1}{2} (\partial_\mu \hat{q})^2$

leads to a propagator:

$$\hat{q}(k) \hat{q}(p) = \frac{1}{k^2} (2\pi)^d \delta^{(d)}(k+p) \Theta(k)$$

$$\Theta(k) = \begin{cases} 1 & b\Lambda \leq |k| \leq \Lambda \\ 0 & \text{otherwise} \end{cases}$$

Represent it as dashed lines - - -



Then treat all the remaining terms as perturbative
hence, expand the exponential and perform
the Wick contractions :

$$-\int d^d x \frac{1}{2} (\partial_\mu \hat{\varphi})^2$$

$$\int d\hat{\varphi} e$$

$$\cdot \left\{ 1 - \int d\omega \left(\frac{1}{2} m^2 \hat{\varphi}^2 + \lambda \left(\frac{1}{6} \hat{\varphi}^3 \hat{\varphi}^1 + \frac{1}{4} \hat{\varphi}^2 \hat{\varphi}^2 + \dots \right) \right) \right.$$

$$+ \frac{1}{2!} \int d\omega_1 \int d\omega_2 \left(\quad \right)_{\omega_1} \left(\quad \right)_{\omega_2}$$

$$\left. + \dots \right\}$$

For example :

$$\int d\hat{\varphi} e \quad - \int d\omega \frac{1}{2} (\partial_\mu \hat{\varphi})^2 \quad - \int d\omega \frac{\lambda}{4!} \hat{\varphi}^2 \hat{\varphi}^2$$

$$= - \int d\omega \frac{\lambda}{4!} \hat{\varphi}^2 \hat{\varphi}^2 \hat{\varphi}^1 \hat{\varphi}^1$$

$$= - \int d\omega \frac{1}{2} \hat{\varphi}^2(\omega) \cdot \frac{\lambda}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2}$$

$$= - \int d\omega \frac{1}{2} \hat{\varphi}^2(\omega) \cdot \mu$$

$b\Lambda \leq k \leq \Lambda$

where

$$\mu = \frac{\lambda}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = \frac{\lambda}{(\zeta\pi)^{d/2} \Gamma\left(\frac{d}{2}\right)} \frac{1-b^{d-2}}{d-2} \Lambda^{d-2}$$

$b\Lambda \leq |k| \leq \Lambda$

The term $-\int d\omega \frac{1}{2} q^2(\omega) \mu$

can be rewritten as a correction to the mass term in $\mathcal{L}(q)$ on the shell $|k| < b\Lambda$:

$$\frac{1}{2} m^2 q^2 \rightarrow \frac{1}{2} (m^2 + \mu) q^2$$

This is an additive correction to m^2 .

There are also multiplicative corrections that rescale m^2 to m'^2 once we rescale:

$$x \rightarrow x' = bx$$

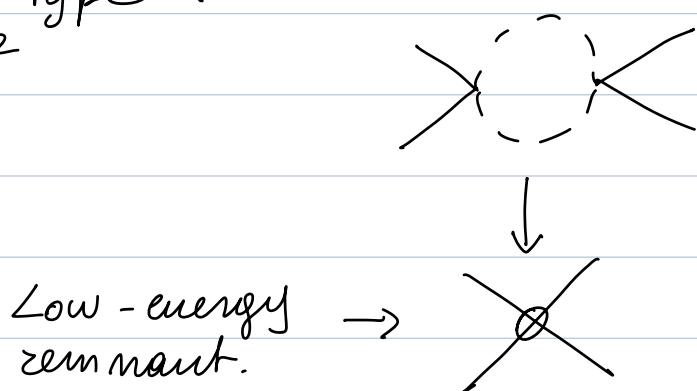
$$k \rightarrow k' = \frac{k}{b}$$

so that the free-field action is always:

$$S = \int d^d x' \frac{1}{2} (\partial_{\mu}' q')^2$$

Analogously, $\lambda \phi^4$ gets corrections from λ^2 terms of the type :

$$(\lambda \phi^2 \bar{\phi}^2)^2$$



Low-energy remnant. \rightarrow