The Bert and Ernie (and Witten) Genus

The "Witten" Genus

$$\hat{A}(R)P_{\mathrm{B}}^{m}(q,\ R)=\eta(q^2)^{-8m}$$
$$\times\exp\left(\sum_{k=1}^{\infty}G_{2k}(q^2)\frac{1}{(2k)!}\mathrm{Tr}(\mathrm{i}R/2\pi)^{2k}\right).$$

A.N.Schellekens and N.P.Warner (Received, June 1986)
*Physics Letters B **177**, 18 September 1986, 317-323*

Bert Schellekens Fest

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Bert and Ernie:

Understanding Gauge and Gravitational Anomalies in String Theory

Green and Scharz: Anomaly cancellation

Direct approach: Apply the Atiyah-Singer Index Theorem in *2n+2 dimensions*:

$$\text{Anomaly} \equiv \hat{A}(R) \sum_i \text{Ch}(F,\ g_i)\ \text{Ch}(R,\ s_i)$$

The pieces

★ The curvature two-forms

$$F_g \;=\; F^a_{\mu\nu}\, T^a_g\, dx^\mu \wedge dx^\nu \qquad\qquad R_s \;=\; R^\alpha{}_{\beta\,\mu\nu}\, (\Lambda_s)^\beta{}_\alpha\, dx^\mu \wedge dx^\nu$$

★ Ch(X , g) = *Chern character* of curvature 2-form, X, acting on representation, g.

$$\text{Ch}(\,X\,,\,g\,) \;\equiv\; \text{Tr}\left(\exp\frac{iX_g}{2\pi}\right)$$

★ Â(*R*) = *"A-roof" genus*

$$\hat{A}(R) = \exp\left(-\sum_{p=1}^{\infty} \frac{B_{2p}}{4p}\frac{1}{(2p)!}\left[\text{Tr}(\mathrm{i}R/2\pi)^{2p}\right]\right)$$

where the B_{2p} are the Bernoulli numbers

Assemble "the anomaly" for all the ***massless chiral fields*** in the theory.

Useful way to encode these forms:

- $\mathbf{x}_\alpha$ = skew 2-form eigenvalues of Riemann tensor, R, acting on the tangent bundle/vector representation; $\alpha = 1,2,\ldots,n+1$

- $\mathbf{y}_\mathbf{a}$ = skew 2-form eigenvalues of Maxwell field, F, acting on the fundamental/vector representation.

The traces in Ch(F , g) and Ch(R , s) can then be written in terms of $\mathbf{x}_\alpha$ and $\mathbf{y}_\mathbf{a}$.

One also has:

$$\widehat{A}(R) \;=\; \prod_\alpha \frac{\frac{x_\alpha}{2}}{\sinh \frac{x_\alpha}{2}}$$

Global Anomalies in 2n+2-dimensions:

$$\text{Anomaly} \equiv \hat{A}(R) \sum_i \text{Ch}(F,\ g_i)\ \text{Ch}(R,\ s_i)\Big|_{\textit{2n+2-form part}}$$

Local Anomalies in 2n+2-dimensions:

$$\text{Anomaly} \equiv \hat{A}(R) \sum_i \text{Ch}(F,\ g_i)\ \text{Ch}(R,\ s_i)\Big|_{\textit{2n+4-form part}}$$

- ***Why does the local anomaly vanish in type II chiral string theories?***

- ***Why does the local anomaly factorize in heterotic strings?***

$$\text{Anomaly}_{(2n+4)-form} = \left[\text{Tr}\, F^2 - \text{Tr}\, R^2 \right] \wedge \text{X}_{2n-form}$$

And do so in precisely the right manner to lead to the Green-Schwarz anomaly cancellation mechanism

Green and Scharz: Anomaly cancellation mechanism

Bert: This has to be a fundamental, universal property of any string theory …

How do you organize a universal calculation of

$$\text{Anomaly} \equiv \hat{A}(R) \sum_i \text{Ch}(F,\ g_i)\ \text{Ch}(R,\ s_i)$$

One needs to figure out all the Maxwell and Lorentz representations in the *chiral, massless sector of the string*…

… and then compute and assemble the Chern Characters of all these pieces.

Partition Functions

Partition functions assemble and count all excitations according to the energy level …

The Ramond sector partition function with $(-1)^f$ counts precisely the contributions of the chiral fields of the theory …

The key insight: Use the character-valued partition functions to keep track of all the representation structure and thus create an

"Anomaly Generating Function"

The Anomaly Generating Function

Partition function: P(q, $\bar{q}$) ➞ P(q, $\bar{q}$, F, R) ↔ P(q, $\bar{q}$, y_a, x_α)

$$A(q,F,R) \;=\; P_{++}(q,x_\alpha,y_a) \;\equiv\; \mathrm{Tr}_{\mathrm{Ramond}}\left[(-1)^{\bar{f}}\, q^{\bar{L}_0-\frac{c}{24}}\, q^{L_0-\frac{c}{24}}\, e^{\frac{iR}{2\pi}}\, e^{\frac{iF}{2\pi}}\right]$$

= *Anomaly Generating Function*

For example:

$$\frac{\vartheta_3(0|\tau)}{\eta(\tau)} \;\equiv\; q^{-\frac{1}{24}}\prod_{n=1}^{\infty}\left(1 + q^{n-\frac{1}{2}}\right)^2 \qquad\qquad q \;\equiv\; e^{2\pi i\tau}$$

$$\to\; q^{-\frac{1}{24}}\prod_{n=1}^{\infty}\left(1+q^{n-\frac{1}{2}}e^{y_a}\right)\prod_{n=1}^{\infty}\left(1+q^{n-\frac{1}{2}}e^{-y_a}\right) \;=\; \frac{\vartheta_3(\frac{iy_a}{2\pi}|\tau)}{\eta(\tau)}$$

Important points:

★ $\lim_{F,R\to 0} A(q,F,R) \;\sim$ Chiral Ramond partition function of the string

★ Local Gauge and Gravitational Anomaly: $A(q,F,R)\Big|$

- Coefficient of q^0 → massless sector
- $2n+4-$form → anomaly form (local anomalies)

A Crucial Ingredient

The left-moving, space-time bosons: $q \equiv e^{2\pi i \tau}$

$$P_B(q) = q^{-\frac{n}{12}} \prod_{m=1}^{\infty} (1-q^m)^{-2n} = \eta(\tau)^{-2n} = \left(\frac{\eta(\tau)}{\vartheta_1'(0,\tau)} \right)^n$$

Â(R) = *"A-roof" genus*

$$\widehat{A}(R) = \prod_\alpha \frac{\frac{x_\alpha}{2}}{\sinh \frac{x_\alpha}{2}}$$

Combine in the "character-valued partition function" as:

$$\widehat{A}(R)\, P_B(q) \rightarrow q^{-\frac{n}{12}} \prod_\alpha \left[\frac{\frac{x_\alpha}{2}}{\sinh(\frac{x_\alpha}{2})} \prod_{m=1}^{\infty} \frac{1}{(1-q^m e^{x_\alpha})(1-q^m e^{-x_\alpha})} \right]$$

$$= \prod_\alpha \left(\frac{x_\alpha\, \eta(\tau)}{\vartheta_1(\frac{x_\alpha}{2\pi i}|\tau)} \right)$$

The elliptic generalization of Â(R) …

A.N.Schellekens, N.P.Warner (June 1986)
Physics Letters B Volume 177, Issues 3–4, 18 September 1986, Pages 317-323
(October 1986) Nuclear Physics B Volume 287, 1987, Pages 317-361

A.N.Schellekens, K.Pilch and N.P.Warner (October 1986)
Nuclear Physics B Volume 287, 1987, Pages 362-380

Don Zagier, Note on the Landweber-Stong elliptic genus
(October 1986)

Edward Witten, (November 1986)
Comm. Math. Phys. 109(4): 525-536 (1987).

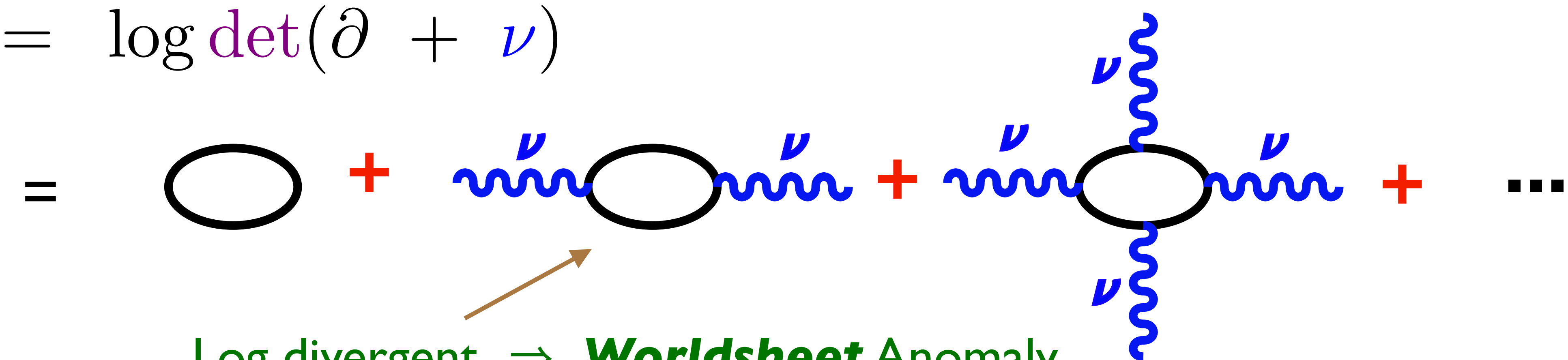
Modular invariance of the string

Space-time momentum integrals + ghosts →

The partition function: $P(q,\bar{q})$ must be a modular function of weight (-n,-n)

The anomaly generating function A(q, F, R) must be a (*nearly*) modular function of weight -n.

$$\log \vartheta(\nu|\tau) \;=\; \log \det(\partial \;+\; \nu)$$



Log divergent ⇒ ***Worldsheet*** Anomaly

$$\frac{\vartheta\left(\frac{\nu}{c\tau+d}\,\middle|\,\frac{a\tau+b}{c\tau+d}\right)}{\eta\left(\frac{a\tau+b}{c\tau+d}\right)} \;\sim\; e^{\frac{i\pi\nu^2}{c\tau+d}}\,\frac{\vartheta(\nu|\tau)}{\eta(\tau)}$$

This leads to:

$$A\left(q\left(\frac{a\tau+b}{c\tau+d}\right), \frac{F}{c\tau+d}, \frac{R}{c\tau+d}\right) \;=\; (c\tau+d)^{-n}\, exp\left[\frac{ic\,(\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2)}{32\,\pi^3\,(c\tau+d)}\right] A(q(\tau), F, R)$$

weight -n

Space-time Gauge and Gravitational Anomalies

Define: $\mathcal{A}(\tau) \equiv A(q, F, R)\Big|_{2n+4-\text{form}}$

Then: $\text{Anomaly} = \mathcal{A}(\tau)\Big|_{\text{Coefficient of } q^0}$ ← Why does this vanish/factorize?

Ignoring the $\text{Tr}\, F^2 - \text{Tr}\, R^2$ term, the modular properties of A(q, F, R)

$$A\left(q\left(\frac{a\tau+b}{c\tau+d}\right), \frac{F}{c\tau+d}, \frac{R}{c\tau+d}\right) = (c\tau+d)^{-n}\, exp\left[\frac{ic\,(\text{Tr}\, F^2 - \text{Tr}\, R^2)}{32\,\pi^3\,(c\tau+d)}\right] A(q(\tau), F, R)$$

imply that *$\mathcal{A}(\tau)$ must be a modular function of weight 2*:

$$\mathcal{A}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 \mathcal{A}(\tau)$$

Simple property of modular functions:

Any (holomorphic) modular function of weight 2 must be the τ-derivative a (holomorphic) modular function of weight 0:

$$\mathcal{A}(\tau) = \frac{d}{d\tau}\mathcal{B}(\tau)$$

$$= \frac{d}{d\tau}\sum_{k=-N}^{\infty} b_k\, q^k = \sum_{k=-N}^{\infty} k\, b_k\, q^k \sim -N\, q^{-N} + \dots + 0\, q^0 + \dots$$

Coefficient of q^0 vanishes

Elementary to incorporate the anomalous modular transformation:

$$A\left(q\left(\frac{a\tau+b}{c\tau+d}\right), \frac{F}{c\tau+d}, \frac{R}{c\tau+d}\right) = (c\tau+d)^{-n}\, exp\left[\frac{ic\,(\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2)}{32\,\pi^3\,(c\tau+d)}\right] A(q(\tau), F, R)$$

to arrive at:

$$\mathcal{A}(\tau)\Big|_{\mathrm{Coefficient\ of\ q^0}} = \left[\,\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2\,\right] \wedge \mathrm{X}_{2n-form}$$

The result

A fundamental, deep connection between world-sheet and space-time physics

Global world-sheet diffeomorphism invariance of string theory
⇒ Modular invariance of the string partition function
⇒ Space-time gravitational and gauge anomalies vanish/factorize

Worldsheet anomalies
⇒ $\mathrm{Tr}\,F^2 - \mathrm{Tr}\,R^2$ factorized space-time anomalies

The story continues…

The Elliptic Genus and the Index of the Dirac-Ramond Operator

The story started with the Atiyah-Singer Index Theorem …

Schellekens and Warner (June 1986), Physics Letters B 177, 18 September 1986, 317

but it rapidly became clear that this computation was all about the (families) *index theorem of the Dirac-Ramond operator* and the loop-space adaptation of the Fujikawa method …

Pilch, Schellekens and Warner (Received October 1986)
Path Integral Calculation of String Anomalies
Nuclear Physics B Volume 287, 1987, Pages 362-380

Don Zagier, *Note on the Landweber-Stong elliptic genus* (preprint October 1986)

Edward Witten, *Elliptic Genera and Quantum Field Theory* (Received: November 1986)
Comm. Math. Phys. 109(4): 525-536 (1987).
Abstract: …. *Some of the relevant calculations have been done previously by Schellekens and Warner in a different context…*

However this enabled "the mathematics narrative"

https://ncatlab.org/nlab/show/Witten+genus

The interpretation of elliptic genera ... has precursors in … A. N. Schellekens, Nicholas P. Warner … but strictly originates in Edward Witten, Elliptic genera and quantum field theory ..

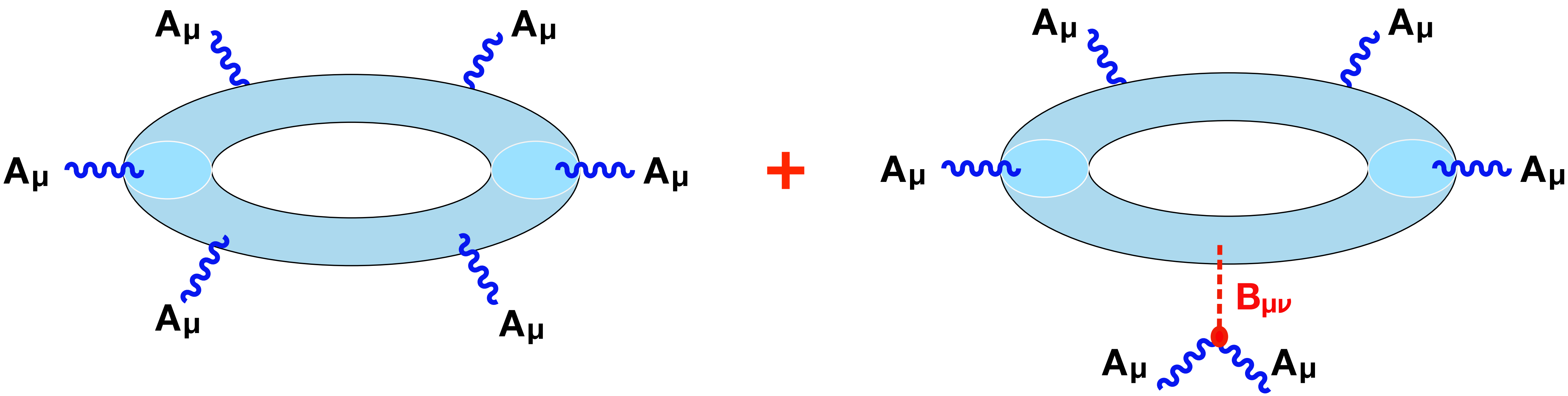
All references to *Pilch, Schellekens, and Warner* "*Path Integral Calculation of String Anomalies*" are omitted from this alternative, and inaccurate narrative…

Important developments

Anomaly factorization "enables" the Green-Schwarz mechanism …

but does the heterotic string actually make this happen?

Is this anomaly free?



Yes!

W. Lerche, B.E.W. Nilsson, A.N. Schellekens (Feb, 1987), *Heterotic String Loop Calculation of the Anomaly Cancelling Term,* Nucl.Phys.B 289 (1987) 609

W. Lerche, B.E.W. Nilsson, A.N. Schellekens, N.P. Warner (June, 1987), *Anomaly Cancelling Terms From the Elliptic Genus Nucl.Phys.B 299 (1988) 91-116*

It all started with Bert …

The heterotic string paper

"Although much work remains to be done there seem to be no insuperable obstacles to deriving all of known physics from the $E_8 \times E_8$ heterotic string."

Anomalies, Characters and Strings

"Perhaps the heterotic string is truly a theory of everything, containing not just all of physics, but also all known mathematics."