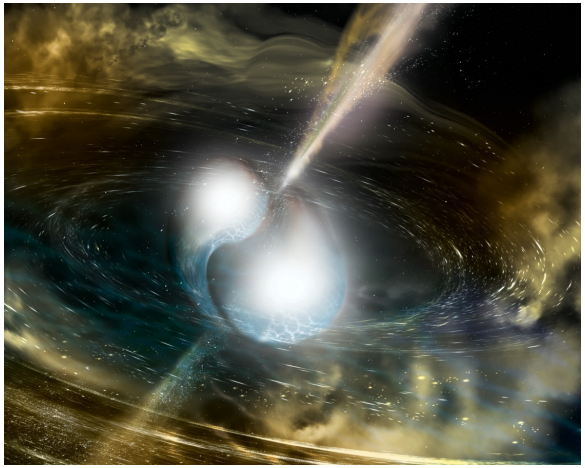

Gravitational waves
Lecture 3: What are gravitational waves?

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Linearized general relativity

- Weak gravitational fields:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

- Restrict coordinate transformations to gauge transformations:

$$x'^{\mu} = x^{\mu} + \xi^{\mu} \qquad \Longrightarrow \qquad h'_{\mu\nu} = h_{\mu\nu} - (\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu})$$

- Define a new tensor

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, h$$

with gauge transformations $\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} - \eta_{\mu\nu}\partial_{\rho}\xi^{\rho})$

- Linearized Einstein equations become

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}$$

where gauge transformations were used to impose the Lorentz gauge

$$\partial^{\mu}\bar{h}_{\mu\nu} = 0$$

Analogy with electromagnetism

- In electromagnetism, the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ in terms of electric potential Φ and magnetic vector potential $\mathbf{A}$:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

- Define the **four-potential**

$$A_\mu = (-\Phi/c, A_x, A_y, A_z)$$

and the **field tensor**

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- One has

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Exercise

Analogy with electromagnetism

- Define **gauge transformations**

$$A'_\mu = A_\mu - \partial_\mu \Lambda$$ for some function $\Lambda(x)$

- **Field tensor** $F_{\mu\nu}$ is invariant under these, because

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\ &= \partial_\mu A_\nu - \partial_\mu \partial_\nu \Lambda - \partial_\nu A_\mu + \partial_\nu \partial_\mu \Lambda \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= F_{\mu\nu} \end{aligned}$$

  - $\mathbf{E}$ and $\mathbf{B}$ preserved: gauge transformations don't change the physics!

Analogy with electromagnetism

- Define **gauge transformations**

$$A'_\mu = A_\mu - \partial_\mu \Lambda \qquad \text{for some function } \Lambda(x)$$

- Field tensor $F_{\mu\nu}$ is invariant under these, because

$$F'_{\mu\nu} = F_{\mu\nu}$$

  - $\mathbf{E}$ and $\mathbf{B}$ preserved: gauge transformations don't change the physics!

- Can always use a gauge transformations to impose the **Lorentz gauge**

$$\partial^\mu A'_\mu = 0$$

  - Indeed, given some four-potential A_μ ,

$$\partial^\mu A'_\mu = \partial^\mu (A_\mu - \partial_\mu \Lambda) = 0 \iff \partial^\mu \partial_\mu \Lambda = \partial^\mu A_\mu \text{ , or } \Box \Lambda(t, \mathbf{x}) = \partial^\mu A_\mu(t, \mathbf{x})$$

Writing $f(t, \mathbf{x}) \equiv \partial^\mu A_\mu(t, \mathbf{x})$, the equation we need to solve is

$$\Box \Lambda(t, \mathbf{x}) = f(t, \mathbf{x})$$

Using the Green's function of the d'Alembertian, $\mathcal{G}(t, \mathbf{x})$:

$$\Lambda(t, \mathbf{x}) = \int dt'\, d^3\mathbf{x}'\, f(t, \mathbf{x}')\, \mathcal{G}(t - t', \mathbf{x} - \mathbf{x}')$$

Analogy with electromagnetism

- Define **gauge transformations**

$$A'_\mu = A_\mu - \partial_\mu \Lambda \qquad \text{for some function } \Lambda(x)$$

- Field tensor $F_{\mu\nu}$ is invariant under these, because

$$F'_{\mu\nu} = F_{\mu\nu}$$

  - $\mathbf{E}$ and $\mathbf{B}$ preserved: gauge transformations don't change the physics!

- Can always use a gauge transformations to impose the **Lorentz gauge** $\partial^\mu A'_\mu = 0$ in which case the Maxwell equations lead to

$$\Box A_\mu = -\mu_0 J_\mu$$

with $J_\mu = (-\rho\, c, j_x, j_y, j_z)$

ρ charge density

$\mathbf{j} = (j_x, j_y, j_z)$ current density

Linearized gravity and electromagnetism

Linearized gravity

- Gauge transformations

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho)$$

- Lorentz gauge

$$\partial^\mu \bar{h}_{\mu\nu} = 0$$

- Field equations

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- General solution (Green's function)

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int_{\mathcal{V}} d^3\mathbf{x}' \, \frac{T_{\mu\nu}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right)}{|\mathbf{x} - \mathbf{x}'|}$$

Electromagnetism

- Gauge transformations

$$A'_\mu = A_\mu - \partial_\mu \Lambda$$

- Lorentz gauge

$$\partial^\mu A_\mu = 0$$

- Field equations

$$\Box A_\mu = -\mu_0 \, J_\mu$$

- General solution (same Green's function!)

$$A_\mu(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d^3\mathbf{x}' \, \frac{J_\mu\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right)}{|\mathbf{x} - \mathbf{x}'|}$$

No instantaneous action at a distance!

- Gravitational waves

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right) \bar{h}_{\mu\nu} = 0$$

- Electromagnetic waves

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right) A_\mu = 0$$

How many polarizations for gravitational waves?

- Even after imposing the Lorentz gauge, there is residual gauge freedom!
  - Lorentz gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$ is not spoiled by transformations that satisfy $\Box \xi_\mu = 0$
  - In the vacuum case (field equations $\Box \bar{h}_{\mu\nu} = 0$), this can be used to impose **four more conditions**:

$$\boxed{\bar{h}_{0\mu} = 0} \qquad \text{and} \qquad \boxed{\bar{h}^{\mu}{}_{\mu} = 0}$$

... so that

$$\boxed{\partial^i \bar{h}_{ij} = 0} \qquad \text{and} \qquad \boxed{\bar{h}^{i}{}_{i} = 0}$$

(and also $\bar{h}_{ij} = h_{ij} - \frac{1}{2}\eta_{ij} h = h_{ij}$)

- Plane gravitational wave traveling in the z direction: $h_{ij} = A_{ij} \cos\left[\omega(t - z/c)\right]$
- Then $\partial^i h_{ij} = \partial^z h_{zj} = (\omega/c)\, A_{zj} \sin\left[\omega(t - z/c)\right] = 0 \implies h_{zj} = 0$, and traceless

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\left[\omega(t - z/c)\right]$$

How do gravitational waves manifest themselves?

- Gravitational waves in the transverse-traceless gauge:

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\left[\omega(t - z/c)\right]$$

- Full spacetime metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{\mathrm{TT}}$:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + h_+ \cos[\omega(t - z/c)] & h_\times \cos[\omega(t - z/c)] & 0 \\ 0 & h_\times \cos[\omega(t - z/c)] & 1 - h_+ \cos[\omega(t - z/c)] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$:

$$\begin{aligned} ds^2 \;=\; & -c^2 dt^2 + \left(1 + h_+ \cos[\omega(t - z/c)]\right) dx^2 + \left(1 - h_+ \cos[\omega(t - z/c)]\right) dy^2 \\ & + \; 2h_\times \cos[\omega(t - z/c)]\, dx dy \;+\; dz^2 \end{aligned}$$

How do gravitational waves manifest themselves?

- If only h_+ :

$$ds^2 = -c^2dt^2 + (1 + h_+ \cos[\omega(t - z/c)])\ dx^2 + (1 - h_+ \cos[\omega(t - z/c)])\ dy^2 + dz^2$$

- Consider neighboring points $(ct, x_0, 0, 0)$, $(ct, x_0 + dx, 0, 0)$
  Then $dx^\mu = (0, dx, 0, 0)$, hence **physical distance**

$$\sqrt{ds^2} = \sqrt{1 + h_+ \cos[\omega(t - z/c)]}dx \simeq \left(1 + \frac{1}{2}h_+ \cos[\omega(t - z/c)]\right) dx$$

- Similarly for separation in the y direction, but with sign difference:

$$\sqrt{ds^2} = \sqrt{1 - h_+ \cos[\omega(t - z/c)]}dy \simeq \left(1 - \frac{1}{2}h_+ \cos[\omega(t - z/c)]\right) dy$$

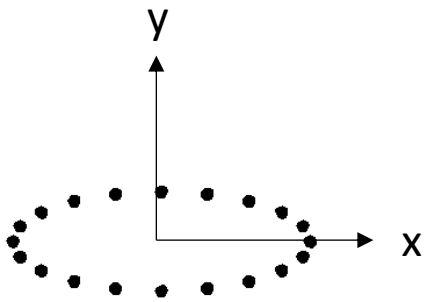
How do gravitational waves manifest themselves?

- "plus" polarization (only h_+):

$$ds^2 = -c^2dt^2 + (1 + h_+ \cos[\omega(t - z/c)])\ dx^2 + (1 - h_+ \cos[\omega(t - z/c)])\ dy^2 + dz^2$$

Physical distances between neighboring particles in free fall:

- x direction: $\sqrt{ds^2} \simeq \left(1 + \frac{1}{2}h_+ \cos[\omega(t - z/c)]\right) dx$
- y direction: $\sqrt{ds^2} \simeq \left(1 - \frac{1}{2}h_+ \cos[\omega(t - z/c)]\right) dy$



- "cross" polarization (only $h_\times$):

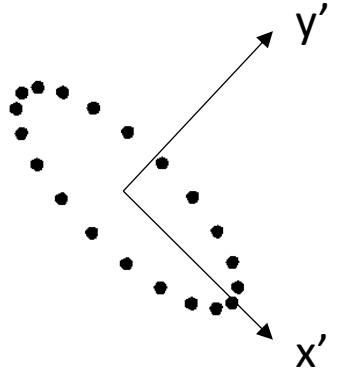
$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + 2h_\times \cos[\omega(t - z/c)]\, dxdy + dz^2$$

Easier to interpret if x, y axes rotated over 45^0 to x', y' axes:

$$ds^2 = -c^2dt^2 + (1 + h_\times \cos[\omega(t - z/c)])\ dx'^2 + (1 - h_\times \cos[\omega(t - z/c)])\ dy'^2 + dz^2$$

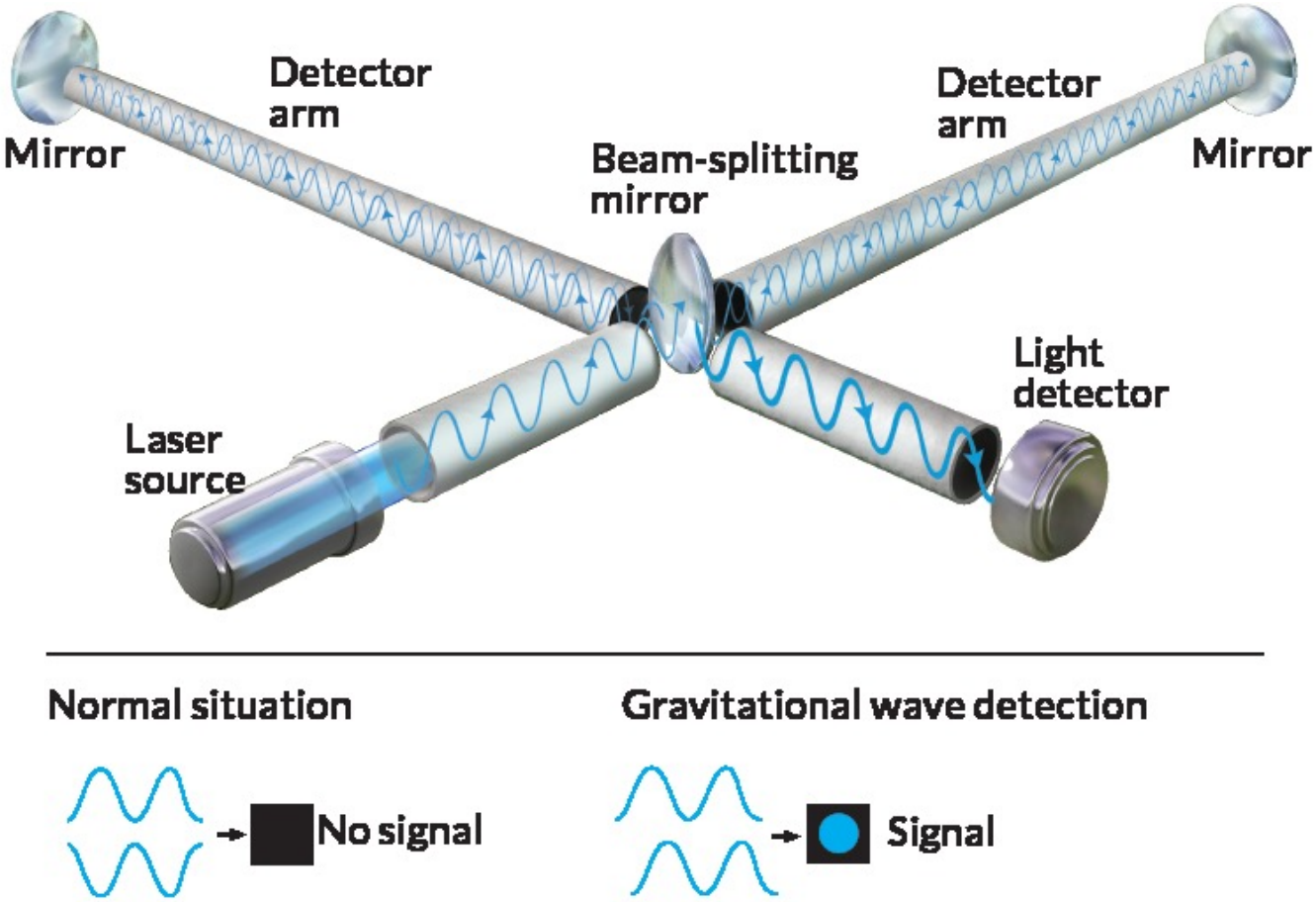
After 45^0 rotation, same form as "plus" polarization, but with

$$h_+ \longrightarrow h_\times$$



Exercise

These effects are exploited by laser interferometers



Response of interferometers to h_+ , $h_\times$ will be investigated in lecture 5

How are gravitational waves generated?

- Linearized Einstein equations

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- Given an energy-momentum tensor $T_{\mu\nu}$, solution is

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int_{\mathcal{V}} d^3\mathbf{x}' \, \frac{T_{\mu\nu}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right)}{|\mathbf{x} - \mathbf{x}'|}$$

- Outside the source we want to be in the transverse-traceless gauge!
  It turns out that there exists a projection operator that takes us there…

  - Define $P_{ij} = \delta_{ij} - n_i n_j$ with n^i unit vector in direction of propagation
    Then
    $n^i P_{ij} = 0$ and $P_i{}^k P_{kj} = P_{ij}$
  - Define $\Lambda_{ij;kl} = P_{ik}P_{jl} - \frac{1}{2} P_{ij}P_{kl}$
    Then
    $n^i \Lambda_{ij;kl} = 0$ and $\Lambda^i{}_{i;kl} = 0$

Hence $h_{ij}^{\mathrm{TT}} = \Lambda_{ij;kl}\, h^{kl}$ indeed satisfies $n^i h_{ij}^{\mathrm{TT}} = 0$, $h^{\mathrm{TT}i}_{\ \ i} = 0$

transverse **traceless**

How are gravitational waves generated?

- Gravitational waves in transverse-traceless gauge:

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{4G}{c^4}\,\Lambda_{ij;kl}(\hat{\mathbf{n}}) \int_{\mathcal{V}} d^3\mathbf{x}'\,\frac{T^{kl}(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

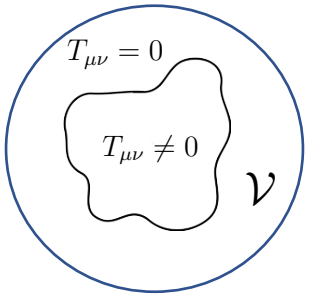
- Far from the source: $|\mathbf{x}-\mathbf{x}'| \simeq r$

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\,\Lambda_{ij;kl}(\hat{\mathbf{n}}) \int_{\mathcal{V}} d^3\mathbf{x}'\,T^{kl}\left(t-\frac{r}{c},\mathbf{x}'\right)$$

- Can we express this in terms of energy density rather than stress tensor?

From stress tensor to energy density

- Assume integration volume $\mathcal{V}$ taken sufficiently large, such that $T_{\mu\nu} = 0$ on the boundary (e.g. very large sphere containing the source)



- Then

$$\begin{aligned}
\int_{\mathcal{V}} d\mathbf{x}\, T^{ij} &= \int_{\mathcal{V}} d\mathbf{x}\, T^{kj}\, \delta^i_k \\
&= \int_{\mathcal{V}} d\mathbf{x}\, T^{kj}\, \partial_k x^i \\
&= -\int_{\mathcal{V}} d\mathbf{x}\, \partial_k T^{kj}\, x^i \qquad \text{and no boundary term because } T^{kj} = 0 \text{ at boundary} \\
&= \int_{\mathcal{V}} d\mathbf{x}\, \partial_0 T^{0j}\, x^i \qquad \text{because } \partial_\mu T^{\mu j} = \partial_0 T^{0j} + \partial_k T^{kj} = 0 \\
&= \int_{\mathcal{V}} d\mathbf{x}\, \partial_0 T^{0l}\, \delta^j_l\, x^i \\
&= \int_{\mathcal{V}} d\mathbf{x}\, \partial_0 T^{0l}\, (\partial_l x^j)\, x^i \\
&= -\int_{\mathcal{V}} d\mathbf{x}\, \partial_l \partial_0 T^{0l}\, x^j\, x^i - \int_{\mathcal{V}} d\mathbf{x}\, \partial_0 T^{0l}\, x^j\, \delta^i_l \\
&= \int_{\mathcal{V}} d\mathbf{x}\, \partial_0 \partial_0 T^{00}\, x^i\, x^j + \int_{\mathcal{V}} d\mathbf{x}\, \partial_k T^{kl}\, x^j\, \delta^i_l \\
&= \frac{1}{c^2} \int_{\mathcal{V}} d\mathbf{x}\, \ddot{T}^{00}\, x^i x^j - \int_{\mathcal{V}} d\mathbf{x}\, T^{ij} \qquad \text{because } \partial_0 = \frac{1}{c}\frac{\partial}{\partial t} \text{, and dot denotes } \frac{\partial}{\partial t}
\end{aligned}$$

- Hence $\int_{\mathcal{V}} d^3\mathbf{x}\, T^{ij} = \frac{1}{2}\ddot{M}^{ij}$ with $M^{ij} \equiv \frac{1}{c^2}\int_{\mathcal{V}} d^3\mathbf{x}\, T^{00} x^i x^j$ the **mass quadrupole moment**

How are gravitational waves generated?

- Gravitational waves in transverse-traceless gauge:

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{4G}{c^4}\,\Lambda_{ij;kl}(\hat{\mathbf{n}}) \int_{\mathcal{V}} d^3\mathbf{x}'\,\frac{T^{kl}(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

- Far from the source: $|\mathbf{x}-\mathbf{x}'| \simeq r$

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\,\Lambda_{ij;kl}(\hat{\mathbf{n}}) \int_{\mathcal{V}} d^3\mathbf{x}'\,T^{kl}\left(t-\frac{r}{c},\mathbf{x}'\right)$$

- In terms of the **mass quadrupole moment**:

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{2G}{c^4}\,\Lambda_{ij;kl}(\hat{\mathbf{n}})\,\ddot{M}^{kl}(t-r/c)$$

where $M^{ij} \equiv \frac{1}{c^2}\int_{\mathcal{V}} d^3\mathbf{x}\,T^{00}x^i x^j$

This is called the **quadrupole formula**

How are gravitational waves generated?

- In terms of the **mass quadrupole moment**:

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{2G}{c^4}\,\Lambda_{ij;kl}(\hat{\mathbf{n}})\,\ddot{M}^{kl}(t-r/c)$$

- Without loss of generality, let the z axis point towards the observer so that it is the direction of propagation

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- With $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, the right hand side becomes

$$\frac{1}{r}\frac{2G}{c^4}\begin{pmatrix} \frac{\ddot{M}^{11}-\ddot{M}^{22}}{2} & \ddot{M}^{12} & 0 \\ \ddot{M}^{12} & -\frac{\ddot{M}^{11}-\ddot{M}^{22}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}\Bigg|_{t-r/c}$$

$$\left.\begin{aligned} h_+ &= \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}^{11}-\ddot{M}^{22}\right) \\ h_\times &= \frac{2}{r}\frac{G}{c^4}\,\ddot{M}^{12} \end{aligned}\right\}$$

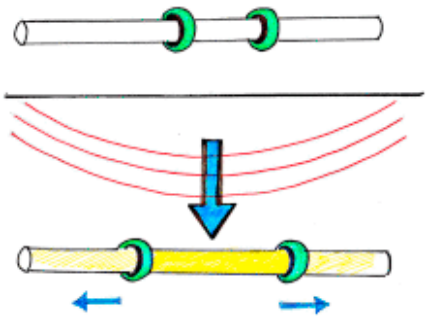
evaluated at the "retarded time" $t-r/c$

Exercise

Energy carried by gravitational waves

- Throughout Einstein's life, reality of gravitational waves was regularly put in doubt
  - By Einstein himself, in 1936! Erroneous calculation seemingly showed that gravitational waves are an artefact of linearization, not present in the full theory
  - What if it could be shown that gravitational waves carried energy?

- Feynman's "sticky bead" argument:
  - Beads on a stick experience friction, hence heat
  - Energy must come from the gravitational wave
  - Gravitational waves carry energy!



- Going from linear order to quadratic order in $h_{\mu\nu}$:
  - Energy in gravitational waves itself causes spacetime curvature
  - Can even derive an energy-momentum tensor for them:

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} \rangle$$

Energy carried by gravitational waves

- In electromagnetism: **Poynting flux**

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

- Electromagnetic waves, far from the source: $\mathbf{E} \perp \mathbf{B}$ and $|\mathbf{B}| = |\mathbf{E}|/c$

$$|\mathbf{S}| \propto |\mathbf{E}|^2 = E_i\, E^i$$

- Convenient to average the flux over a wave period:

$$\mathcal{F} \propto \langle E_i\, E^i \rangle$$

- In terms of the field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$E_i = c\, F_{i0} = c\,(\partial_i A_0 - \partial_0 A_i) = -c\,\partial_0 A_i$$

- This suggests energy flux for gravitational waves:

$$\mathcal{F}_{\rm GW} \propto \langle \partial_0 h_{ij}^{\rm TT} \partial_0 h_{ij}^{\rm TT} \rangle \propto \langle \dot{h}_{ij}^{\rm TT} \dot{h}_{ij}^{\rm TT} \rangle$$

- Detailed calculation gives the proportionality factor:

$$\boxed{\mathcal{F}_{\rm GW} = \frac{c^3}{32\pi G} \langle \dot{h}_{ij}^{\rm TT} \dot{h}_{ij}^{\rm TT} \rangle}$$

Energy carried by gravitational waves

- Gravitational energy flux:

$$\mathcal{F}_{\rm GW} = \frac{c^3}{32\pi G}\langle \dot{h}_{ij}^{\rm TT}\dot{h}_{ij}^{\rm TT}\rangle$$

- Energy per unit time passing through a sphere of radius r :

$$\begin{aligned}\frac{dE}{dt} &= \int_{\mathcal{S}} dA\, \mathcal{F}_{\rm GW} \\ &= \frac{c^3 r^2}{32\pi G}\int_0^{\pi}\sin(\theta)\,d\theta\int_0^{2\pi} d\phi\,\langle \dot{h}_{ij}^{\rm TT}\dot{h}_{ij}^{\rm TT}\rangle \\ &= \frac{c^3 r^2}{16\pi G}\int_0^{\pi}\sin(\theta)\,d\theta\int_0^{2\pi} d\phi\,\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle\end{aligned}$$

$$\dot{h}_{ij}^{\rm TT} = \begin{pmatrix} \dot{h}_+ & \dot{h}_\times & 0 \\ \dot{h}_\times & -\dot{h}_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

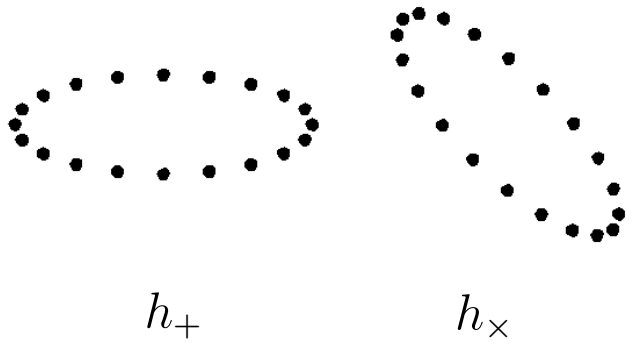
Hermann Bondi
"Gravitational waves are real, you could use them to boil water!"

Summary

- Linearized gravity is like electromagnetism, but with twice the indices!

- Gravitational waves in the transverse-traceless gauge:

$$h_{ij}^{\rm TT} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\left[\omega(t - z/c)\right]$$



- Quadrupole formula describing how they are generated:

$$h_{ij}^{\rm TT}(t, \mathbf{x}) = \frac{1}{r}\frac{2G}{c^4}\,\Lambda_{ij;kl}(\hat{\mathbf{n}})\,\ddot{M}^{kl}(t - r/c)$$

where

$$M^{ij} \equiv \frac{1}{c^2}\int_{\mathcal{V}} d^3\mathbf{x}\, T^{00} x^i x^j$$

- Gravitational wave energy flux:

$$\mathcal{F}_{\rm GW} = \frac{c^3}{32\pi G}\langle \dot{h}_{ij}^{\rm TT}\dot{h}_{ij}^{\rm TT}\rangle$$