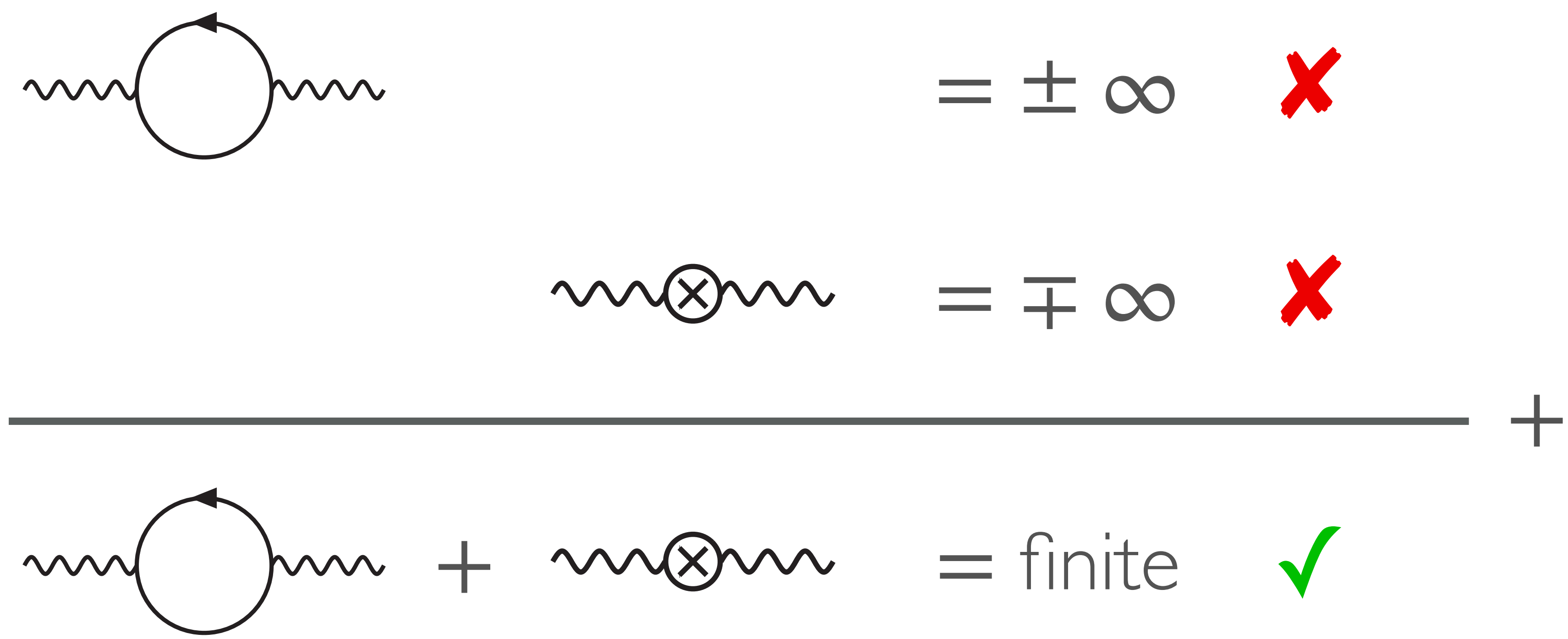
TAMING INFINITIES
A THEORIST JOB

Coenraad Marinissen
c.marinissen@nikhef.nl

Supervisors: Eric Laenen & Marcel Vonk

INFINITIES IN QFT

- Renormalisation

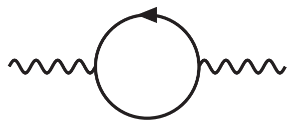
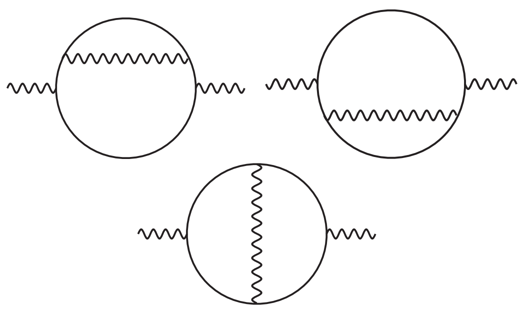


- Infinite number of counter terms ✗
- Finite number of counter terms ✓

➡ Renormalizable field theories

INFINITIES IN QFT

- For renormalizable field theories, perturbation theory looks fine

$\mathcal{O}(\alpha) =$  $+$  $+$  $+ \quad \ldots \quad +$ counter terms

INFINITIES IN QFT

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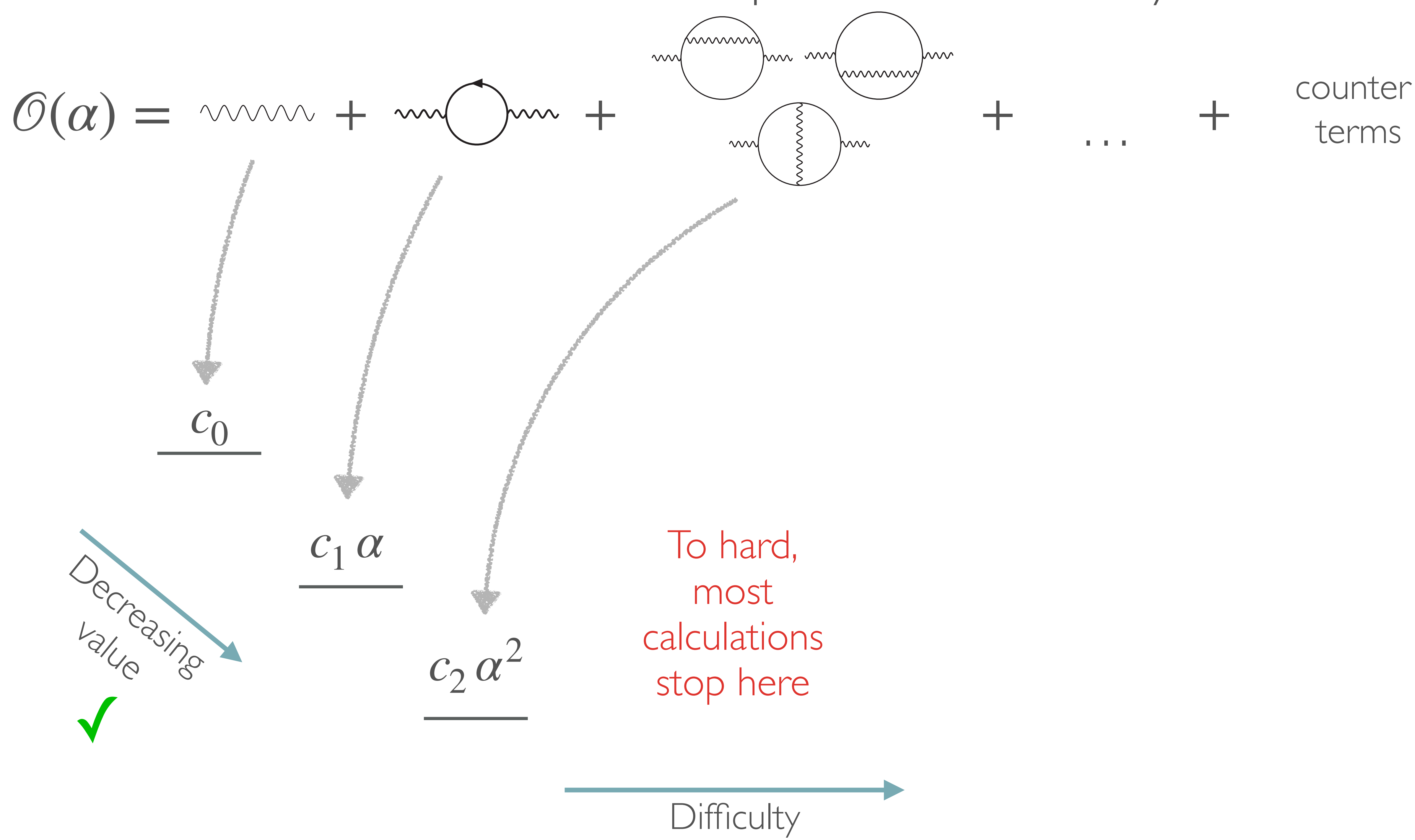

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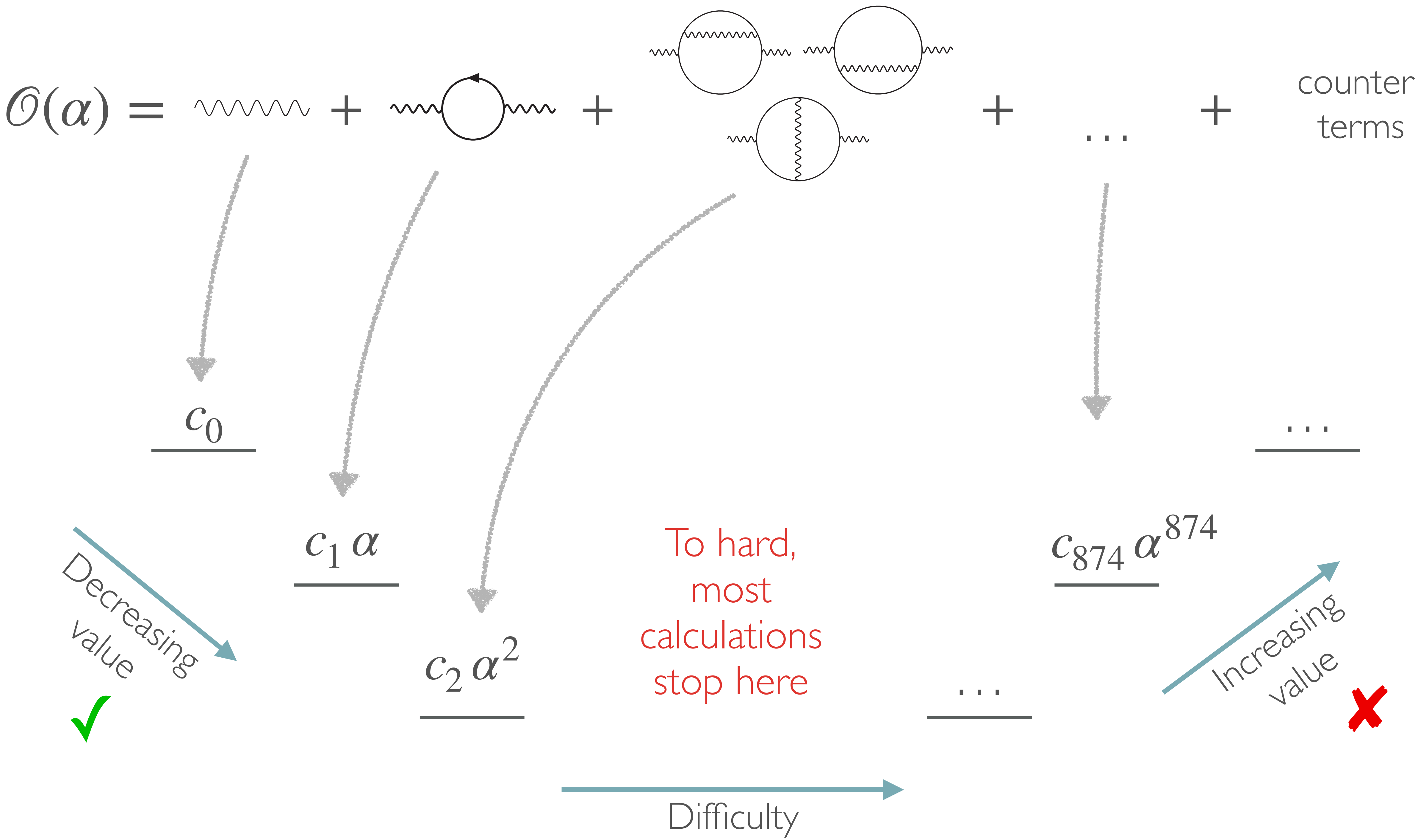

PERTURBATION THEORY

- Perturbative expansions in QFT:

$$\mathcal{O} = \sum_{n=0}^{\infty} c_n \, \alpha^n, \quad \text{with} \quad c_n \sim n!$$

- Problem: divergent for all $\alpha \neq 0$ ➡ *asymptotic* series

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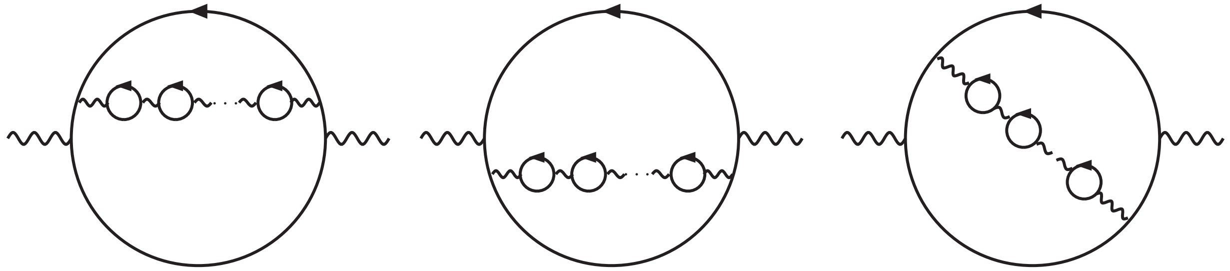
- Problem: divergent for all $\alpha \neq 0$ → *asymptotic* series

1. Numerical value?
2. Reason? → Non-perturbative effects are missing
3. Source? { Instantons → #Feynman diagrams; Renormalons → Bubble diagrams ['t Hooft '77]



SAVING PERTURBATION THEORY

Strategy 1

"Naive" approach → Neglect increasing terms

- Works reasonable well
- Example: Standard model

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Strategy 2

Math approach → *Resurgence*

- Mathematical theory to study asymptotic series

[J. Écalle 1985]

RESURGENCE

Example: $x^2 \frac{df}{dx} = -x + f$

Try perturbative Ansatz: $f(x) = \sum_{n=0}^{\infty} c_n \, x^{n+1}$ ➡ $c_n = n!$

Two problems:

1. Divergent for all $x \neq 0$
2. first order ODE: free parameter?

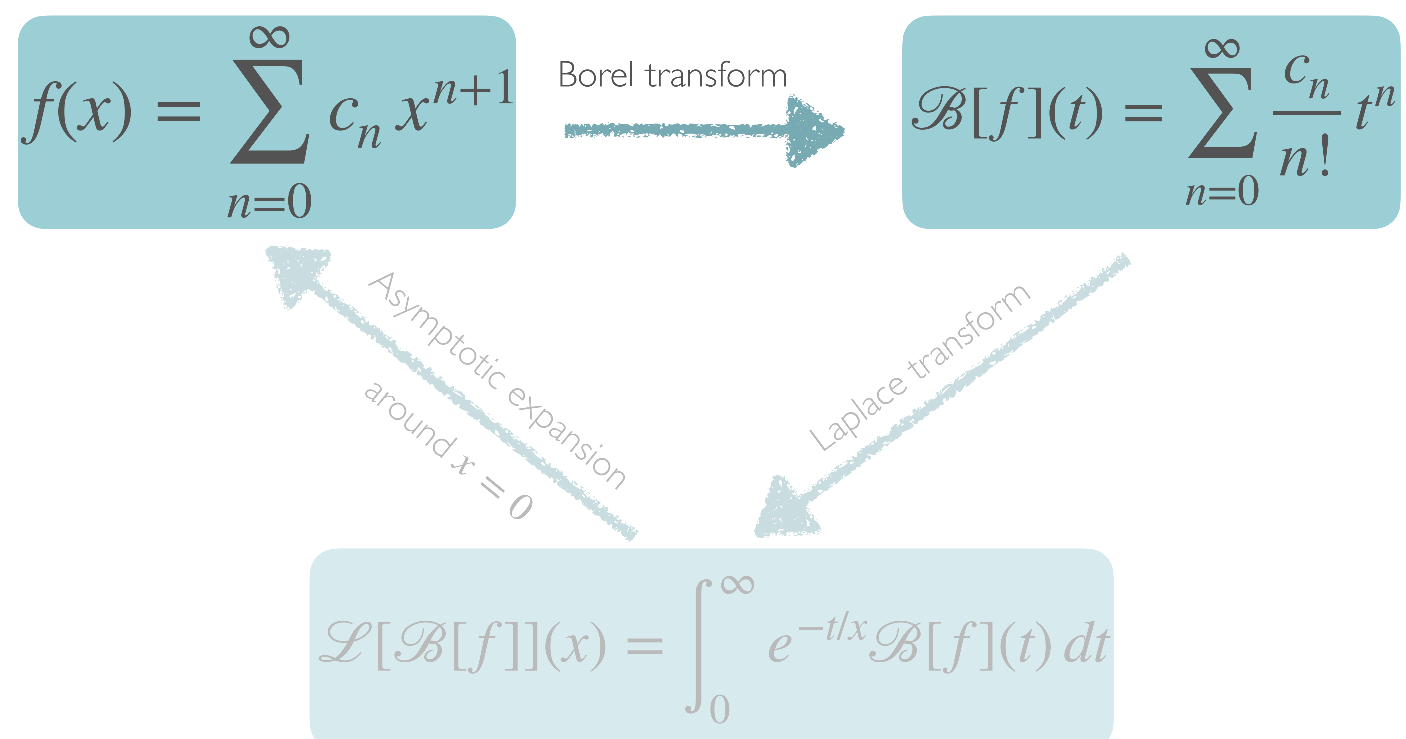
Hom. eq. $x^2 \frac{dg}{dx} = g$ ➡ $g(x) = C\, e^{-1/x}$

RESURGENCE

$$x^2 \frac{df}{dx} = -x + f \quad \begin{cases} \text{Perturbative solution:} & f(x) = \sum_{n=0}^{\infty} n!\, x^{n+1} \\ \text{Homogeneous solution:} & g(x) = C\, e^{-1/x} \end{cases}$$

Apply Borel summation

1. $\mathcal{B}[f](t) = \sum_{n=0}^{\infty} t^n = \dfrac{1}{1-t}$



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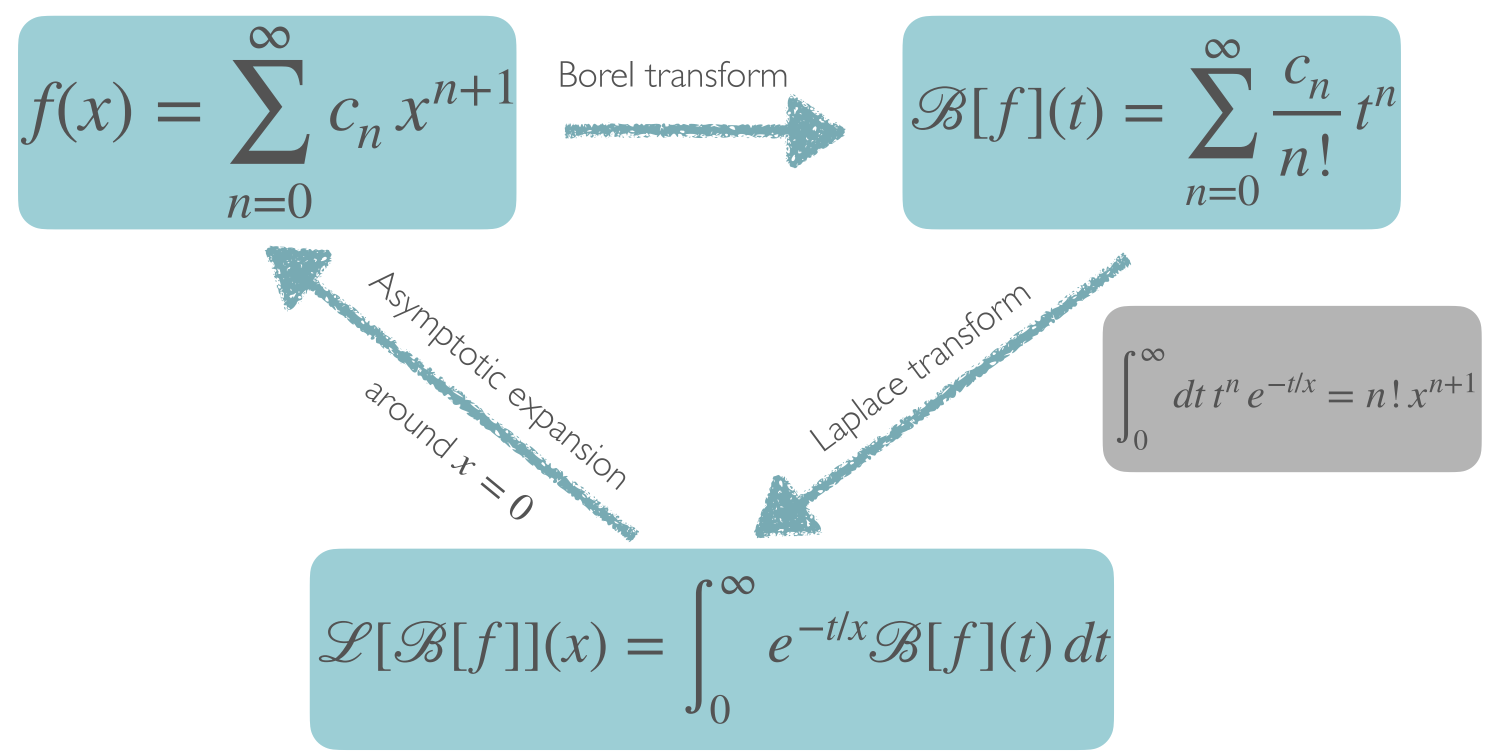

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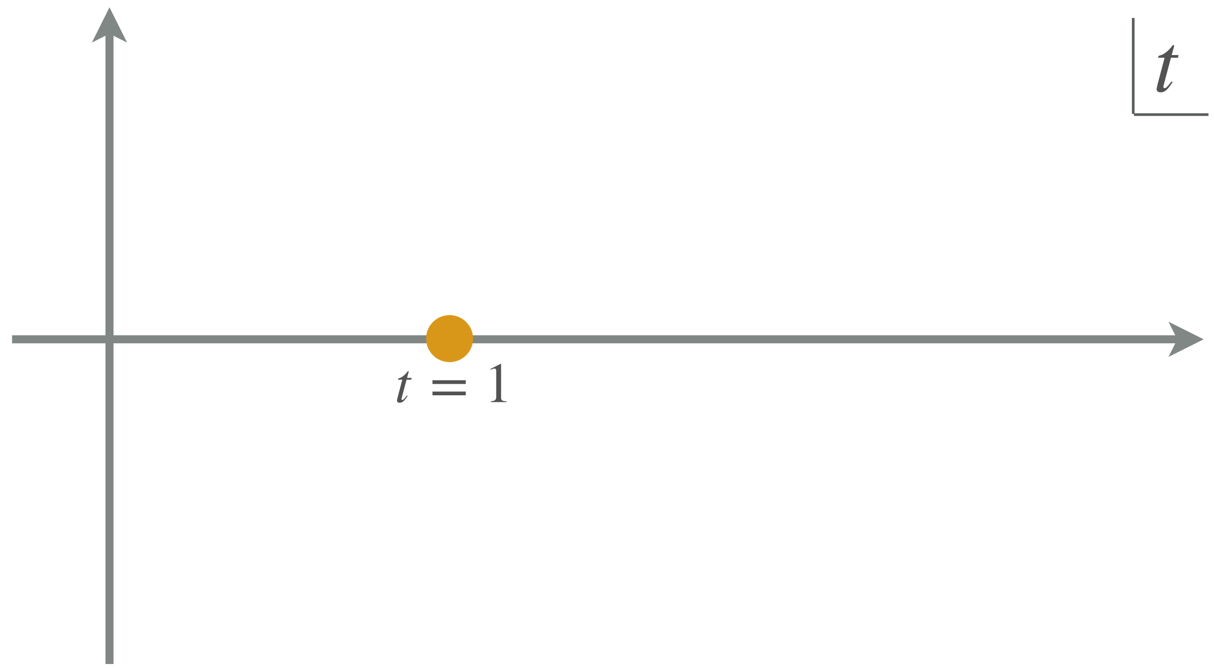

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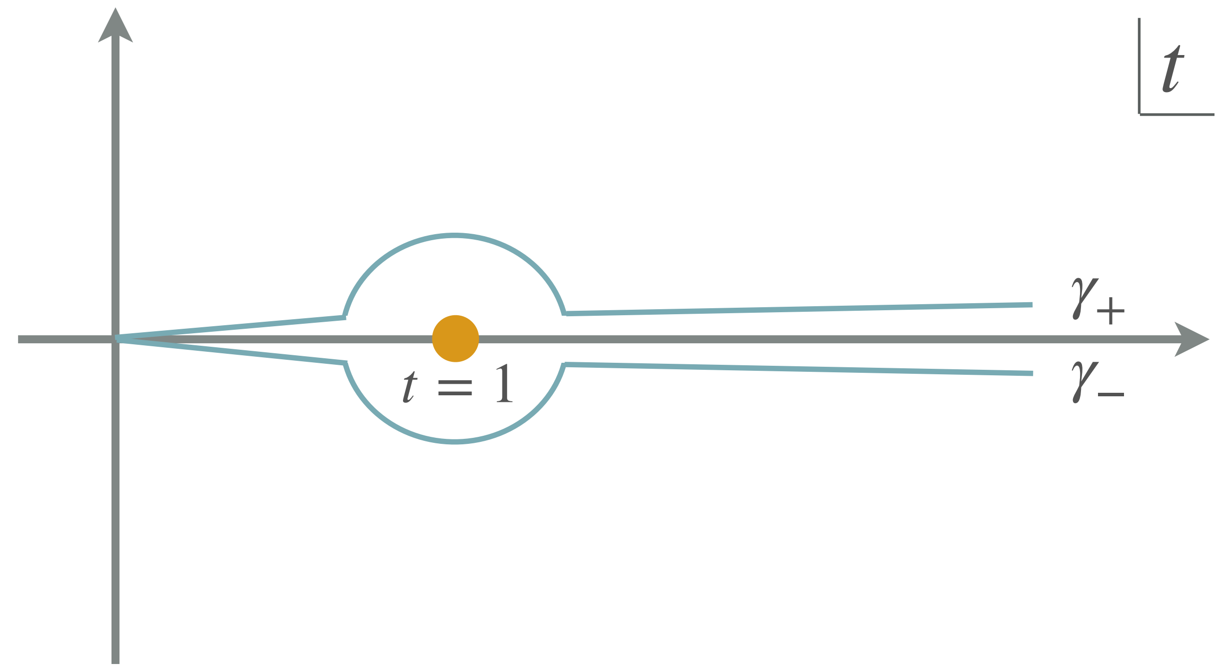

Ambiguity

$$\int_{\gamma_+ - \gamma_-} dt \frac{e^{-t/x}}{1-t} = \oint dt \frac{e^{-t/x}}{1-t} = 2\pi i e^{-1/x}$$

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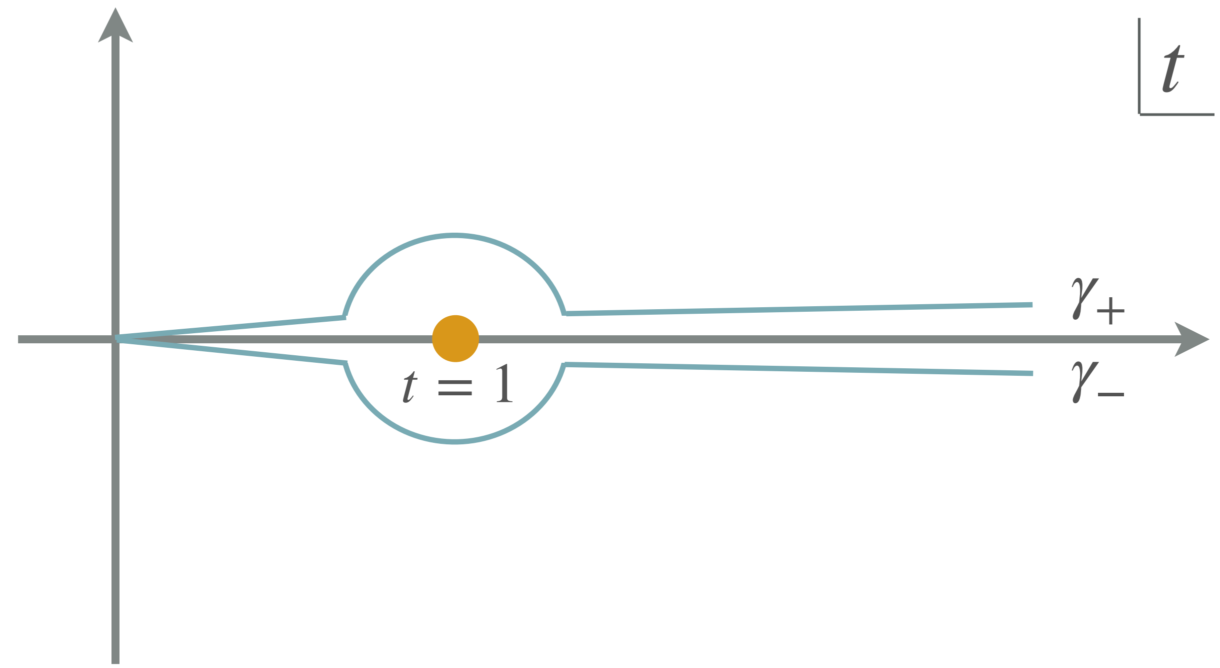

Ambiguity reveals that perturbative solution is part of a larger class of solutions:

$$f(x, C) = \int_{\gamma_\pm} \frac{e^{-t/x}}{1-t} dt + C\, e^{-1/x}$$

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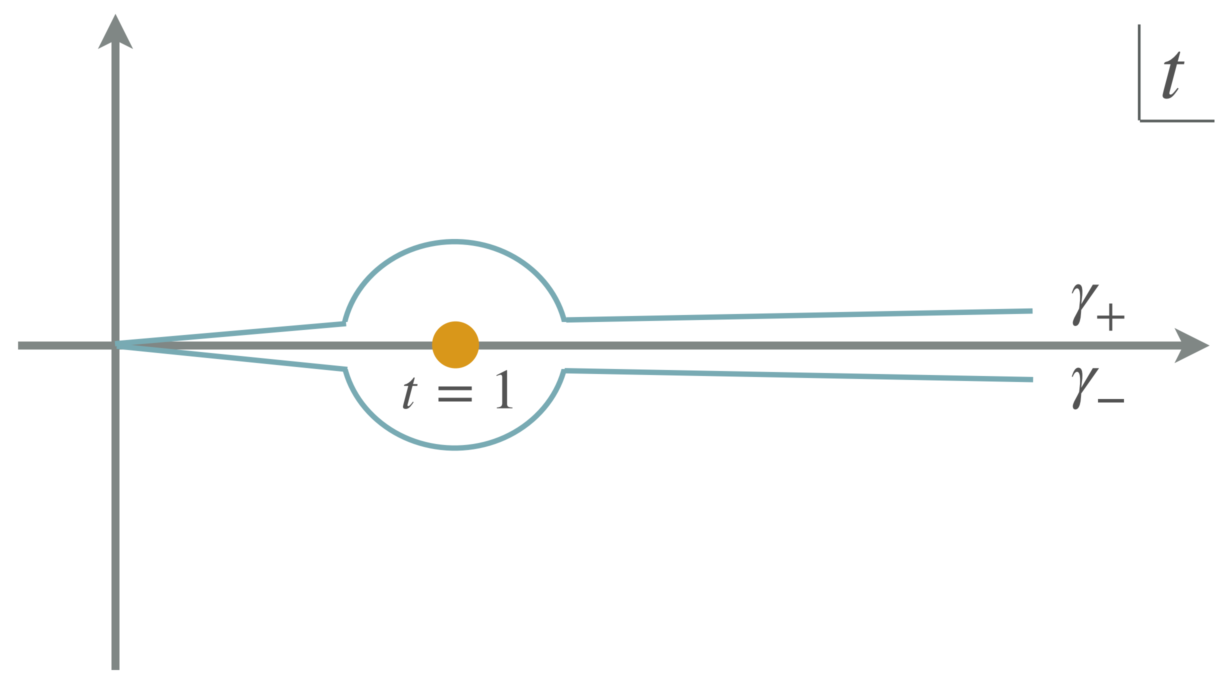

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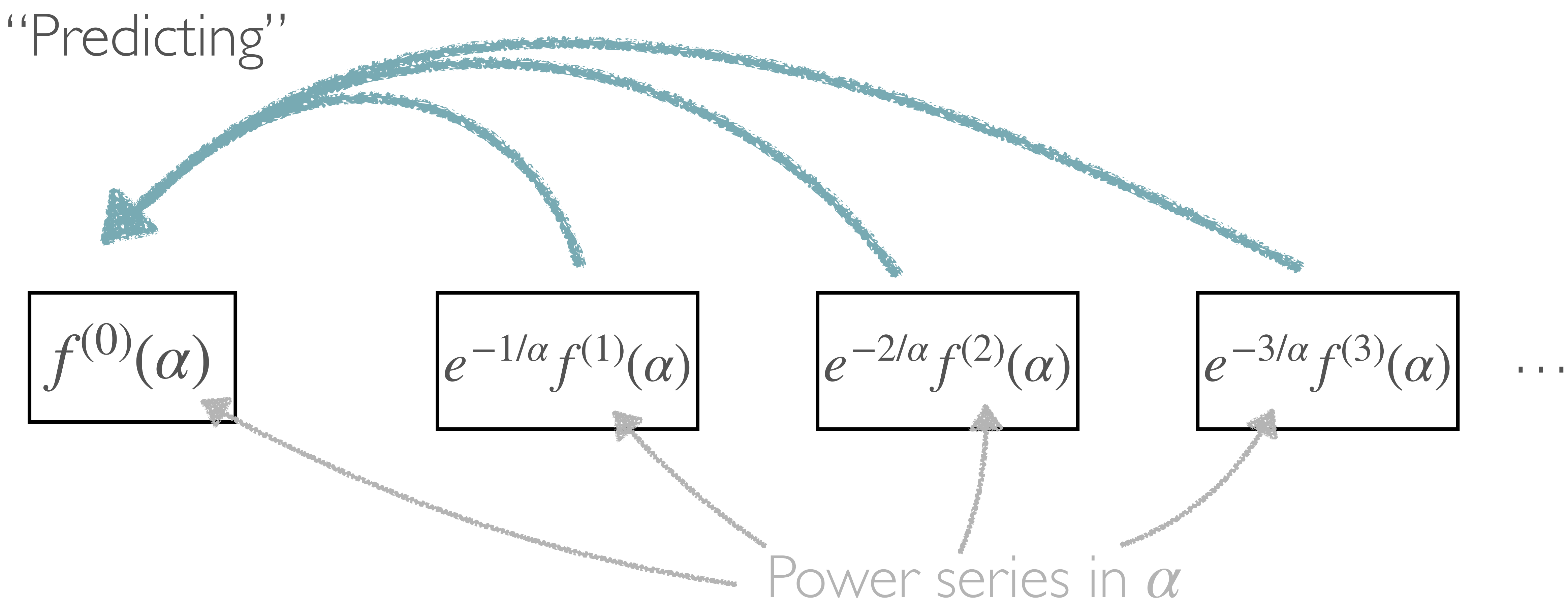
Non-perturbative term *resurged*

LARGE ORDER RELATIONS



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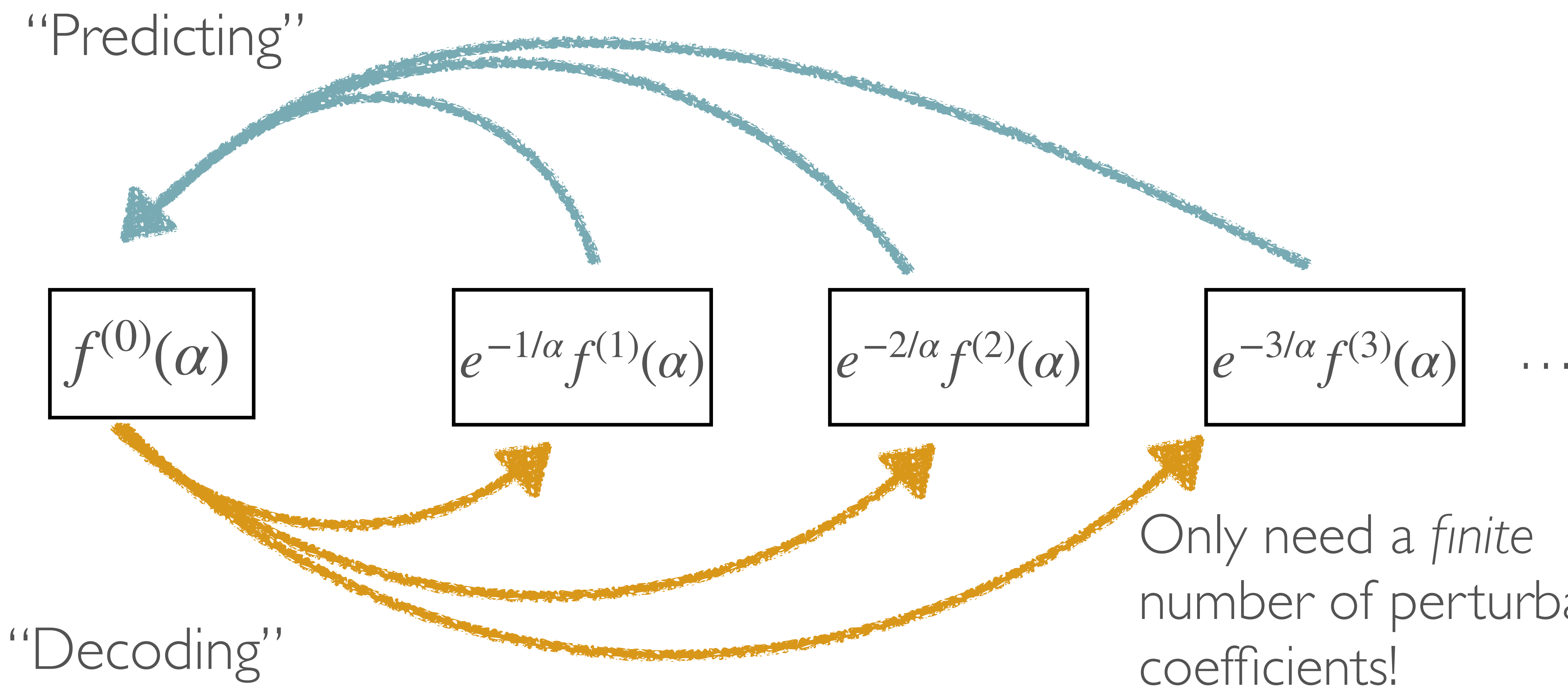
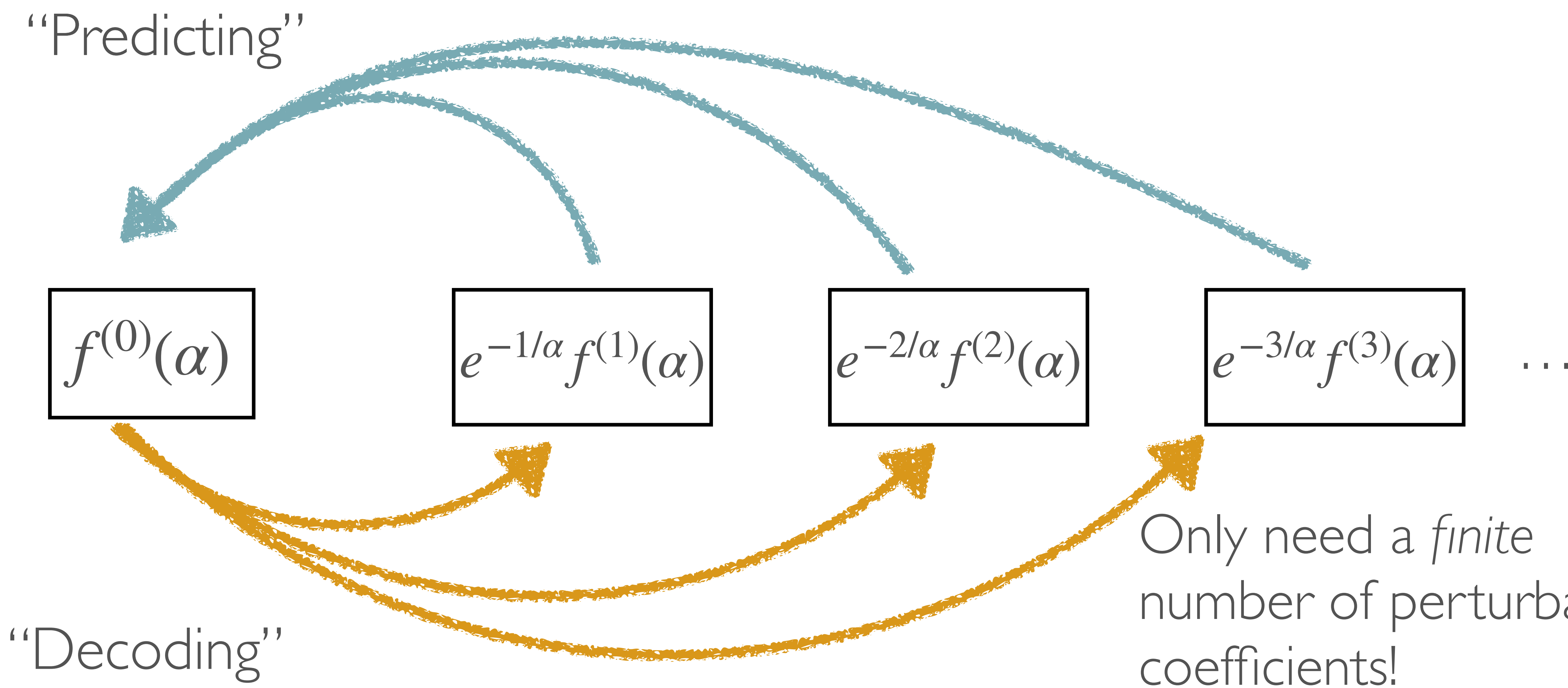


Only need a *finite* number of perturbative coefficients!

LARGE ORDER RELATIONS



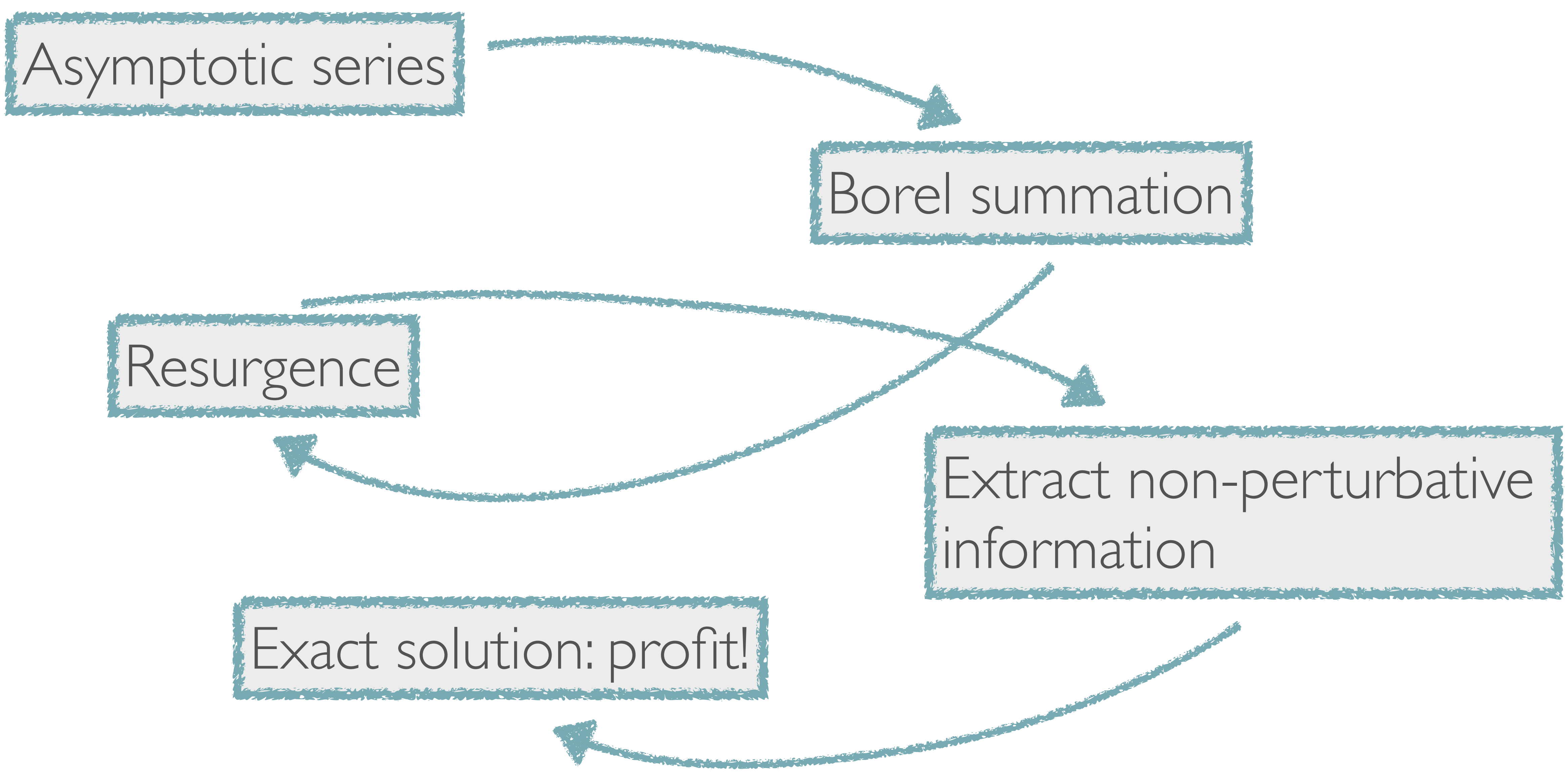
Only need a *finite* number of perturbative coefficients!

To do:

1. Compute enough coefficients
2. Learn to do resurgence with only a few perturbative coefficients

CONCLUSION

- Non-perturbative information is hidden in perturbative coefficients
- Asymptotic growth: it can be tamed, nothing to be afraid of!

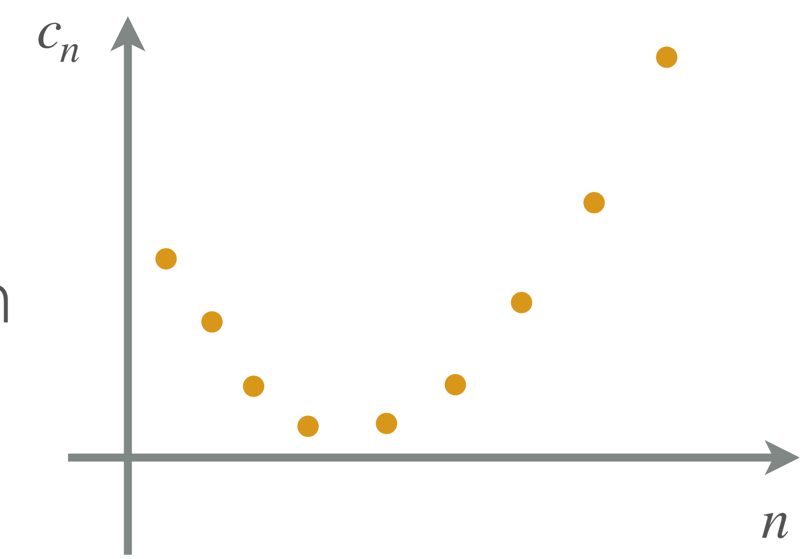


OPTIMAL TRUNCATION

- In practice: we only have a few coefficients of the asymptotic series.

$$\mathcal{O}(g) = \sum_{n=0}^{N} c_n g^n$$ ➡ Why is it still a good estimate of experiment?

- Consider $c_n = \dfrac{n!}{A^n}$



- Use Stirling approximation to find optimal truncation

$$|c_n x^n| = n! \left|\frac{x}{A}\right|^n \approx \exp\left(n \log n - n - n \log\left|\frac{x}{A}\right|\right)$$

- This has a saddle given at $N = \left|\dfrac{A}{x}\right|$

- Evaluating the next term gives the error made in the optimal truncation

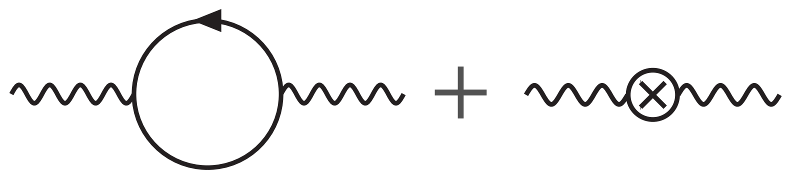
$$c_{N+1} |x|^{N+1} \sim e^{-|A/x|}$$

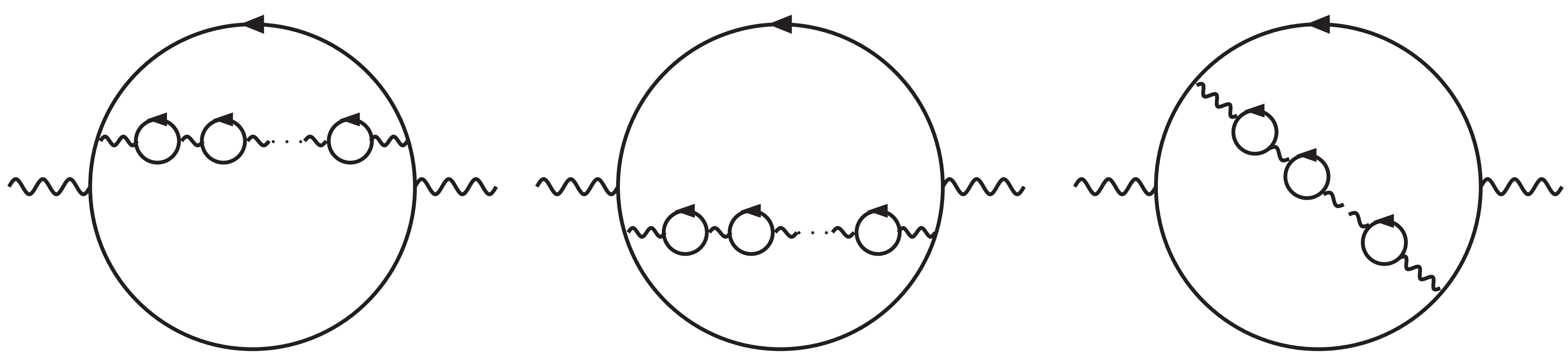
- Conclusion: Borel summation and optimal truncation agree up to (small) non-perturbative exponential factors

RENORMALONS

- First discovered by 't Hooft ['t Hooft '77]
- Classes of diagrams that causes perturbative coefficients to grow as $c_n \sim n!$
- Often related to so called bubble diagrams

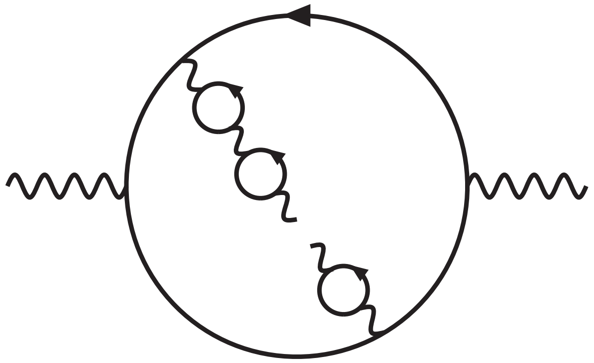
(not #diagrams $= n!$)

- Ingredient:  $\sim \log(k^2)$



RENORMALONS

- Schematic computation



$$\sim \sum_{n=0}^{\infty} \alpha \int_0^{\infty} dk^2 F(k^2) \left[\alpha \log(k^2)\right]^n$$

RENORMALONS

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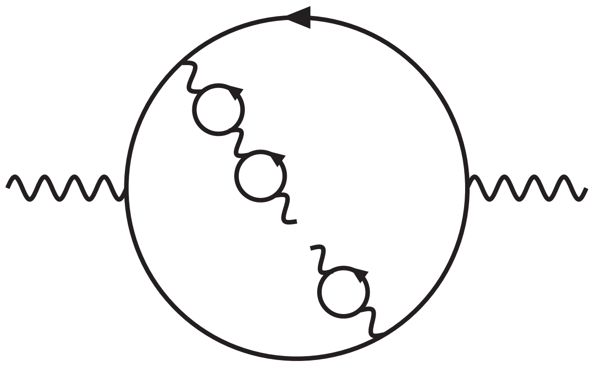

$$\sim \sum_{n=0}^{\infty} \alpha \int_0^\infty dk^2 F(k^2) \left[\alpha \log(k^2)\right]^n$$

- IR: $k^2 \ll 1$
- UV: $k^2 \gg 1$

$$F(k^2) = \begin{cases} 1 + k^2 + \ldots, & k^2 \ll 1 \\ \frac{1}{k^4} + \ldots, & k^2 \gg 1 \end{cases}$$

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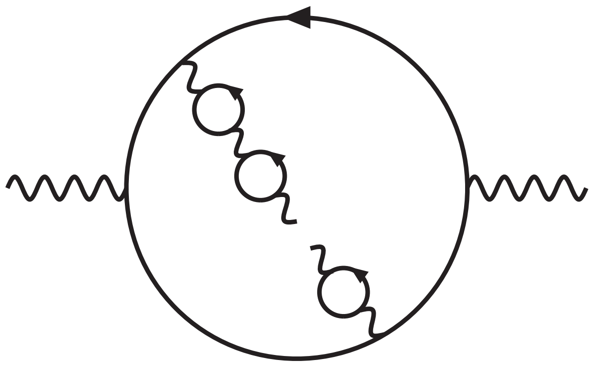

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$k^2 = e^z$

$$\sim \int_0^1 dk^2 \left[\log(k^2)\right]^n + \int_1^{\infty} dk^2 \frac{1}{k^4} \left[\log(k^2)\right]^n = \int_{-\infty}^0 dz\, z^n e^z + \int_0^{\infty} dz\, z^n e^{-z}$$

$$= \underbrace{(-1)^n \Gamma(n+1)}_{\text{IR}} + \underbrace{\Gamma(n+1)}_{\text{UV}}$$

BOREL SUMMATION

$$f(x) = \sum_{n=0}^{\infty} c_n \, x^{n+1}$$

Borel transform

$$\mathscr{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

Laplace transform

$$\mathscr{L}[\mathscr{B}[f]](x) = \int_0^{\infty} e^{-t/x} \mathscr{B}[f](t) \, dt$$

Asymptotic expansion around $x = 0$

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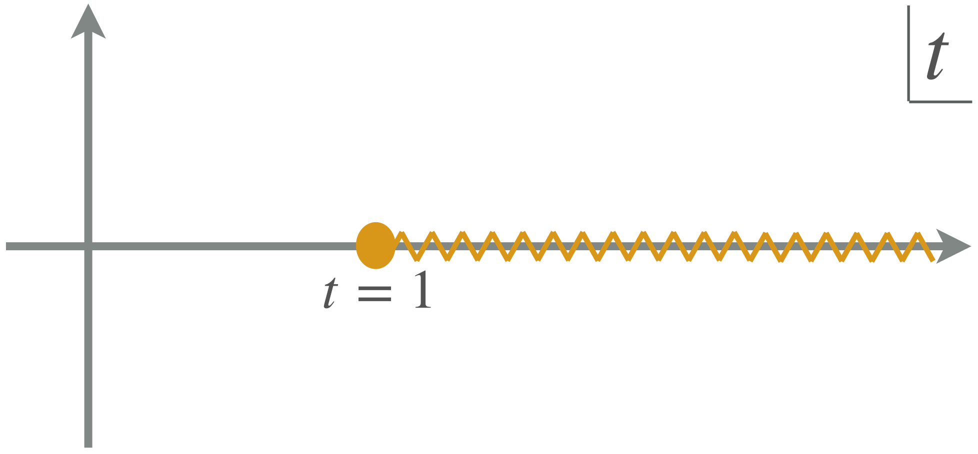
RESURGENCE

- More general: $f_n \sim n!\left(1 + \frac{a}{n} + \frac{b}{n^2} + \ldots\right)$

$$\mathscr{B}[f](t)\Big|_{t=1} = \frac{a}{t-1} + \psi(t-1)\log(t-1)$$



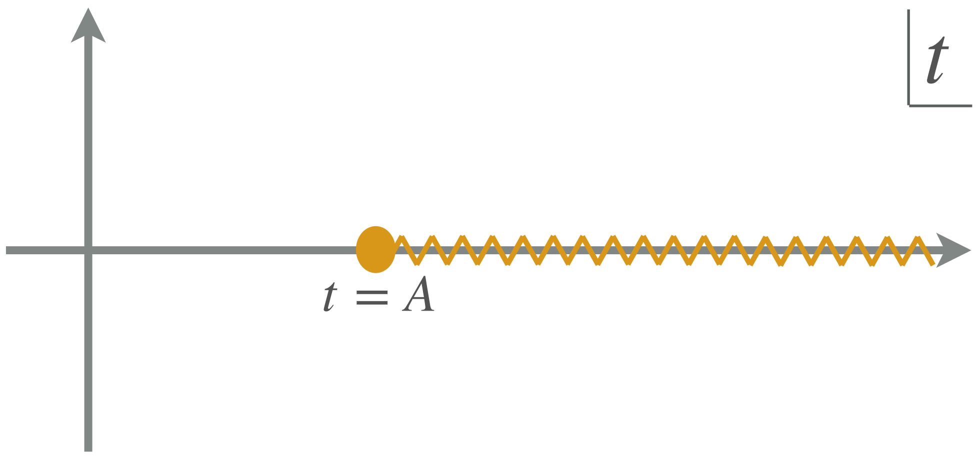
- Writing $\psi(t) = \mathscr{B}[f^{(1)}](t)$, where $f^{(1)} = a + \sum_{n=0}^{\infty} f_n^{(1)}\, x^{n+1}$

➡ $f(x,\sigma) = f^{(0)}(x) + \ \sigma\, e^{-1/x} f^{(1)}(x)$

RESURGENCE

- More general: $f_n \sim n!\left(1+\frac{a}{n}+\frac{b}{n^2}+\dots\right)$

$$\mathscr{B}[f](t)\Big|_{t=A} = \frac{a}{t-A} + \psi(t-A)\log(t-A)$$



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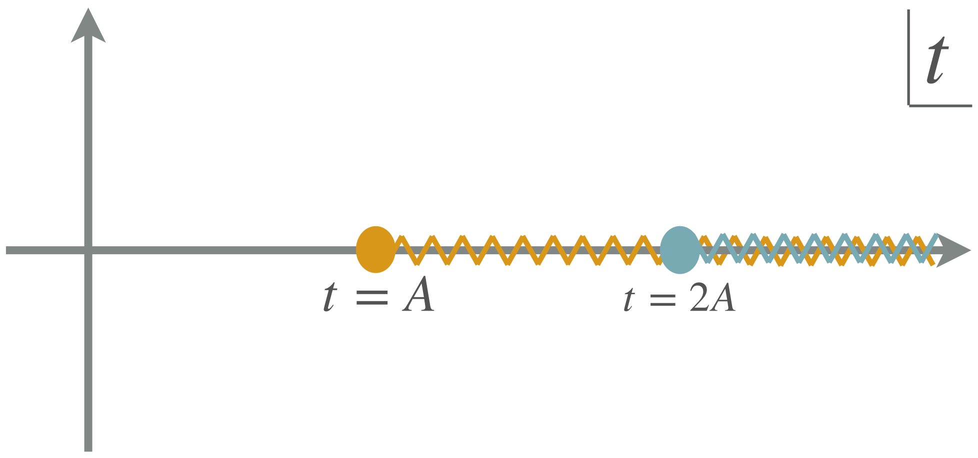

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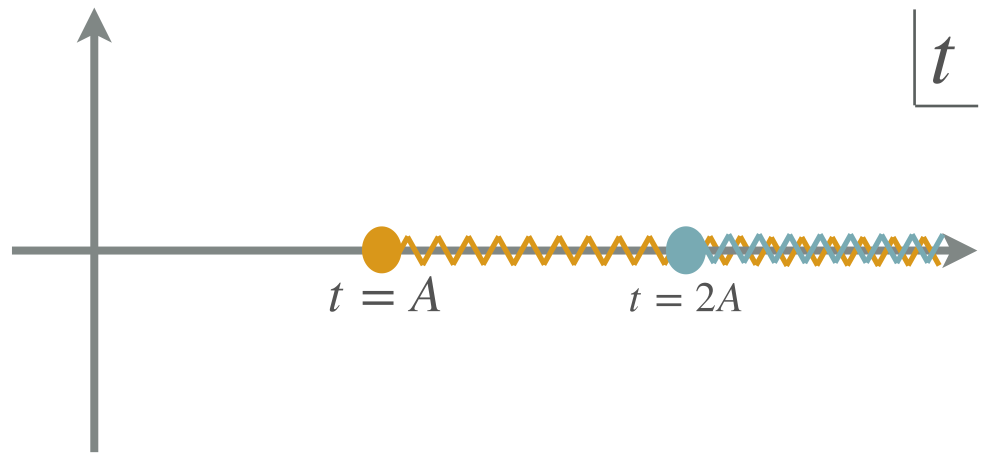

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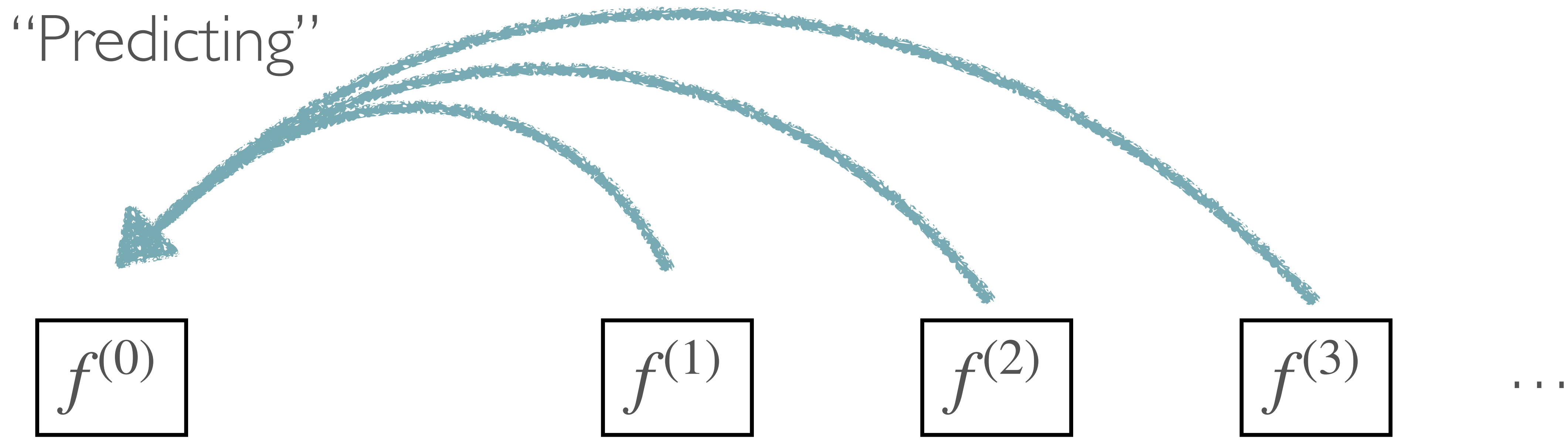
- Transseries: $f(x,\sigma) = f^{(0)}(x) + \sum_{n=1}^{\infty} \sigma^n e^{-nA/x} f^{(n)}(x)$

Perturbative sectors

Non-perturbative sectors

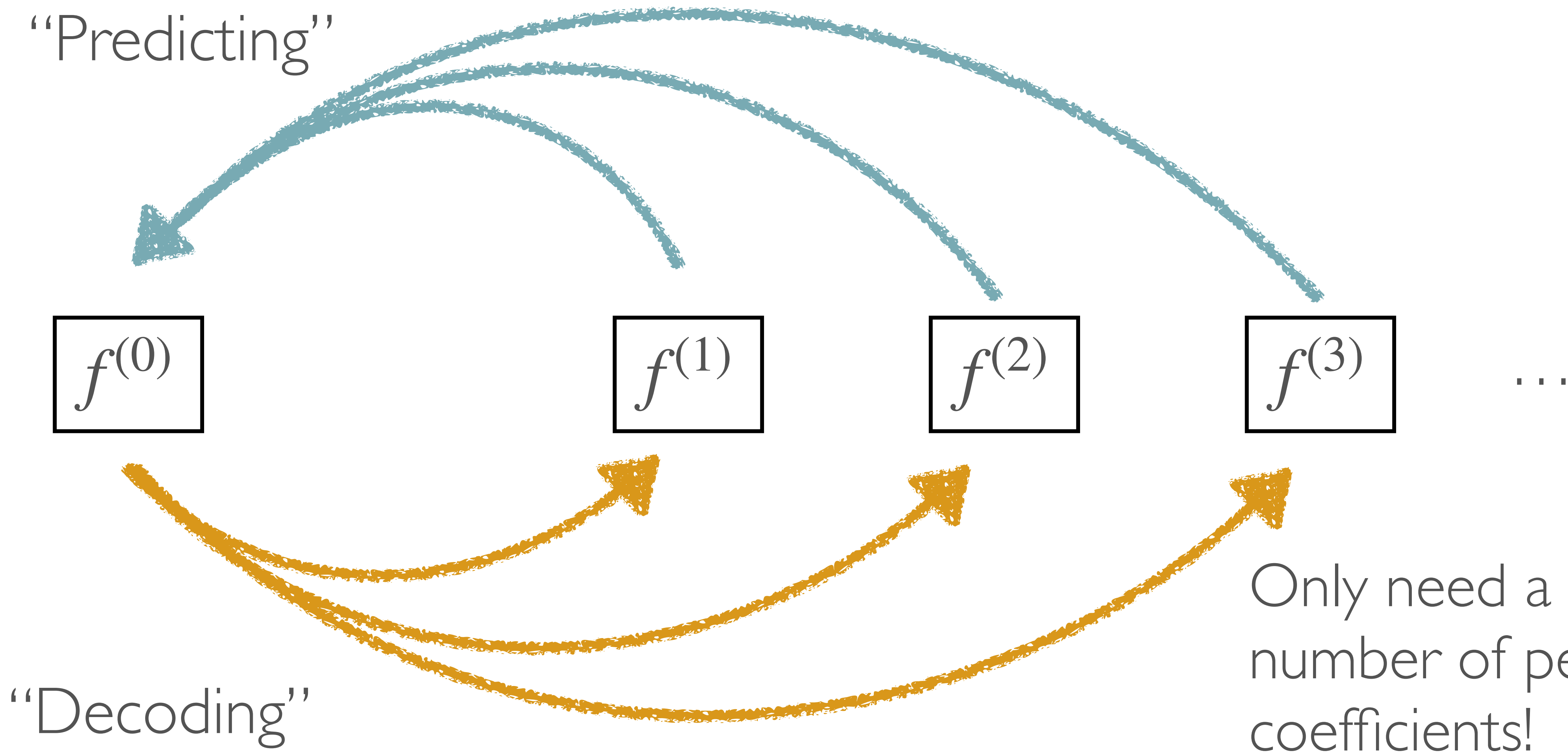
LARGE ORDER RELATIONS



Large order relations (true in large n limit)

$$f_n^{(0)} \sim \sum_{h=0}^{\infty} \frac{(n-h)!}{A^{n-h}} f_h^{(1)} \quad + \quad \sum_{h=0}^{\infty} \frac{(n-h)!}{(2A)^{n-h}} f_h^{(2)} \quad + \quad \sum_{h=0}^{\infty} \frac{(n-h)!}{(3A)^{n-h}} f_h^{(3)} \quad + \quad \mathcal{O}\left(4^{-n}\right)$$

LARGE ORDER RELATIONS



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Large order relations (true in large n limit)

$$f_n^{(0)} \sim \sum_{h=0}^{\infty} \frac{n!}{A^n} \left(f_0^{(1)} + \frac{A\, f_1^{(1)}}{n} + \dots \right) \quad + \quad \sum_{h=0}^{\infty} \frac{n!}{(2A)^n} \left(f_0^{(2)} + \frac{2A\, f_1^{(2)}}{n} + \dots \right) \quad + \quad \mathcal{O}\left(3^{-n}\right)$$

ALIEN DERIVATIVES

[J. Écalle 1985]
[D. Sauzin, 1405.0356]

- Resurgence ⟷ singularity structure in the Borel plane:

$$\mathscr{B}[F](t)\Big|_{t=\omega} = \frac{a}{t-\omega} + \mathscr{B}[G](t-\omega)\log(t-\omega) + \text{regular terms}$$

- Underlying mathematical structure of resurgence can be captured by Alien derivatives:

$$\Delta_\omega F = a + G$$

If ω is not a singular point of $\mathscr{B}[F]$, then $\Delta_\omega F = 0$

Properties: $\Delta_\omega(F\,G) = F\,(\Delta_\omega G) + (\Delta_\omega F)\,G$

- For a one-parameter transseries → Écalle's bridge equation

$$f(x,\sigma) = \sum_{n=0}^{\infty} \sigma^n e^{-nA/x} f^{(n)}(x) \;\rightarrow\; \Delta_{\ell A} f^{(n)} = \begin{cases} 0 & \ell > 1 \\ (n+\ell) S_\ell f^{(n+\ell)} & \ell \leq 1, \quad \ell \neq 0 \end{cases}$$

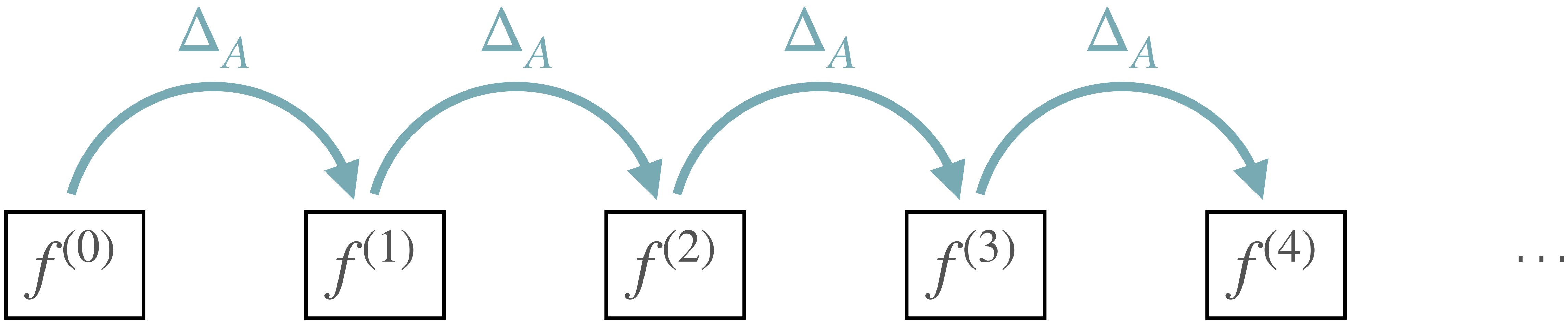
Stokes constants

ALIEN CHAIN

[Aniceto, Basar, Schiappa, 1802.10441]

"Standard" resurgence picture

Forward motions



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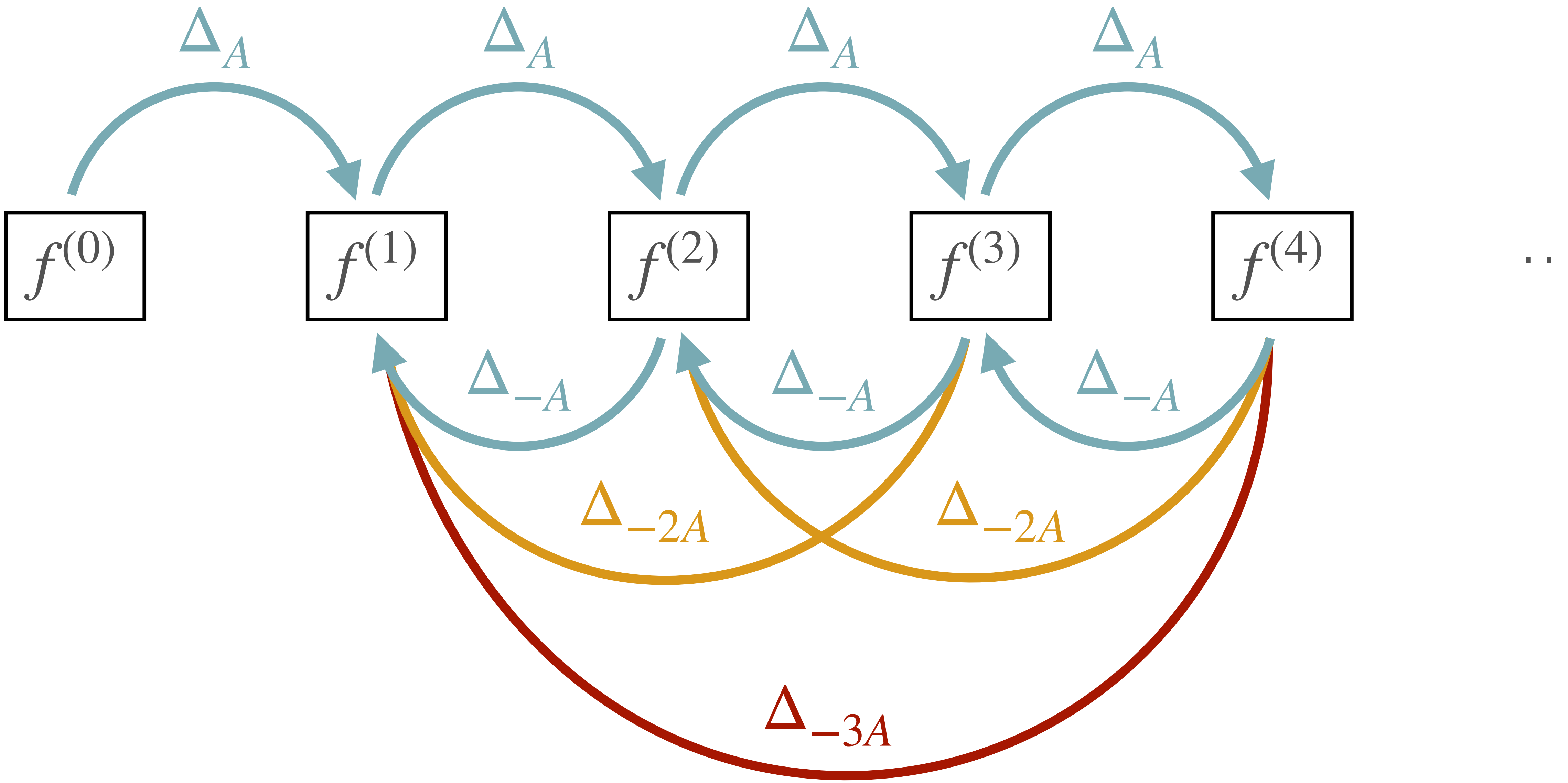


Backward motions

TWO PARAMETER TRANSSERIES

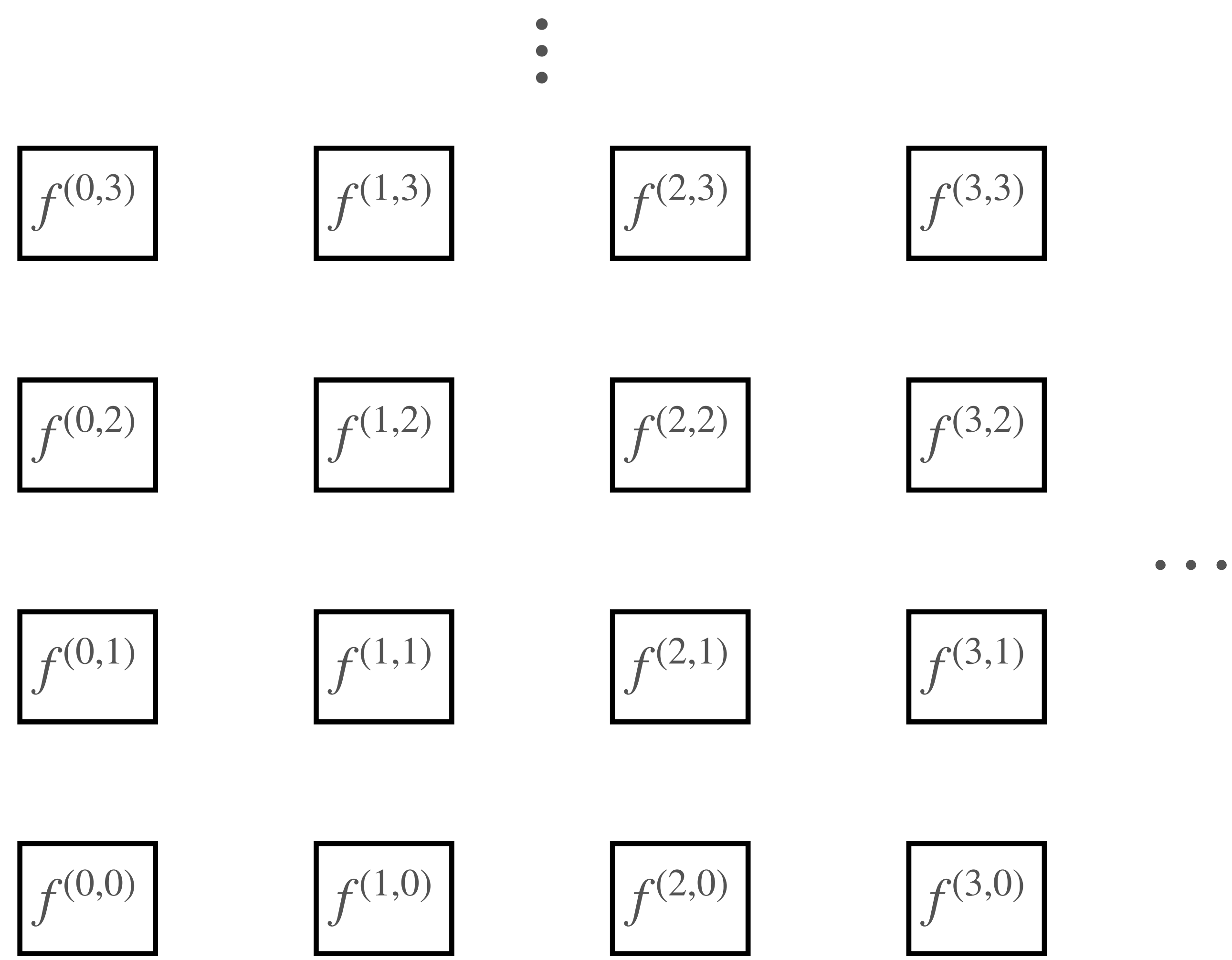
- More than one non-perturbative exponent $e^{-A_1/x}$ and $e^{-A_2/x}$

➜ Two parameter transseries:

$$f(x,\sigma_1,\sigma_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \sigma_1^n\, \sigma_2^m\, e^{-nA_1/x}\, e^{-mA_2/x} f^{(n,m)}(x)$$

- Alien lattice
- Richer structure of allowed alien motions

[Aniceto, Basar, Schiappa, 1802.10441]



TWO PARAMETER TRANSSERIES

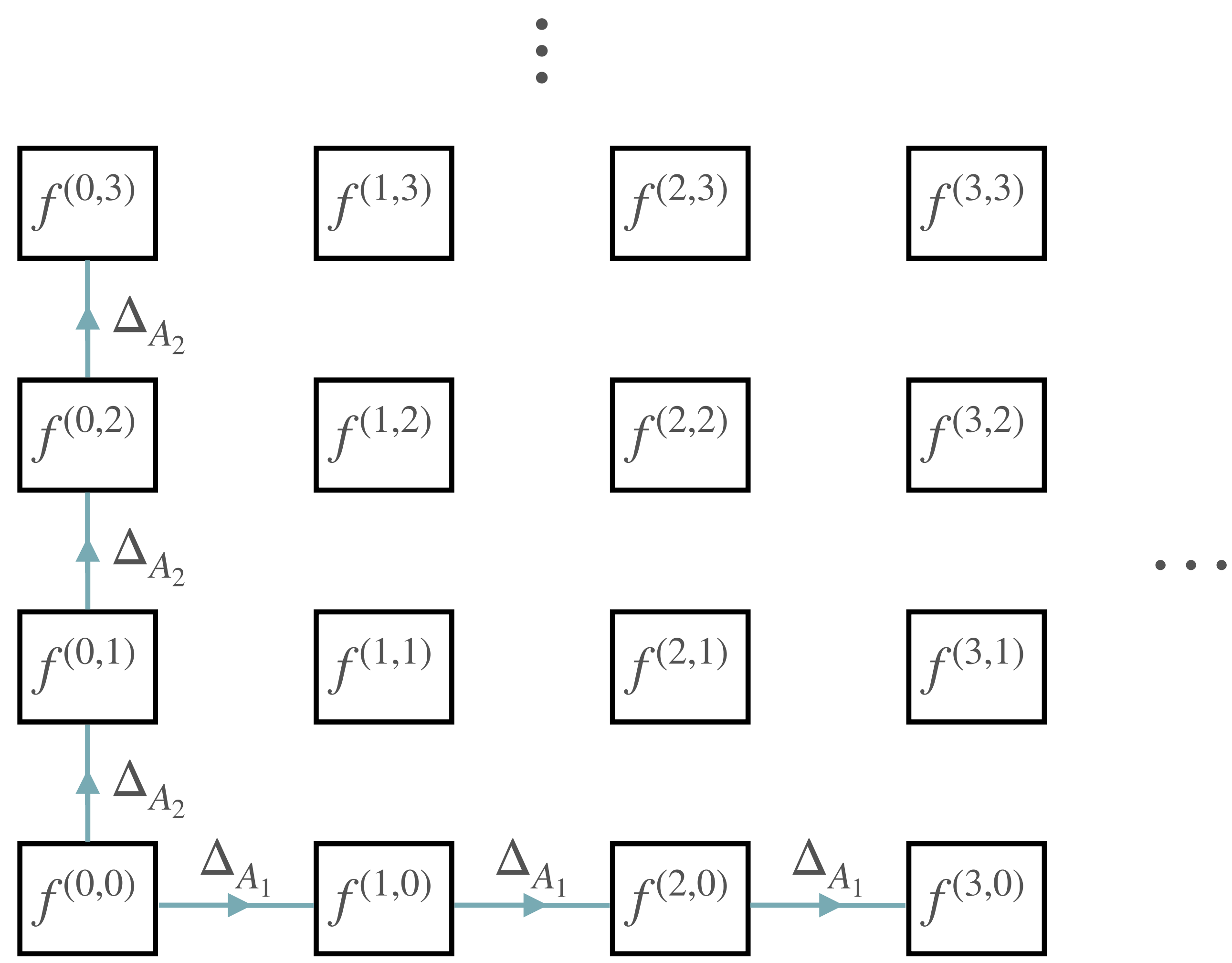
- More than one non-perturbative exponent $e^{-A_1/x}$ and $e^{-A_2/x}$

➜ Two parameter transseries:

$$f(x,\sigma_1,\sigma_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \sigma_1^n\,\sigma_2^m\,e^{-nA_1/x}\,e^{-mA_2/x}\,f^{(n,m)}(x)$$

- Alien lattice
- Richer structure of allowed alien motions

[Aniceto, Basar, Schiappa, 1802.10441]



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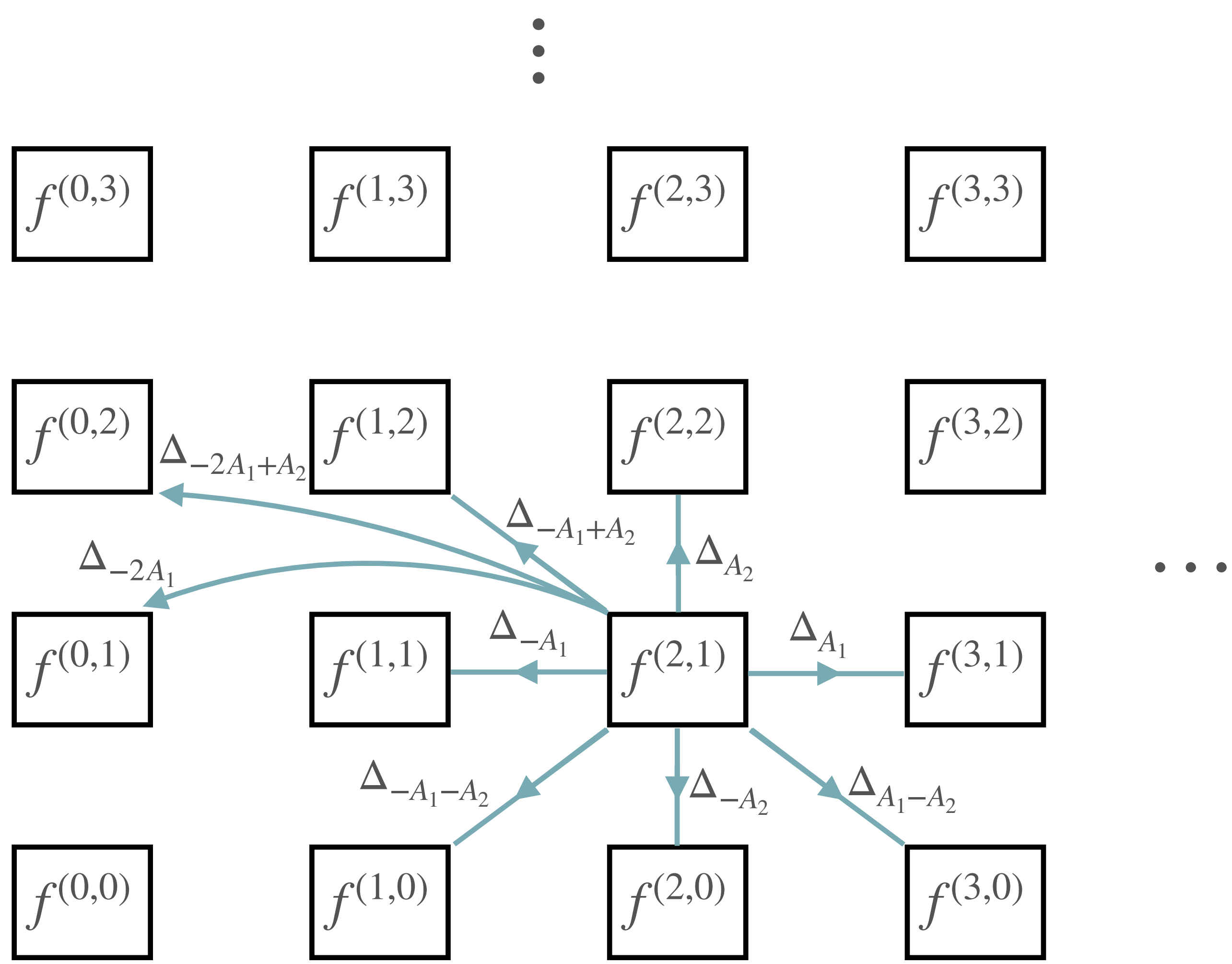
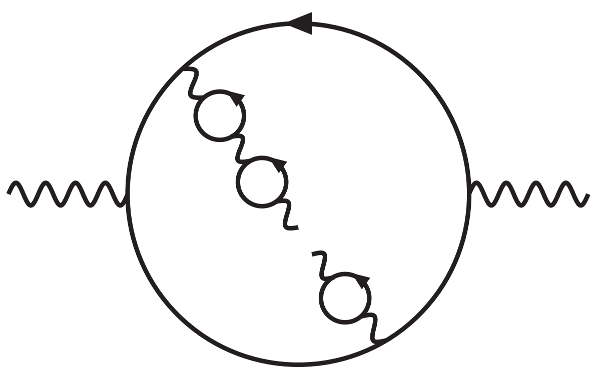


RENORMALONS

- Schematic computation



$$\sim \sum_{n=0}^{\infty} \alpha \int_0^{\infty} dk^2 F(k^2) \left[\alpha \log(k^2)\right]^n$$

- IR: $k^2 \ll 1$
- UV: $k^2 \gg 1$

$$F(k^2) = \begin{cases} 1 + k^2 + \ldots, & k^2 \ll 1 \\ \frac{1}{k^4} + \ldots, & k^2 \gg 1 \end{cases}$$

- Renormalons: $n!$ growth from a single class of diagrams
  - IR renormalons: $(-1)^n n!$
  - UV renormalons: $n!$
- Will see later that this is the QED picture, in QCD the role of UV and IR renormalons will be switched
- Related to non-perturbative power corrections: $\left(\frac{\Lambda}{Q}\right)^p$