Series expansion methods for Feynman integrals, and their application to Higgs plus jet integrals

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Work on Higgs plus jet in collaboration with:

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Introduction

Outline of the talk

- Introduction
- The method of differential equations
  - Basic definitions
  - Canonical basis
  - Boundary terms
- Series expansion method
  - For canonical bases
  - For coupled systems (elliptic & more)
- Results:
  - Higgs + jet families F and G
  - 4-Mass banana graphs
- Conclusion

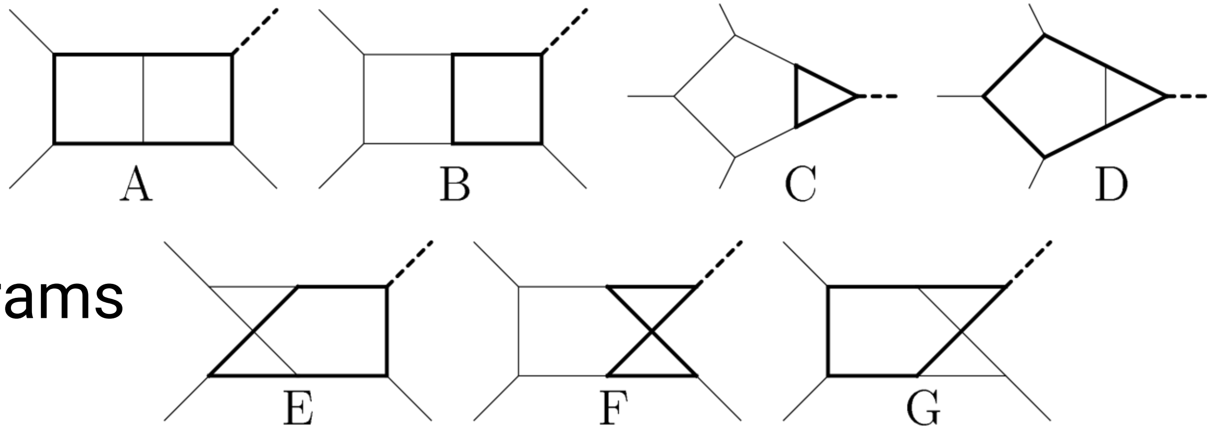
Introduction

- For some phenomenological processes, the bottleneck in the computation of the amplitudes and cross-section is the evaluation of the master integrals
- One example, is production of the Higgs boson @ LHC via gluon-gluon fusion
- The Higgs particle does not couple directly to gluons: Interaction is mediated by a heavy quark loop, so that NLO @ 2-loop
- To this date, no NLO computation is available of the whole p_T-spectrum, including quark-mass effects for all quark flavors
  - An NLO computation including the top-quark mass but neglecting bottom-quark mass has been performed using sector decomposition for the integrals [Jones, Kerner, Luisoni, 2018]
  - Various computations have also been done in HEFT (some up to N^3LO) e.g. [Chen, Gehrmann, Glover, Jaquier, 2016]

Introduction

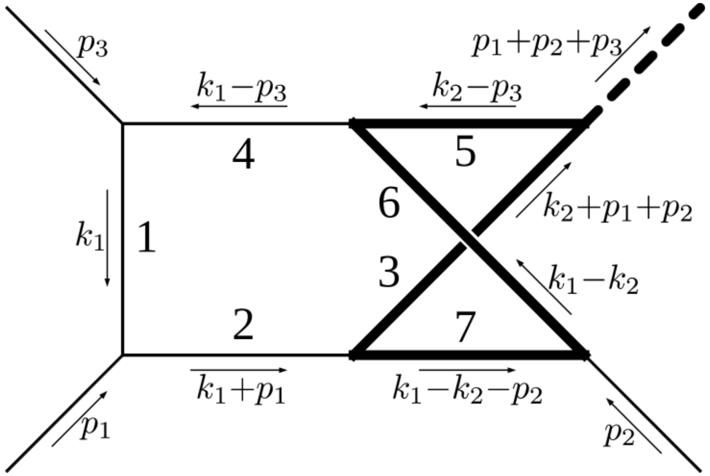
[Bonciani et al, 1609.06685]

- Amplitude computation:
  - $\mathcal{O}(300)$ Feynman diagrams
  - Dirac algebra $\Rightarrow \mathcal{O}(20000)$ scalar diagrams
  - The diagrams fit into 7 topologies.
  - Non-planar families:



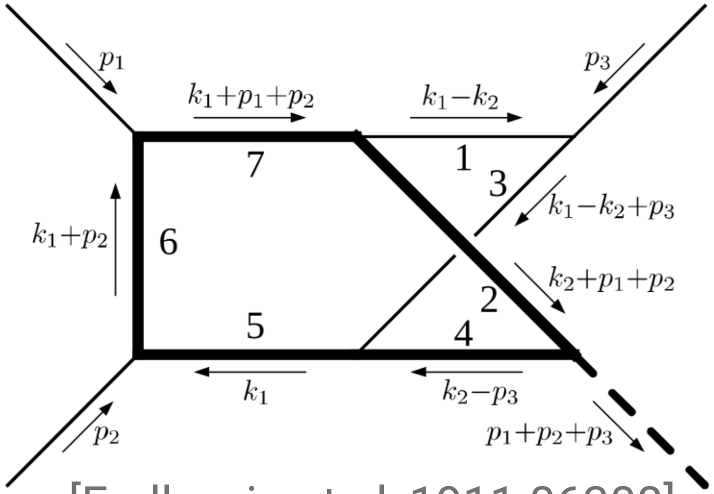
$$s = (p_1+p_2)^2, \qquad t = (p_1+p_3)^2,$$
$$p_4^2 = (p_1+p_2+p_3)^2 = s+t+u.$$

F:



[Bonciani et al, 1907.13156]

G:



[Frellesvig et al, 1911.06308]

Family F
Master integrals

- IBP-reduction:
  - 73 master integrals
  - Default FIRE basis: $\mathcal{O}(1\ \text{GB})$
  - More suitable (pre-canonical) basis: $\mathcal{O}(100\ \text{MB})$
  - Possible using either FIRE or KIRA

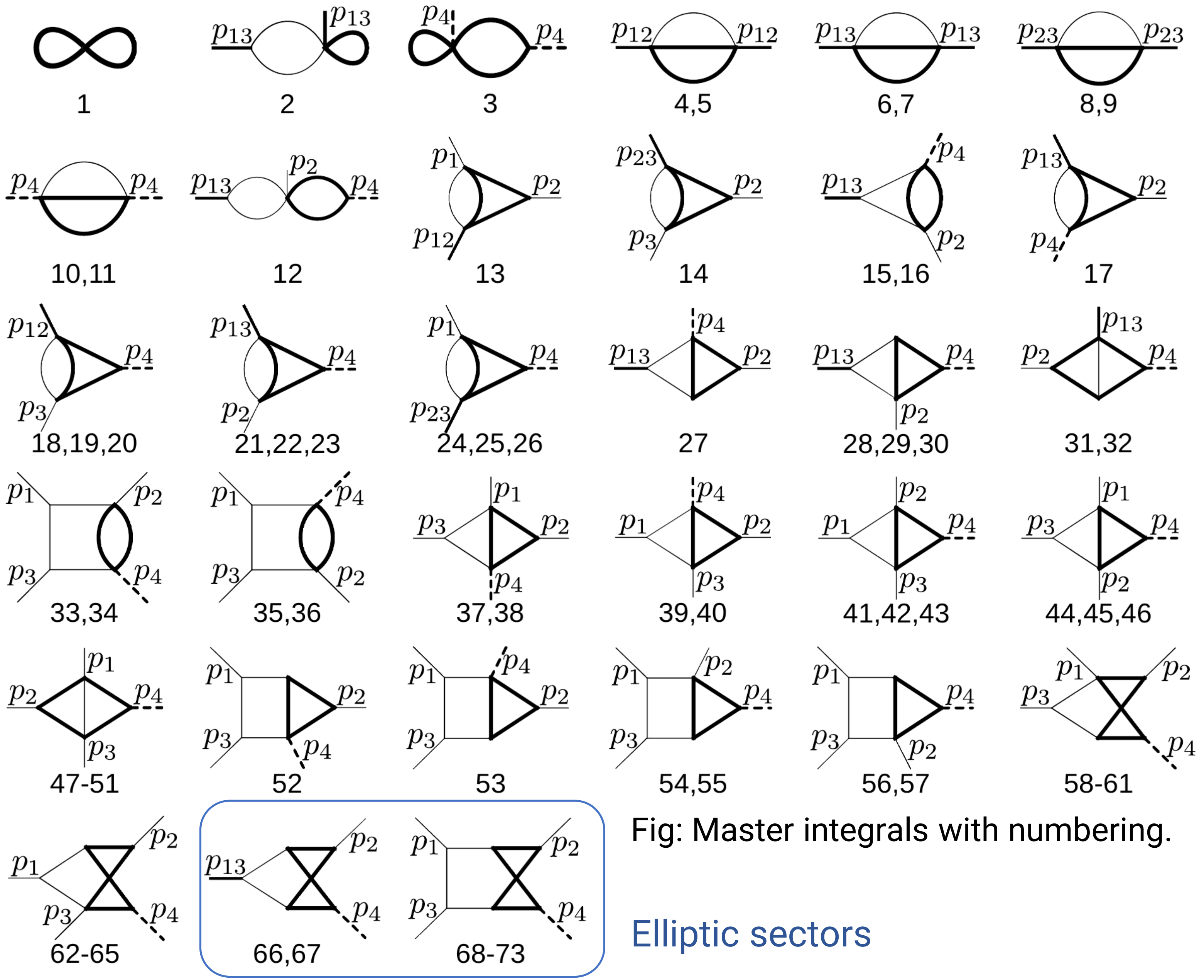


Fig: Master integrals with numbering.

Family F

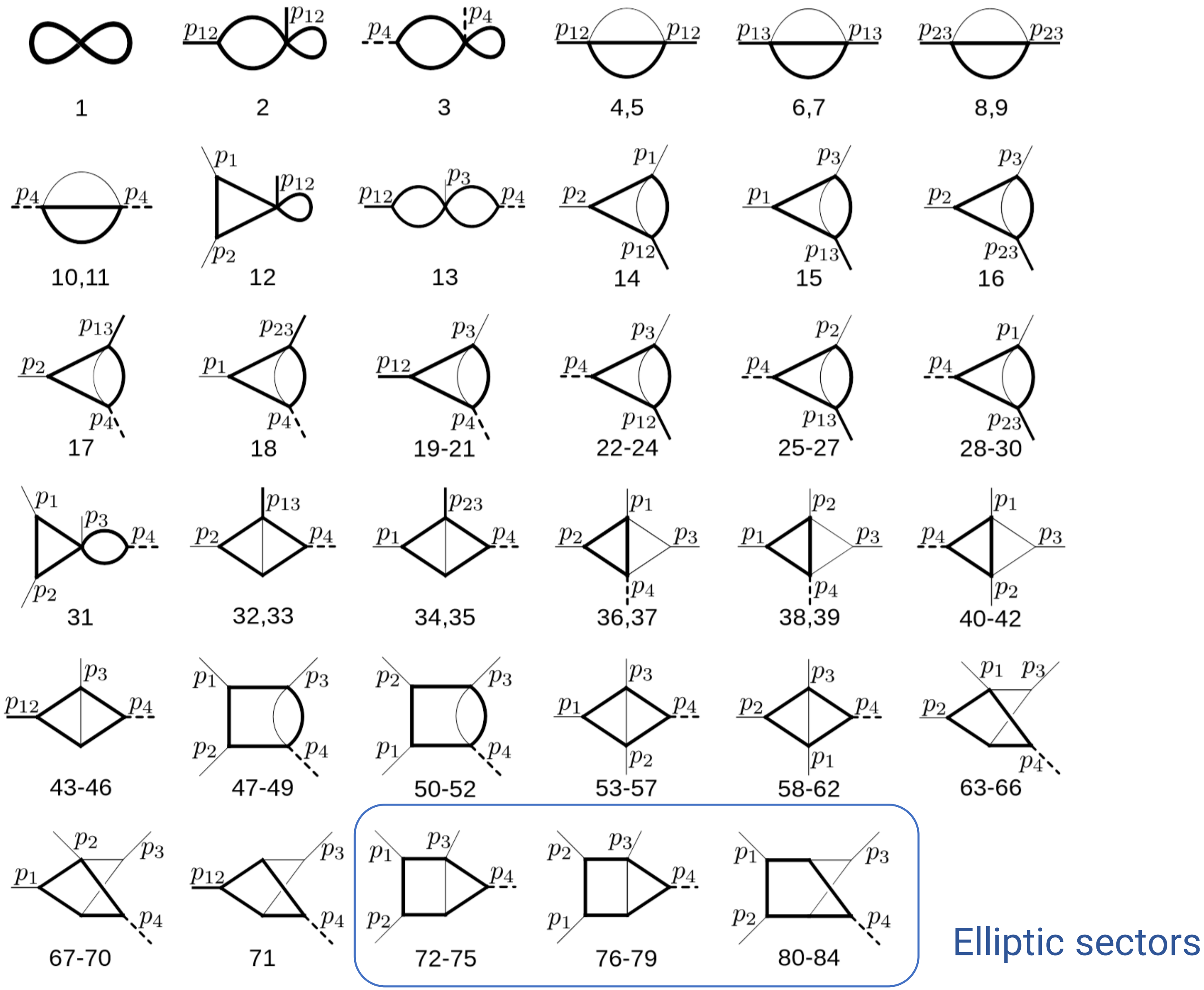
- There are two elliptic sectors. Their associated maximal cuts are:

$$I_{011111100} \to \int \frac{dz}{(p_4^2-t)\sqrt{z\,(z+p_4^2-t)\,\left(z^2+(p_4^2-t)z-4m^2t\right)}}$$

$$I_{111111100} \to \int \frac{dz}{t\,(z+s)\sqrt{z\,(z+p_4^2-t)\,\left(z^2+(p_4^2-t)z-4m^2t\right)}}$$

Family G
Master integrals

- IBP-reduction:
  - 84 master integrals
  - Default FIRE basis: $\mathcal{O}(1\text{ GB})$
  - More suitable (pre-canonical) basis: $\mathcal{O}(100\text{ MB})$
  - Possible using either FIRE or KIRA



The method of differential equations

Basic notions

- Partial derivatives of Feynman integrals are combinations of Feynman integrals within the same family
- Thus, given a family of master integrals $\vec{f}$, and using IBP-reduction we may write:

$$d\vec{f} = \sum_{s \in S} \mathbf{M}_s \vec{f} ds$$

[Kotikov, 1991], [Remiddi, 1997]
[Gehrmann, Remiddi, 2000]

- Properties of the differential equations:

$$0 = d^2\vec{f} \Rightarrow \quad \partial_{s_1}\mathbf{M}_{s_2} - \partial_{s_2}\mathbf{M}_{s_1} + [\mathbf{M}_{s_1}, \mathbf{M}_{s_2}] = 0 \quad \text{for all } s_1, s_2 \in S$$

$\sum_{s \in S} s\mathbf{M}_s = \mathbf{\Gamma}$ Where $\mathbf{\Gamma}$ is diagonal matrix containing the mass dimensions

Canonical basis

- Things simplify considerable in a so-called canonical basis
- Let's take a look at what happens under a change of basis
- Let $\vec{B} = \mathbf{T}\vec{f}$. Then we have: $\frac{\partial}{\partial s_i}\vec{B} = \left[(\partial_{s_i}\mathbf{T})\,\mathbf{T}^{-1} + \mathbf{T}\mathbf{M}_{s_i}\mathbf{T}^{-1}\right]\vec{B}\,.$
- The canonical basis conjecture claims that $\exists \mathbf{T} :\ d\vec{B} = \epsilon d\tilde{\mathbf{A}}\vec{B}$ [Henn, 2013]
- And that for families expressible in terms of multiple polylogarithms we have:

$$d\tilde{\mathbf{A}} = \sum_{i \in \mathcal{A}} \mathbf{A}_i d\log(l_i)$$

Canonical basis

- The formal solution can be given in terms of Chen's iterated integrals: [Chen, 1977]

$$d\vec{B} = \epsilon \left(d\tilde{\mathbf{A}} \right) \vec{B} \quad \Rightarrow \quad \vec{B} = \mathbb{P} \exp \left[\epsilon \int_\gamma d\tilde{\mathbf{A}} \right] \vec{B}_{\text{boundary}}$$

$$\vec{B} = \sum_{k\geq 0} \epsilon^k \sum_{j=1}^k \int_0^1 \gamma^*(d\tilde{\mathbf{A}})(t_1) \int_0^{t_1} \gamma^*(d\tilde{\mathbf{A}})(t_2) \ldots \int_0^{t_{j-1}} \gamma^*(d\tilde{\mathbf{A}})(t_j)\, \vec{B}^{(k-j)}_{\text{boundary}}$$

- The symbol of the integrals is given by:

$$\mathcal{S}\left(B_i^{(k)}\right) = \sum_j \mathcal{S}\left(B_j^{(k-1)}\right) \otimes d\tilde{\mathbf{A}}_{ij}$$

- The iterated integrals may yield MPL's, iterated integrals of Eisenstein series / modular forms, …

[For non-polylogarithmic examples, see works Adams & Weinzierl, and also 1907.01251]

Canonical basis

$$\vec{B} = \sum_{k\geq 0} \epsilon^k \sum_{j=1}^{k} \int_0^1 \gamma^*(d\tilde{\mathbf{A}})(t_1) \int_0^{t_1} \gamma^*(d\tilde{\mathbf{A}})(t_2) \dots \int_0^{t_{j-1}} \gamma^*(d\tilde{\mathbf{A}})(t_j)\, \vec{B}^{(k-j)}_{\mathrm{boundary}}$$

- Note that even when $$d\tilde{\mathbf{A}} = \sum_{i\in\mathcal{A}} \mathbf{A}_i d\log(l_i)\,,$$

the iterated integrals might not be expressible in terms of MPLs! (Or at least known how to.)

- This happens when there are multiple non-simultaneously rationalizable square roots. In that case it may not be manifestly possible to obtain the form:

$$G(a_1,\dots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\dots,a_n;t)$$

- But sometimes an ansatz-based approach works

Canonical basis

$$\vec{B} = \sum_{k\geq 0} \epsilon^k \sum_{j=1}^{k} \int_0^1 \gamma^*(d\tilde{\mathbf{A}})(t_1) \int_0^{t_1} \gamma^*(d\tilde{\mathbf{A}})(t_2) \dots \int_0^{t_{j-1}} \gamma^*(d\tilde{\mathbf{A}})(t_j)\, \vec{B}^{(k-j)}_{\text{boundary}}$$

- Note that it is not necessary to express the integrals in terms of multiple polylogarithms, even when possible.
- If we expand $$\gamma^*(d\tilde{\mathbf{A}})(t) = t^r \left[\sum_{p=0}^{k} \mathbf{C}_p t^p + \mathcal{O}\left(t^{k+1}\right)\right] dt\,,$$

where r is integer or half-integer, then all integrations can be performed analytically. This is essentially the basis of the series expansion methods central in this talk.

- There is more to consider: radius of convergence, analytic continuation, non-canonical bases. But let's not get ahead of ourselves!

Finding the canonical basis

[Lee, 1411.0911]
[Prausa, 1701.00725]
[Meyer, 1705.06252]
[Gituliar, Magerya, 1701.04269]
[Dlapa, Henn, Yan, 2002.02340]

- Many publicly available algorithms (Epsilon, Fuchsia, Canonica, ..)
- Canonical basis can often be computed "manually"
  - First find a "pre-canonical" basis: $d\vec{f} = d\left(\tilde{\mathbf{A}}_0 + \epsilon\tilde{\mathbf{A}}_1\right)\vec{f}$
  - Find a period matrix for fixed integer dimension: $d\vec{\mathbf{P}} = d\tilde{\mathbf{A}}_0\vec{\mathbf{P}}$
  - Then note: $d\left(\mathbf{P}^{-1}\vec{f}\right) = \epsilon\underbrace{\mathbf{P}^{-1}d\tilde{\mathbf{A}}_1\mathbf{P}}_{d\tilde{\mathbf{A}}}\left(\mathbf{P}^{-1}\vec{f}\right)$
  - If we work <u>on</u> the maximal cut of a given sector (i.e. modulo its subtopologies), then ***P*** can be directly derived <u>from</u> its maximal cuts, which always solve the homogeneous part of the differential equations.

Finding the canonical basis

- Using the previous equations, the diagonal blocks can be put in canonical form. By systematically shifting out terms from sub-topologies, we may also make the full system canonical.

[Gehrmann, von Manteuffel, Tancredi, Weihs, 1404.4853]

$$d\begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} \varepsilon d\mathbf{A} & d\mathbf{D} & d\mathbf{E} & \cdots \\ 0 & \varepsilon d\mathbf{B} & d\mathbf{F} & \cdots \\ 0 & 0 & \varepsilon d\mathbf{C} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix}$$

- For example, suppose that the (i, j)-th entry of the differential equation matrix has the form $\mathrm{R} + \epsilon\, S$. Then, shift, $B_i \to B_i + \alpha(\ldots)B_j$, where α depends on the external scales. This returns a differential equation for $\alpha(\ldots)$, which may be solved to put the ϵ^0 term to zero. Repeating this leads to: $d\vec{B} = \epsilon d\tilde{\mathbf{A}}\vec{B}$

Finding the canonical basis

- The result may end up being quite complicated! For example, for family "F" of the Higgs + jet integrals, we found:

$$
\begin{aligned}
B_{58} &= \epsilon^4 r_5 r_{15} \mathrm{I}_{1,0,1,1,1,1,1,0,0}\,,\\
B_{59} &= \epsilon^4 \left(p_4^2 - t\right)\left(\mathrm{I}_{1,0,1,1,0,1,1,0,0} - \mathrm{I}_{1,0,1,1,1,1,1,-1,0}\right)\,,\\
B_{60} &= \frac{s^2 - p_4^2 s + t^2 - t p_4^2}{p_4^2 - s}\mathrm{I}_{1,0,1,0,1,1,1,0,0}\epsilon^4 + \left(-p_4^2 + s + t\right)\left(\mathrm{I}_{1,-1,1,1,1,1,1,0,0} + \right.\\
&\quad \left. + t\mathrm{I}_{1,0,1,1,1,1,1,0,0}\right)\epsilon^4 + \frac{t}{p_4^2 - s}\left(\frac{1}{4}(B_6 + B_{10}) + \frac{1}{2}(B_8 - B_{13} - B_{14} + \right.\\
&\quad \left. + B_{18} + B_{21}) - B_{22} - B_{44} + B_{46} + B_{50} - B_{59}\right)\,,\\
B_{61} &= \epsilon^3 r_1 r_6 \left(\left(-s - t + p_4^2\right)\left((-2\epsilon)\mathrm{I}_{1,0,1,1,1,1,1,0,0} - \mathrm{I}_{1,0,1,1,1,0,2,0,0}\right) + s\mathrm{I}_{1,0,2,1,0,1,1,0,0} + \right.\\
&\quad \left. + \left(t - p_4^2\right)\mathrm{I}_{1,0,2,1,1,1,1,-1,0}\right)\,,\\
B_{62} &= \epsilon^4 r_2 r_{14} \mathrm{I}_{1,1,1,0,1,1,1,0,0}\,,\\
B_{63} &= \epsilon^4 \left(p_4^2 - t\right)\left(\mathrm{I}_{1,1,1,0,1,1,0,0,0} - \mathrm{I}_{1,1,1,0,1,1,1,0,-1}\right)\,,\\
B_{64} &= s\mathrm{I}_{1,1,1,-1,1,1,1,0,0}\epsilon^4 + (st)\mathrm{I}_{1,1,1,0,1,1,1,0,0}\epsilon^4 + \frac{t}{s+t}\left(\frac{1}{4}(-B_6 - B_{10}) + B_{22} + \right.\\
&\quad \left. + \frac{1}{2}(-B_4 + B_{13} + B_{14} - B_{21} - B_{24}) - B_{31} + B_{41} - B_{43} - B_{50}\right) + \\
&\quad + \frac{1}{s+t}\left(\left(-s^2 - ts - 2t^2 + 2tp_4^2\right)\mathrm{I}_{1,0,1,0,1,1,1,0,0}\epsilon^4 + sB_{63}\right)\,,
\end{aligned}
$$

$$
\begin{aligned}
r_1 &= \sqrt{-p_4^2}\,, & r_2 &= \sqrt{-s}\,,\\
r_3 &= \sqrt{-t}\,, & r_4 &= \sqrt{t - p_4^2}\,,\\
r_5 &= \sqrt{s + t - p_4^2}\,, & r_6 &= \sqrt{4m^2 - p_4^2}\,,\\
r_7 &= \sqrt{4m^2 - s}\,, & r_8 &= \sqrt{4m^2 - t}\,,\\
r_9 &= \sqrt{4m^2 - p_4^2 + t}\,, & r_{10} &= \sqrt{4m^2 - p_4^2 + s + t}\,,\\
r_{11} &= \sqrt{4m^2(p_4^2 - s - t) + st}\,, & r_{12} &= \sqrt{4m^2 t + s(p_4^2 - s - t)}\,,\\
r_{13} &= \sqrt{4m^2 s + t(p_4^2 - s - t)}\,, & r_{14} &= \sqrt{4m^2 t(s + t - p_4^2) - \left(p_4^2\right)^2 s}\,,\\
r_{15} &= \sqrt{-4m^2 st + \left(p_4^2\right)^2\left(s + t - p_4^2\right)}\,, & r_{16} &= \sqrt{16m^2 t + \left(p_4^2 - t\right)^2}\,.
\end{aligned}
$$

- In particular, the first 65 integrals can be written in canonical $d\log$-form, while the remaining integrals are in elliptic sectors

Finding the canonical basis

- Note, I conveniently wrote everything in terms total differentials. But how do we actually find $\tilde{\mathbf{A}}$, such that: $\partial_{s_i}\tilde{\mathbf{A}} = \mathbf{A}_{s_i}$?
- For this we can let:

$$\tilde{\mathbf{A}}_1 := \int \mathbf{A}_{s_1} ds_1 \,,$$

$$\tilde{\mathbf{A}}_i := \int \left(\mathbf{A}_{s_i} - \partial_{s_i} \sum_{j=1}^{i-1} \mathbf{A}_j \right) ds_i \,, \qquad i = 2, \ldots, 4 \,.$$

$$\tilde{\mathbf{A}} = \sum_i \tilde{\mathbf{A}}_i$$

- $\tilde{\mathbf{A}}_i$ should not depend on the variables s_j, with $j < i$, and we can plug in numbers for those to easy the integration

Analytic integration (Family F)

- Generate an ansatz of basis functions, in the manner of Duhr-Gangl-Rhodes:

[Duhr, Gangl, Rhodes, 1110.0458]

$$\mathrm{Li}_2\left(\pm l_i l_j\right), \mathrm{Li}_2\left(\pm \frac{l_i}{l_j}\right), \mathrm{Li}_2\left(\pm \frac{1}{l_i l_j}\right) \quad \text{for } l_i, l_j \in \mathcal{A}_2 \cup \{l_{33}, l_{38}, l_{41}\},$$

$$\log(\pm l_i) \log(\pm l_j)$$

- Require $1 - x \in \mathrm{Span}_{\mathbb{Q}}(\mathcal{A})$ for each $\mathrm{Li}_2(x)$
- Furthermore, we require $-\infty < x \leq 1$, so not to cross branch cuts of $\mathrm{Li}_2(x)$
- Then, we match the ansatz at the symbol level: $\mathcal{S}\left(B_i^{(k)}\right) = \sum_j \mathcal{S}\left(B_j^{(k-1)}\right) \otimes d\tilde{\mathbf{A}}_{ij}$.

Analytic integration of Family F

- For example, B_{65} at weight 2, in region $\mathcal{R}$ is given by:

$$
\begin{aligned}
B_{65}^{(2)} = & -2\zeta_2 - 4\mathrm{Li}_2\left(-l_{27}^{-1}\right) - 4\mathrm{Li}_2\left(l_{27}^{-1}\right) - 2\mathrm{Li}_2\left(-l_{25}l_{27}^{-1}\right) + 2\mathrm{Li}_2\left(-l_{26}l_{27}^{-1}\right) + 2\mathrm{Li}_2\left(l_{28}l_{27}^{-1}\right) \\
& - 2\mathrm{Li}_2\left(-l_{25}^{-1}l_{27}^{-1}\right) + 2\mathrm{Li}_2\left(-l_{26}^{-1}l_{27}^{-1}\right) + 2\mathrm{Li}_2\left(l_{27}^{-1}l_{28}^{-1}\right) - \log^2\left(l_{25}\right) + \log^2\left(l_{26}\right) - \log^2\left(-l_{27}\right) \\
& + \log^2\left(-l_{28}\right) + 2\log\left(l_{43}\right)\log\left(l_{25}\right) - 2\log\left(l_1\right)\log\left(-l_{27}\right) + 2\log\left(-l_2\right)\log\left(-l_{27}\right) \\
& - 2\log\left(-l_5\right)\log\left(-l_{27}\right) + 2\log\left(-l_6\right)\log\left(-l_{27}\right) + 2\log\left(-l_7\right)\log\left(-l_{27}\right) \\
& - 2\log\left(-l_8\right)\log\left(-l_{27}\right) - 2\log\left(l_{26}\right)\log\left(l_{44}\right) - 2\log\left(-l_{28}\right)\log\left(-l_{48}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}: \quad & t < -4m^2 \,\&\, s < -4m^2 \,\&\, \left(\left(s \leq t \,\&\, \frac{4m^2(s+t) - st}{4m^2} < p_4^2 < \frac{-4m^2 s + st + t^2}{t}\right) \,||\right. \\
& \left.\left(t < s \,\&\, \frac{4m^2(s+t) - st}{4m^2} < p_4^2 < \frac{-4m^2 t + s^2 + st}{s}\right)\right) \,\&\, m^2 > 0\,.
\end{aligned}
$$

Expressions for weight 3 and 4

- Weight 3 can be written as a one-fold integral:

$$\vec{B}^{(i)}(\gamma(1)) = \int_\gamma d\tilde{\mathbf{A}}\vec{B}^{(i-1)} + \vec{B}^{(i)}(\gamma(0))\,.$$

- For weight 4, use an IBP-identity:

$$\begin{aligned}\vec{B}^{(i)}(\gamma(1)) &= \left[\tilde{\mathbf{A}}\vec{B}^{(i-1)}\right]_{\gamma(0)}^{\gamma(1)} - \int_\gamma \tilde{\mathbf{A}}d\vec{B}^{(i-1)} + \vec{B}^{(i)}(\gamma(0))\,,\\ &= \int_\gamma \left(\tilde{\mathbf{A}}(\gamma(1))d\tilde{\mathbf{A}} - \tilde{\mathbf{A}}d\tilde{\mathbf{A}}\right)\vec{B}^{(i-2)} + [\tilde{\mathbf{A}}]_{\gamma(0)}^{\gamma(1)}\vec{B}^{(i-1)}(\gamma(0)) + \vec{B}^{(i)}(\gamma(0))\,,\end{aligned}$$

Boundary conditions

Boundary conditions

- To solve a system of differential equations, we need to compute boundary conditions at some suitable kinematic point or limit
- It is convenient to take a point where most of the external scales vanish, and where the Feynman integrals will simplify considerably
- However, we can't plug singular kinematic point into the Feynman parametrization. For example:

$$\frac{e^{\gamma_E \epsilon}}{i\pi^{1-\epsilon}} \int d^d k_1 \frac{1}{(-k_1^2+m^2)\left(-(k_1+p)^2+m^2\right)} = \frac{2\log\left(\frac{-\sqrt{-p^2}-\sqrt{4m^2-p^2}}{\sqrt{-p^2}-\sqrt{4m^2-p^2}}\right)}{\sqrt{-p^2}\sqrt{4m^2-p^2}} + \mathcal{O}(\epsilon)$$

- In the limit $m^2 = x$, with $x \downarrow 0$ we have at order ϵ^0: $-\frac{2\left(\log\left(-p^2\right)-\log(x)\right)}{p^2} + \mathcal{O}(x)^1$

Boundary conditions

$$-\frac{2\left(\log\left(-p^2\right)-\log(x)\right)}{p^2}+\mathcal{O}(x)^1$$

- Now, suppose we had started directly in the massless limit. We'd find:

$$e^{\gamma_E \epsilon}\left(i\pi^{d/2}\right)^{-1}\int d^d k_1 \frac{1}{\left(-k_1^2\right)\left(-\left(k_1+p\right)^2\right)}=\frac{2}{p^2\epsilon}-\frac{2\log\left(-p^2\right)}{p^2}+\mathcal{O}(\epsilon)$$

- The kinematic singularity has been transformed into a dimensionally regulated pole! We therefore can't use the above expression to fix boundary conditions for the generic case.
- So, how do we obtain boundary conditions without computing the generic mass configuration integral first? – defeating the purpose of choosing a simple boundary point

Boundary conditions

[See works by Beneke and Smirov]

- The solution is to use the method of expansions by regions.
- There is a particularly simple formulation in the parametric representation, which is implemented in the publicly available Mathematica package asy.m

See e.g. [Jantzen, Smirnov, Smirnov, 1206.0546]

- Recall the Feynman parametrization:

$$I_{G,\bar{a}}(S) = \left(i\pi^{\frac{d}{2}}\right)^{l} \Gamma\left(a - \frac{ld}{2}\right) \int_{\mathbb{RP}^{n-1}} \left[d^{n-1}\vec{\alpha}\right] \left(\prod_{i=1}^{n} \frac{\alpha_i^{a_i-1}}{\Gamma\left(a_i\right)}\right) \mathcal{U}^{a-\frac{d}{2}(l+1)} \mathcal{F}^{-a+\frac{ld}{2}}$$

- Where, $\left[d^{n-1}\vec{\alpha}\right] := \sum_{j=1}^{n} (-1)^j \alpha_j d\alpha_1 \wedge \cdots \wedge \widehat{d\alpha_j} \wedge \cdots \wedge d\alpha_n$
- Cheng-Wu: $\int \left[d^{n-1}\vec{\alpha}\right] \to \int_{\mathbb{R}_+^n} d^n\vec{\alpha}\,\delta\left(1 - \sum_{j\in J} \alpha_j\right)$

Expansion by regions

- Suppose we are interested in a kinematic limit $s_i \to x^{\gamma_i} s_i$ for $i = 1, \ldots, p$
- Then there exists a set of regions $\{R_i\}$, where $R_i = (r_{i1}, \ldots, r_{im})$ is a vector of rational numbers.
- For each region R_i we consider the Feynman parametrized integral with the rescaling: $\alpha_j \to x^{R_{ij}} \alpha_j \,, \quad d\alpha_j \to x^{R_{ij}} d\alpha_j \,, \quad s_j \to x^{\gamma_j} s_j$
- The asymptotic limit is then given by summing over the contributions for each region.

Expansion by regions

- Let's have another look at the bubble. We have the Feynman parametrization:

$$\frac{e^{\gamma_E \epsilon}\Gamma(\epsilon+1)}{i\pi^{1-\epsilon}} \int_\Delta d\alpha_1 d\alpha_2 \, (\alpha_1+\alpha_2)^{2\epsilon} \left(\alpha_1^2 m^2 + \alpha_2^2 m^2 + 2\alpha_1\alpha_2 m^2 - \alpha_1\alpha_2 p^2\right)^{-1-\epsilon}$$

- We feed asy.m the $\mathcal{U}$ and $\mathcal{F}$ polynomials, and obtain the regions:

$$R_1 = \{0,0\}, \quad R_2 = \{0,-1\}, \quad R_3 = \{0,1\}$$

- Leading to: $\frac{e^{\gamma_E \epsilon}\Gamma(\epsilon+1)}{i\pi^{1-\epsilon}} \int_\Delta d\alpha_1 d\alpha_2 \Big(x^{-\epsilon} (x\alpha_1+\alpha_2)^{2\epsilon} \left(x^2\alpha_1^2 - p^2\alpha_1\alpha_2 + 2x\alpha_1\alpha_2 + \alpha_2^2\right)^{-1-\epsilon}$
$+ (\alpha_1+\alpha_2)^{2\epsilon} \left(x\alpha_1^2 - p^2\alpha_1\alpha_2 + 2x\alpha_1\alpha_2 + x\alpha_2^2\right)^{-1-\epsilon}$
$+ x^{-\epsilon} (\alpha_1+x\alpha_2)^{2\epsilon} \left(\alpha_1^2 - p^2\alpha_1\alpha_2 + 2x\alpha_1\alpha_2 + x^2\alpha_2^2\right)^{-1-\epsilon} \Big)$

- For the purpose of computing boundary conditions, we often only need the leading term in the expansion in the line parameter

Expansion by regions

- Therefore we obtain:

$$\frac{e^{\gamma_E \epsilon}\Gamma(\epsilon+1)}{i\pi^{1-\epsilon}} \int_\Delta d\alpha_1 d\alpha_2 \Big(x^{-\epsilon}\alpha_2^{-1+\epsilon}\left(-p^2\alpha_1 + m^2\alpha_2\right)^{-1-\epsilon} + \\ +\alpha_1^{-\epsilon-1}\alpha_2^{-\epsilon-1}\left(\alpha_1+\alpha_2\right)^{2\epsilon}\left(-p^2\right)^{-1-\epsilon} + x^{-\epsilon}\alpha_1^{\epsilon-1}\left(\alpha_1 m^2 - \alpha_2 p^2\right)^{-\epsilon-1}\Big)$$

- Although we have a sum of terms, it is clear that each piece is simpler than the Feynman parametrization for the massive bubble. We may perform the integrations and obtain:

$$\frac{\epsilon\left(-p^2\right)^{-\epsilon-1}\Gamma(-\epsilon)^2\Gamma(\epsilon)}{\Gamma(-2\epsilon)} - \frac{2x^{-\epsilon}\Gamma(\epsilon)}{p^2} = -\frac{2\left(\log\left(-p^2\right)-\log(x)\right)}{p^2} + \mathcal{O}(\epsilon)$$

- Which agrees with the result we found before!

Boundary conditions of family F

- Note that the method is not restricted to simple integrals! Take the following master integral from family F:

$$B_{73} = t\epsilon^4 \left(\mathrm{I}_{1,1,1,1,1,1,1,-2,0} + \frac{4s\mathrm{I}_{1,1,1,1,1,1,1,-1,-1}}{2s+t-p_4^2} + \mathrm{I}_{1,1,1,1,1,1,1,0,-2} \right) + \\ - \frac{t\epsilon^4 \left(-4s-t+p_4^2\right)}{4} \left(\mathrm{I}_{1,1,1,1,1,1,1,-1,0} + \mathrm{I}_{1,1,1,1,1,1,1,0,-1} \right) .$$

- And consider the limit $(s, t, p_4^2, m^2) \to (xs, xt, xp_4^2, m^2)$

- Asy:

$$S_1 : \quad \alpha_i \to \alpha_i \, ,$$
$$S_2 : \quad \alpha_i \to \alpha_i \quad \text{for } i = \{1,2,4\} \, , \; \alpha_i \to x\alpha_i \quad \text{for } i = \{3,5,6,7\} \, ,$$

- Scaling:

$$\lim_{x\to 0} \mathrm{I}_{1,1,1,1,1,1,1,\sigma_1,\sigma_2} \sim \mathrm{I}^{(1)}_{1,1,1,1,1,1,1,\sigma_1,\sigma_2} + x^{-\epsilon-1}\mathrm{I}^{(2)}_{1,1,1,1,1,1,1,\sigma_1,\sigma_2} \, ,$$
$$\text{for } (\sigma_1,\sigma_2) \in \{(-2,0),(-1,0),(-1,-1),(0,-1),(0,-2)\} \, ,$$

Boundary conditions of family F

- Hence:

$$\lim_{x\to 0} B_{73} \sim \epsilon^4 x^{-\epsilon} \left[-\frac{4st\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-1,-1}}{p_4^2 - 2s - t} + t\left(\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0} + \mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,0,-2}\right)\right].$$

- (Terms $x^{a+b\epsilon}$ with $a > 0$ have been put to zero, since:

$$x^{a+b\epsilon} = x^a + b\, x^a \log(x)\epsilon + \frac{1}{2} b^2 x^a \log(x)^2\, \epsilon^2 + \cdots$$

and, $\lim_{x\to 0} x^a \log(x) \to 0$ for $a > 0$)

- It remains to compute the leading orders $I^{(2),(x=0)}_{1,1,1,1,1,1,1,\sigma_1,\sigma_2}$

Boundary conditions of family F

- We work out the example: $\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0}$

- Symanzik polynomials:

$$\mathcal{U}^{(2),(x=0)}_{1,1,1,1,1,1,1,0,0} = (\alpha_1+\alpha_2+\alpha_4)\,(\alpha_3+\alpha_5+\alpha_6+\alpha_7)\ ,$$
$$\mathcal{F}^{(2),(x=0)}_{1,1,1,1,1,1,1,0,0} = (\alpha_1+\alpha_2+\alpha_4)\,(\alpha_3+\alpha_5+\alpha_6+\alpha_7)^2\, m^2 - \alpha_2\alpha_4\,(\alpha_3+\alpha_5+\alpha_6+\alpha_7)\, t\,.$$

- Cheng-Wu theorem: $\alpha_3 \to 1-\alpha_5-\alpha_6-\alpha_7\,,$ $\quad \displaystyle\int_0^1\int_0^{1-\alpha_7}\int_0^{1-\alpha_6-\alpha_7} d\alpha_5\, d\alpha_6\, d\alpha_7 = \frac{1}{6}\,.$

$$\begin{aligned}\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0} = {}& \frac{1}{6}\Gamma(2\epsilon+1)e^{2\gamma\epsilon}\left(\prod_{i\in\{1,2,4\}}\int_0^\infty d\alpha_i\right)\Big(8(\epsilon+1)(2\epsilon+1)(m^2)^2\mathcal{F}^{-2\epsilon-3}\mathcal{U}^{3\epsilon+1}\\
&-2(2\epsilon+1)(3\epsilon-1)\,(\alpha_1+\alpha_2+\alpha_4)\left(2\,(\alpha_1+\alpha_2+\alpha_4)\,m^2-\alpha_2\alpha_4 t\right)\mathcal{F}^{-2\epsilon-2}\mathcal{U}^{3\epsilon-2}\\
&-8(\epsilon+1)(2\epsilon+1)m^2\alpha_2\alpha_4 t\mathcal{F}^{-2\epsilon-3}\mathcal{U}^{3\epsilon}-2(2\epsilon+1)m^2\,(\alpha_1+\alpha_2+\alpha_4)\,\mathcal{F}^{-2\epsilon-2}\mathcal{U}^{3\epsilon-1}\\
&+2(\epsilon+1)(2\epsilon+1)t^2\alpha_2^2\alpha_4^2\mathcal{F}^{-2\epsilon-3}\mathcal{U}^{3\epsilon-1}+(3\epsilon-2)(3\epsilon-1)\,(\alpha_1+\alpha_2+\alpha_4)^2\,\mathcal{F}^{-2\epsilon-1}\mathcal{U}^{3\epsilon-3}\Big)\,,\end{aligned}$$

Boundary conditions of family F

- Integrating out any of the remaining 3 parameters naively leads to hypergeometric ${}_2F_1$'s
- Homogenize / projectivize the integrand by letting $\alpha_i \to \alpha_i/\alpha_8$ for $i = 1,2,4$, by including an overall $1/\alpha_8^4$ and a delta function $\delta\left(1 - \sum_{i\in\{1,2,4,8\}} \alpha_i\right)$
- Now pick the Cheng-Wu transform $\alpha_1 \to 1 - \alpha_2 - \alpha_4$
- $$\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0} = \frac{1}{6}\Gamma(2\epsilon+1)e^{2\gamma\epsilon}\int_0^1 d\alpha_4 \int_0^{1-\alpha_4} d\alpha_2 \int_0^\infty d\alpha_8 \Big(\alpha_8^{\epsilon-1}\left(\alpha_8 m^2 - \alpha_2\alpha_4 t\right)^{-2\epsilon-3}$$
$$\times\left(\alpha_8^2 m^4(\epsilon+3)(\epsilon+4) + 2\alpha_2\alpha_4\alpha_8 m^2 t(\epsilon-2)(\epsilon+4) + \alpha_2^2\alpha_4^2 t^2(\epsilon-3)(\epsilon-2)\right)\Big).$$

Boundary conditions of family F

- The remaining integrations can be performed in terms of Γ-functions using:

$$\int (1-\alpha_1)^{-1+n_2}\,\alpha_1^{-1+n_1} d\alpha_1 = \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1+n_2)}\,,\ \text{for}\ \mathrm{Re}(n_1)>0\ \text{and}\ \mathrm{Re}(n_2)>0$$

$$\int \frac{\alpha_1^{n_1-1}}{(1+\alpha_1)^{n_2}} d\alpha_1 = \frac{\Gamma(n_1)\Gamma(n_2-n_1)}{\Gamma(n_2)}\,,\ \text{for}\ \mathrm{Re}\,(n_1)<\mathrm{Re}\,(n_2)\ \ \text{and}\ \ \mathrm{Re}\,(n_1)>0$$

- The final result is given by: $\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0} = \dfrac{2e^{2\gamma\epsilon}\left(m^2\right)^{-\epsilon}(-t)^{-\epsilon-1}\Gamma(-\epsilon)^2\Gamma(\epsilon)\Gamma(\epsilon+2)}{(2\epsilon+2)\Gamma(1-2\epsilon)}\,.$

- In fact, explicit computation shows:

$$\mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-2,0} = \mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,-1,-1} = \mathrm{I}^{(2),(x=0)}_{1,1,1,1,1,1,1,0,-2}\,.$$

- Hence: $\lim\limits_{x\to 0} B_{73} \sim x^{-\epsilon}\left(-4\pi e^{2\gamma\epsilon}\epsilon^3\dfrac{(p_4^2-4s-t)}{(p_4^2-2s-t)}(m^2)^{-\epsilon}(-t)^{-\epsilon}\Gamma(2\epsilon)\cot(\pi\epsilon)\right)\,.$

Boundary conditions of family F

- All boundary conditions for family F:

$$\lim_{x\to 0} B_1 = e^{2\gamma\epsilon}\Gamma(\epsilon+1)^2(m^2)^{-2\epsilon}\,,$$

$$\lim_{x\to 0} B_2 \sim x^{-\epsilon}\left(\pi e^{2\gamma\epsilon}\epsilon(m^2)^{-\epsilon}(-t)^{-\epsilon}\Gamma(2\epsilon+1)\cot(\pi\epsilon)\right)\,,$$

$$\lim_{x\to 0} B_i = 0 \quad \text{for } i = 3,\ldots,72\,.$$

$$\lim_{x\to 0} B_{73} \sim x^{-\epsilon}\left(-4\pi e^{2\gamma\epsilon}\epsilon^3\frac{(p_4^2-4s-t)}{(p_4^2-2s-t)}(m^2)^{-\epsilon}(-t)^{-\epsilon}\Gamma(2\epsilon)\cot(\pi\epsilon)\right)\,.$$

- Requires computation of numerous integrals:

Boundary conditions of family F

```
IntSteps = Association[{
    G[6, {0, 0, 2, 0, 1, 0, 2, 0, 0}]1 → {{"CW", α5 → 1 - α3}, {"Int", {α7, 0, ∞}}, {"Int", {α3, 0, 1}}, {"Save"}},
    G[6, {0, 0, 2, 1, 0, 0, 2, 0, 0}]1 → {{"CW", α7 → 1 - α3}, {"Int", {α4, 0, ∞}}, {"Int", {α3, 0, 1}}, {"Save"}},
    G[6, {0, 0, 2, 1, 0, 2, 0, 0, 0}]1 → {{"CW", α6 → 1 - α3}, {"Int", {α4, 0, ∞}}, {"Int", {α3, 0, 1}}, {"Save"}},
    G[6, {0, 0, 2, 2, 0, 1, 0, 0, 0}]1 → {{"CW", α6 → 1 - α3}, {"Int", {α4, 0, ∞}}, {"Int", {α3, 0, 1}}, {"Save"}},
    G[6, {0, 1, 0, 0, 2, 0, 2, 0, 0}]1 → {{"CW", α5 → 1 - α7}, {"Int", {α2, 0, ∞}}, {"Int", {α7, 0, 1}}, {"Save"}},
    G[6, {0, 1, 0, 0, 2, 2, 0, 0, 0}]1 → {{"CW", α6 → 1 - α5}, {"Int", {α2, 0, ∞}}, {"Int", {α5, 0, 1}}, {"Save"}},
    G[6, {0, 2, 0, 0, 1, 0, 2, 0, 0}]1 → {{"CW", α5 → 1 - α7}, {"Int", {α2, 0, ∞}}, {"Int", {α7, 0, 1}}, {"Save"}},
    G[6, {1, 0, 0, 0, 2, 0, 2, 0, 0}]1 → {{"CW", α5 → 1 - α7}, {"Int", {α1, 0, ∞}}, {"Int", {α7, 0, 1}}, {"Save"}},
    G[6, {1, 0, 2, 0, 0, 2, 0, 0, 0}]1 → {{"CW", α6 → 1 - α3}, {"Int", {α1, 0, ∞}}, {"Int", {α3, 0, 1}}, {"Save"}},
    G[6, {0, 0, 1, 1, 1, 0, 2, 0, 0}]1 → {{"CW", α3 → 1 - α5 - α7}, {"Int", {α4, 0, ∞}}, {"Int", {α5, 0, 1 - α7}}, {"Int", {α7, 0, 1}}, {"Save"}},
    G[6, {0, 0, 1, 1, 1, 0, 3, 0, 0}]1 → {{"CW", α3 → 1 - α5 - α7}, {"Int", {α4, 0, ∞}}, {"Int", {α5, 0, 1 - α7}}, {"Int", {α7, 0, 1}}, {"Save"}},
    G[6, {0, 0, 2, 1, 1, 0, 2, 0, 0}]1 → {{"CW", α3 → 1 - α5 - α7}, {"Int", {α4, 0, ∞}}, {"Int", {α5, 0, 1 - α7}}, {"Int", {α7, 0, 1}}, {"Save"}},
    G[6, {0, 1, 0, 1, 1, 0, 2, 0, 0}]1 → {{"CW", α7 → 1 - α5}, {"Int", {α2, 0, ∞}}, {"Int", {α4, 0, ∞}}, {"Int", {α5, 0, 1}}, {"Save"}},
    G[6, {0, 1, 0, 1, 1, 0, 2, 0, 0}]2 → {{"CW", α5 → 1 - α7}, {"Int", {α7, 0, 1}}, {"Resc", α8}, {"CW", α2 → 1 - α4}, {"Int", {α8, 0, ∞}}, {"Int", {α4, 0, 1}}, {"Save"}},
    G[6, {0, 1, 1, 0, 1, 2, 0, 0, 0}]1 → {{"CW", α3 → 1 - α5}, {"Int", {α2, 0, ∞}}, {"Int", {α5, 0, 1}}, {"Int", {α6, 0, ∞}}, {"Save"}},
    G[6, {0, 1, 1, 0, 1, 3, 0, 0, 0}]1 → {{"CW", α3 → 1 - α5}, {"Int", {α2, 0, ∞}}, {"Int", {α5, 0, 1}}, {"Int", {α6, 0, ∞}}, {"Save"}},
    G[6, {0, 1, 1, 0, 2, 2, 0, 0, 0}]1 → {{"CW", α3 → 1 - α5}, {"Int", {α2, 0, ∞}}, {"Int", {α5, 0, 1}}, {"Int", {α6, 0, ∞}}, {"Save"}},
    G[6, {0, 1, 1, 1, 0, 2, 0, 0, 0}]1 → {{"CW", α6 → 1 - α3}, {"Int", {α4, 0, ∞}}, {"Int", {α2, 0, ∞}}, {"Int", {α3, 0, 1}}, {"Save"}},
    G[6, {0, 1, 1, 1, 0, 2, 0, 0, 0}]2 → {{"CW", α6 → 1 - α3}, {"Int", {α3, 0, 1}}, {"Resc", α8}, {"CW", α2 → 1 - α4}, {"Int", {α8, 0, ∞}}, {"Int", {α4, 0, 1}}, {"Save"}},
    G[6, {0, 1, 1, 2, 0, 2, 0, 0, 0}]2 → {{"CW", α6 → 1 - α3}, {"Int", {α3, 0, 1}}, {"Resc", α8}, {"CW", α2 → 1 - α4}, {"Int", {α8, 0, ∞}}, {"Int", {α4, 0, 1}}, {"Save"}},
    G[6, {0, 2, 2, 1, 1, 0, 0, 0, 0}]1 → {{"CW", α3 → 1 - α5}, {"Int", {α5, 0, 1}}, {"Resc", α8}, {"CW", α2 → 1}, {"Int", {α8, 0, ∞}}, {"Int", {α4, 0, ∞}}, {"Save"}},
    G[6, {1, 0, 1, 0, 1, 0, 2, 0, 0}]1 → {{"CW", α3 → 1 - α5 - α7}, {"Int", {α1, 0, ∞}}, {"Int", {α5, 0, 1 - α7}}, {"Int", {α7, 0, 1}}, {"Save"}},
    G[6, {1, 0, 1, 0, 1, 0, 3, 0, 0}]1 → {{"CW", α3 → 1 - α5 - α7}, {"Int", {α1, 0, ∞}}, {"Int", {α5, 0, 1 - α7}}, {"Int", {α7, 0, 1}}, {"Save"}},
    G[6, {1, 0, 1, 0, 1, 2, 0, 0, 0}]1 → {{"CW", α3 → 1 - α5 - α6}, {"Int", {α1, 0, ∞}}, {"Int", {α5, 0, 1 - α6}}, {"Int", {α6, 0, 1}}, {"Save"}},
    G[6, {1, 0, 1, 0, 1, 3, 0, 0, 0}]1 → {{"CW", α3 → 1 - α5 - α6}, {"Int", {α1, 0, ∞}}, {"Int", {α5, 0, 1 - α6}}, {"Int", {α6, 0, 1}}, {"Save"}},
    G[6, {1, 0, 1, 0, 2, 0, 2, 0, 0}]1→ {{"CW", α3 → 1 - α5 - α7}, {"Int", {α1, 0, ∞}}, {"Int", {α5, 0, 1 - α7}}, {"Int", {α7, 0, 1}}, {"Save"}},
    G[6, {1, 0, 2, 0, 1, 2, 0, 0, 0}]1 → {{"CW", α3 → 1 - α5 - α6}, {"Int", {α1, 0, ∞}}, {"Int", {α5, 0, 1 - α6}}, {"Int", {α6, 0, 1}}, {"Save"}},
    G[6, {1, 1, 0, 1, 1, 0, 1, 0, 0}]2 → {{"CW", α5 → 1 - α7}, {"Int", {α7, 0, 1}}, {"Resc", α8}, {"CW", α2 → 1 - α4}, {"Int", {α8, 0, ∞}}, {"Int", {α4, 0, 1}}, {"Int", {α1, 0, ∞}}, {"Save"}},
    G[6, {1, 1, 0, 1, 2, 0, 1, 0, 0}]2 → {{"CW", α5 → 1 - α7}, {"Int", {α7, 0, 1}}, {"Resc", α8}, {"CW", α2 → 1 - α4}, {"Int", {α8, 0, ∞}}, {"Int", {α4, 0, 1}}, {"Int", {α1, 0, ∞}}, {"Save"}},
    G[6, {1, 1, 1, 1, 0, 1, 0, 0, 0}]2 → {{"CW", α6 → 1 - α3}, {"Int", {α3, 0, 1}}, {"Resc", α8}, {"CW", α2 → 1 - α4}, {"Int", {α8, 0, ∞}}, {"Int", {α4, 0, 1}}, {"Int", {α1, 0, ∞}}, {"Save"}},
    G[6, {1, 1, 1, 1, 0, 2, 0, 0, 0}]2 → {{"CW", α6 → 1 - α3}, {"Int", {α3, 0, 1}}, {"Resc", α8}, {"CW", α2 → 1 - α4}, {"Int", {α8, 0, ∞}}, {"Int", {α4, 0, 1}}, {"Int", {α1, 0, ∞}}, {"Save"}},
    G[6, {1, 1, 1, 1, 1, 1, 1, -2, 0}]2 → {{"CW", α3 → 1 - α5 - α6 - α7}, {"Int", {α5, 0, 1 - α6 - α7}}, {"Int", {α6, 0, 1 - α7}}, {"Int", {α7, 0, 1}}, {"Resc", α8}, {"CW", α1 → 1 - α2 - α4}, {"Int", {α8, 0, ∞}}, {"Int", {α2, 0, 1 - α4}}, {"Int", {α4, 0, 1}}, {"Save"}},
    G[6, {1, 1, 1, 1, 1, 1, 1, -1, -1}]2 → {{"CW", α3 → 1 - α5 - α6 - α7}, {"Int", {α5, 0, 1 - α6 - α7}}, {"Int", {α6, 0, 1 - α7}}, {"Int", {α7, 0, 1}}, {"Resc", α8}, {"CW", α1 → 1 - α2 - α4}, {"Int", {α8, 0, ∞}}, {"Int", {α2, 0, 1 - α4}}, {"Int", {α4, 0, 1}}, {"Save"}},
    G[6, {1, 1, 1, 1, 1, 1, 1, 0, -2}]2 → {{"CW", α3 → 1 - α5 - α6 - α7}, {"Int", {α5, 0, 1 - α6 - α7}}, {"Int", {α6, 0, 1 - α7}}, {"Int", {α7, 0, 1}}, {"Resc", α8}, {"CW", α1 → 1 - α2 - α4}, {"Int", {α8, 0, ∞}}, {"Int", {α2, 0, 1 - α4}}, {"Int", {α4, 0, 1}}, {"Save"}}
}];
```

Series expansion methods

Series expansions of Feynman integrals

- We will follow the series expansion strategy of F. Moriello's paper [1907.13234], which was applied in [1907.13156] (family F) and [1911.06308] (family G), and discuss some additional optimizations.
- Main steps:
  - Write down a sequence of line segments to a kinematic point.
  - Series expand the differential equations along each segment
  - Solve the differential equations in terms of series expansions, along each path, and use the result to fix the boundary conditions for the next path.
- May be used to obtain high-precision numerical results, including stable results near threshold singularities

Series expansions

- Note also the range of previous literature on series expansions. For single scale problems, see e.g.:

S. Pozzorini and E. Remiddi, *Precise numerical evaluation of the two loop sunrise graph master integrals in the equal mass case*, *Comput. Phys. Commun.* **175** (2006) 381–387, [hep-ph/0505041].

U. Aglietti, R. Bonciani, L. Grassi, and E. Remiddi, *The Two loop crossed ladder vertex diagram with two massive exchanges*, *Nucl. Phys.* **B789** (2008) 45–83, [arXiv:0705.2616].

R. Mueller and D. G. Öztürk, *On the computation of finite bottom-quark mass effects in Higgs boson production*, *JHEP* **08** (2016) 055, [arXiv:1512.08570].

B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, *JHEP* **05** (2018) 028, [arXiv:1802.00833].

R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Solving differential equations for Feynman integrals by expansions near singular points*, *JHEP* **03** (2018) 008, [arXiv:1709.07525].

R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Evaluating elliptic master integrals at special kinematic values: using differential equations and their solutions via expansions near singular points*, *JHEP* **07** (2018) 102, [arXiv:1805.00227].

R. Bonciani, G. Degrassi, P. P. Giardino, and R. Gröber, *A Numerical Routine for the Crossed Vertex Diagram with a Massive-Particle Loop*, *Comput. Phys. Commun.* **241** (2019) 122–131, [arXiv:1812.02698].

- For multi-scale problems, series expansions have been considered before in special kinematic limits. See e.g.:

K. Melnikov, L. Tancredi, and C. Wever, *Two-loop $gg \to Hg$ amplitude mediated by a nearly massless quark*, *JHEP* **11** (2016) 104, [arXiv:1610.03747].

K. Melnikov, L. Tancredi, and C. Wever, *Two-loop amplitudes for $qg \to Hq$ and $q\bar{q} \to Hg$ mediated by a nearly massless quark*, *Phys. Rev.* **D95** (2017), no. 5 054012, [arXiv:1702.00426].

R. Bonciani, G. Degrassi, P. P. Giardino, and R. Grober, *Analytical Method for Next-to-Leading-Order QCD Corrections to Double-Higgs Production*, *Phys. Rev. Lett.* **121** (2018), no. 16 162003, [arXiv:1806.11564].

R. Bruser, S. Caron-Huot, and J. M. Henn, *Subleading Regge limit from a soft anomalous dimension*, *JHEP* **04** (2018) 047, [arXiv:1802.02524].

J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, *Double-Higgs boson production in the high-energy limit: planar master integrals*, *JHEP* **03** (2018) 048, [arXiv:1801.09696].

J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, *Double Higgs boson production at NLO in the high-energy limit: complete analytic results*, *JHEP* **01** (2019) 176, [arXiv:1811.05489].

Series expansions

- Canonical basis: $d\vec{B} = \epsilon d\tilde{\mathbf{A}}\vec{B}\,, \quad \vec{B} = \sum_{i\geq 0} \vec{B}^{(i)}\varepsilon^i$
- Consider a contour $\gamma(x) : [0,1] \to \mathbb{C}^p$ where p is the number of external scales.
- Let $\vec{B}(\gamma(x)) = \vec{B}(x)$
- Then order-by-order in ϵ we have:

$$\begin{cases} \dfrac{\partial \vec{B}^{(i)}(x)}{\partial x} = \mathbf{A}_x \vec{B}^{(i-1)}(x). \\ \vec{B}^{(i)}(\gamma(1)) = \displaystyle\int_0^1 \mathbf{A}_x \vec{B}^{(i-1)} dx + \vec{B}^{(i)}(\gamma(0))\,. \end{cases}$$

- Upon series expanding, each integration is of the form:

$\int x^w \log(x)^n\,,$ for $n \in \mathbb{Z}_{\geq 0}$ and $w \in \mathbb{Q}$,

whose primitives have the same form (use integration by parts), e.g.:

$$\int x^{-3/5}\log^2(x)dx = \tfrac{5}{4}x^{2/5}\left(2\log^2(x) - 10\log(x) + 25\right)$$

Basic integration strategy

- Suppose we are on a line (segment) with line parameter x
- Next, suppose that $\mathbf{A}_x = \sum_{j=-m}^{\infty} \mathbf{A}_{x,j} x^{j/n}$, and that we expand up to order $\mathcal{O}(x^{50})$.
- We seek to find a maximal positive point x_δ such that:

$$\left| \left(\mathbf{A}_x + O\left(x^{50}\right)\right)\big|_{x=x_\delta} - \mathbf{A}_x\left(x_\delta\right) \right| \leq \delta$$

  , where δ indicates some desired precision of the matrix expansions.

- It's hard to find x_δ exactly, but we can find an estimate $0 < x'_\delta < x_\delta$, by rescaling the line parameter such that the nearest singularity in the complex plane has distance ≥ 1 from the origin, and looking at the magnitude of the highest order terms

Basic integration strategy

- So, we solve the equation $|\mathbf{A}_{x_\delta,50n}|x_\delta^{50} = \delta$ to get an estimate for x_δ
- Note: we can verify the estimate explicitly, and if it is incorrect decrease x_δ
- We chose δ as a bound on the derivative matrix, and typically the integrated Feynman integrals have a slightly lower precision:
  - Poles may add up and decrease the order of the expansion
  - The coefficients of the Feynman integrals are generally not monotonically decreasing. For example, the coefficients may alternate.
- We can interpret δ as a rough estimate of the final precision on the segment.

Basic integration strategy

- We may now evaluate our integrated results at $\gamma(x_\delta)$, and consider a new line segment which is centered at $\gamma(x_\delta)$, iterating the procedure until we reach the desired endpoint.
- However, to cross singularities and branch-cuts, we have to center expansions on them directly. We will then obtain series that contains terms such as x^{-k}, $\log(x)\, x^k$ and $x^{k/2}$, capturing the analytic behaviour.
- To estimate how close we can go towards a singularity, we can first expand around the singularities and compute the respective x_δ's.

Integration strategy improvements

- Using Mobius transformations we may improve the convergence of the expansions. For example, consider: $f(x) = \dfrac{1}{1/10 + x} + \dfrac{1}{1-x}$
- Then: $f(x) = 9 - 101x + 999x^2 - 10001x^3 + 99999x^4 - 1000001x^5 + \mathcal{O}(x)^6$
- Next, consider the Mobius transformation: $x = \frac{2y}{11-9y}$, so that for $y \in [-1,1]$, we have $x \in [-1/10,1]$.
- We then have: $f(y) = 9 - \dfrac{202y}{11} + 18y^2 - \dfrac{202y^3}{11} + 18y^4 - \dfrac{202y^5}{11} + \mathcal{O}(y)^6$
- And numerically we find: $S_{100} f(y = 11/13) = -0.335377$

$$f(x = 1/2) = 1/3, \quad S_{100} f(x = 1/2) = -1.31477 \ldots \cdot 10^{70},$$

Integration strategy improvements

- Thus, we may improve the integration strategy in the following way:
  - Find the singularity whose real part is nearest on the left of the origin
  - Find the singularity whose real part is nearest on the right of the origin
  - Map these respective singularities to -1, and 1.
- Lastly, we may use (diagonal) Pade approximants to accelerate the convergence of our series. These are rational functions, whose series expansion matches the original series. For example:

$$\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\frac{7x^5}{256}-\frac{21x^6}{1024}+\frac{33x^7}{2048}-\frac{429x^8}{32768}+\frac{715x^9}{65536}-\frac{2431x^{10}}{262144}+O\left(x^{11}\right)$$

$$S_{10}(\sqrt{1+x})|_{x=1/2} - \sqrt{1+1/2} = -2.72 \cdot 10^{-6}$$

- Pade approximant

$$\sqrt{1+x} \approx \frac{1+\frac{22x}{9}+\frac{33x^2}{16}+\frac{11x^3}{16}+\frac{55x^4}{768}}{1+\frac{35x}{18}+\frac{175x^2}{144}+\frac{25x^3}{96}+\frac{25x^4}{2304}-\frac{x^5}{4608}}$$

$$(P_{4,5}(\sqrt{1+x})|_{x=1/2} - \sqrt{1+1/2}) = -3.47 \cdot 10^{-10}$$

Elliptic sectors of family F

- Now that we understand how to setup up expansions for a canonical basis, let's focus on elliptic (and higher order coupled) sectors.

- For family F the elliptic sectors are given by:

$$
\begin{aligned}
B_{66} &= s\epsilon^4 r_2 \mathrm{I}_{0,1,1,1,1,1,1,0,0}\,,\\
B_{67} &= \epsilon^4 r_2 \mathrm{I}_{-2,1,1,1,1,1,1,0,0}\,,\\
B_{68} &= t\epsilon^4\left(p_4^2 - t\right)\left(\mathrm{I}_{1,1,1,1,1,1,1,-1,0} - \mathrm{I}_{1,1,1,1,1,1,1,0,-1}\right)\,,\\
B_{69} &= t\epsilon^4\left(\mathrm{I}_{1,1,1,1,1,1,1,-2,0} - \mathrm{I}_{1,1,1,1,1,1,1,0,-2} + s\left(\mathrm{I}_{1,1,1,1,1,1,1,-1,0} - \mathrm{I}_{1,1,1,1,1,1,1,0,-1}\right)\right)\,,\\
B_{70} &= t\epsilon^4 r_{16}\left(\mathrm{I}_{1,1,1,1,1,1,1,-1,0} + \mathrm{I}_{1,1,1,1,1,1,1,0,-1}\right)\,,\\
B_{71} &= \frac{t\epsilon^4\left(p_4^2 - t\right)^2}{\left(2s + t - p_4^2\right) r_{16}}\mathrm{I}_{1,1,1,1,1,1,1,-1,-1}\,,\\
B_{72} &= t\epsilon^4 r_2 r_5 r_{12} \mathrm{I}_{1,1,1,1,1,1,1,0,0}\,,\\
B_{73} &= t\epsilon^4\Bigg(\mathrm{I}_{1,1,1,1,1,1,1,-2,0} + \frac{4s}{-p_4^2 + 2s + t}\mathrm{I}_{1,1,1,1,1,1,1,-1,-1} + \mathrm{I}_{1,1,1,1,1,1,1,0,-2} +\\
&\quad + \frac{1}{4}\left(4s + t - p_4^2\right)\left(\mathrm{I}_{1,1,1,1,1,1,1,-1,0} + \mathrm{I}_{1,1,1,1,1,1,1,0,-1}\right)\Bigg)
\end{aligned}
$$

Elliptic sectors of family F

- The differential equations are:

$$\frac{\partial}{\partial x_i}\vec{B}_{66-73}(\vec{x},\epsilon) = \sum_{j=0}^{\infty} \epsilon^j \mathbf{A}_{x_i}^{(j)}(\vec{x})\vec{B}_{66-73}(\vec{x},\epsilon) + \vec{G}_{66-73}(\vec{x},\epsilon)$$

- The homogeneous matrix has the following schematic form:

$$\mathbf{A}_{\lambda}^{(0)} = \left(\begin{array}{cc|cccccc} 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & * & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 & 0, \end{array}\right)$$

- We see that integrals 66,67 and 70,71 are <u>coupled</u>.

Solving non-canonical systems

- So far, we have only considered canonical systems of differential equations, which have no homogeneous components. Next, let us consider a coupled system:

Lower order terms, and subtopology terms

$$\partial_x \vec{f} = M \vec{f} + \vec{b}$$

k coupled integrals in some sector

- First we seek to solve the homogeneous system, $\partial_x \vec{g} = M \vec{g}$
- This may be done by combining the system into a k-th order differential equation for any of the f_i, and using the Frobenius method.

Solving non-canonical systems

- Let $\partial = \partial_x$, $g^{(j)} \equiv \partial^j \vec{g}$, and $\vec{g}^{(j)} \equiv M^{(j)} \vec{g}$
- Then we have: $M^{(0)} = 1,$ $M^{(j+1)} = \partial M^{(j)} + M^{(j)} M^{(1)}$ for all $j \geq 1$
- Next, let's obtain a single differential equation for g_1: $\sum_{j=0}^{n} c_j g_1^{(j)} = 0$
- First let: $\tilde{M}_{ij} = M_{1j}^{(i-1)}$, and $\vec{g}^{\partial} = (g_1, \partial g_1, \ldots, \partial^{k-1} g_1)$
- Then $\vec{g}^{\partial} = \tilde{M} \vec{g}$ and $\vec{g} = \tilde{M}^{-1} \vec{g}^{\partial}$
- Similarly, consider the $(k+1) \times k$ – matrix $\left(\tilde{M}_+\right)_{ij} = M_{1j}^{(i-1)}$

  and $(k+1)$ – vector $\vec{g}_+^{\partial} = (g_1, \partial g_1, \ldots, \partial^k g_1)$
- Then: $\vec{g}_+^{\partial} = \tilde{M}_+ \vec{g}$

Solving non-canonical systems

- Using standard algorithms we may find a vector c^T in the left null-space of $\widetilde{M}_+$.
- Then we have $c^T \vec{g}_+^\partial = c^T \tilde{M}_+ \vec{g}^\partial = 0$

  which defines the differential equation we were looking for: $\sum_{j=0}^{n} c_j g_1^{(j)} = 0$
- According to the Frobenius method, we can always find one solution of the form:

$$g_1(x) = x^r s(x), \quad s(x) = \sum_{m=0}^{\infty} s_m x^m$$

  where r is a rational number.
- This solution is found by plugging it as an ansatz into $\sum_{j=0}^{n} c_j g_1^{(j)} = 0$,

  and solving the resulting linear system order-by-order in x.

Solving non-canonical systems

- The leading order defines a polynomial equation for r called the indicial equation. In order for our series solution to be valid, we have to let r be the maximal root of the equation.
- The remaining coefficients s_j may be solved using a recursion relation.
- Next, how do we find the remaining solutions?
- For convenience, let $D_1 = \sum_{i=0}^{k} c_i \partial^i$, and let h denote our Frobenius solution.
- Suppose we have another solution written as $h \times \mu$. Then:

$$0 = D_1(h\mu) = \sum_{i=0}^{k} c_i \partial^i (h\mu) = \sum_{i=0}^{k} \sum_{n=0}^{i} c_i \begin{pmatrix} i \\ n \end{pmatrix} \left(\partial^{i-n} h\right) \left(\partial^n \mu\right)$$

Solving non-canonical systems

$$0 = D_1(h\mu) = \sum_{i=0}^{k} c_i \partial^i (h\mu) = \sum_{i=0}^{k} \sum_{n=0}^{i} c_i \begin{pmatrix} i \\ n \end{pmatrix} \left(\partial^{i-n} h\right) \left(\partial^n \mu\right)$$

- For the coefficient of μ in the above equation, we have simply:

$$\sum_{i=0}^{k} c_i \partial^k h = D_1 h = 0$$

- Thus, we obtained a differential equation for $\partial_x \mu$ of order $k-1$! We may again find one solution for this differential equation using the Frobenius method.
- We can continue recursively this way, until we reach a differential equation of order 1, for which the only solution is given by the Frobenius method.

Solving non-canonical systems

- Suppose now that we have found k solutions for D_1, and consider the Wronskian:

$$W = \begin{vmatrix} h_1 & \cdots & h_k \\ \partial h_1 & \cdots & \partial h_k \\ \vdots & \ddots & \vdots \\ \partial^{h-1} h_1 & \cdots & \partial^{k-1} h_k \end{vmatrix}$$

- Then we have: $G = \tilde{M}^{-1} W, \quad \partial G = MG$

- If we sum over the columns of G, multiplying them by constants, we obtain the most general solution to the homogeneous differential equation $\partial_x \vec{g} = M \vec{g}$

- But, we are interested in the inhomogeneous equation: $\partial_x \vec{f} = M \vec{f} + \vec{b}$

- We can solve it using the same multiplicative trick as before.

Solving non-canonical systems

- Consider the matrix $B = \frac{1}{k}(\vec{b}, \ldots, \vec{b})$
- Furthermore, suppose that $F = GH$, and that:

$$\partial F = MF + B$$

- Then we find: $$F \partial H = B \Rightarrow H = \int G^{-1} B + C$$

  where C is any constant matrix. In particular we may let $C = \mathrm{diag}\,(c_1, \ldots, c_k)$

- Then the most general solution to $\partial_x \vec{f} = M\vec{f} + \vec{b}$ is given by:

$$\vec{g} = \sum_{j=1}^{k} F_k, \quad F = G\left(\int G^{-1} B + C\right)$$

Results

Results for H+j family F

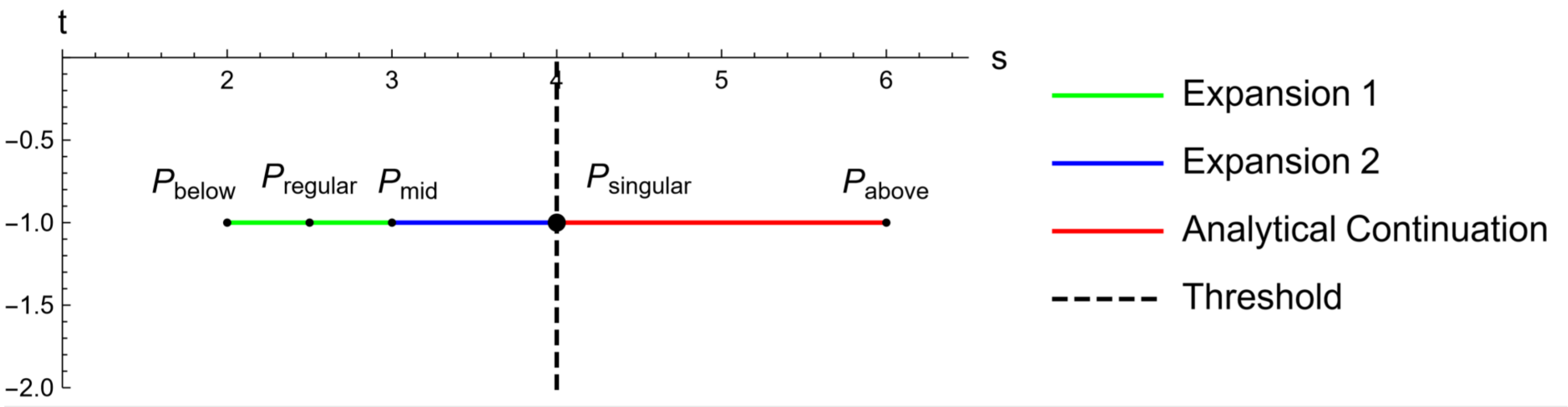
[1907.13156]

- Example: We consider a path $\mathrm{P}_{\text{below}} = (s = 2, t = -1, p_4^2 = 13/25, m^2 = 1)$

$$\Big\downarrow \gamma$$

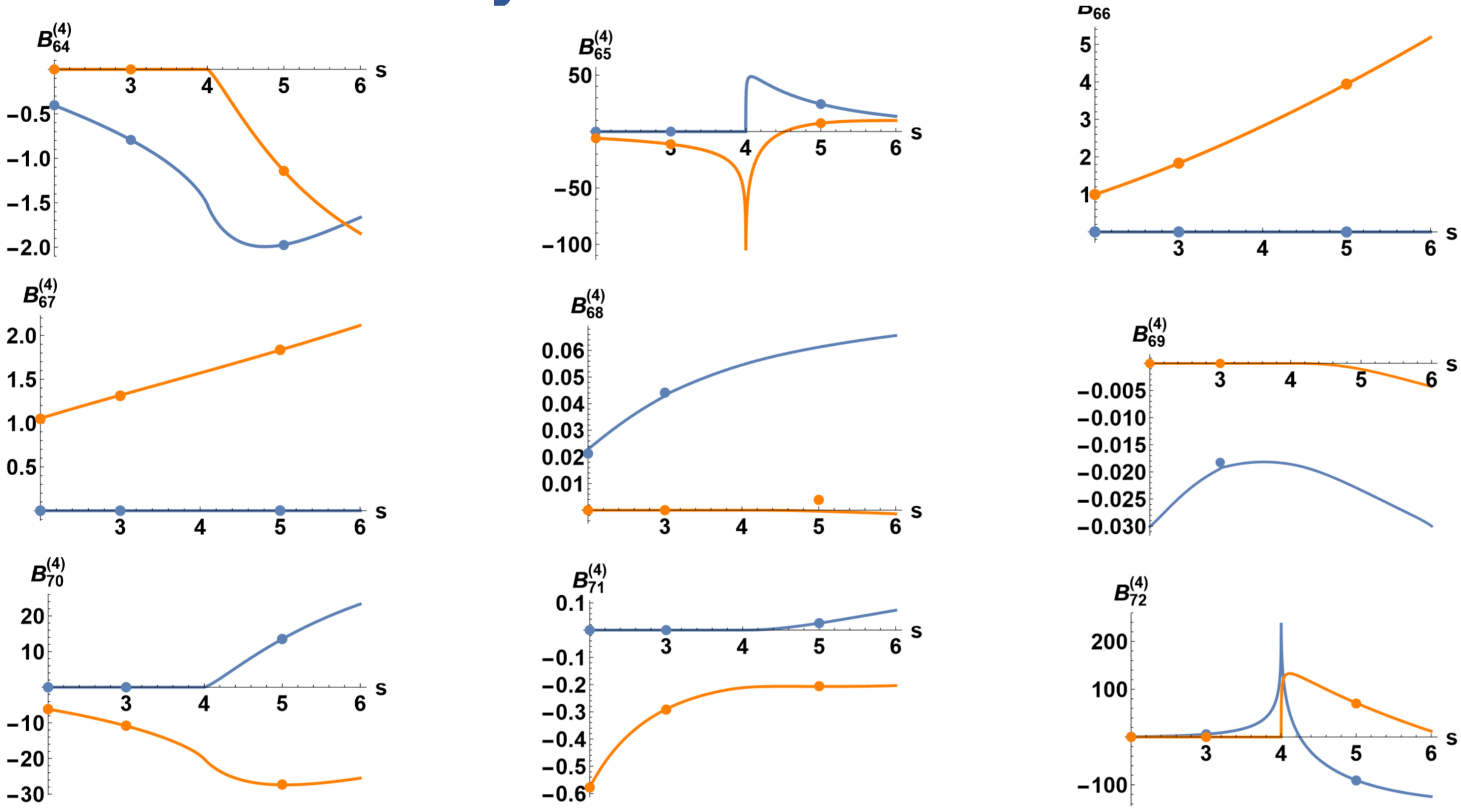
$$\mathrm{P}_{\text{above}} = (s = 6, t = -1, p_4^2 = 13/25, m^2 = 1)$$

which crosses a particle production threshold. Along the path we defined two expansions, one centered at P_{regular} and one at P_{singular}, which are matched at P_{mid}:



Plots for family F

The real part of the integrals is in blue, the imaginary part is orange.



Results for family G

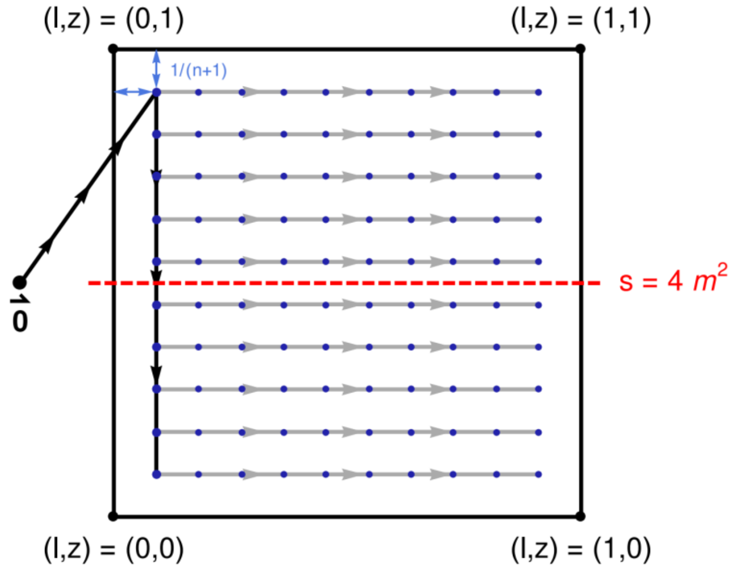
[1911.06308]

- We can also obtain 3-dimensional plots, if we sample enough points.

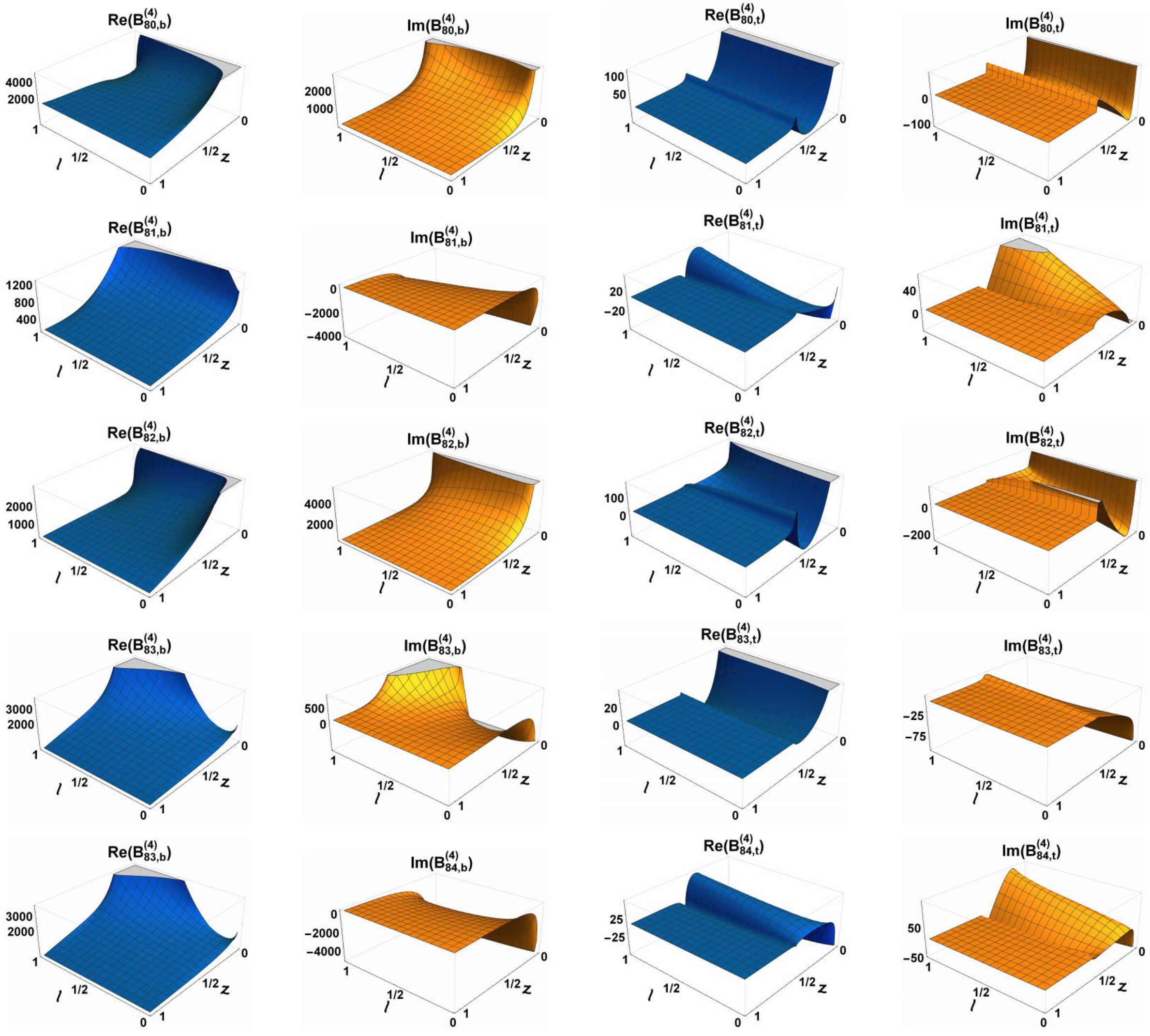
  Consider the parametrization:

$$
\begin{aligned}
&\text{top} \quad (l,z)_t: \quad s = \frac{87-74z}{25z}, \qquad t = \frac{87\,l\,(z-1)}{25z}, \qquad p_4^2 = \frac{13}{25}, \\
&\text{bottom} \quad (l,z)_b: \quad s = \frac{323761}{361z}, \qquad t = \frac{323761\,l\,(z-1)}{361z}, \qquad p_4^2 = \frac{323761}{361}.
\end{aligned}
$$

- Which maps the physical regions of the top quark and bottom quark contributions to the unit square:



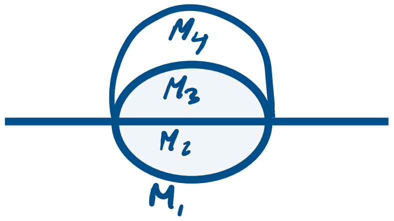
Plots for family G



- These results were sampled from 10000 points.
- We performed numerous internal cross-checks at high precision:

Line(s).	Evaluated at	#Segments ($k=2$)	Max relative error
$\vec{0} \to \left(\frac{1}{101}, \frac{1}{101}\right)_t$	Endpoint	16	$\mathcal{O}(10^{-28})$
$\vec{0} \to \left(\frac{1}{101}, \frac{1}{101}\right)_b$	Endpoint	31	$\mathcal{O}(10^{-26})$
$\left(\frac{x}{101}, \frac{100}{101}\right)_t \to \left(\frac{x}{101}, \frac{1}{101}\right)_t$ for x=1,...,100	$\left(\frac{x}{101}, \frac{y}{101}\right)_t$ for x,y=1,...,100	2568	$\mathcal{O}(10^{-25})$
$\left(\frac{45}{101}, \frac{45}{101}\right)_b \to \left(\frac{1}{101}, \frac{100}{101}\right)_t$	Endpoint	21	$\mathcal{O}(10^{-27})$
$\vec{0} \to \begin{pmatrix} s=53 \\ t=-11 \\ p_4^2=23 \end{pmatrix} \to \left(\frac{100}{101}, \frac{45}{101}\right)_b$	Endpoint	47	$\mathcal{O}(10^{-25})$

Results for 4-mass banana graph



- We can also compute expansions for highly coupled systems.

- First consider the equal-mass case. $\vec{B}^{\text{banana}} = (I^{\text{banana}}_{2211}, I^{\text{banana}}_{2111}, I^{\text{banana}}_{1111}, I^{\text{banana}}_{1110})$

$$\partial_x \vec{B}^{\text{banana}} = \begin{pmatrix} -\frac{x^2(\epsilon+1)+2x(8\epsilon-1)+64(\epsilon+1)}{x(x^2-20x+64)} & \frac{2(x+20)\left(6\epsilon^2+5\epsilon+1\right)}{(x-16)(x-4)x} & -\frac{6\left(24\epsilon^3+26\epsilon^2+9\epsilon+1\right)}{(x-16)(x-4)x} & -\frac{2\epsilon^3}{(x-16)x} \\ \frac{3}{x-4} & \frac{2x\epsilon+x+16\epsilon+4}{4x-x^2} & \frac{12\epsilon^2+7\epsilon+1}{(x-4)x} & 0 \\ 0 & \frac{4}{x} & -\frac{3\epsilon+1}{x} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \vec{B}^{\text{banana}}$$

- With $x = p_1^2/m^2$.

- We consider boundary conditions in the limit $x \to -\frac{1}{\tilde{x}}$, with $\tilde{x} \downarrow 0$,

  which is equivalent to the limit of vanishing mass.

Results for 4-mass banana graph

$$I^{\text{banana}}_{1111} = ie^{3\gamma\epsilon}\Gamma(3\epsilon+1)\left(m^2\right)^{-3\epsilon-1}\tilde{x}^{3\epsilon+1}\int_{\Delta_4} d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \left(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_4\alpha_3+\alpha_2\alpha_4\alpha_3+\right.$$
$$\left.\alpha_1\alpha_2\alpha_4\right)^{4\epsilon}\left(\alpha_2\alpha_3\alpha_1^2\tilde{x}+\alpha_2\alpha_4\alpha_1^2\tilde{x}+\alpha_3\alpha_4\alpha_1^2\tilde{x}+\alpha_2\alpha_3^2\alpha_1\tilde{x}+\alpha_2\alpha_4^2\alpha_1\tilde{x}+\alpha_3\alpha_4^2\alpha_1\tilde{x}+\right.$$
$$\alpha_2^2\alpha_3\alpha_1\tilde{x}+\alpha_2^2\alpha_4\alpha_1\tilde{x}+\alpha_3^2\alpha_4\alpha_1\tilde{x}+4\alpha_2\alpha_3\alpha_4\alpha_1\tilde{x}+\alpha_2\alpha_3\alpha_4^2\tilde{x}+\alpha_2\alpha_3^2\alpha_4\tilde{x}+\alpha_2^2\alpha_3\alpha_4\tilde{x}+$$
$$\left.+\alpha_2\alpha_3\alpha_4\alpha_1\right)^{-3\epsilon-1}.$$

- Asy:

$$\begin{aligned}
&R_1=\{0,-1,-1,-1\}, && R_2=\{0,-1,-1,0\}, && R_3=\{0,0,0,0\},\\
&R_4=\{0,0,0,-1\}, && R_5=\{0,1,1,0\}, && R_6=\{0,0,1,0\},\\
&R_7=\{0,-1,0,-1\}, && R_8=\{0,-1,0,0\}, && R_9=\{0,0,0,1\},\\
&R_{10}=\{0,1,1,1\}, && R_{11}=\{0,0,1,1\}, && R_{12}=\{0,1,0,0\},\\
&R_{13}=\{0,0,-1,-1\}, && R_{14}=\{0,1,0,1\}, && R_{15}=\{0,0,-1,0\}.
\end{aligned}$$

$$\begin{aligned}
&\mathrm{I}^{R_1}_{1111}\sim xe^{3\gamma\epsilon}\Gamma(\epsilon)^3 && \mathrm{I}^{R_2}_{1111}\sim\frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)} && \mathrm{I}^{R_3}_{1111}\sim\frac{3e^{3\gamma\epsilon}\epsilon x^{3\epsilon+1}\Gamma(-\epsilon)^4\Gamma(3\epsilon)}{\Gamma(-4\epsilon)}\\
&\mathrm{I}^{R_4}_{1111}\sim\frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)} && \mathrm{I}^{R_5}_{1111}\sim\frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)} && \mathrm{I}^{R_6}_{1111}\sim xe^{3\gamma\epsilon}\Gamma(\epsilon)^3\\
&\mathrm{I}^{R_7}_{1111}\sim\frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)} && \mathrm{I}^{R_8}_{1111}\sim\frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)} && \mathrm{I}^{R_9}_{1111}\sim xe^{3\gamma\epsilon}\Gamma(\epsilon)^3\\
&\mathrm{I}^{R_{10}}_{1111}\sim\frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)} && \mathrm{I}^{R_{11}}_{1111}\sim\frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)} && \mathrm{I}^{R_{12}}_{1111}\sim xe^{3\gamma\epsilon}\Gamma(\epsilon)^3\\
&\mathrm{I}^{R_{13}}_{1111}\sim\frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)} && \mathrm{I}^{R_{14}}_{1111}\sim\frac{e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)} && \mathrm{I}^{R_{15}}_{1111}\sim\frac{2e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)}
\end{aligned}$$

Results for 4-mass banana graph

- Then we obtain:

$$I_{1111}^{\text{banana}} \overset{\tilde{x}\downarrow 0}{\sim} \frac{6e^{3\gamma\epsilon}\epsilon x^{\epsilon+1}\Gamma(-\epsilon)^2\Gamma(\epsilon)^3}{\Gamma(-2\epsilon)} + \frac{8e^{3\gamma\epsilon}\epsilon x^{2\epsilon+1}\Gamma(-\epsilon)^3\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(-3\epsilon)} + \frac{3e^{3\gamma\epsilon}\epsilon x^{3\epsilon+1}\Gamma(-\epsilon)^4\Gamma(3\epsilon)}{\Gamma(-4\epsilon)} + 4xe^{3\gamma\epsilon}\Gamma(\epsilon)^3 + \mathcal{O}(\tilde{x}^2)\,.$$

$$I_{1110}^{\text{banana}} = e^{3\gamma\epsilon}\Gamma(\epsilon)^3\,.$$

- We may compute the expansions using a (soon to be released) Mathematica package:

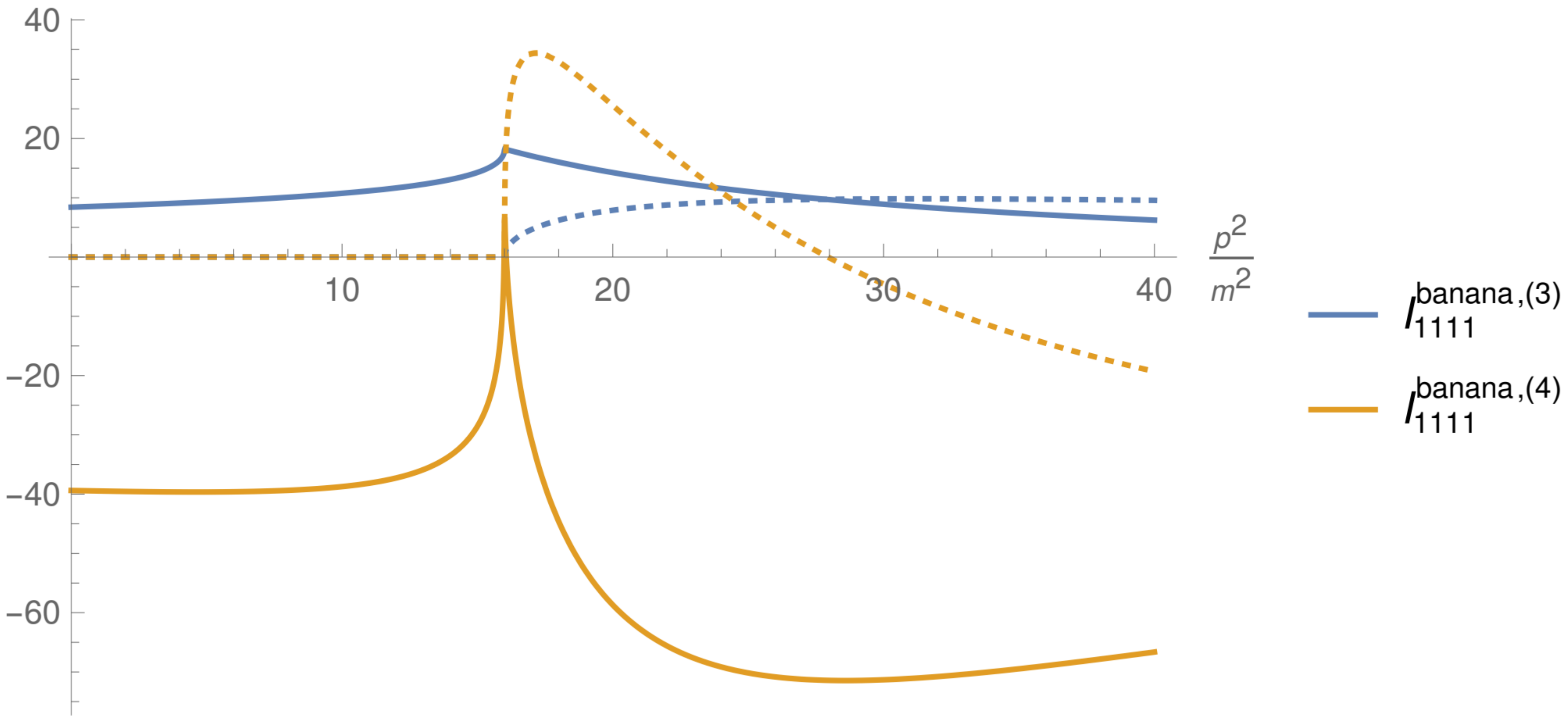
```
BananaBoundaryConditions = {
   "IGNORE",
   "IGNORE",
   eps^3 E^(3*eps*EulerGamma)*(4*x*Gamma[eps]^3 +
    (6*eps*x^(1 + eps)*Gamma[-eps]^2*Gamma[eps]^3)/Gamma[-2*eps] +
    (8*eps*x^(1 + 2*eps)*Gamma[-eps]^3*Gamma[eps]*Gamma[2*eps])/Gamma[-3*eps] +
    (3*eps*x^(1 + 3*eps)*Gamma[-eps]^4*Gamma[3*eps])/Gamma[-4*eps]),
   E^(3*eps*EulerGamma)*Gamma[eps]^3 eps^3
}// PrepareBoundaryConditions[#, <|t -> -1/x|>] &;

Results1 = TransportBoundaryConditions[BananaBoundaryConditions, <|t -> -1/x|>, 1];
Results2 = TransportBoundaryConditions[Results1, <|t -> x|>, 40, {"SaveExpansions"}];

ResultsFunction = ToPiecewiseFunction[Results2[[2]]];

ReImPlot[{ResultComplete[[3, 4]], ResultComplete[[3, 5]]}, {x, 0, 40},
    WorkingPrecision -> 40]
```

Results for 4-mass banana graph



Results for 4-mass banana graph

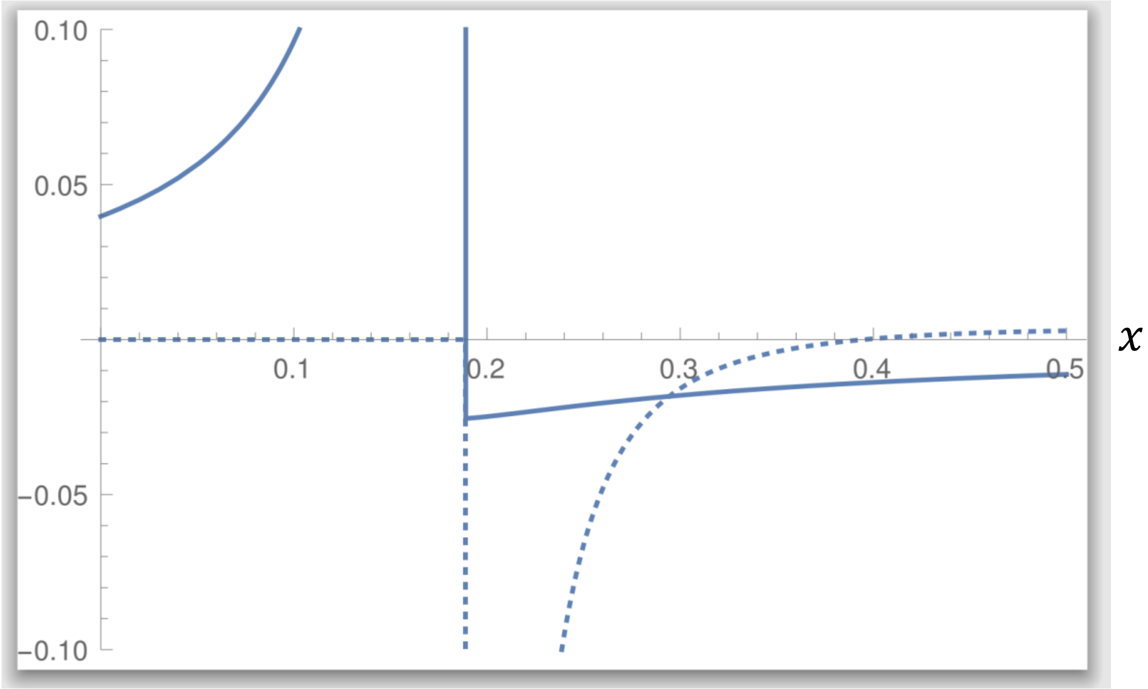
- We can also produce plots in the unequal mass case.
- We choose the basis:

```
DEqBasis = {
   G[1, {1, 1, 2, 2, 0, 0, 0, 0, 0}],
   G[1, {1, 2, 1, 2, 0, 0, 0, 0, 0}],
   G[1, {1, 2, 2, 1, 0, 0, 0, 0, 0}],
   G[1, {2, 1, 1, 2, 0, 0, 0, 0, 0}],
   G[1, {2, 1, 2, 1, 0, 0, 0, 0, 0}],
   G[1, {2, 2, 1, 1, 0, 0, 0, 0, 0}],
   (1 + 3 ϵ) G[1, {1, 1, 1, 2, 0, 0, 0, 0, 0}],
   (1 + 3 ϵ) G[1, {1, 1, 2, 1, 0, 0, 0, 0, 0}],
   (1 + 3 ϵ) G[1, {1, 2, 1, 1, 0, 0, 0, 0, 0}],
   (1 + 3 ϵ) G[1, {2, 1, 1, 1, 0, 0, 0, 0, 0}],
   (1 + 4 ϵ) (1 + 3 ϵ) G[1, {1, 1, 1, 1, 0, 0, 0, 0, 0}],
   ϵ^2 G[1, {0, 1, 1, 1, 0, 0, 0, 0, 0}],
   ϵ^2 G[1, {1, 0, 1, 1, 0, 0, 0, 0, 0}],
   ϵ^2 G[1, {1, 1, 0, 1, 0, 0, 0, 0, 0}],
   ϵ^2 G[1, {1, 1, 1, 0, 0, 0, 0, 0, 0}]
  };
```

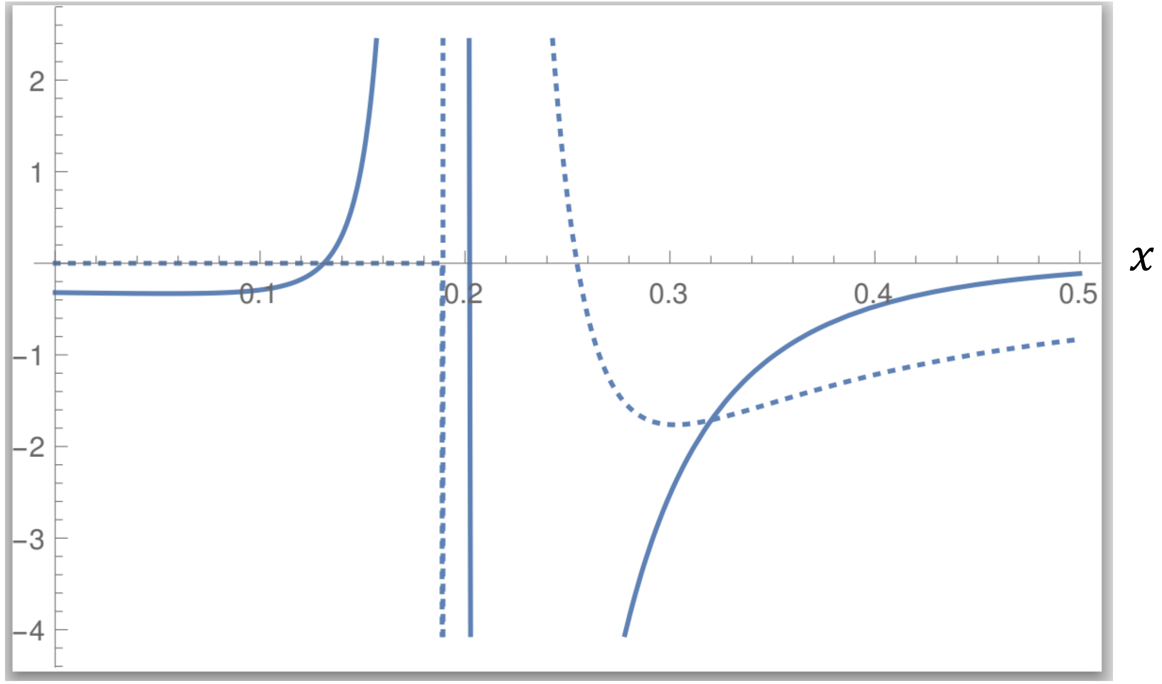
Results for 4-mass banana graph (preliminary)

- Then we find for $p^2 = 200x, m_1^2 = 4, m_2^2 = 3, m_3^2 = 2, m_4^2 = 1$

$B_5^{(1)} = S_{2121}^{(0)}$



$B_5^{(4)}$



Conclusion

- We reviewed the method of differential equations for Feynman integrals
- We discussed how to solve the differential equations in terms of series expansions
- We discussed applications of these methods to:
  - Non-planar Higgs + jet families F and G
  - Beyond-elliptic Feynman integrals

Thank you for listening!