On the classification of holomorphic vertex operator algebras of central charge 24

Nils Scheithauer

May 13, 2022

Overview

1. Historical introduction
2. Schellekens' list
3. The proof of Schellekens' conjecture for $V_1 \neq 0$
4. A geometric classification

1979: Conway and Norton formulate the moonshine conjecture

The monster is the largest sporadic group. Conway and Norton conjecture that there is a graded module $V = \bigoplus_{n=0}^{\infty} V_n$ for the monster such that the graded traces $T_g(\tau) = \sum_{n=0}^{\infty} \operatorname{tr} g|_{V_n} \, q^{n-1}$ are hauptmoduln for certain explicitly given genus 0 groups.

J. H. Conway, S. P. Norton, Monstrous moonshine, Bull. London Math. Soc. 11 (1979), 308–339

1986: Borcherds defines the notion of a vertex algebra

Inspired by 2-dimensional quantum field theory Borcherds defines the notion of a vertex algebra.

R. E. Borcherds, Vertex algebras, Kac-Moody algebras and the Monster, Proc. Nat. Acad. Sci. USA 83 (1986), 3068–3071

Historical introduction

1988: Frenkel, Lepowsky and Meurman construct the moonshine module

Frenkel, Lepowsky and Meurman construct a holomorphic vertex operator algebra $V^\natural$ of central charge 24 which carries an action of the monster and satisfies the moonshine conjecture for all elements in the centralizer of a certain involution.

I. Frenkel, J. Lepowsky, A. Meurman, Vertex operator algebras and the Monster, Academic Press, 1988

1992: Borcherds proves the moonshine conjecture

Borcherds shows that $V^\natural$ satisfies the moonshine conjecture for all elements in the monster using generalized Kac-Moody algebras and the no-ghost theorem.

R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, Invent. Math. 109 (1992), 405–444

Historical introduction

1993: Schellekens' list

The weight-1 subspace of a holomorphic vertex operator algebra $V = \bigoplus_{n=0}^{\infty} V_n$ of central charge 24 is a Lie algebra. Schellekens shows that there are at most 71 possibilities for this Lie algebra and conjectures that each of them is realized by a unique vertex operator algebra.

A. N. Schellekens, Meromorphic $c = 24$ conformal field theories, Comm. Math. Phys. 153 (1993), 159–185

1996: Zhu proves the modularity of characters

Zhu shows that the characters of a rational vertex operator algebra and its modules are invariant under $\mathrm{SL}_2(\mathbb{Z})$.

Y. Zhu, Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc. 9 (1996), 237–302

Historical introduction

2008: Huang proves the Verlinde formula for vertex operator algebras

Huang shows that the matrix describing the transformation of the characters of a rational vertex operator algebra under $\tau \mapsto -1/\tau$ diagonalizes the fusion rules.

Y.-Z. Huang, Vertex operator algebras and the Verlinde conjecture, Commun. Contemp. Math. 10 (2008), 103–154

2020: Proof of Schellekens' conjecture for $V_1 \neq 0$

By the combined efforts of many authors Schellekens' conjecture is proved for $V_1 \neq 0$. The main tools are orbifold theory for the existence part and inverse orbifolding for the uniqueness.

Schellekens' list

A vertex operator algebra is a graded vector space $V = \bigoplus_{n=0}^{\infty} V_n$ together with a state-field correspondence

$$V \to \mathrm{End}(V)[\![z, z^{-1}]\!]$$
$$a \mapsto Y(a,z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

satisfying a number of axioms. In particular the fields are local to each other, i.e.

$$(z-w)^n \, [Y(a,z), Y(b,w)] = 0$$

for $n >> 0$ and V carries an action of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n} C \, .$$

Schellekens' list

In his classification of holomorphic vertex operator algebras of central charge 24 Schellekens assumed the modularity of characters which was proven 3 years later by Zhu. He showed:

Theorem

Let V be a holomorphic vertex operator algebra of central charge 24. Then V_1 is reductive and exactly one of the following holds:

i) $V_1 = 0$.

ii) $\dim V_1 = 24$. In this case V_1 is abelian and V is isomorphic to V_Λ where Λ is the Leech lattice.

iii) V_1 is semisimple and $\mathrm{rk}(V_1) \leq 24$.

Schellekens' list

Theorem

Let V be a holomorphic vertex operator algebra of central charge 24. Suppose V_1 is semisimple. Then V_1 is one of 69 explicitly given Lie algebras.

The above 69 Lie algebras together with 0 and $\mathbb{C}^{24}$ form Schellekens' list.

We sketch the proof. The character $\mathrm{ch}_V : H \times \mathfrak{h} \to \mathbb{C}$ defined by

$$\mathrm{ch}_V(\tau, z) = \mathrm{tr}_V\, e^{2\pi i z_0} q^{L_0 - 1}$$

is holomorphic on $H \times \mathfrak{h}$ and transforms as a Jacobi form of weight 0 and lattice index L for a certain lattice L. The function

$$P(\tau, z) = \exp\left(-(2\pi i)^2 \frac{\langle z, z\rangle}{24} E_2(\tau)\right) \Delta(\tau)\, \mathrm{ch}_V(\tau, z)$$

Schellekens' list

transforms under $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ as

$$P\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^{12} P(\tau, z)\,.$$

This implies that the m-th coefficient in the Taylor expansion of P in z is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $12 + m$. From this Schellekens derives 7 trace identities. He shows that they have exactly 69 solutions.

The lowest order identity is

$$\frac{h_i^\vee}{k_i} = \frac{\dim(V_1) - 24}{24}\,.$$

Schellekens' list

Conjecture (1993)

Each of the 71 Lie algebras on Schellekens' list is realized by a unique holomorphic vertex operator algebra of central charge 24.

Theorem (2020)

Each of the 70 non-trivial Lie algebras on Schellekens' list is realized by a unique holomorphic vertex operator algebra of central charge 24.

The only missing case is the uniqueness of the moonshine module $V^\natural$ which has trivial weight-1 subspace.

The proof of Schellekens' conjecture for $V_1 \neq 0$

Existence

There is a uniform existence proof by Höhn. He also shows that Schellekens' list naturally decomposes into 12 families.

G. Höhn, On the genus of the moonshine module, arXiv:1708.05990

Another approach is based on orbifold theory. Let V be a holomorphic vertex operator algebra and $g \in \mathrm{Aut}(V)$ an automorphism of order n. Define $G = \langle g \rangle$. Then V^G can be extended to a holomorphic vertex operator algebra. For $i \in \mathbb{Z}/n\mathbb{Z}$ let $V(g^i)$ be the unique twisted V-module corresponding to g^i.

The proof of Schellekens' conjecture for $V_1 \neq 0$

Then

$$V^{\mathrm{orb}(g)} = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} V(g^i)^G = V^G \oplus V(g)^G \oplus \ldots \oplus V(g^{n-1})^G$$

is a holomorphic vertex operator algebra.

Now all Lie algebras on Schellekens' list can be realized by orbifolding suitable lattice vertex operator algebras.

Uniqueness

So far the uniqueness proofs all use the method of inverse orbifolding. Under the above conditions there is an automorphism ζ of $V^{\mathrm{orb}(g)}$ such that

$$(V^{\mathrm{orb}(g)})^{\mathrm{orb}(\zeta)} \cong V .$$

The proof of Schellekens' conjecture for $V_1 \neq 0$

Let L be a lattice and V the associated vertex operator algebra. Suppose we want to prove the uniqueness of $V^{\mathrm{orb}(g)}$.
Let W be another vertex operator algebra with $W_1 \cong V_1^{\mathrm{orb}(g)}$. We consider an orbifold $W^{\mathrm{orb}(\sigma_h)}$ with $h \in W_1$ semisimple. If $W_1^{\mathrm{orb}(\sigma_h)} \cong V_1$ then $W^{\mathrm{orb}(\sigma_h)} \cong V$ by the uniqueness of lattice vertex operator algebras. Let ζ be the inverse orbifold of σ_h. Then the orbifolds of g and ζ, which are both in $\mathrm{Aut}(V)$, have isomorphic weight-1 spaces. If we can show that they are conjugate to each other then

$$W \cong (W^{\mathrm{orb}(\sigma_h)})^{\mathrm{orb}(\zeta)} \cong V^{\mathrm{orb}(\zeta)} \cong V^{\mathrm{orb}(g)} .$$

A geometric classification

Theorem (Borcherds, 1985)

There is a 1-to-1 correspondence between the deep holes in the Leech lattice and the Niemeier lattices with roots.

Theorem (Conway, Parker, Sloane, 1982)

There are exactly 23 deep holes in the Leech lattice. The class of a deep hole is uniquely determined by its hole diagram.

This provides a geometric classification of the Niemeier lattices.

The argument can be extended to the holomorphic vertex operator algebras of central charge 24.

A geometric classification

Let V be a holomorphic vertex operator algebra of central charge 24 and $g \in \mathrm{Aut}(V)$ of order n. Then

$$\dim(V_1^{\mathrm{orb}(g)}) \leq 24 + \sum_{d|n} c_n(d) \dim(V_1^{g^d}) .$$

If we have equality and g satisfies two more conditions then we call g a generalized deep hole.

Let V_Λ denote the vertex operator algebra associated with the Leech lattice Λ.

A geometric classification

Theorem

The orbifold construction $g \mapsto V_\Lambda^{\mathrm{orb}(g)}$ defines a 1-to-1 correspondence between the algebraic conjugacy classes of generalized deep holes g in $\mathrm{Aut}(V_\Lambda)$ with $(V_\Lambda^g)_1 \neq 0$ and the isomorphism classes of holomorphic vertex operator algebra of central charge 24 with $V_1 \neq 0$.

Theorem

There are exactly 70 conjugacy classes of generalized deep holes in $\mathrm{Aut}(V_\Lambda)$ with $(V_\Lambda^g)_1 \neq 0$. The class of a generalized deep hole is uniquely determined by its generalised hole diagram.

The proof uses $h_i^\vee / k_i = (\dim(V_1) - 24)/24$ but is apart from that purely geometric and independent of Schellekens' results.