Perturbative and colorful lectures on **Strong Interactions**

lecture 2/4

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collinear approximation

$$\frac{d^2\sigma_{q\bar{q}g}}{dx_q dx_{\bar{q}}} = \frac{\alpha_s}{2\pi} C_F \frac{x_q^2 + x_{\bar{q}}^2}{(1-x_q)(1-x_{\bar{q}})} \sigma_{q\bar{q}}$$

- let's express the cross section as a function of $z = x_g$ and θ_{qg}
- in the limit of small angles θ_{qg} (such as $1 - \cos\theta \simeq \theta^2/2$), on finds:

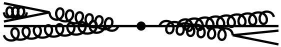
$$\mathrm{d}^2\sigma_{q\bar{q}g} = \sigma_{q\bar{q}}\, C_F \frac{\alpha_s}{2\pi} \left[1 + (1-z)^2\right] \frac{\mathrm{d}z}{z} \frac{\mathrm{d}\theta^2}{\theta^2}$$

- the singularity $\theta \to 0$ is avoided by the non zero quark masses
- dead cone effect: radiations are supressed in a cone $\theta < \theta_{min}$
- we integrate over the angle, and keep only the leading term, corresponding to $\theta_{min} = m_q/E_e$ we get the a famous Leading Log approximation (LL):

$$\mathrm{d}\sigma_{q\bar{q}g} = \sigma_{q\bar{q}}\, C_F \frac{\alpha_s}{2\pi} \left[1 + (1-z)^2\right] \ln \frac{E_e^2}{m_q^2} \frac{\mathrm{d}z}{z}$$

- this approximation is used in many Monte Carlo simulation for the parton shower

Jet algorithms

jet {  } jet

- obs: hadrons are collimated along the parton they originate from
- jet algorithm: mapping between n particles and m jets

$$\underset{n \text{ particles}}{\{p_i\}} \xrightarrow{\text{jet algo}} \underset{m \text{ jets}}{\{j_k\}}$$

- the way to cluster them is arbitrary
- need to be infra-red and collinear safe
- depend on a resolution parameter

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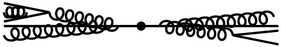
jet {  } jet

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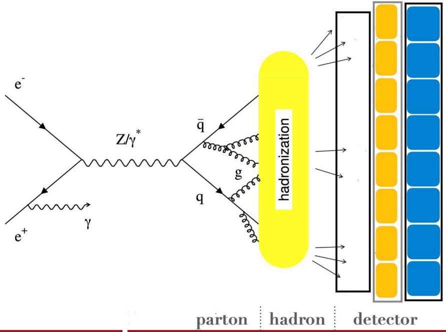
- the way to cluster them is arbitrary
- need to be infra-red and collinear safe
- depend on a resolution parameter
- can be applied at different levels:
- usually data and MC are compared at hadron level



Example 1: **JADE** algorithm

- **JADE** algorithm:

  i and *j* are in a same jet if $M_{ij}^2 = (p_i + p_j)^2 < y_{cut}\, s$

  $\rightarrow$ they are gathered in a forming jet with $p = p_i + p_j$

  if not, they belong to different jets

  - ends up when looped on all particles

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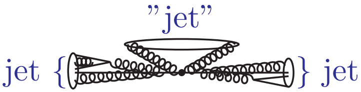
- weak point: some wrong associations: 2 low energy particles at large angle can be associated in a jet.



Jet algorithms: anti-kt

- many other algorithms have been developed
- some are more adapted to e^+e^- collisions, others to pp collisions
- at the LHC, the anti-k_t algorithm is widely used:
  - compute distances between i and j (R is a parameter)

$$d_{ij} = \min\left[1/k_{t,i}^2, 1/k_{t,j}^2\right] \Delta_{ij}^2/R^2 \qquad \Delta_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

  - compute distance between i and the beam

$$d_{iB} = 1/k_{t,i}^2$$

  - for i, if *min* is $d_{iB} \rightarrow i$ is a jet
  - if *min* is d_{ij} *to* the two particles are combined

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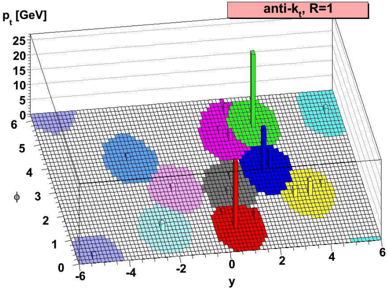
  - if *min* is d_{ij} *to* the two particles are combined
  - very nice conical jets not depending on low p_t particles
  - LHC uses $R = 0.4$ (or 0.8 for fat jets)



Jet multiplicity

- let's come back to $e^+e^- \to q\bar{q}g$ and to **JADE** algo
- there are 3 jets if for the 3 pairs :

$$M_{ij}^2 = \frac{s}{2} x_i x_j (1 - \cos\theta_{ij}) = \frac{s}{4}\left[(x_i + x_j)^2 - x_k^2\right] = s(1 - x_k) > s\, y_{cut}$$

$$\text{e.i.} \quad x_q < 1 - y_{cut} \qquad x_{\bar{q}} < 1 - y_{cut} \qquad x_g < 1 - y_{cut}$$

- the 3 jet rate defined as $f_3 = \sigma_{3jet}/\sigma_{tot}$ can be computed

$$\sigma_{tot}\ f_3 \quad = \quad \int d\sigma_{q\bar{q}g} \quad = \quad C_F \frac{\alpha_S}{2\pi} \int_{2y_{cut}}^{1-y_{cut}} \frac{dx_q}{1 - x_q} \int_{1+y_{cut}-x_q}^{1-y_{cut}} \frac{dx_{\bar{q}}(x_q^2 + x_{\bar{q}}^2)}{1 - x_{\bar{q}}}$$

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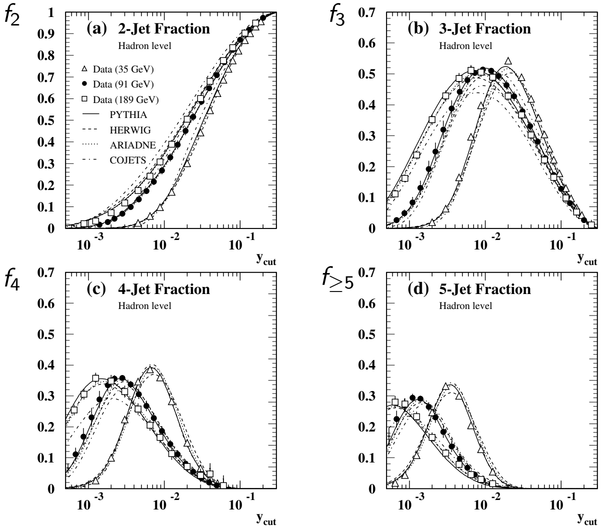
$$= C_F \frac{\alpha_S}{2\pi} \left(4 Li_2\left(\frac{y_{cut}}{1 - y_{cut}}\right) + (3 - 6y_{cut}) \log\left(\frac{y_{cut}}{1 - 2y_{cut}}\right) + 2\log^2\left(\frac{y_{cut}}{1 - y_{cut}}\right) - 6y_{cut} - \frac{9}{2} y_{cut}^2 - \frac{\pi^2}{3} + \frac{5}{2} \right)$$

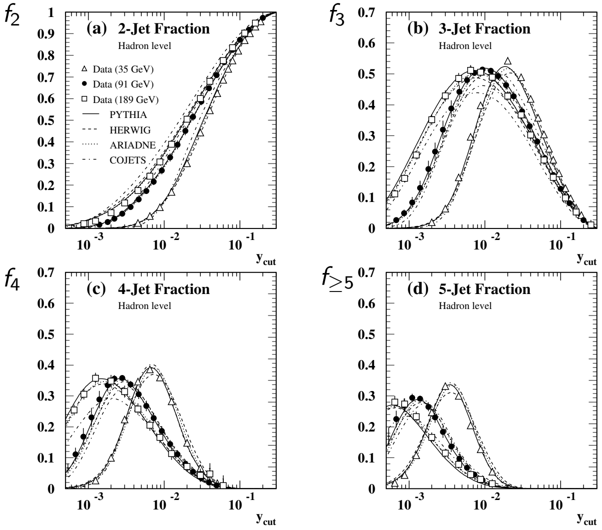


$f_i = \sigma_{i\,jets} / \sigma_{tot}$

- $N_{jet} \geq 2$
- if $y_{cut} \nearrow \Rightarrow N_{jet} \searrow$

(because each jet includes a larger fraction of s)

- depends on s ! - why ?



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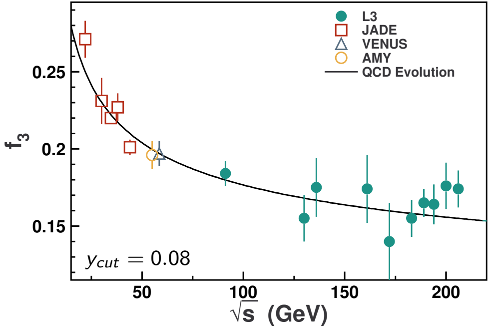
$\Rightarrow$ way to measure α_S!

- sensitivity to $\alpha_S \nearrow$ with $N_{jet} \nearrow$ but stat $\searrow$



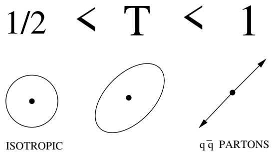
- energy scale dependence in good agreement with the QCD predicted evolution

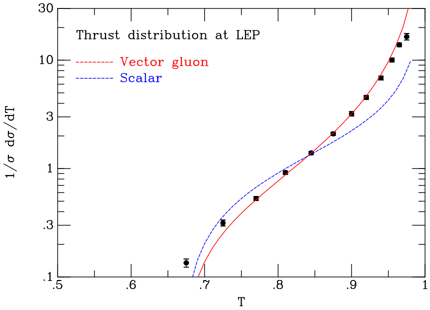
$$\alpha_s(Q^2) = \frac{\alpha_s(\mu_R^2)}{1 + \frac{23}{6\pi}\alpha_s(\mu_R^2)\ln\frac{Q^2}{\mu_R^2}}.$$

Event Shape

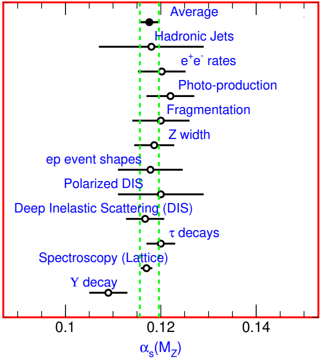
- jet independent observables also provide a sensitivity to α_S
- example : Thrust

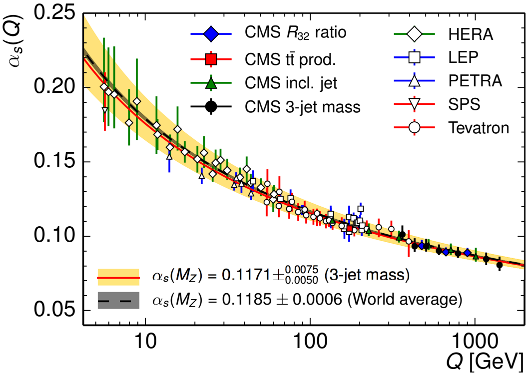
$$T = \max_{\vec{n}} \left(\frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|} \right)$$





Comparison of worldwide α_S measurements



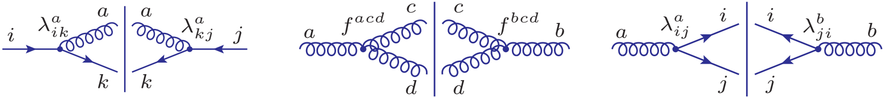


great success

- precision at ‰ level
- it has been measured that α_S is independent of the quark flavor.

Test of QCD gauge structure

- what are the free parameters of the different QCD couplings ?



$$(1)\ \sum_a \left(\lambda^a_{ik}\lambda^a_{kj}\right) = 4\,C_F\,\delta_{ij} \qquad C_F = T_F\frac{N^2-1}{N} = \frac{4}{3}$$

$$(2)\ \sum_{cd} f^{acd} f^{bcd} = C_A\,\delta_{ab} \qquad C_A = 2\,T_F\,N = 3$$

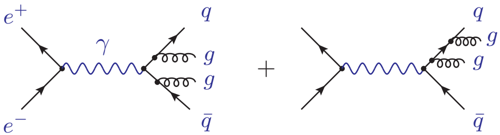
$$(3)\ \mathrm{Tr}\left(\lambda^a\lambda^b\right) = 4\,T_F\,\delta^{ab} \qquad T_F = \frac{1}{2}$$

- to confirm that QCD has the correct gauge structure

$\Rightarrow$ to access the gauge parameters C_F, C_A and T_F, we need to study 4-jet events.

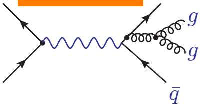
4-jet events

$\mathcal{A}$:

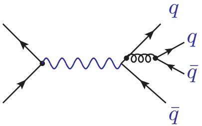


non-abelian

$\mathcal{B}$:

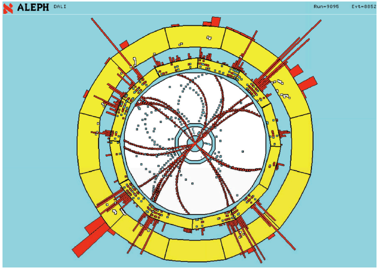


$\mathcal{C}$:



$$d\sigma = \left(\frac{\alpha_S}{2\pi}\right)^2 \left[C_F^2\, \mathcal{A} + C_F C_A\, \mathcal{B} + C_F T_F n_f\, \mathcal{C} \right]$$

- $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are kinematic functions corresponding to the diagrams



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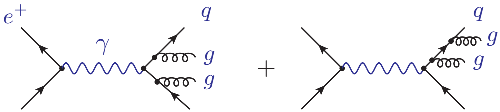
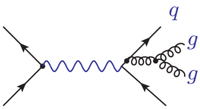


spin $1/2 \to 1/2 + 1$

$\mathcal{B}$:



spin $1 \to 1 + 1$

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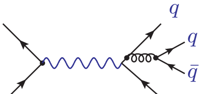
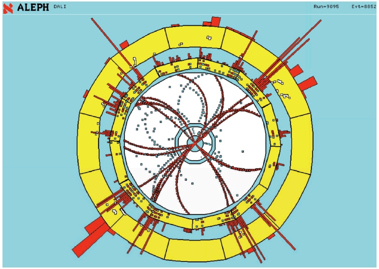


spin $1 \to 1/2 + 1/2$

$$d\sigma = \left(\frac{\alpha_S}{2\pi}\right)^2 \left[C_F^2\,\mathcal{A} + C_F C_A\,\mathcal{B} + C_F T_F n_f\,\mathcal{C} \right]$$

- $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are kinematic functions corresponding to the diagrams
- constrains on the gauge parameters can be obtain from angular distributions



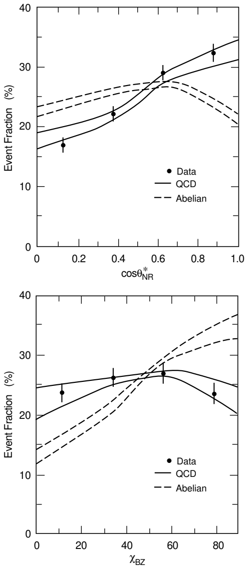
4-jet events

$$E_1 > E_2 > E_3 > E_4$$

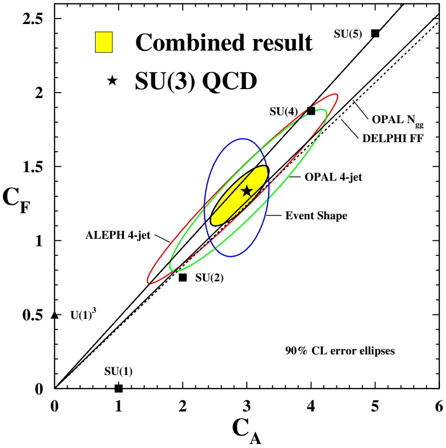
$$\cos\Theta_{NR^*} = \left| \frac{(\vec{p}_1 - \vec{p}_2)\cdot(\vec{p}_3 - \vec{p}_4)}{|\vec{p}_1 - \vec{p}_2|\,|\vec{p}_3 - \vec{p}_4|} \right|$$

$$\cos\chi_{\rm BZ} = \left| \frac{(\vec{p}_1 \times \vec{p}_2)\cdot(\vec{p}_3 \times \vec{p}_4)}{|\vec{p}_1 \times \vec{p}_2|\,|\vec{p}_3 \times \vec{p}_4|} \right|$$

⇒ confirms the need a the non-abelian term



Constrains on gauge parameters



- combination of 4-jets angle measurements with other measurement
- abelian $U(1)$ excluded by 12 sigmas
- only $SU(3)$ is compatible !

- N_{gg}: limit from fraction of gluon jets
- FF: limit coming from Fragmentation Functions

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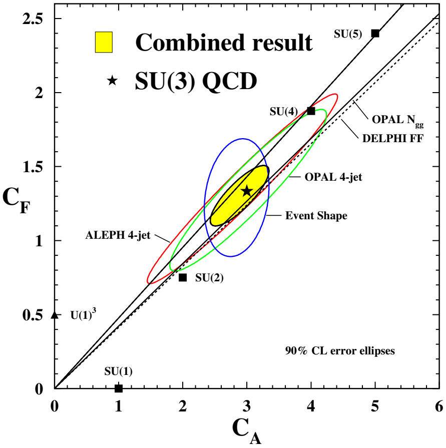

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wonderful success

section conclusions

- ✓ quark electric charges
- ✓ quark spin 1/2
- ✓ gluon spin 1
- ✓ α_S running
- ✓ need non-abelian term
- ✓ $SU(3)$ jauge parameters

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complete success !

$\Rightarrow$ QCD is a (the ?) good theory to describe strong interactions

$\Rightarrow$ can it be used to study the hadron structure (next section)

$\Rightarrow$ is it portable in hadron-hadron interactions ? (next-to-next section)

- Part 3 -

Structure of hadrons

Structure of the proton

- proton in the quark model: **2 up quarks, 1 down quark.**

- the picture seems consistent: up-charge $= +\frac{2}{3}$; down charge $= -\frac{1}{3}$

$$2 \times \frac{2}{3} - 1 \times \frac{1}{3} = +1$$

- but is this **right?** Is it **all?**

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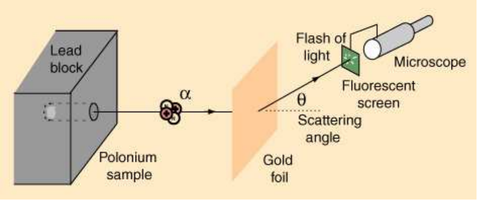
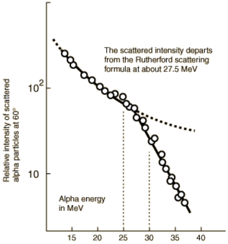
- but is this **right?** Is it **all?**

  - do we see interactions between them or are they free particles?
  - the colour field (gluons) between the quarks fluctuates in $q\bar{q}$ pairs
    $\Rightarrow$ **gluons and sea quarks**
    they carry a small fraction of the proton momentum
    $\Rightarrow$ **small x physics**
    i.e. the study of a colour field inside the proton.
  - How can we study that? $\Rightarrow$ **scattering experiments**

- 1909: Geiger and Marsden and Rutherford : Rutherford scattering - Atomic nucleus



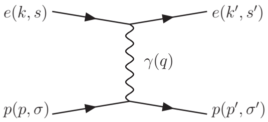


$\langle r_{nucleus} \rangle < 10^{-14} m$

$\Rightarrow$ let see use the same principle but using an elementary particle (e^-) scattered off a p

$$e^- + p \rightarrow e^- + p$$

Mott Scattering: point-like spin 1/2 off point-like spin 1/2



$$\mathcal{M} = \langle k', s' | J_e^\mu | k, s \rangle \frac{g_{\mu\nu}}{q^2} \langle p', \sigma' | J_p^\nu | p, \sigma \rangle$$

$$J_e^\mu = -ie\, \bar{u}(k', s')\, \gamma^\mu\, u(k, s)$$

$$J_p{}^\mu = -ie\, \bar{u}(p', \sigma')\, \gamma^\mu\, u(p, \sigma)$$

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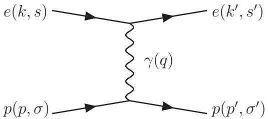

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$$J_p{}^\mu = -ie\, \bar{u}(p', \sigma')\, \gamma^\mu\, u(p, \sigma)$$

$$\overline{|\mathcal{M}|^2} = \frac{1}{4} \sum_{s,s',\sigma,\sigma'} |\mathcal{M}|^2 = \frac{e^4}{q^4} L^{\mu\nu}\, W_{\mu\nu}$$

$$L^{\mu\nu} = \frac{1}{2} \sum_{s,s'} J_e^\mu J_e^\nu$$

$$= 2\left(k'^\mu k^\nu + k^\mu k'^\nu - (k \cdot k' + m_e^2) g^{\mu\nu}\right)$$

$$W^{\mu\nu} = 2\left(p'^\mu p^\nu + p^\mu p'^\nu - (p \cdot p' + m_p^2) g^{\mu\nu}\right)$$

Using $p' = p + k - k'$ and $q^2 = -2\,k \cdot k'$ $(m_e \ll k, k')$ the tensor product gives:

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}\left[\frac{-q^2}{2}(k - k') \cdot p + 2(k \cdot p)(k' \cdot p) + \frac{m_p^2 q^2}{2}\right]$$

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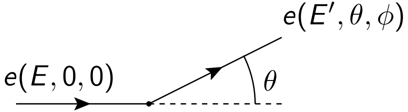
in the lab frame (fixed target - $p = (m_p, 0, 0, 0)$)



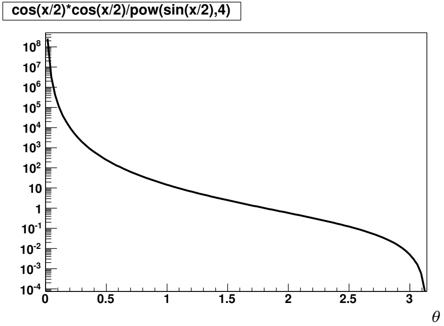
$p \cdot k = m_p\, E, \qquad p \cdot k' = m_p\, E'$
$k \cdot k' = E\, E'\,(1 - \cos\theta)$
and $E - E' = -\frac{q^2}{2m_p}$

$$\overline{|\mathcal{M}|^2} = \frac{16e^4}{q^4} E E' m_p^2 \left(\cos^2(\theta/2) - \frac{q^2}{2m_p^2}\sin^2(\theta/2)\right)$$

$$\Rightarrow \boxed{\frac{d^2\sigma}{dE'\,d\Omega} = \frac{4\alpha^2}{Q^4}E'^2\left[\cos^2(\theta/2) - \frac{q^2}{2m_p^2}\sin^2(\theta/2)\right]\delta(E - E' + \frac{q^2}{2m_p})}$$

Note 1: the first term corresponds to the classic calculation called the Mott scattering cross section:

$$\frac{d\sigma_{Mott}}{d\Omega} = \frac{4\alpha^2}{Q^4} E'^2 \cos^2(\theta/2) = \frac{\alpha^2}{4E^2} \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}$$



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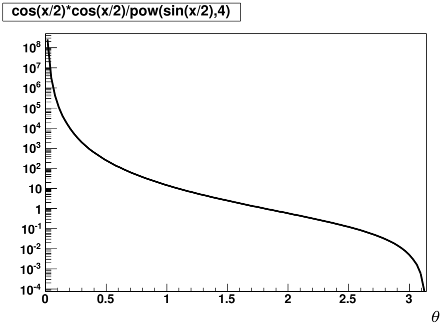


the second term vanishes for $Q^2 \to 0$, i.e. real photons (purely transverse).

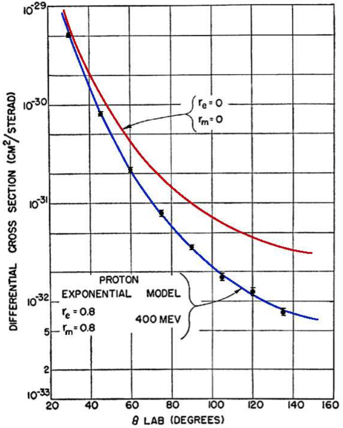
Note 2: in the ultra relativistic limit, using Mandelstam variables:

$$\begin{aligned} s &= (k+p)^2 = 2\,p\cdot k, \\ t &= (k-k')^2 = q^2 = -2\,k\cdot k', \\ u &= (k-p')^2 = (k'-p)^2 = -2\,p\cdot k', \end{aligned}$$

$$\overline{|\mathcal{M}|^2} = 2e^4\,\frac{s^2+u^2}{t^2}$$

Form Factors: elastic point-like spin 1/2 off hadronic target

1953 Hofstadter et al. (SLAC)



$$e^- + N \to e^- + N$$

Deviation w.r.t point-like scattering

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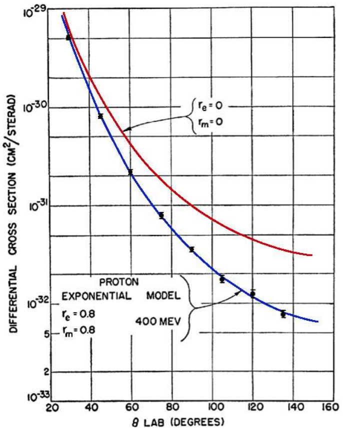

$$e^- + N \to e^- + N$$

Deviation w.r.t point-like scattering

Experimentalist way to introduce the FF:

$$\frac{d\sigma}{d\Omega_e} = \frac{d\sigma_{Mott}}{d\Omega_e}$$

$$x \quad \left[G_E(Q)^2 + \frac{\tau}{\epsilon} G_M(Q)^2 \right] \frac{1}{1+\tau}$$

$\tau = \frac{Q^2}{4mp^2}$ and ϵ : virtual photon polarisation

$\frac{1}{1+\tau} = \frac{Ee}{Ee'}$: recoil factor

$Q^2 = -q^2 = -(k' - k)^2 \simeq 2E\,E'(1 - \cos\theta)$

G_E: Fourier transform of the target electric charge distribution (resp. G_M: distribution of the magnetisation current densities)

$$G_E(Q) = \int \mathrm{d}^3 r\, \rho(r)\, e^{ir\cdot Q} \qquad \text{with} \qquad G_E(0) = 1$$

for small $|q|$

$$G_E(Q) = \int \left(1 + iQ\cdot r - \frac{(Q\cdot r)^2}{2} + \dots\right) \rho(r)\, \mathrm{d}^3 r$$

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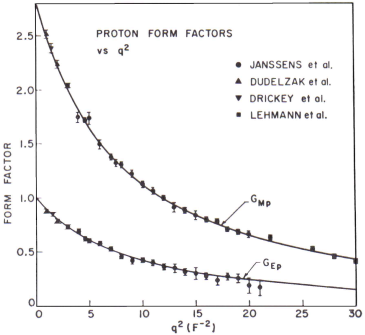
Average squared radius :

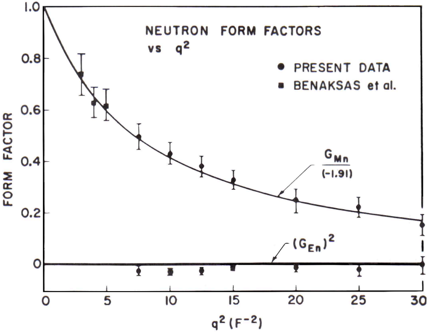
$$\langle r^2 \rangle = \int r^2 \rho(r)\, \mathrm{d}^3 r = 4\pi \int r^2 \rho(r)\, r^2\, \mathrm{d}r$$

Assuming spherical symmetry $\Rightarrow \rho(r) = \rho(-r)$:

$$G_E(Q) \simeq 1 + 0 - \frac{Q^2}{6} \langle r^2 \rangle + \dots .$$

$$\langle r^2 \rangle^{1/2} \simeq 0.74 \pm 0.24\ 10^{-15}\ \mathrm{m}$$

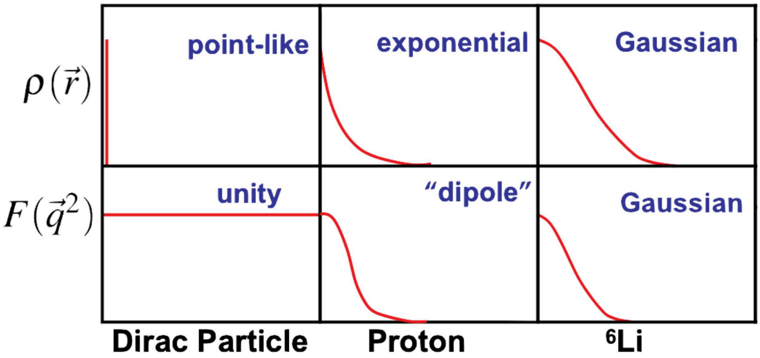




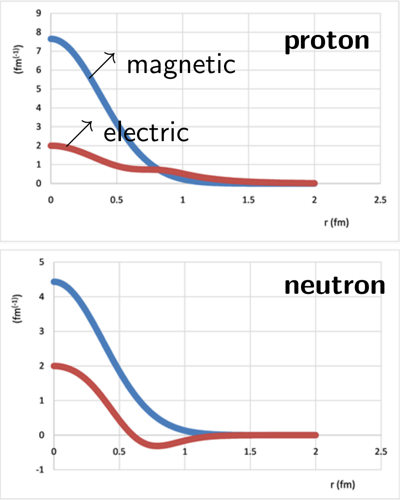
Fitted with a dipole shape:

$$|G_D(Q)| \sim \frac{1}{(1 + Q^2/\mu_0^2)^2}$$

with $\mu_0^2 \simeq 0.71$ GeV^2


$\rho(\vec{r})$
$F(\vec{q}^2)$
point-like
exponential
Gaussian
unity
"dipole"
Gaussian
Dirac Particle
Proton
^{6}Li

Example of recent work: Kurai [2020]



transverse charge distributions as a function of the impact parameter b

$\Rightarrow$ negative charge densities at the neutron center

$\Rightarrow$ contradicts the negative pion cloud surrounding a positively charged core.

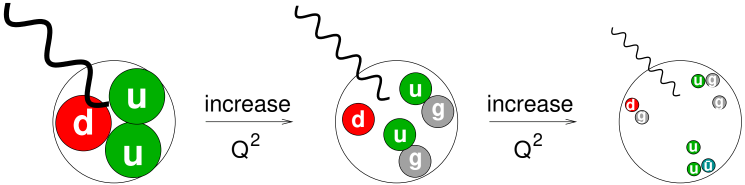
Compton wave length

To resolve small distances (Δr) we need small de Broglie wave length:

$$\lambda\!\!\!^{-} = \lambda/2\pi = \Delta r/2\pi = \frac{\hbar\, c}{p} \quad \Rightarrow \quad \Delta r = \frac{h\, c}{p} \qquad (1)$$

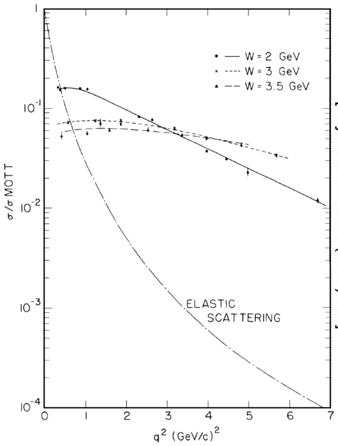
As $E^2 = c^2p^2 + m^2c^4$:

$$\lambda\!\!\!^{-} = \frac{\hbar\, c}{\sqrt{E^2 - m^2c^4}} \overset{E^2 \ll -m^2c^4}{\simeq} \frac{\hbar\, c}{Q}$$

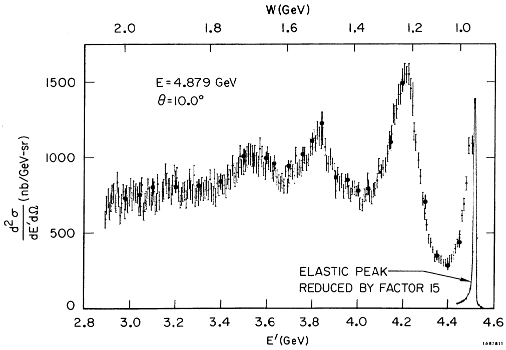


Deep Inelastic Scattering

1967 SLAC - e beam up to 21 GeV



[Phys. Rev. Lett. 23 (1969) 935]



$$W^2 = (q+p)^2 = X^2 = q^2 + 2p\cdot q + m_p^2$$

$$x = \frac{Q^2}{2\,p\cdot q} \Rightarrow W^2 = m_p^2 + \frac{1-x}{x}Q^2$$

- Elastic case: $\delta(E - E' + q^2/2m_p)$

$$W^2 = m_p^2 + \frac{1-x}{x} Q^2 = m_p^2 \qquad \rightarrow \qquad x = 1$$

- deep inelastic case: $x \ll 1$

- in between: resonances

$\Rightarrow$ in the deep inelastic (continuum) region need to parametrise the $\gamma^* - p$ interaction in the most general form

That's all for today