

The squashed seven-sphere and the swampland

Bengt E.W. Nilsson

Chalmers University of Technology, Göteborg

Talk at the Bert-fest May 13, 2022

Some historic referencies:

- Salam and Strathdee, "On Kaluza-Klein theory" (1982)
- Breitenlohner and Freedman, "Positive energy in AdS backgrounds...", (PLB 1982)
- BN and Pope, "Scalar and Dirac eigenfunctions on the squashed seven-sphere" (PLB, 1983)
- Bais, Nicolai and van Nieuwenhuizen, "Geometry of coset spaces and massless modes on the squashed S^7 " (NPB, 1983)
- Duff, BN and Pope, "The criterion for vacuum stability in Kaluza-Klein supergravity" (PLB, 1984)
- Duff, BN and Pope, "Kaluza-Klein supergravity" (Phys Rep, 1986)

Recent relevant work

Issue: The exact operator spectrum on the squashed seven-sphere

Recent progress:

- Master theses by Aspman, Padellaro, Ekhammar and Karlsson (Chalmers)
- BN, Padellaro and Pope, "The role of singletons ...", (hep-th 1811.06228, in JHEP)
- Ekhammar and BN, "On the squashed S^7 operator spectra" (hep-th 2105.05229, in JHEP)
- Karlsson and BN, in preparation

Other works related to the AdS stability question

- Vafa and Ooguri, "Non-supersymmetric AdS and the swampland" (hep-th 1610.01533)
- Witten: "Instability of the KK vacuum" (NPB 1982)
- Berkooz and Rey: "Non-susy stable vacua in M-theory" (hep-th 9807200)
- Murugan: Thesis (Princeton) (hep-th 1610.03166):
Issues in it pointed out to me by Mike Duff

Introduction

The AdS swampland conjecture

- We will discuss aspects of the **right-squashed** S^7 with no susy aiming at a determination of its stability properties.
- It is known since the 1980s that it is BF stable since it is a skew-whiffed version of the right-squashed S^7 with $\mathcal{N} = 1$ susy (Duff, BN and Pope, 1984) :
 - Skew-whiffing (i.e., orientation reversal) does not alter the spectrum of 2nd order operators which eliminates the only problematic issue connected to the Lichnerowicz operator.

Are there other potentially fully stable non-supersymmetric vacua contradicting the AdS conjecture?

- Tomasiello et al (see 2112.10795): Using "fake susy" to establish stability but seems not to be applicable to the squashed S^7
- Samtleben et al (see 2112.11966): Susy breaking deformations in S-folds might avoid all known decay modes (??)

Motivation

There has appeared over the last years several proposals of stable non-supersymmetric vacua in various supergravity theories:

- Some such vacua were discovered as extrema of gauged supergravity theories in four or five dimensions but almost all of them were later found to be unstable by uplifting to ten or eleven dimensions where BF-unstable higher modes appear (see Brane-jet analysis by Warner et al and Nicolai et al)
- One case that still seems to be up in the air was discussed recently by Giambrone et al (hep-th 2112.11966). It uses the existence of two supersymmetry breaking deformation parameters that are locally but not globally possible to eliminate by coordinate transformations. This feature might make this vacuum stable against all known decay modes according to the authors.

Motivation

Motivation for looking at the squashed S^7 case

- The squashed S^7 enjoys the property of space invaders which means that it is not a solution of any supergravity theory in AdS_4 and thus not an up-lift of a gauged supergravity vacuum. The issue of stability here is therefore entirely different from the previous ones.

Is it possible to analyse the stability of left-squashed S^7 with the present knowledge of its spectrum?

- Non-trivial decay modes: (Berkooz-Ray, Murugan): Tadpoles in the bulk or exactly marginal operators in the field theory require detailed understanding of the spectrum (looking for dangerous marginal single trace or multi-trace operators)
- The best we can do is to analyse the decay modes we do have knowledge about.
- Below we will review the present situation!

The squashed seven sphere geometry

Basics

- $ds^2 = d\mu^2 + \frac{1}{4} \sin^2 \mu \omega_i^2 + \frac{1}{4} \lambda^2 (\nu_i + \cos \mu \omega_i)^2$,
 - where $\nu_i = \sigma_i + \Sigma_i$ and $\omega_i = \sigma_i - \Sigma_i$
(here σ_i, Σ_i are two $SU(2)$ 1-forms)
 - Einstein for two values of the squashing parameter λ :
Round S^7 : $\lambda = 1$, squashed S^7 : $\lambda = 1/\sqrt{5}$
 - this metric can be derived from a distance sphere in the projective quaternionic 2-plane HP^2 .
- Riemann tensor in terms of structure constants of G/H

$$R_{cda}{}^b = f_{cd}{}^i f_{ia}{}^b + \frac{1}{2} f_{cd}{}^e f_{ea}{}^b + \frac{1}{2} f_{[c|a|}{}^e f_{d]}{}^e{}^b$$
 - where $f_{abc} = -\frac{1}{\sqrt{5}} a_{abc}$, in terms of octonionic structure constants.
 - The integrability condition for the existence of Killing spinors is given by the Weyl tensor in terms of the holonomy group G_2
 - Since 8 of $SO(7)$ goes to $1 \oplus 7$ under G_2 we find that there can be one supersymmetry: left-squashing
 - Skew-whiffing (orientation reversal) eliminates this Killing spinor in the right-squashed case

The coset master equation: Basics

Linearizing and diagonalizing the M-theory supergravity equations gives the masses of the AdS_4 fields in terms of operators on the seven-spheres: The eigenmodes are denoted Y (indices suppressed) On a coset G/H (with coordinates y) a unique G element $L^{-1}(y)$ can be associated to the Fourier modes Y by

$$Y(y) \equiv L^{-1}(y) \rightarrow hL^{-1}(y)g^{-1} = hY(y)g^{-1} \quad (1)$$

Then $L^{-1}dL = e^a T_a + \Omega^{\bar{a}} T_{\bar{a}} \Rightarrow$ the coset master equation

$$\nabla_a Y + \frac{1}{2} f^{ab}{}_c \Sigma_{bc} Y = T_a Y \quad (2)$$

where T_a are generators in G not in H and in our case

$$G/H = Sp(2) \times Sp(1)^C / Sp(1)^A \times Sp(1)^{B+C} \quad (3)$$

- ∇_a is $SO(7)$ covariant and Σ_{ab} are $SO(7)$ generators
- the left-hand side is an H -covariant derivative: define $\hat{\nabla}_a = \partial_a + \Omega_a^{\bar{b}} T_{\bar{b}} = \nabla_a Y + \frac{1}{2} f^{ab}{}_c \Sigma_{bc} Y$

The coset master equation: Applications

The splitting of tangent space irreps into H-irreps goes via G_2 :

- $SO(7) \rightarrow G_2 \rightarrow H \Rightarrow \hat{\nabla}_a a_{bcd} = \hat{\nabla}_a c_{bcde} = 0$
 - Ex: 2-forms harmonics split as
 $21 \rightarrow 7 \oplus 14 \rightarrow ((1, 1) \oplus (0, 2)) \oplus ((0, 2) \oplus (1, 3) \oplus (2, 0))$
- Octonionic structure constants also useful in writing out d=7 gamma matrices: $(\Gamma^a)_{bc} = a_{abc}$, $(\Gamma^a)_{b8} = \delta_{ab}$

In the following we will utilise the G_2 features as much as possible (Ekhammar-BN):

- Eigenvalues of Δ_0 : squaring coset master eq gives
 $\Delta_0 Y = -\square Y = T_a T_a Y = (C_G - C_H)Y = C_G Y = \kappa_0^2 Y$
- Eigenvalues of Δ_1 : $\Delta_1 Y_a = -\square Y_a + 6m^2 Y_a = \kappa_1^2 Y_a$
- However, squaring the coset master eq gives a ∇Y -term that requires a further squaring

$$-\square Y_a - \frac{1}{\sqrt{5}} a_{abc} \nabla_b Y_c + \frac{3}{10} Y_a = -(T_a T_a Y)_a. \quad (4)$$

The coset master equation: Eigenvalues of Δ_1

- Eliminating the \square term and using $C_H = \frac{12}{5}$ for the two H-pieces irreps in the SO(7) irrep **7** we get

$$a_{abc} \nabla_b Y_c = \sqrt{5}(C_G - \kappa^2)Y_a \quad (5)$$

which needs to be squared again.

- Setting $DY_a = a_{abc} \nabla_b Y_c \Rightarrow$

$$D^2 Y_a = \frac{6}{\sqrt{5}} DY_a + \kappa_1^2 \Rightarrow 5(\kappa_1^2 - C_G) = 6(\kappa_1^2 - C_G) + \kappa_1^2 \quad (6)$$

$$\Rightarrow \kappa_1^2 = C_G + \frac{7}{10} + \sqrt{\frac{1}{5}C_G + \frac{49}{100}} \quad (7)$$

Introducing the dimensionfull constant m from the Freund-Rubin ansatz by $m^2 = 9/20$ this can be written

$$\Delta_1 = \frac{m^2}{9} (\sqrt{20C_G + 49} \pm 1)^2 - 4m^2 \quad (8)$$

The coset master equation: Eigenvalues of Δ_1

This result is nice since inserting it into the 1-form mass formula

$$M^2(1_{1,2}^-) = (\sqrt{\Delta_1 + 4m^2} \pm 3m^2)^2 - m^2 \quad (9)$$

gives the SO(3,2) irrep $(E_0, spin^{parity})$ value (in units of m^2)

$$E_0(1^-) = \frac{3}{2} + \frac{1}{2}\sqrt{M^2 + 1} = \dots \quad (10)$$

This set of E_0 values fit with supersymmetry and the number of cross diagrams derived in Padellaro, BN and Pope (2019) and the degeneracy of G-irreps (max 2 here).

The coset master equation: Application to 2-forms

- Eigenvalues of Δ_2 : Repeating the steps above leads to symmetric products of covariant derivatives! (\Rightarrow hard to see what to do).
- Better approach is to use the G_2 properties: Eliminating the \square term gives now

$$\begin{aligned}
 & (\Delta_2 - 12m^2 - (C_G - C_H))Y_{ab} + W_{abcd}Y^{cd}Y_{ab} \\
 & = -\frac{4}{3}m^2(P_{14}Y)_{ab} + \frac{4}{3}m(a_{cd}[\hat{\nabla}_{|c}Y_{d|b}]). \quad (11)
 \end{aligned}$$

- The Weyl tensor term: A bit tricky to obtain eigenvalues of the Weyl tensor but possible.
- An important feature: the sum $C_H + \text{Weyl}$ eigenvalues respect G_2 although H-irreps have different values for both C_H and Weyl.
- This is needed to keep the G_2 structure of the equations (Ekhammar-BN, explained in Karlsson-BN (to appear)).

The coset master equation: Application to 2-forms

- Now one can decompose the whole eigenvalue equation into its G_2 pieces: set $Y_{ab}^{(14)} = (P_{14}Y)_{ab}$ and $Y_a = a_{abc}(P_7Y)_{bc}$

$$\mathbf{7} : \kappa_2^2 Y_a = \frac{m^2}{9} (20C_G + 72) Y_a - \frac{4m}{3} (a_{abc} \nabla_b Y_c) \quad (12)$$

$$\mathbf{14} : \kappa_2^2 Y_{ab}^{(14)} = \frac{m^2}{9} 20C_G Y_{ab}^{(14)} + \frac{2m}{3} (P_{14} \nabla Y)_{ab} \quad (13)$$

These two equations mix but can be solved together giving

- Four towers of eigenvalues (Ekhammar-BN (2021))

$$\Delta_2 = \frac{m^2}{9} 20C_G \quad (14)$$

$$\Delta_2 = \frac{m^2}{9} (20C_G + 72) \quad (15)$$

$$\Delta_2 = \frac{m^2}{9} (\sqrt{20C_G + 49} \pm 2)^2 - m^2 \quad (16)$$

Application: 2-forms and beyond

- The mass operator for these 2-form modes is related to spin 1^+ vector fields in AdS_4 via

$$M^2(1^+) = \Delta_2 \quad (17)$$

which gives $SO(3, 2)$ irrep values (in units of m^2)

$$E_0(1^+) = \frac{3}{2} + \frac{1}{2} \sqrt{\Delta_2/m^2 + 1}. \quad (18)$$

- again we see that a double square roots are avoided.
- a funny feature is that the number under the square root is always the square of an odd integer.

Recent results

All remaining operators on the squashed S^7 can be treated with these methods (with quite a bit of algebra):

- The representation spectrum in AdS_4 was derived in 2019 (BN, Padellaro and Pope):
 - There is a fermionic singleton in the right-squashed spectrum but not in the left-squashed one
 - There is a Higgsing/de-Higgsing going on (involving singletons) when the round S^7 is broken to the squashed ones
 - The first example of this was the "space invaders scenario" (Duff, BN and Pope, 1983) =>
 - The squashed sphere vacua are not part of the truncated $\mathcal{N} = 8$ gauged supergravity theory in AdS_4
- The operator spectra of Δ_L , $Q \equiv \star d$ were found recently (Ekhammar-BN, 2021) and of $\mathcal{D}_{3/2}$ (Karlsson-BN, to appear)
- Much more efficient group theory formulas will appear soon (Karlsson-BN, to appear)

Susy and skew-whiffing: Summary of the present situation

The left-squashed case with $\mathcal{N} = 1$:

- All derived eigenvalues fit into Heidenreich type $\mathcal{N} = 1$ supermultiplets

Effects of skew-whiffing to the right-squashed case with no susy

- Bosonic operators quadratic in derivatives are not effected
- Fermionic operators and $Q \equiv \star d$ acting on 3-forms are linear in derivatives =>
positive and negative parts of the spectra are interchanged
- => BF-stable for both orientations!

Conclusions

Conclusions

The degeneration pattern in the cross-diagrams (BN, Padellaro, Pope) indicates that there are still some missing eigenvalues corresponding to a small set of $\mathcal{N} = 1$ susy irreps (Karlsson-BN, work in progress)

- \Rightarrow Not yet possible to rule out tadpoles dual to dangerous marginal single trace or multi-trace operators after skew-whiffing
 - even if these were to occur a full analysis requires adding $1/N$ corrections
- Concerning Witten's "bubbles of nothing" instability: No conclusions yet either way

Thus more work is required to settle the issue of stability of the non-susy squashed S^7 vacuum in relation to presently known decay modes.

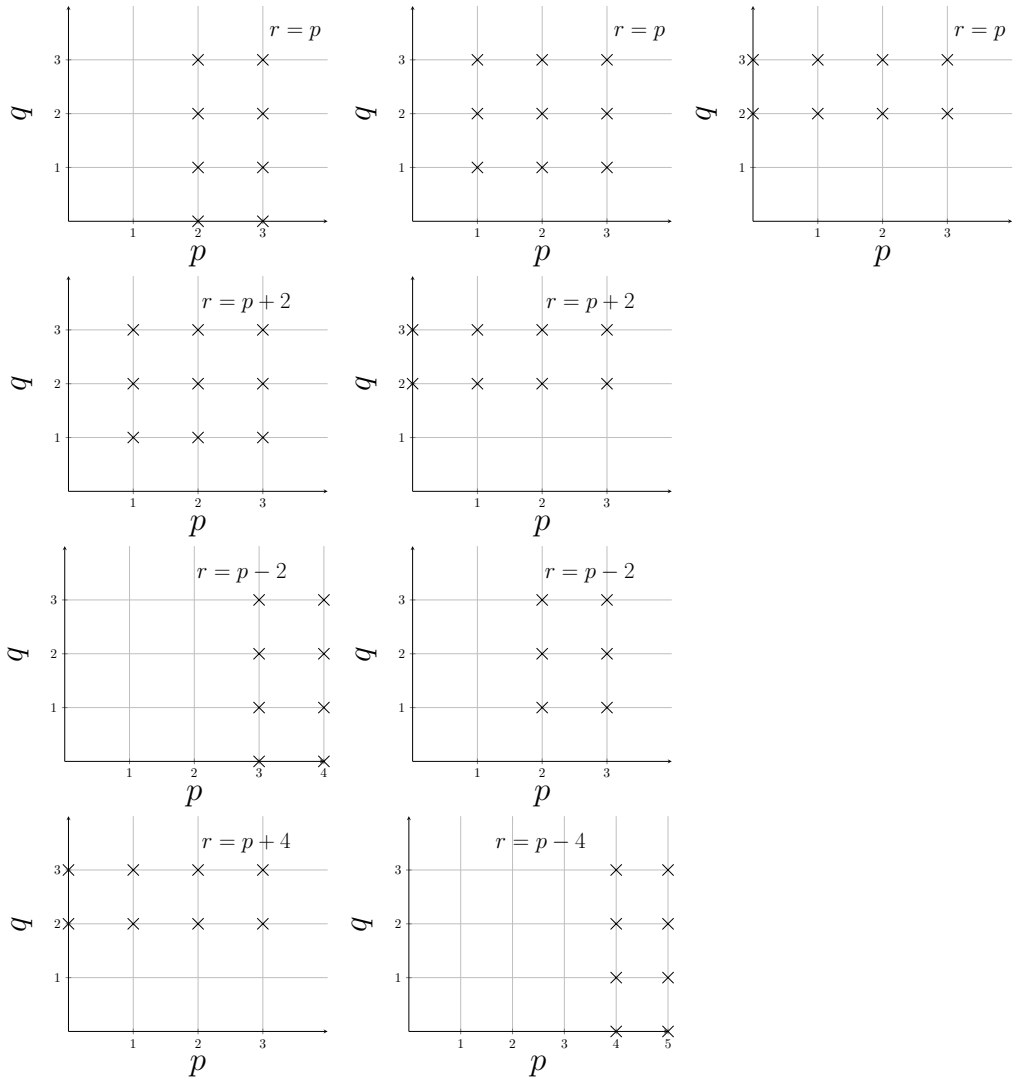


Figure 7: (2;2) towers