

RG Flow

Exercises

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1) Conformal / dilatation invariance

The 2-point correlator $\langle O(x) O(0) \rangle$ preserves dilations if it is solution of the Ward Identity:

$$0 = \left(2\Delta + x^\mu \frac{\partial}{\partial x^\mu} \right) \langle O(x) O(0) \rangle$$

where Δ is the dimension of O

a) Show that $\langle O(x) O(0) \rangle = \frac{1}{x^{2\Delta}}$ is indeed a solution

b) show that $\langle O(x) O(0) \rangle = \frac{1}{x^{2\Delta}} \frac{1}{\log^2(x^2/\Lambda_{\text{cso}}^2)}$

is not a solution

2) Beta function & RG Fixed Points (FPs)

The existence of FPs also depends on the number of space-time dimensions.

In fact, in $d = 4 - 2\epsilon$ the beta function is:

$$\beta(g, \epsilon) = -\epsilon g + \beta(g)$$

where $\beta(g)$ is the one in $d=4$ dimensions

a) Given the perturbative expansion of $\beta(g)$ for small g :

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 + \dots$$

Find the zeroes of $\beta(g, \epsilon)$ for g and ϵ small, i.e. find g^* that solves:

$$\beta(g^*, \epsilon) = 0$$

b) Find for which sign of β_0 $g^* \neq 0$ is real and positive.

3) Dimensional analysis and renormalizability

$$S = \int d^d x \mathcal{L}(\varphi, \partial\varphi)$$

↑ ↑ ↑
Action Lagrangian
density Fields
 & derivatives

In Natural Units:

$$\hbar = c = 1$$

$$[\hbar] = [c] = 0$$

Since $c \sim \text{length} \times \text{time}^{-1}$

$$\hbar \sim \text{energy} \times \text{time}$$

it follows in Natural Units :

$$\text{energy} \sim \text{length}^{-1} \sim \text{time}^{-1}$$

Hence : $[S] = 0$

that implies : $\mathcal{O} = [d^{\frac{d}{2}}x] + [\mathcal{L}]$
 $= -d + [\mathcal{L}]$

so that : $[\mathcal{L}] = d$

Thus all terms in \mathcal{L} must have (energy)
dimension of in d-dimensional spacetime

a) Derive the canonical dimension
of each field in \mathcal{L} from the free
Lagrangian , employing that:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \quad [\partial_\mu] = [m] = 1$$

Convince yourself that :

$$\text{For any } \underline{\text{bosonic}} \text{ field } B : [B] = \frac{d-2}{2}$$

$$\text{For any } \underline{\text{fermionic}} \text{ field } F : [F] = \frac{d-1}{2}$$

b) Given the dimensions of the fields,
determine the dimension of each coupling
in the interaction Lagrangian density
($L = L_{\text{free}} + L_I$)

e.g. $L = \underbrace{\frac{1}{2}(\partial_\mu q)^2}_{\text{(Euclidean)}} + \underbrace{\frac{1}{2}m^2 q^2}_{= L_{\text{free}}} + \underbrace{\frac{\lambda}{4!} q^4}_{= L_I}$

- Derive $[\lambda]$.

c) Renormalizability: An interaction is renormalizable if the corresponding coupling is dimensionless, i.e.:

$$[\text{coupling}] = 0$$

This statement depends on the number of space-time dimensions of !

- For which value of λ is $L_I = \frac{\lambda}{4!} q^4$ renormalizable?

Classification of interactions :

[coupling] > 0 superrenormalizable

[coupling] $= 0$ renormalizable

[coupling] < 0 nonrenormalizable

If we expand around the massless free-field point, this classification coincides with the Wilson-flow classification of perturbations in:

relevant

marginal

irrelevant

respectively -

- Convince yourself that all interactions in the SM Lagrangian density are renormalizable -

- Is a Pauli term $L_I = \bar{\psi} \sigma_{\mu\nu} \gamma_5 \psi F_{\mu\nu}$ renormalizable in $d=4$?

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \sigma_{\mu\nu} = \frac{1}{2} [\gamma_\mu, \gamma_\nu]$$