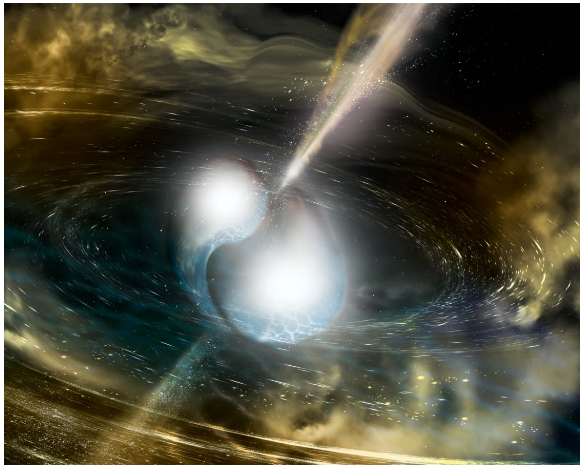


Gravitational waves
Lecture 1: Introduction

Utrecht University

Chris Van Den Broeck

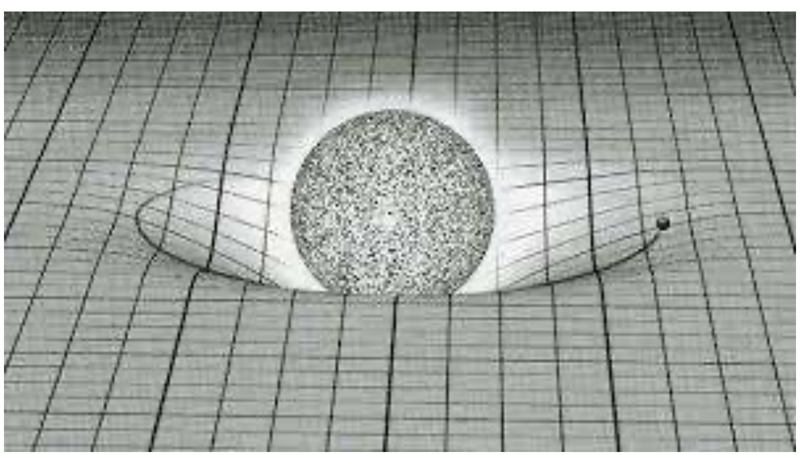
c.f.f.vandenbroeck@uu.nl

Einstein's theory of gravity

- 1915: Albert Einstein proposes the general theory of relativity
- Gravity as curvature of spacetime
- Einstein field equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

  - $G_{\mu\nu}$ the Einstein tensor, which encodes spacetime geometry
  - $T_{\mu\nu}$ the energy-momentum tensor, which gives the flow of matter and energy

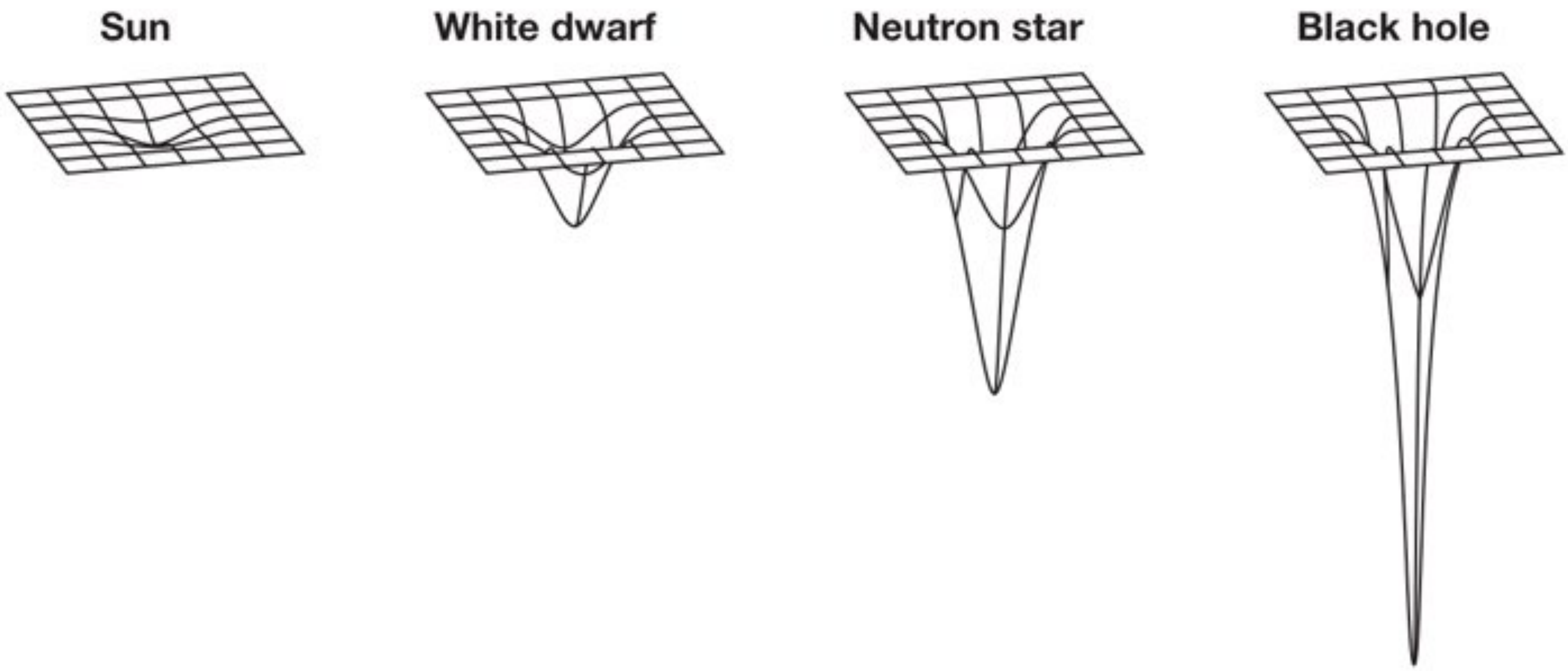


"Matter tells spacetime how to curve, spacetime tells matter how to move"

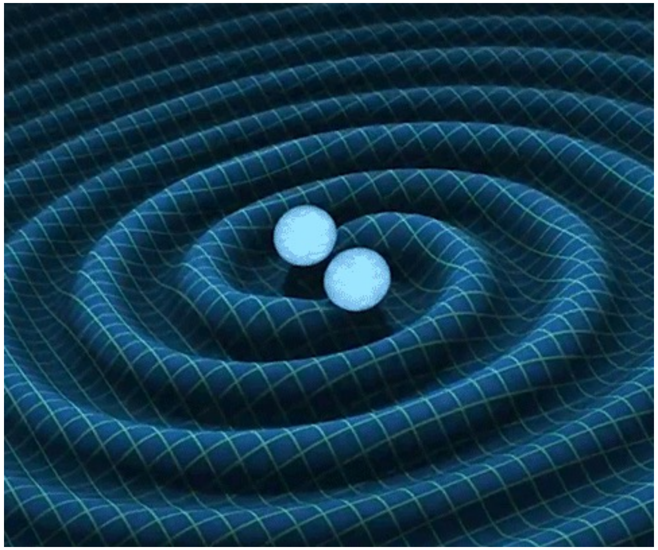
Einstein's theory of gravity

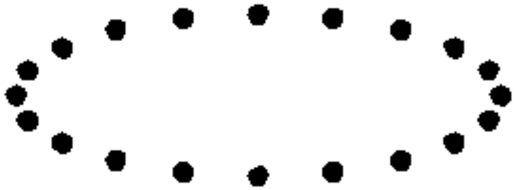
➢ More compact objects cause larger spacetime curvature

$R \sim 7 \times 10^5\,\mathrm{km}$ $\quad$ $R \sim 7 \times 10^3\,\mathrm{km}$ $\quad$ $R \sim 10\,\mathrm{km}$ $\quad$ $R = \frac{2GM}{c^2}$

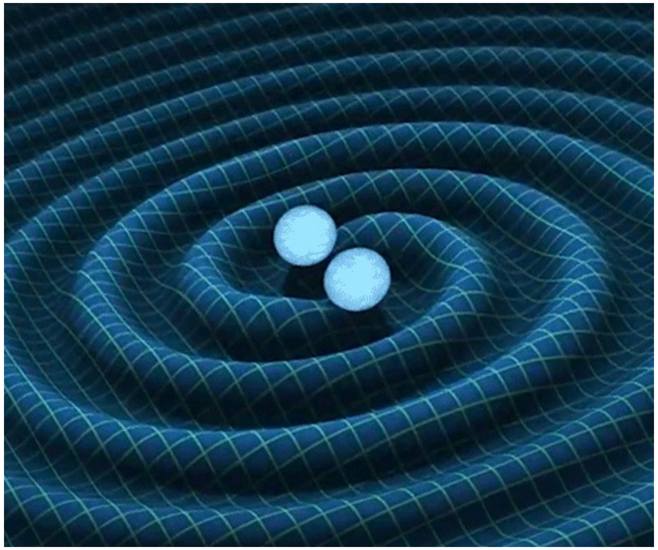


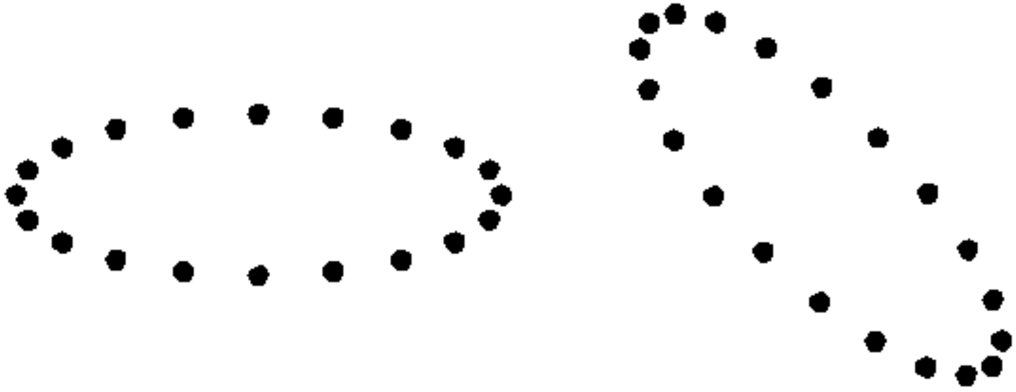
Gravitational waves



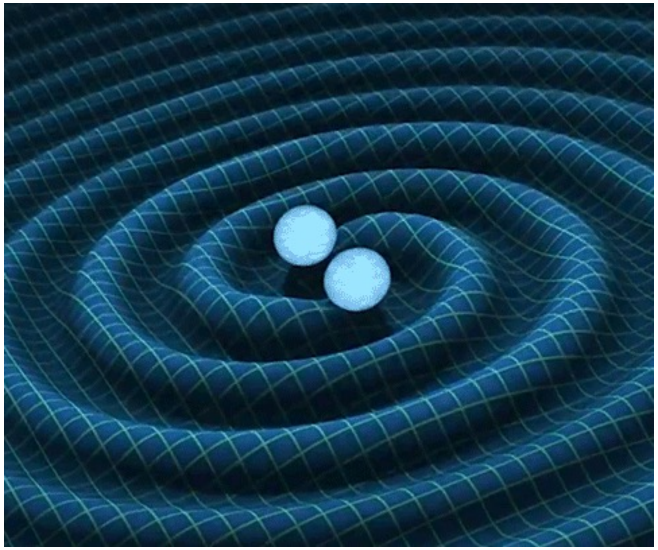


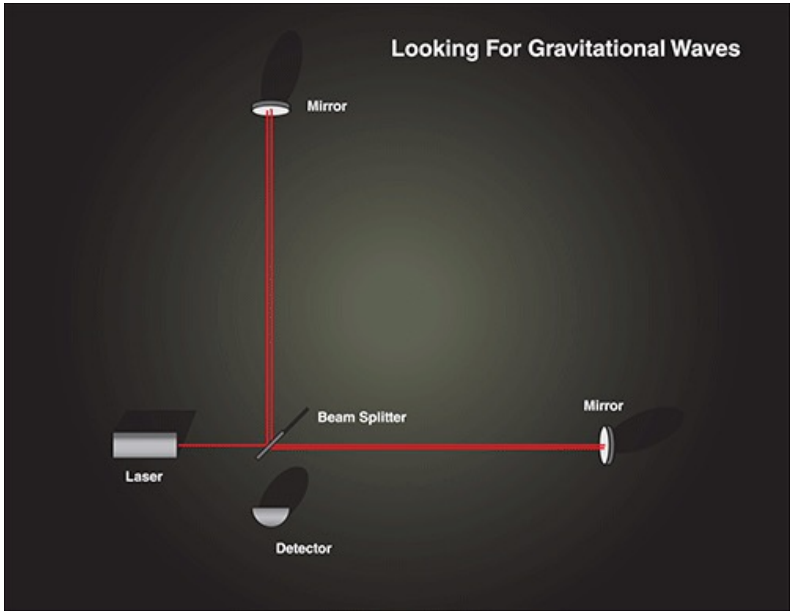
Gravitational waves

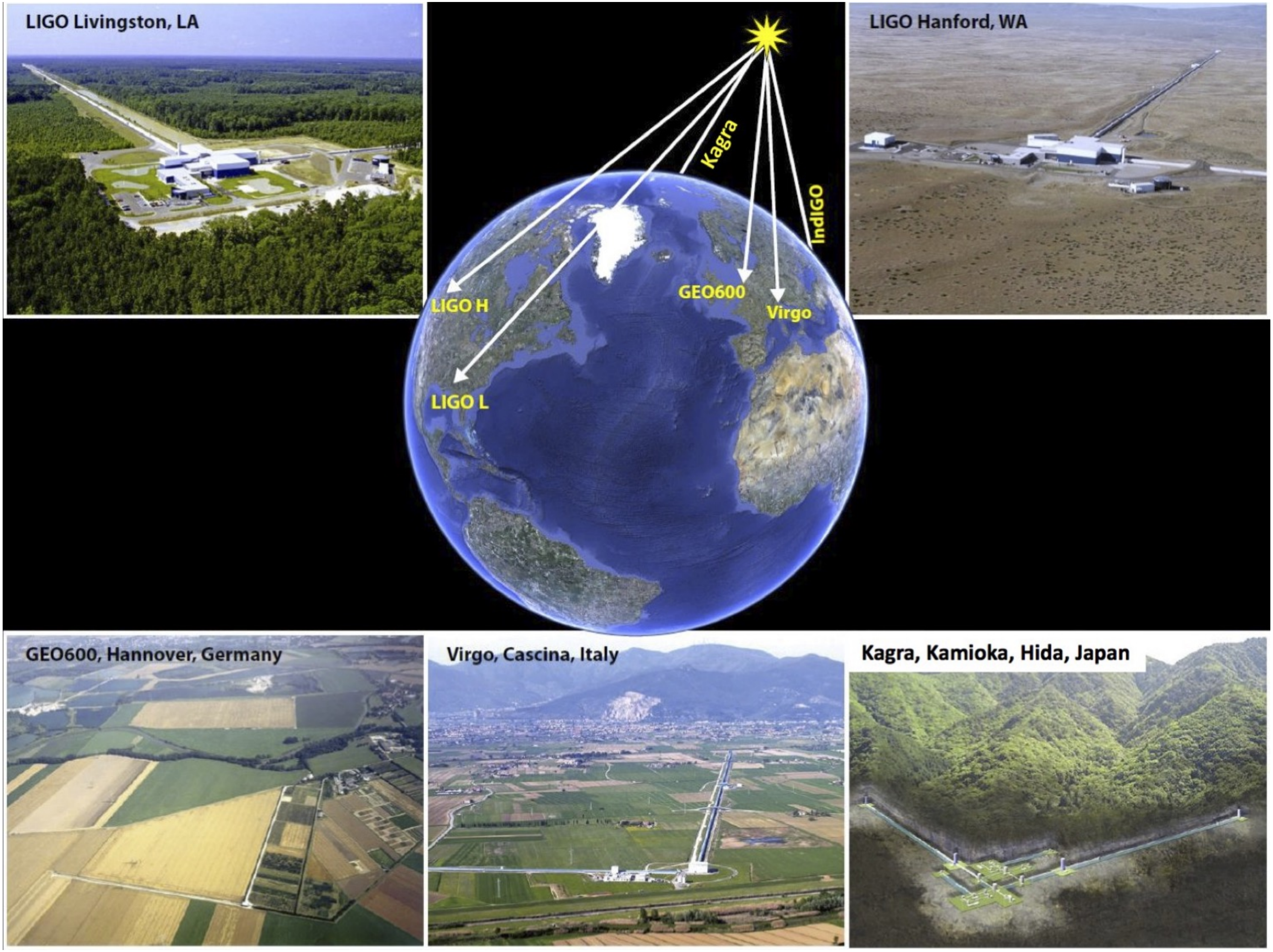




Laser interferometers

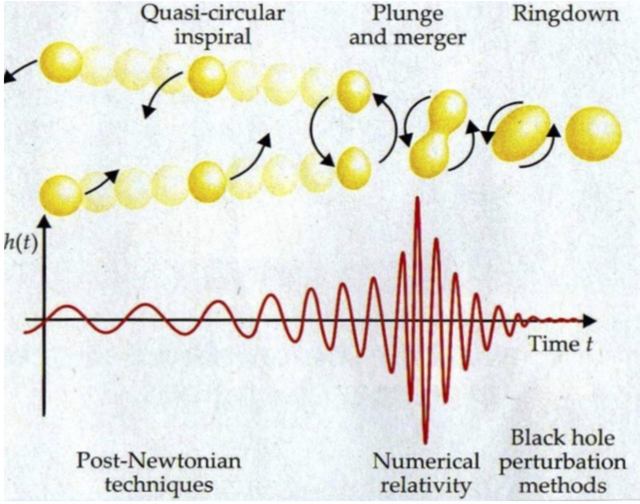




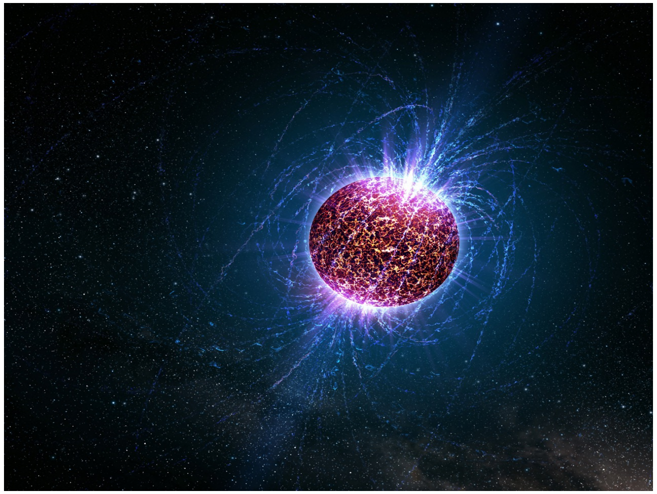

LIGO Livingston, LA
LIGO Hanford, WA
Kagra
IndIGO
GEO600
Virgo
LIGO H
LIGO L
GEO600, Hannover, Germany
Virgo, Cascina, Italy
Kagra, Kamioka, Hida, Japan

Detectable astrophysical sources

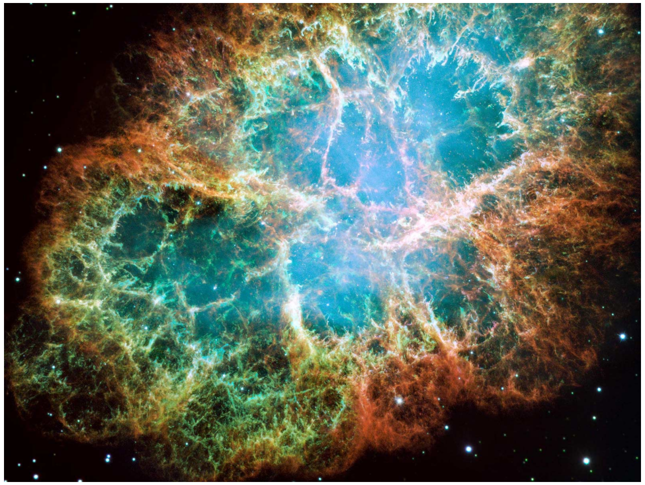
Merging neutron stars, black holes



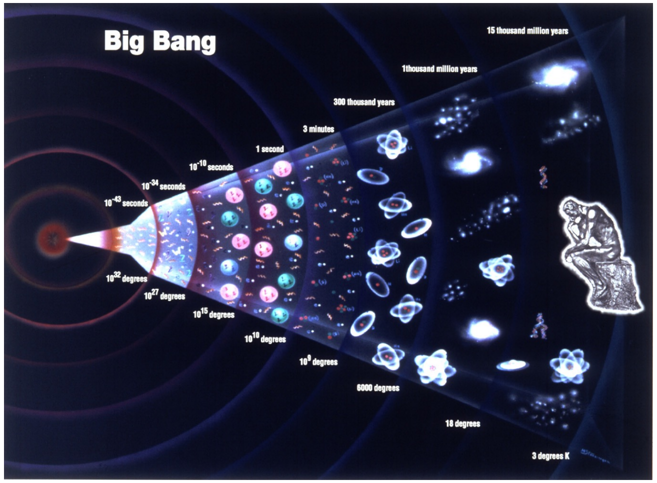
Fast-spinning neutron stars



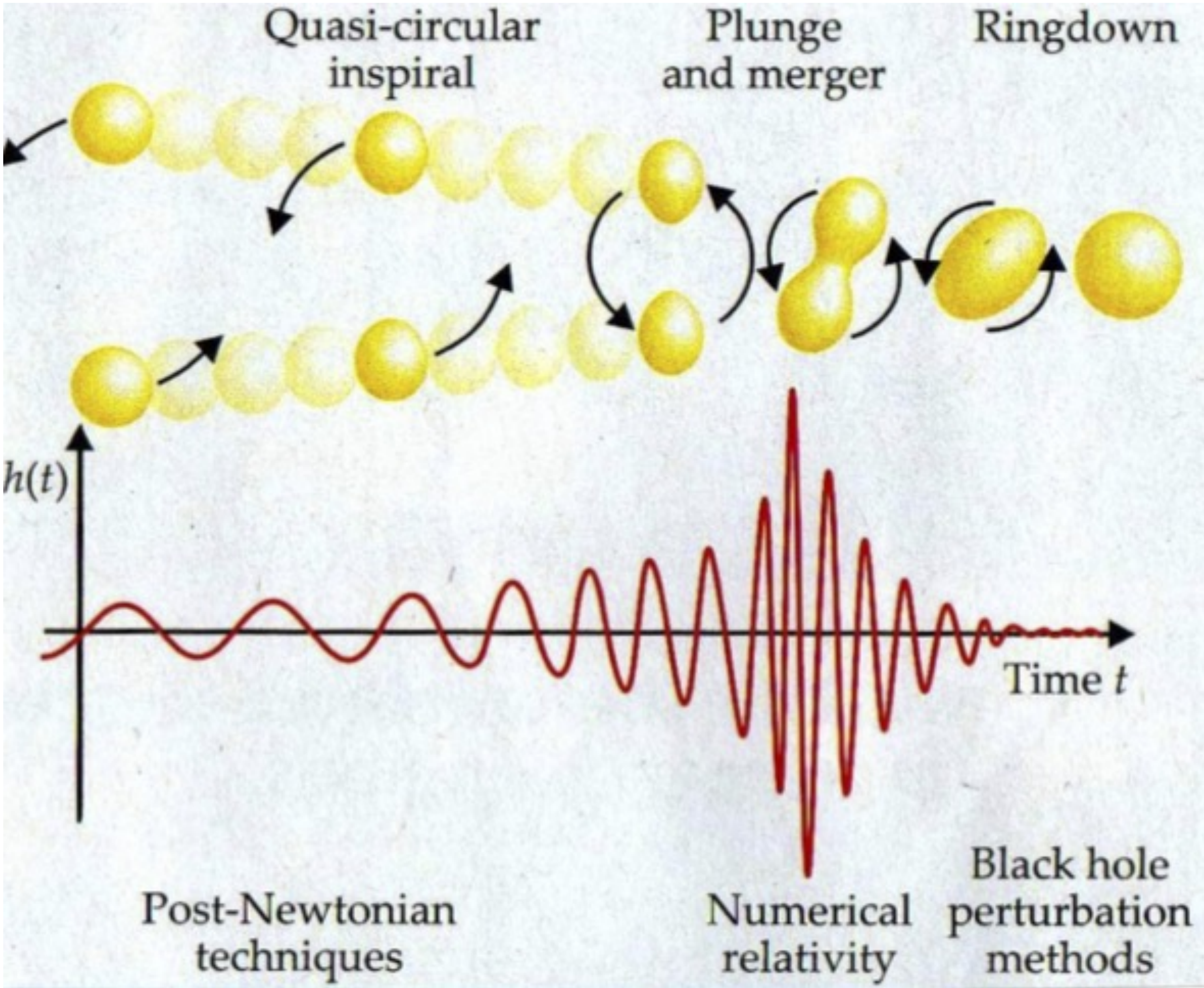
Supernovae



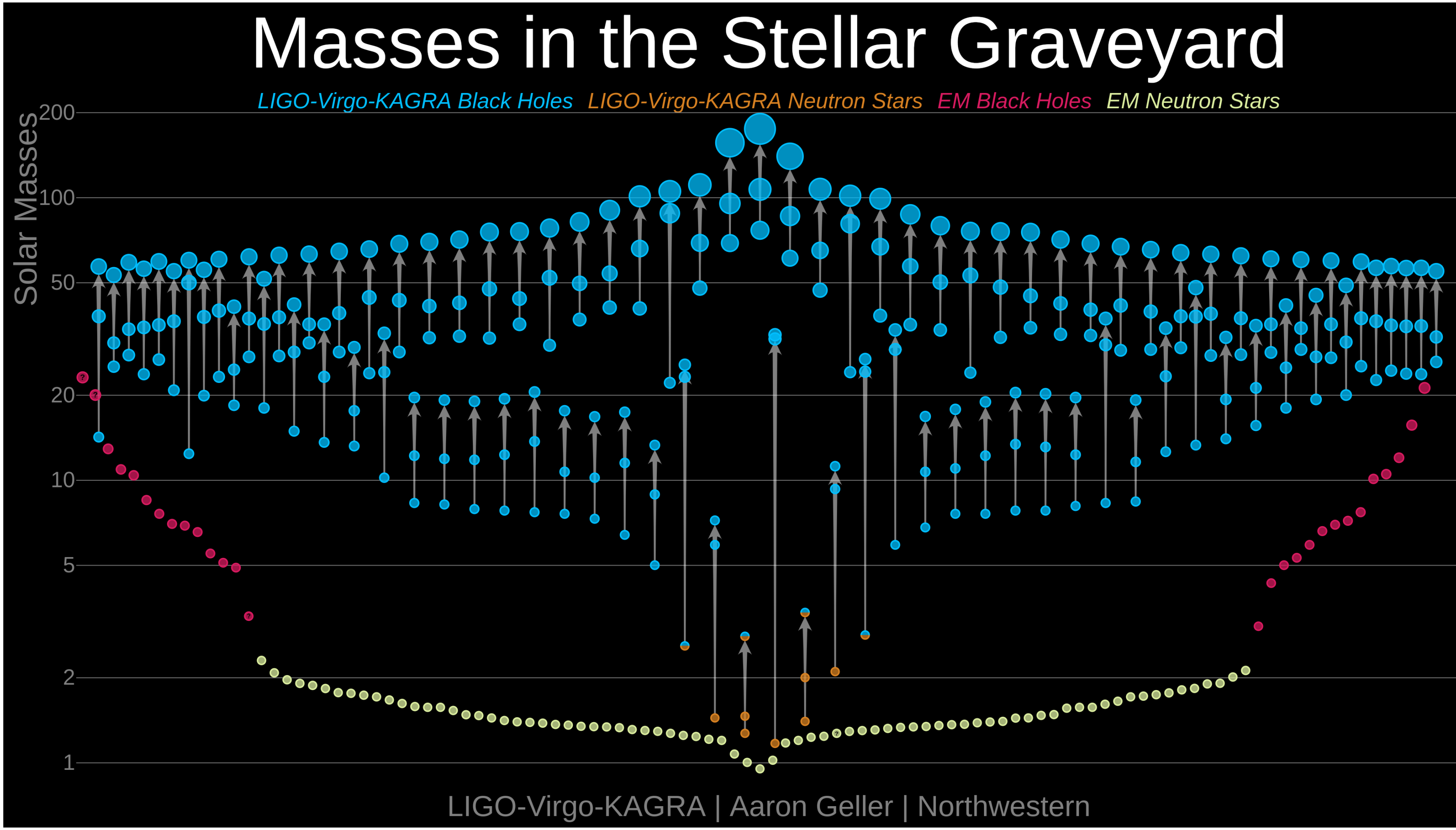
Primordial gravitational waves



The coalescence of compact objects

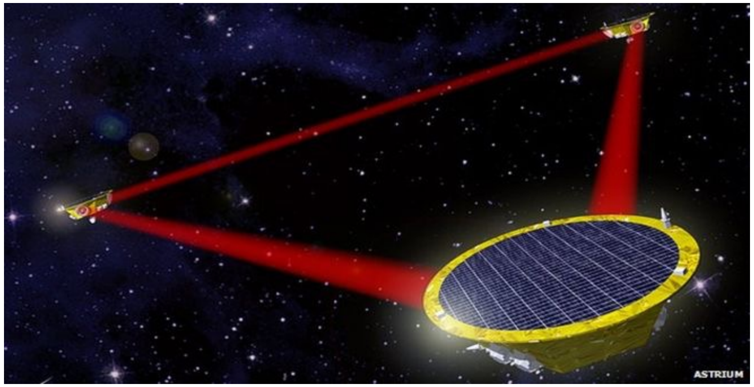


Gravitational wave detections are now routine!

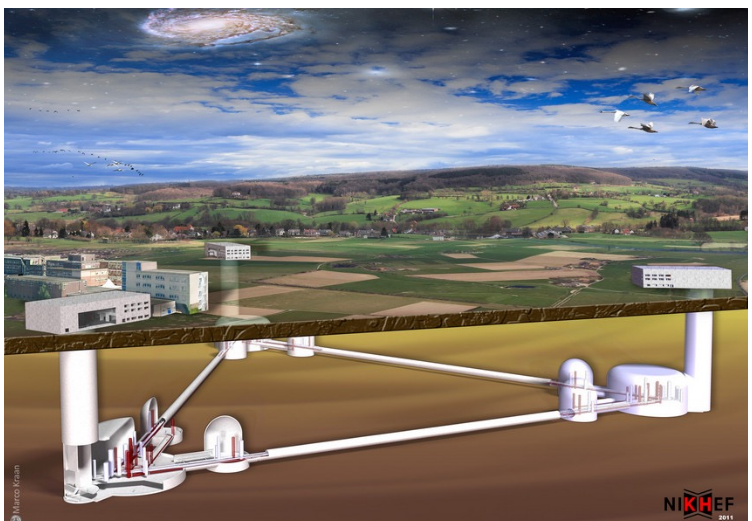


- 90 detections so far
  - Majority are from binary black holes, but also binary neutron stars and mixed neutron star-black hole mergers

Future gravitational wave detectors



- 2034: Laser Interferometer Space Antenna (LISA)
  - 3 probes in orbit around the Sun, ~1 million kilometers between them
  - Mergers of supermassive black holes



- ~2035: Einstein Telescope
  (and in USA: Cosmic Explorer)
  - O(10^5) detections per year
  - Covers the entire visible Universe
  - *Might be built in the border region of Belgium, the Netherlands, Germany!*

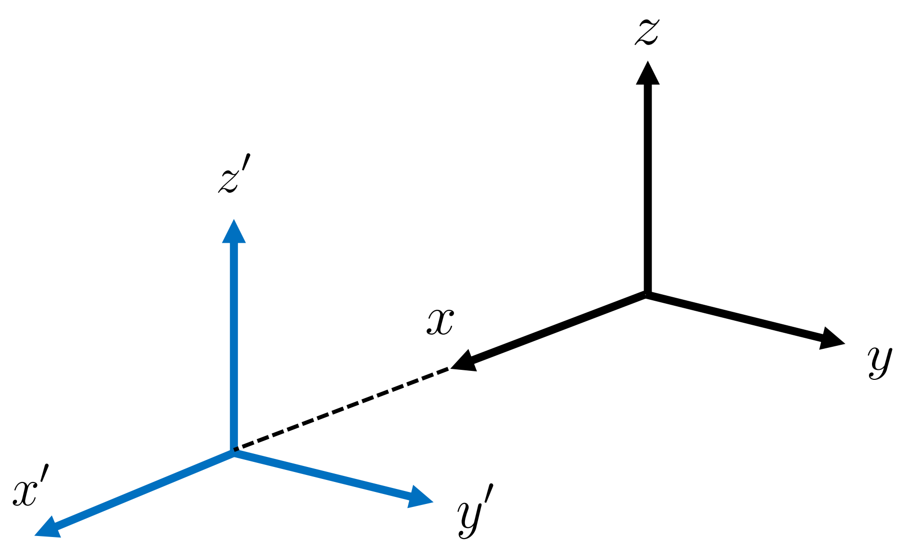
Questions answered in these lectures

- *What are the dynamics of spacetime?*
- *What are gravitational waves?*
- *What do gravitational waves from binary neutron stars and black holes look like?*
- *How are gravitational waves detected?*
- *What kind of science do gravitational waves enable?*

Relativity revisited

Galilean transformations

- Consider two inertial reference frames moving with respect to each other at constant velocity v :



- If time flows at the same rate in the two frames: $t' = t$

- If origins coincided at $t' = t = 0$:

$$\boxed{\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}}$$

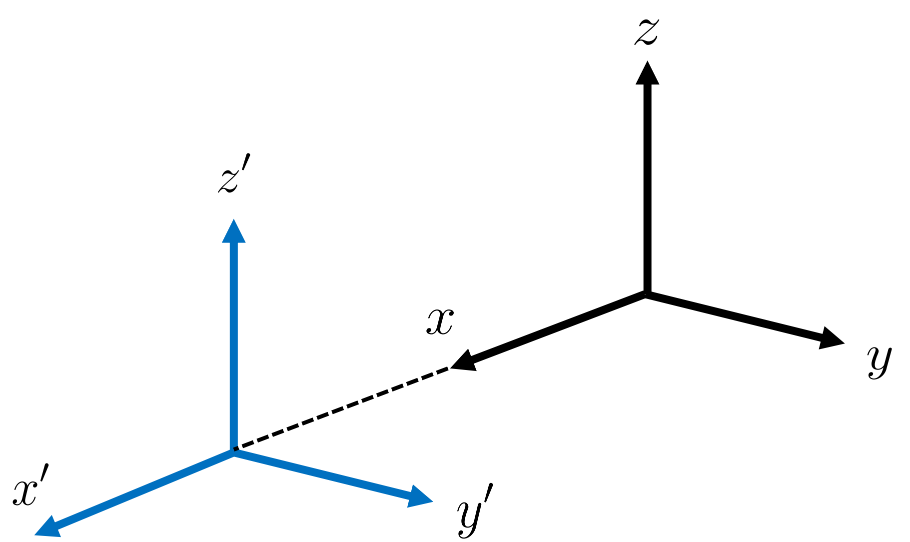
- Velocities of particles:
  $\vec{u}' = \vec{u} - v\,\hat{e}_x$
- Accelerations of particles:
  $\vec{a}' = \vec{a}$

Special relativity

- Einstein (1905) formulates the **special theory of relativity**
  - Something strange about Maxwell's laws of electromagnetism: *Don't remain unchanged under Galilean transformations!*
  - Measurement by Michelson and Morley (1887): *Speed of light seemed the same in different inertial frames*
- **Postulates** of special relativity:
  - The equations describing the basic laws of physics are the same in all inertial frames of reference
  - The speed of light in vacuum has the same value in all inertial frames of reference

Special relativity

- Consider two inertial reference frames moving with respect to each other at constant velocity v :



- Let a pulse of light be emitted at $t' = t = 0$, spreading out at the speed of light

- Point on the wavefront at a later time $t > 0$ in the unprimed frame:

$$c^2 t^2 = x^2 + y^2 + z^2$$

- Point on the wavefront at corresponding time t' in the primed frame:

$$c^2 t'^2 = x'^2 + y'^2 + z'^2$$

Special relativity

- Point on the wavefront at a later time $t > 0$ in the unprimed frame:

$$c^2t^2 = x^2 + y^2 + z^2$$

- Point on the wavefront at corresponding time t' in the primed frame:

$$c^2t'^2 = x'^2 + y'^2 + z'^2$$

- These expressions are not consistent with Galilean transformations!

$$c^2t'^2 = x'^2 + y'^2 + z'^2 \quad \Rightarrow \quad \cancel{\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}} \quad \Rightarrow \quad c^2t^2 = x^2 + y^2 + z^2$$

- However, they are consistent with **Lorentz transformations**:

$$\boxed{\begin{aligned} x' &= \gamma\,(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\,(t - vx/c^2) \end{aligned}}$$

where $\boxed{\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}}$

"Lorentz factor"

Exercise

The metric

- For light:

$$c^2\Delta t^2 \;=\; \Delta x^2 + \Delta y^2 + \Delta z^2$$

$$c^2\Delta t'^2 \;=\; \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

so that

$$0 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -c^2\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

- Easy to show that for **any** Δt , Δx , Δy , Δz

$$-c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -c^2\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

Exercise

(though in general not zero)

- Notion of **spacetime distance**:

$$(\Delta s)^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

or in infinitesimal form:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

Since this expression defines distances in spacetime, is called the **metric**

The metric

- Metric to compute spacetime distances:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

- Notation that will be convenient later:

$$(x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

so that the metric becomes

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

- Can be written in terms of a **metric tensor** $\eta_{\mu\nu}$:

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu$$

$$\boldsymbol{\eta} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Metric tensor

- Metric

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu}\, dx^\mu\, dx^\nu$$

- **Einstein summation convention**:

Whenever an index appears twice in the same term, once "up" and once "down", it should be considered summed over.

… hence

$$\boxed{ds^2 = \eta_{\boldsymbol{\mu}\boldsymbol{\nu}}\, dx^{\boldsymbol{\mu}}\, dx^{\boldsymbol{\nu}}}$$

where the object $\eta_{\mu\nu}$ is called the **metric tensor**

The inverse of the metric tensor

- **Metric tensor** $\eta_{\mu\nu}$

  where
  - $\eta_{00} = -1, \quad \eta_{11} = 1, \quad \eta_{22} = 1, \quad \eta_{33} = 1$
  - $\eta_{\mu\nu} = 0$ when $\mu \neq \nu$

$$\boldsymbol{\eta} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

- **Inverse of the metric tensor:**

$$\boldsymbol{\eta}^{-1} \cdot \boldsymbol{\eta} = \mathbf{1}$$

$$\boldsymbol{\eta}^{-1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Using index notation:

$$\eta^{\mu\rho}\,\eta_{\rho\nu} = \delta^{\mu}{}_{\nu}$$

where
- $\delta^{\mu}{}_{\nu} = 1$ when $\mu = \nu$
- $\delta^{\mu}{}_{\nu} = 0$ when $\mu \neq \nu$

Tensors in general

- $\eta_{\mu\nu}$, $\eta^{\mu\nu}$, dx^μ , $\delta^\mu{}_\nu$ are examples of **tensors**

  A tensor is a collection of numbers called **components**, labeled by **indices**, where an index is placed either "up" or "down"

  - Tensors can have more than two indices, e.g. $T^{\mu\nu\rho}$

- "Up" or "down" placement of indices matters!

  New tensors will be defined by **lowering** or **raising** indices with the metric tensor or its inverse

  - Example: from a tensor A^μ we can define a new tensor A_μ through
    $A_\mu = \eta_{\mu\boldsymbol{\alpha}} A^{\boldsymbol{\alpha}}$
    and from a tensor B_μ we can define a new tensor B^μ through
    $B^\mu = \eta^{\mu\boldsymbol{\alpha}} B_{\boldsymbol{\alpha}}$
  - Similarly for more general tensor $T^{\mu\nu\rho}$:
    $\eta_{\mu\boldsymbol{\alpha}} T^{\boldsymbol{\alpha}\nu\rho} = T_\mu{}^{\nu\rho}$

  - **Note**: a tensor like $C^{\mu\nu}$ is usually **not** the inverse of $C_{\mu\nu}$
    This is only the case for the metric tensor!

Tensors in general

- A^μ and $A_\mu = \eta_{\mu\alpha} A^\alpha$ don't have the same components! For example,

$$A_0 = \eta_{0\alpha} A^\alpha = \eta_{00} A^0 = -A^0$$

(although $A_1 = \eta_{1\alpha} A^\alpha = A^1$, and similarly A_2 , A_3)

- The names of dummy indices don't matter! For example,

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = \eta_{\alpha\beta}\, dx^\alpha dx^\beta \qquad \eta^{\mu\alpha} B_\alpha = \eta^{\mu\beta} B_\beta$$

- Tensors can be added component by component:

$$C_\mu{}^{\nu\rho} = D_\mu{}^{\nu\rho} + E_\mu{}^{\nu\rho}$$

  - Need to have the same free indices appearing "up" and "down" in every term!

- Free indices can be renamed, if done consistently in every term:

$$C_\kappa{}^{\nu\rho} = D_\kappa{}^{\nu\rho} + E_\kappa{}^{\nu\rho}$$

is the same set of equations as above

- Greek indices $\mu, \nu, \rho, ... = 0, 1, 2, 3$

  When we want to refer only to spatial components: Latin indices $i, j, k, ... = 1, 2, 3$

From special to general relativity

- Physical spacetime distance in **special relativity**:

$$\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu}$$

- Physical spacetime distance in **general relativity**:

$$\boxed{ds^2 = g_{\mu\nu} dx^\mu dx^\nu}$$

  - Metric tensor is symmetric:

    $$g_{\mu\nu} = g_{\nu\mu}$$

  - Inverse metric denoted $g^{\mu\nu}$, so that

    $$g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu$$

  - Indices lowered and raised with metric and its inverse; for example

$$A_\mu = g_{\mu\alpha} A^\alpha \qquad B^\mu = g^{\mu\alpha} B_\alpha \qquad T_\mu{}^{\nu\rho} = g_{\mu\alpha} T^{\alpha\nu\rho}$$

From special to general relativity

- We have seen that the proper distance in special relativity is preserved under **Lorentz transformations**

$$
\begin{aligned}
x'^0 &= \gamma(x^0 - (v/c)\, x^1) \\
x'^1 &= \gamma(-(v/c)\, x^0 + x^1) \\
x'^2 &= x^2 \\
x'^3 &= x^3
\end{aligned}
$$

where $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

- General relativity allows for (almost)* any coordinate transformations:

$$
\begin{aligned}
x'^0 &= x'^0(x^0, x^1, x^2, x^3) \\
x'^1 &= x'^1(x^0, x^1, x^2, x^3) \\
x'^2 &= x'^2(x^0, x^1, x^2, x^3) \\
x'^3 &= x'^3(x^0, x^1, x^2, x^3)
\end{aligned}
$$

or more compactly $x'^\mu = x'^\mu(x)$, where x is shorthand for (x^0, x^1, x^2, x^3)

- This is the same as saying that proper distance is preserved under general coordinate transformations:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = g'_{\alpha\beta}\, dx'^\alpha\, dx'^\beta$$

* We do require $x'^\mu(x)$ to be invertible, so that we can express $x^\mu(x')$, and also that it be differentiable.

General coordinate transformations

- In general relativity, proper distance is preserved under any coordinate transformations:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = g'_{\alpha\beta}\, dx'^\alpha\, dx'^\beta$$

- This tells us how the components of $g_{\mu\nu}$ change under coordinate transformations!
  - From the chain rule:

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^0} dx'^0 + \frac{\partial x^\mu}{\partial x'^1} dx'^1 + \frac{\partial x^\mu}{\partial x'^2} dx'^2 + \frac{\partial x^\mu}{\partial x'^3} dx'^3 = \frac{\partial x^\mu}{\partial x'^\alpha} dx'^\alpha$$

$$dx^\nu = \frac{dx^\nu}{dx'^0} dx'^0 + \frac{dx^\nu}{dx'^1} dx'^1 + \frac{dx^\nu}{dx'^2} dx'^2 + \frac{dx^\nu}{dx'^3} dx'^3 = \frac{dx^\nu}{dx'^\beta} dx'^\beta$$

  - Therefore

$$\boxed{\frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}} dx'^\alpha dx'^\beta = \boxed{g'_{\alpha\beta}} dx'^\alpha dx'^\beta$$

  - From this we read off:

$$\boxed{g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}}$$

The light cone

- In special relativity, the metric is

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

  - For particles moving slower than speed of light:

    $$dx^2 + dy^2 + dz^2 < c^2 dt^2$$

    so that

    $$ds^2 < 0$$

  - For photons:

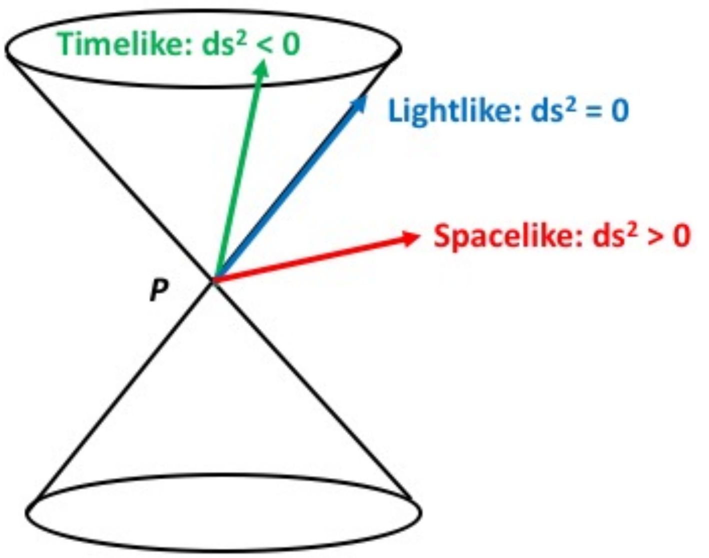
    $$ds^2 = 0$$

  - For hypothetical particles moving faster than speed of light:

    $$ds^2 > 0$$

- This leads to concept of **light cone**
  - Distinction between timelike, lightlike, spacelike is independent of coordinate system
  - Concept carries over to general relativity, with

    $$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$



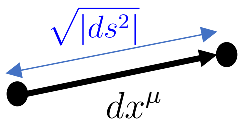
Physical spacetime distances

- The metric is

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

which has dimensions (length)^2

- $\sqrt{|ds^2|}$ is the **physical distance** between points separated by coordinate vector dx^μ

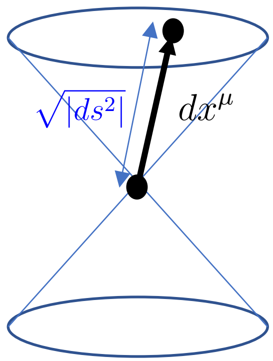


  - For spacelike separations, this has the familiar meaning of distance
  - What does "distance" mean in a timelike direction?

    Write

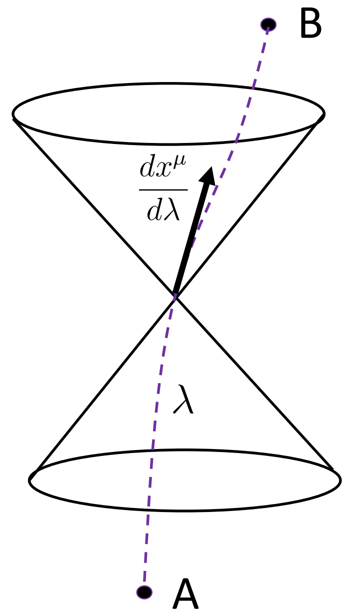
$$\sqrt{|ds^2|} = c\, d\tau$$

The quantity $d\tau$ is the **proper time elapsed according to an observer who moves by** dx^μ



Timelike curves

- Consider particle moving on a timelike path $x^\mu(\lambda)$ parameterized by λ



- Proper time $d\tau$ elapsed over a short parameter interval $d\lambda$:

$$c\,d\tau = \sqrt{|ds^2|} = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$$

- Proper time $\Delta\tau_{AB}$ elapsed between points A and B:

$$c\Delta\tau_{\mathrm{AB}} = \int_A^B \sqrt{|ds^2|} = \int_A^B \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \int_{\lambda_A}^{\lambda_B} \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$$

- One can parameterize the curve using proper time: $\lambda = \tau$

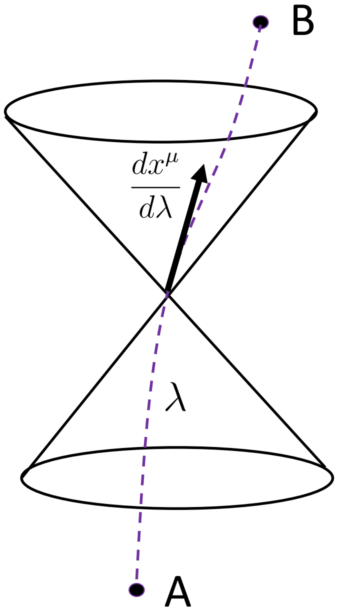
$$c\,\cancel{d\tau} = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\,\cancel{d\tau} \quad\Longrightarrow\quad c = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \quad\Longrightarrow\quad g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2$$

  - $V^\mu = \dfrac{dx^\mu}{d\tau}$ is the **tangent vector** to the curve called **four-velocity**
  - **Norm** of the four-velocity vector: $V_\mu V^\mu = g_{\mu\nu}\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} = -c^2$

Timelike geodesics

- Proper time elapsed in traveling from A to B:

$$c\Delta\tau_{AB} = \int_{\lambda_A}^{\lambda_B} \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$$



- A **geodesic** is a path which minimizes $\Delta\tau_{\mathrm{AB}}$

  This is the path of a **particle in free fall**

- Can be found by extremalizing the "action"

$$S = \int_A^B L(x^\mu, \dot{x}^\mu)\, d\lambda$$

  with "Lagrangian" $L(x^\mu, \dot{x}^\mu) = \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}$

  where dots denote derivatives w.r.t. λ

- This leads to the **geodesic equation** in terms of proper time τ :

$$\boxed{\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0}$$

  where

$$\Gamma^\beta_{\mu\nu} = \frac{1}{2}g^{\beta\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})$$

  with the notation $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$

Timelike geodesics

- Geodesic equation, which describes the motion of free-falling particles:

$$\boxed{\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0}$$

where $\Gamma^\beta_{\mu\nu} = \frac{1}{2}g^{\beta\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})$, with $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$

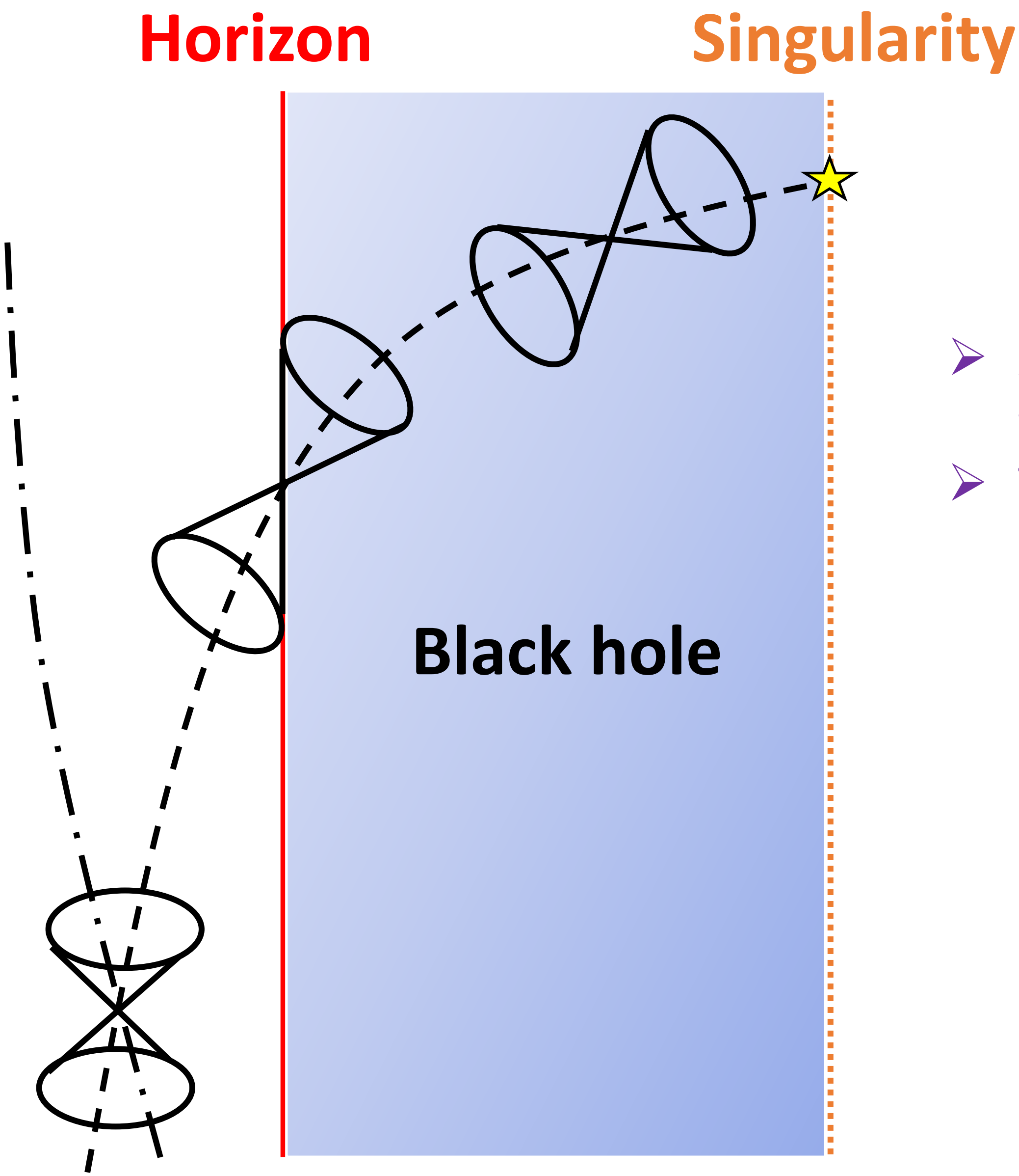
- For flat metric, $g_{\mu\nu} = \eta_{\mu\nu}$, all derivatives zero, $\partial_\mu g_{\rho\sigma} = 0$, hence $\Gamma^\beta_{\mu\nu} = 0$

$$\boxed{\frac{d^2x^\beta}{d\tau^2} = 0}$$

Thus, in flat spacetime the timelike geodesics are straight lines!

- General metric depends on spacetime: $g_{\mu\nu}(x)$
  - Dramatic example: spacetime of a black hole
  - “Tilting” of light cones prevents any timelike curve from being straight line

Timelike curves near a black hole



- At horizon: future lightcone tangent to horizon
- The horizon lies along a lightlike direction
  - Will be the case in any coordinate system!
  - No escape once inside

The right hand side of the Einstein equations

- Einstein field equations:

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

  - Left hand side: curvature of spacetime, given by the metric $g_{\mu\nu}$
  - Right hand side: energy-momentum tensor

- Meaning of the energy-momentum tensor?

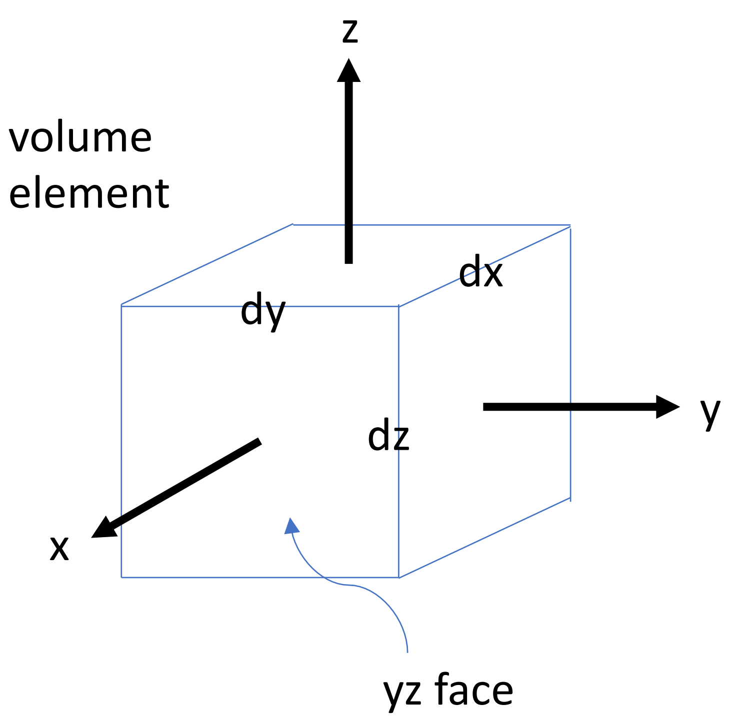
$$\begin{array}{cc} & \begin{array}{cc} \text{energy density} & \text{(momentum density) x c} \end{array} \\ T^{\mu\nu} = & \begin{pmatrix} \boxed{T^{00}} & \boxed{T^{0x} \quad T^{0y} \quad T^{0z}} \\ T^{x0} & T^{xx} \quad T^{xy} \quad T^{xz} \\ T^{y0} & T^{yx} \quad T^{yy} \quad T^{yz} \\ T^{z0} & T^{zx} \quad T^{zy} \quad T^{zz} \end{pmatrix} \\ & \text{stress tensor} \end{array}$$

- Notation: T^{00} , T^{0i} , T^{ij}

  where Latin indices denote spatial components: $i, j, k = 1, 2, 3$

The energy-momentum tensor

- Assume a matter distribution with density ρ
  - $T^{00} = \rho\, c^2$ is the energy density
  - $T^{0i}/c = \rho\, v^i$ is the momentum density in the *i*th direction
  - T^{ij} is the *i*th component of force per unit area across surface with normal in the direction *j*



- Consider volume element $dx\, dy\, dz$
  - T^{xx} is the x-component of force per unit area on the yz face
    = pressure on yz face:
    $$P^x = \frac{F^x}{dydz} = \frac{dp^x/dt}{dydz} = \frac{dp^x}{dtdydz}$$
  - Momentum $dp^x = dm\, v^x = \rho\, dxdydz\, v^x$
  - Hence
    $$T^{xx} = \frac{\rho\, dx\cancel{dy}\cancel{dz}}{dt\cancel{dy}\cancel{dz}} v^x$$
    $$= \rho\, \frac{dx}{dt}\, v^x = \rho\, (v^x)^2$$
  - More generally
    $$T^{ij} = \rho\, v^i v^j$$

Energy-momentum conservation

- In flat spacetime:

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

- For $\nu = 0$: $\partial_0 T^{00} + \partial_i T^{i0} = 0$

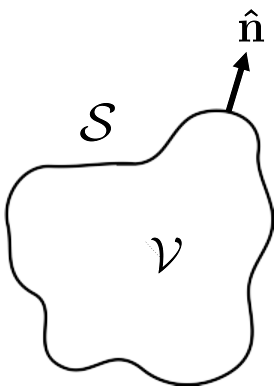
  Define
  - $\epsilon \equiv T^{00}$ energy density
  - $\pi^i \equiv T^{i0}/c$ momentum density

  Since $\partial_0 = \frac{\partial}{\partial x^0} = \frac{\partial}{c\partial t}$ and $\partial_i \pi^i = \frac{\partial}{\partial x^i}\pi^i = \boldsymbol{\nabla}\cdot\boldsymbol{\pi}$, one has

$$\frac{\partial \epsilon}{\partial t} = -\boldsymbol{\nabla}\cdot\boldsymbol{\pi}\, c^2$$

  Note that that $\boldsymbol{\pi}\, c^2$ is energy flux! For example, in the *x* direction:

$$\pi^x c^2 = \rho v^x\, c^2 = \frac{dm}{\cancel{dx}dydz}\frac{\cancel{dx}}{dt}c^2 = \frac{d(m\, c^2)}{dtdydz}$$



- Integrate both sides over volume bounded by a closed surface:

$$\int_{\mathcal{V}} \frac{\partial \epsilon}{\partial t}\, dV = -\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\boldsymbol{\pi}\, c^2\, dV \quad\Longrightarrow\quad \frac{dE}{dt} = -\int_{\mathcal{S}} \hat{\mathbf{n}}\cdot\boldsymbol{\pi}\, c^2\, dA$$

Conservation of energy

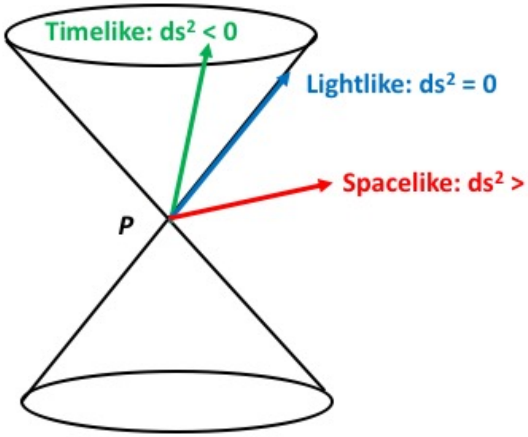
Summary

- In general relativity, physical spacetime distances are given by a **metric tensor** $g_{\mu\nu}$:

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu}$$

  - The metric and its inverse are used to lower and raise indices on other tensors
  - In going to a different coordinate system, the metric tensor transforms as

$$\boxed{g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}}$$



- The **light cone**: surface defined by $ds^2 = 0$
  - Paths of massive particles must stay within this cone

- Particles in free fall move on **geodesics**

$$\boxed{\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0}$$

where $\Gamma^\beta_{\mu\nu} = \frac{1}{2} g^{\beta\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})$

- The **energy-momentum tensor** $T^{\mu\nu}$
  - Energy density, momentum density, the stresses inside a mass distribution