Resummation of small-x double logarithms in QCD: inclusive deep-inelastic scattering

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Nikhef Theory Seminar

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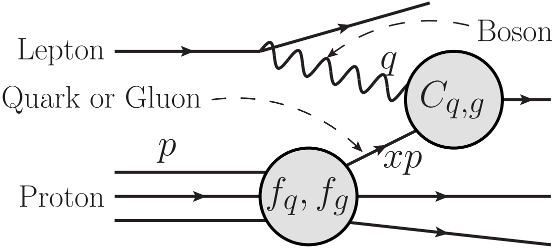
Introduction

Deep Inelastic Scattering: a lepton scatters from a proton.



Boson: γ, H, Z^0 ("Neutral Current") or $W^{\pm}$ ("Charged Current")

Cross-section: $\sigma \sim F_a(x, Q^2 = -q^2 > 0) = C_{a,q} \otimes f_q + C_{a,g} \otimes f_g$

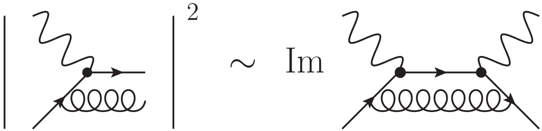
F_a – Structure Function
x – Collinear momentum fraction
Q – Exchanged momentum

$C_{a,j}$ – Coefficient Function $(a = 2, 3, L, \phi)$
$\otimes$ – Mellin Convolution
f_j – Parton Distribution Function (PDF)

Inclusive Deep-Inelastic Scattering

Integrate over all final states:

- to compute $C_{a,q}$, $C_{a,g}$, use the **optical theorem.**
- compute **forward scattering amplitudes.**



Loop integrals:

- use Dimensional Regularization ($d = 4 - 2\varepsilon$).
- divergences manifest as poles in ε.

Renormalization removes UV poles, but "collinear" poles remain:

$$(p-k)^2 \to -2|\vec{p}||\vec{k}|\,(1-\cos\vartheta) : \quad \text{propagator diverges as } \cos\vartheta \to 1 .$$

Collinear/Mass Factorization

To deal with these collinear poles, renormalize the PDF:

- factorize $\tilde{F}_{a,j}$: $C_{a,j}$ is finite. Z_{ji} contains only poles in ε.

$$F_a = \tilde{F}_{a,j} \otimes \tilde{f}_j = C_{a,j} \otimes Z_{ji}\left(x, a_s, \mu_f^2, \varepsilon\right) \otimes \tilde{f}_i = C_{a,j} \otimes f_j \,.$$

Factorization at scale μ_f^2, implies f_j has scale dependence:

$$\frac{d}{d\ln\mu_f^2} f_j = \frac{d}{d\ln\mu_f^2} Z_{ji} \otimes \tilde{f}_i = \left[\frac{d}{d\ln\mu_f^2} Z_{jk} \otimes Z^{-1}_{ki}\right] \otimes f_i = \left[P_{ji}\right] \otimes f_i \,.$$

- this is the DGLAP evolution equation
- P_{ji} are the Splitting Functions

Know Z_{ji} from calculation of $\tilde{F}_{a,j}$, so we can extract P_{ji}.

PDFs are universal to all hadron interactions; Splitting Functions are also.

Splitting Functions

DGLAP evolution: system of $2n_f+1$ coupled equations.

By defining the distributions

$$q_s = \sum_{i=1}^{n_f}(f_i + \bar{f}_i), \qquad q^{\pm}_{ns,ij} = (f_i \pm \bar{f}_i) - (f_j \pm \bar{f}_j), \qquad q_V = \sum_{i=1}^{n_f}(f_i - \bar{f}_i),$$

we have evolution equations in terms of 7 splitting functions:

$$\frac{d}{d\ln\mu_{\mathrm{f}}^2}\begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix},$$

$$\frac{d}{d\ln\mu_{\mathrm{f}}^2} q^{\pm}_{ns,ij} = P^{\pm}_{ns} \otimes q^{\pm}_{ns,ij}, \qquad \frac{d}{d\ln\mu_{\mathrm{f}}^2} q_V = P_V \otimes q_V .$$

Mellin-N Space

Taking a **Mellin transform**, convolutions ($\otimes$) become products,

$$f(N,Q^2) = \int_0^1 \mathrm{d}x \; x^{N-1} f(x,Q^2)\,,$$

- $F_a = C_{a,j} \otimes Z_{ji} \otimes \tilde{f}_i \to C_{a,j}\, Z_{ji}\, \tilde{f}_i$

Computing in N space, quantities “live” on even or odd **moments**:

- even N
  - F_2, F_L for e.m., $(\nu+\bar{\nu})$ DIS, F_3 for $(\nu-\bar{\nu})$ DIS, F_ϕ for scalar-exchange
  - P_{ns}^{+}, P_{ij}
- odd N
  - F_2, F_L for $(\nu-\bar{\nu})$ DIS, F_3 for $(\nu+\bar{\nu})$ DIS
  - P_{ns}^{-}, P_V

Perturbative Series

Expand as a series in $a_s = \alpha_s/(4\pi)$:

$$P = a_s^1\, P^{(0)} + a_s^2\, P^{(1)} + a_s^3\, P^{(2)} + a_s^4\, P^{(3)} + \cdots$$

$$C = \underbrace{\underbrace{\underbrace{a_s^0\, C^{(0)} + a_s^1\, C^{(1)}}_{NLO} + a_s^2\, C^{(2)}}_{N^2LO} + a_s^3\, C^{(3)}}_{N^3LO} + \cdots$$

N^{2}LO: known [Moch, Vermaseren, Vogt '04]

N^{3}LO: partially known

- large-n_f [Davies, Ruijl, Ueda, Vermaseren, Vogt '16]
- large-n_c [Moch, Ruijl, Vermaseren, Vogt '17]
- numerical approx. based on *Mellin moments*

N^{4}LO: $P^{(4)}$, a few moments only [Herzog, Moch, Ruijl, Ueda, Vermaseren, Vogt '19]

Expansion in ε: $C^{(n)} = c^{(n,0)} + \varepsilon c^{(n,1)} + \varepsilon^2 c^{(n,2)} + \cdots$.

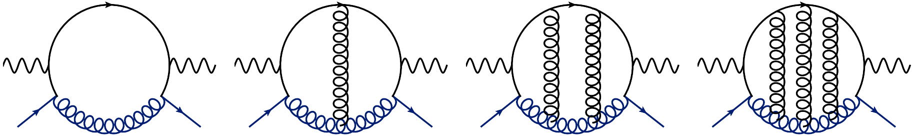
Computation

Compute N dependence directly (done at 3 loops, but not 4).

Or compute Mellin moments of $\tilde{F}_{a,j}$, for even or odd N

- expansion about two-point (propagator) integrals, $(q^2 \gg q \cdot p)$
- compute with MINCER (to 3 loops), FORCER (to 4 loops)

[Larin,Tkachov,Vermaseren ‘91][Ruijl,Ueda,Vermaseren ‘17]



- try to find N dependence from some moments, and extra information

x-space expressions recovered via **Inverse Mellin Transform.**

End-point behaviour

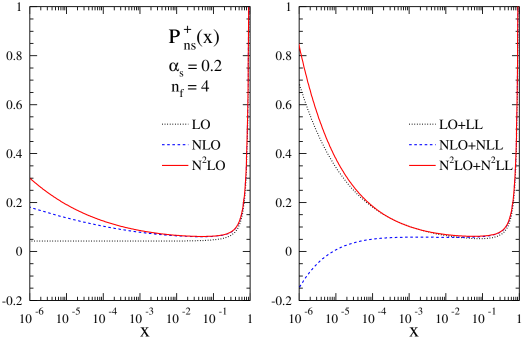
Coefficient and splitting functions are **logarithmically enhanced**:

- high-energy ($x \to 0$): $\ln(x)$
- threshold ($x \to 1$): $\ln(1-x)$

These logarithms spoil the convergence of perturbation theory.

Resum to all orders in a_s?

- $x \to 1$
  [Almasy, Lo Presti, Vogt '16]
- $x \to 0$ (discuss here)
  [Davies, Kom, Moch, Vogt '22]



Small-x behaviour

Power series in x, $\ln(x)$: (N space: $x^m \ln^k(x) \leftrightarrow (-1)^k k!/(N+m)^{k+1}$)

$$P_{ns}^{(n),+} \sim + x^0\left(\ln^{2n}(x) + \ln^{2n-1}(x) + \cdots + \text{const}\right) + \mathcal{O}(x^1),$$
$$C_{a,ns}^{(n),+} \sim + x^0\left(\ln^{2n-1-\delta_{a,L}}(x) + \ln^{2n-2-\delta_{a,L}}(x) + \cdots + \text{const}\right) + \mathcal{O}(x^1)$$

$$P_{ij}^{(n)} \sim + x^{-1}\left(\ln^{n-1}(x) + \ln^{n-2}(x) + \cdots + \text{const}\right) + x^0\left(\ln^{2n}(x) + \ln^{2n-1}(x) + \cdots + \text{const}\right) + \mathcal{O}(x^1),$$
$$C_{a,i}^{(n)} \sim + x^{-1}\left(\ln^{n-2}(x) + \ln^{n-3}(x) + \cdots + \text{const}\right) + x^0\left(\ln^{2n-1-\delta_{a,L}}(x) + \ln^{2n-2-\delta_{a,L}}(x) + \cdots + \text{const}\right) + \mathcal{O}(x^1)$$

x^{-1} single logs: resummed by BFKL formalism

- not covered by the discussion here: double logs only

"Unfactorized" Structure Functions

Recall the parton-level structure function, before factorization,

$$\tilde{F} = C\,Z\,. \qquad \text{(suppressing indices and working in } N \text{ space)}$$

Inverting the definition $P = \frac{d\,Z}{d\ln\mu_{\mathrm{f}}^2} Z^{-1} = \beta_{a_s} \frac{d\,Z}{da_s} Z^{-1}$,

$$Z = 1 - a_s \frac{1}{\varepsilon} P^{(0)} + a_s^2 \Big\{ \frac{1}{2\varepsilon^2} (P^{(0)} + \beta_0) P^{(0)} - \frac{1}{2\varepsilon} P^{(1)} \Big\} - a_s^3 \Big\{ \frac{1}{6\varepsilon^3} (P^{(0)} + \beta_0)(P^{(0)} + 2\beta_0) P^{(0)}$$
$$- \frac{1}{6\varepsilon^2} \Big[(P^{(0)} + 2\beta_0) P^{(1)} + 2(P^{(1)} + \beta_1) P^{(0)} \Big] - \frac{1}{3\varepsilon} P^{(2)} \Big\} + \mathcal{O}\left(a_s^4\right)$$

At a_s^n :

$$\varepsilon^{-n} \; : \; P^{(0)}, \beta_0,$$
$$\varepsilon^{-n+1} \; : \; P^{(0)}, \beta_0,\; P^{(1)}, \beta_1,$$
$$\vdots$$
$$\varepsilon^{-1} \; : \; P^{(n-1)}$$

N^{m}LO knowledge ($P^{(m)}$, β_m) gives leading $(m+1)$ ε poles of Z, and so also $\tilde{F}$, to all a_s orders.

Resummation Ansatz

At a_s^n: $\tilde{F}^{(n)} = \frac{1}{\varepsilon^{2n-1}} x^p \sum_{l=1}^{n} x^{l\varepsilon} \left(A_p^{(n,l)} + \varepsilon B_p^{(n,l)} + \varepsilon^2 C_p^{(n,l)} + \cdots \right)$

$2 \to n+1$ real-emission phase space:

- poles up to ε^{-2n+1}
- logarithmic factor $x^{n\varepsilon} = 1 + n\varepsilon \ln(x) + {n^2}/{2}\, \varepsilon^2 \ln^2(x) + \mathcal{O}(\varepsilon^3)$

Mixed real-virtual contributions:

- poles up to ε^{-2n+1}
- logarithmic factors $x^{\varepsilon}, x^{2\varepsilon}, \ldots, x^{(n-1)\varepsilon}$

After ε expansion, A, B, C give LL, NLL, N^{2}LL contributions to $\tilde{F}^{(n)}$.

Shift x^p gives sub-leading terms in x expansion.

$\left[\text{In } N \text{ space: } \tilde{F}^{(n)} = \frac{1}{\varepsilon^{2n-1}} \sum_{l=1}^{n} \frac{1}{N + l\varepsilon + p} \left(A_p^{(n,l)} + \varepsilon B_p^{(n,l)} + \varepsilon^2 C_p^{(n,l)} + \cdots \right) \right]$

Resummation Ansatz

Now we have two representations for $\tilde{F}$, which we can equate.

- double poles $\varepsilon^{-2n+1}, \dots, \varepsilon^{-n-1}$ have to cancel! KLN theorem.
- once A,B,C are determined, further expansion in ε yields predictions

Example, consider $\tilde{F}_{2,ns}$ at LO, LL accuracy: ($P^{(1)}$ is unknown)

$$\tilde{F}_{2,ns} = C_{2,ns} Z_{ns} = 1 + a_s \frac{1}{\varepsilon}\frac{A^{(1,1)}}{N+\varepsilon} + a_s^2 \frac{1}{\varepsilon^3}\left\{\frac{A^{(2,1)}}{N+\varepsilon} + \frac{A^{(2,2)}}{N+2\varepsilon}\right\} + a_s^3 \frac{1}{\varepsilon^5}\sum_{l=1}^{3}\frac{A^{(3,l)}}{N+l\varepsilon} + \cdots$$

$$= 1 + a_s\left\{-\frac{1}{\varepsilon}P^{(0)} + \varepsilon^0 c^{(1,0)} + \cdots\right\}$$
$$+ a_s^2\left\{\frac{1}{2\varepsilon^2}\left(P^{(0)}\beta_0 + {P^{(0)}}^2\right) - \frac{1}{2\varepsilon}\left(2c^{(1,0)}P^{(0)} + P^{(1)}\right) + \cdots\right\}$$

$$= 1 + a_s\left\{\frac{N^{-1}}{\varepsilon}A^{(1,1)} - \varepsilon^0 N^{-2}A^{(1,1)} + \cdots\right\}$$
$$+ a_s^2\left\{\frac{N^{-1}}{\varepsilon^3}[A^{(2,1)} + A^{(2,2)}] + \frac{N^{-2}}{\varepsilon^2}[-2A^{(2,1)} - A^{(2,2)}] + \frac{N^{-3}}{\varepsilon}[4A^{(2,1)} + A^{(2,2)}] + \cdots\right\}$$

a_s^3 : 3 unknown $A^{(3,l)}$, but ε^{-5} and ε^{-4} coefficients must vanish. ε^{-3} known from $C\,Z$.

When does this work?

Recall that in N space, we can compute either even or odd N values.

- for even-N based quantities, ansatz holds for shifts x^p with p even.
- for odd-N based quantities, it holds for shifts x^p with p odd.

For the "wrong powers", can't consistently determine A,B,C constants.

For singlet structure functions, system is coupled. E.g,

$$\begin{pmatrix} \tilde{F}_{2,q} & \tilde{F}_{2,g} \\ \tilde{F}_{\phi,q} & \tilde{F}_{\phi,g} \end{pmatrix} = \begin{pmatrix} C_{2,q} & C_{2,g} \\ C_{\phi,q} & C_{\phi,g} \end{pmatrix} \begin{pmatrix} Z_{qq} & Z_{qg} \\ Z_{gq} & Z_{gg} \end{pmatrix},$$

where

$$\begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} = \beta_{a_s} \frac{d}{da_s} \left[\begin{pmatrix} Z_{qq} & Z_{qg} \\ Z_{gq} & Z_{gg} \end{pmatrix} \right] \begin{pmatrix} Z_{qq} & Z_{qg} \\ Z_{gq} & Z_{gg} \end{pmatrix}^{-1}.$$

- Method works in the same way.

Procedure

N^{m}LO coefficient, splitting functions (small-x)

$\downarrow$

N^{m}LL $\tilde{F} = C\,Z$, to "all" a_s, ε

$\downarrow$

N^{m}LL coefficient, splitting functions, to "all" a_s

- we don't seek a closed expression/generating function for $\tilde{F}$
  - work at the level of the coefficient, splitting functions
- "all" a_s: computer-limited order ($C\,Z$ becomes large)
  - non-singlet: a_s^{60}, singlet: a_s^{20}

Non-singlet splitting function, N space

Using the above procedure, produce LL expansion:

$$P_{ns}^{+} = -\frac{2a_sC_F}{N} - \frac{4a_s^2C_F^2}{N^3} - \frac{16a_s^3C_F^3}{N^5} - \frac{80a_s^4C_F^4}{N^7} - \frac{448a_s^4C_F^5}{N^9} + \cdots$$

- OEIS (`https://oeis.org`) [A025225]
- `FindGeneratingFunction` (`Mathematica`)
- …

$$= -\frac{N}{2}\left(\sqrt{1 - 4\frac{2a_sC_F}{N^2}} - 1\right) = -\frac{N}{2}(S-1),$$

where $S = \sqrt{1-4\xi}$, $\xi = 2a_sC_F/N^2$.

Non-singlet splitting function, N space

Including also NLL and N^{2}LL terms,

$$P^+_{ns} = -\frac{2a_s C_F}{N} - 2a_s^2 C_F\left(\frac{2C_F}{N^3} + \frac{6C_F - 11C_A + 2n_f}{3N^2} - \frac{[18+36\zeta_2]C_F - 151C_A + 22n_f}{9N}\right) + \cdots$$

guess a basis of all-order functions: $\{1, S, S^{-1}, S^{-3}\}$.

Then:

$$\begin{aligned}P^+_{ns} = &-\frac{N}{2}(S-1) + \frac{a_s}{2}(2C_F - \beta_0)(S^{-1}-1)\\ &+ \frac{1}{96\,C_F}\,a_s N\,\{([156 - 960\,\zeta_2]\,C_F^2 - [80 - 1152\,\zeta_2]\,C_A C_F - 360\,\zeta_2\,C_A^2\\ &\qquad -100\,\beta_0 C_F + 3\beta_0^2)(S-1) + 2\,([12 - 576\,\zeta_2]\,C_F^2\\ &\qquad +[40 + 576\,\zeta_2]\,C_A C_F - 180\,\zeta_2\,C_A^2 + 56\,\beta_0\,C_F - 3\beta_0^2)(S^{-1}-1)\\ &\qquad +3\,(2\,C_F - \beta_0)^2(S^{-3}-1)\}\,.\end{aligned}$$

$n_f \to \beta_0$: more compact typesetting.

Non-singlet splitting function, x space

We can write P^{+}_{ns} in x space in terms of "modified Bessel functions":

$$\widetilde{I}_n(z) = \left(\frac{2}{z}\right)^n I_n(z) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k}, \qquad \text{here: } z = \sqrt{8C_F a_s}\ln\frac{1}{x},$$

$$\begin{aligned}\frac{P^{+}_{ns}}{2a_sC_F} = & \left\{1 + (2\,C_F - \beta_0)\,a_s\,\ln\frac{1}{x} + \frac{1}{2}\,(2\,C_F - \beta_0)^2 a_s^2\,\ln^2\frac{1}{x}\right\}\widetilde{I}_1(z) \\ & + \left\{\frac{1}{3}\,(11\beta_0 + 10\,C_A - 6\,C_F) - 4\,C_F\zeta_2\right\} a_s\,\widetilde{I}_0(z) \\ & + \left\{8\,C_F^2 - 2\,\zeta_2(\,15\,C_A^2 - 48\,C_F C_A + 44\,C_F^2\,)\right\} a_s^2\,\ln^2\frac{1}{x}\,\widetilde{I}_2(z)\,.\end{aligned}$$

There are some interesting structures here—come back to it later.

- expression is not unique: recurrence relations between $\widetilde{I}_n(z)$.

Non-singlet coefficient functions, N space

Similarly, produce LL, NLL, N^{2}LL expansions of the coefficient functions.

Can be written in terms of $F = S^{-1/2} = \left(1 - 4\frac{2a_s C_F}{N^2}\right)^{-1/4}$: (+ odd powers)

$$
\begin{aligned}
C^{+}_{2,ns} = F + \frac{1}{192 C_F} N \Big\{ & -3(32C_F + 11\beta_0)(F^{-1} - 1) + 4(18C_F + 11\beta_0)(F - 1) + 6\beta_0(F^3 - 1) \\
& + 12(2C_F - \beta_0)(F^5 - 1) - 5\beta_0(F^7 - 1) \Big\} \\
+ \frac{1}{9216 C_F} a_s \Big\{ & -128\Big([333 - 1368\,\zeta_2] C_F^2 - [60 - 1728\,\zeta_2] C_A C_F - 540\,\zeta_2 C_A^2 \\
& - 87\beta_0 C_F - 10\beta_0^2 \Big) \frac{1}{\xi}(F^{-3} - F^{-1} + 2\xi) + \cdots \Big\}
\end{aligned}
$$

$C^{+}_{L,ns}$ and $C^{+}_{3,ns}$ have similar forms.

In x space, F can be written in terms of ${}_1F_2(\cdots)$.

- not investigated in any detail...

Large-n_c Limit: all x powers

Recall that $P_{ns}^{(n),\pm} \sim +x^0\big(\ln^{2n}(x)+\cdots\big)+x^1\big(\ln^{2n}(x)+\cdots\big)+\mathcal{O}\big(x^2\big)$,

- for $P_{ns}^{(n),+}$, resum only x^{even}, for $P_{ns}^{(n),-}$, resum only x^{odd}.

In the large-n_c limit ($C_A \to n_c$, $C_F \to n_c/2$): $P_{ns}^{(n),+} = P_{ns}^{(n),-}$.

- we know all x powers in this limit
- order-by-order in a_s, can reconstruct coefficients of $\ln(x)$
  - fit to basis of $\{$Harmonic Polylogarithms, $(1-x)^{-1}$, $x^{\{0,1,2,3\}}\}$

$$P_{ns,L}^{(3)} = +\ln^6(x)\left[n_c^3 C_F\left\{\frac{5}{24}\left(1-\frac{16}{15}(1-x)^{-1}+x\right)\right\}\right]$$
$$+\ln^5(x)\left[n_c^3 C_F\left\{-\frac{4}{3}(2(1-x)^{-1}-1-x)H_1+\frac{22}{9}\left(1-\frac{13}{11}(1-x)^{-1}+\frac{17}{11}x\right)\right\}\right.$$
$$\left.+n_c^2 C_F n_f\left\{-\frac{7}{9}\left(1-\frac{8}{7}(1-x)^{-1}+x\right)\right\}\right]+\ln^4(x)\big[\cdots\big]+\ln^3(x)\big[\cdots\big]+\cdots$$

N^2LL result helped determine analytic N^3LO large-n_c $P_{ns,L}^{(3)} \to N^3LL$.

Works for $C_{2,ns,L}$, $C_{L,ns,L}$ and $C_{3,ns,L}$ also.

Singlet Quantities: Splitting Functions

Singlet splitting functions: $P_{qq}, P_{qg}, P_{gq}, P_{gg}$. Separate diagonal P:

$$P_{qq} = P_{ns}^{+} + P_{qq}^{ps},$$
$$P_{gg} = P_{gg}^{+} + P_{gg}^{ps}.$$

Then LL $P_{gg}^{+} = -\frac{N}{2}(S' - 1)$, where $S' = \sqrt{1-4\xi'}$, $\xi' = -\frac{4C_A a_s}{N^2}$.

The rest, at LL accuracy, a_s^{n+1}: $(\rho = n - k - 2i - 1)$

$$P_{qq}^{ps(n)} = \frac{2\mathcal{C}_n 2^n}{N^{2n+1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-1-2i} (-2)^{i+1+k} (n_f C_F)^{i+1} C_A^k C_F^{\rho} \binom{k+i}{k} \binom{\rho+i+1}{\rho},$$

and $P_{qg}^{(n)}$, $P_{gq}^{(n)}$, $P_{gg}^{ps(n)}$ can be written in a similar form.

$2\mathcal{C}_n$ are the expansion coefficients of $S = \sqrt{1-4\xi}$ (?!)

NLL, N^{2}LL? We can only compute order-by-order.

Singlet Quantities: Coefficient Functions

Similarly for the coefficient functions $C_{2,q}$, $C_{2,g}$, $C_{L,q}$, $C_{L,g}$, $C_{\phi,q}$, $C_{\phi,g}$,

$$C_{a,q} = C^{+}_{a,ns} + C_{a,ps}, \qquad a = \{2, L, \phi\}$$

where $C^{+}_{2,ns}$ etc. were computed above.

The pure-singlet parts, at LL accuracy, a_s^n: $(\rho' = n - k - 2i - 2)$

$$C^{(n)}_{2,ps} = \frac{\mathcal{D}_n 2^n}{N^{2n}} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \sum_{k=0}^{n-2-2i} (-2)^{i+1+k} (n_f C_F)^{i+1} C_A^k C_F^{\rho'} \binom{k+i}{k} \binom{\rho'+i+1}{\rho'},$$

and $C^{(n)}_{2,g}$, $C^{(n)}_{L,ps}$, $C^{(n)}_{L,g}$ can be written in a similar form.

$\mathcal{D}_n = \frac{1}{n!} \prod_{k=0}^{n-1} (1 + 4k)$ are the exp. coeffs. of $F = S^{-1/2}$. (?!)

NLL, N^2LL? We can only compute order-by-order.

Generalized Double-Logarithmic Equation

An alternative small-x description of P^+_{ns}: [Kirschner,Lipatov '76][Velizhanin '14]

$$P^+_{ns}\Big(P^+_{ns} - N + \beta/a_s\Big) = R = \mathcal{O}(1),$$

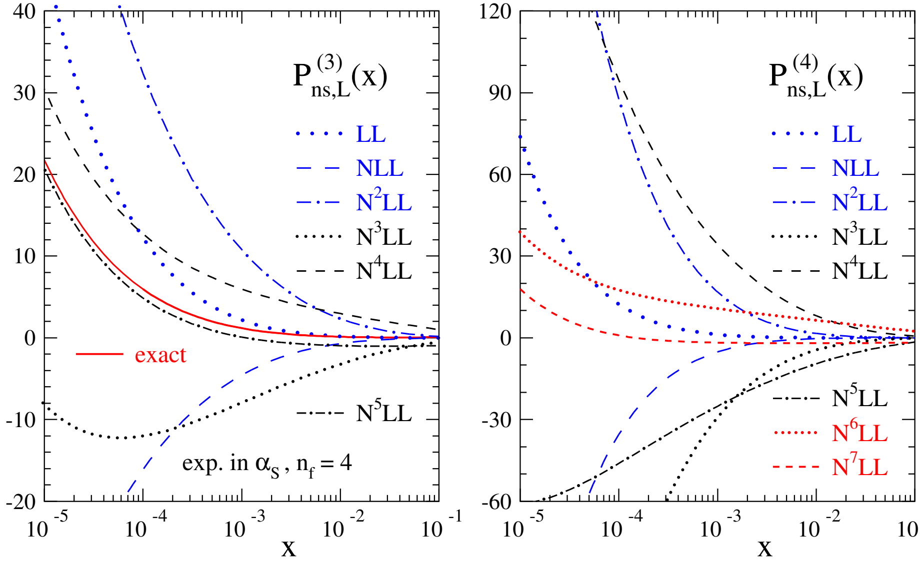
$$\rightarrow P^+_{ns} = -\frac{\beta}{2a_s} - \frac{N}{2}\left(\sqrt{1 + \frac{4R}{N^2} - \frac{2\beta}{N\,a_s} + \frac{\beta^2}{N^2 a_s^2}} - 1\right)$$

- solving order-by-order yields N^{2m+1}LL, given N^{m}LO input.
- caveat: fails at N^{2}LO, for terms $\sim \zeta_2(C_A - 2C_F)$ (??)
  - vanishes for large-n_c, so try the known N^{3}LO $P^{(3)}_{ns,L} \implies$ N^{7}LL $P^{(4)}_{ns,L}$

The N^{7}LL resummation in x space, lets us further investigate the apparent exponential structure:

$$\frac{P^+_{ns}}{2a_s C_F} = \exp\Big[-(\beta_0 - n_c)a_s \ln(1/x)\Big]\Big(\widetilde{I}_1(z) + \frac{1}{3}(11\beta_0 + 13n_c - 18\zeta_2 n_c)a_s \widetilde{I}_0(z)\Big) + \cdots$$

Generalized Double-Logarithmic Equation



Conclusion and Outlook

The structure of the unfactorized structure functions provides a way to obtain deep expansions of coefficient, splitting functions at small x,

- for non-singlet quantities we can find all-order forms.

Fixed-order predictions of the logs have already been used to help determine (parts of the) N^3LO non-singlet splitting function,

- they will also help determine (or check) the remaining colour factors.

For the future...

- how to write the LL singlet quantities in a “nice” way?
  - can this be generalized to NLL, N^2LL?
- what is this apparent exponentiation in the non-singlet splitting function's x space expression?
- can we access the “wrong” x powers outside the large-n_c limit?
  - here, the leading (x^0) behaviour of odd-N quantities is missing.
- what is wrong with the double-logarithmic equation?