

Contiguous relations for Feynman integrals

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[arXiv:1712.09215]

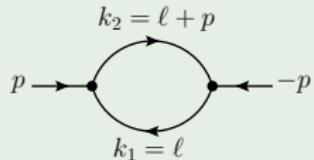
Definition: integral family \mathcal{I}

- loop momenta ℓ_1, \dots, ℓ_L
- denominators D_1, \dots, D_N : quadratic or linear in ℓ

$$\mathcal{I}(s_1, \dots, s_N; d) := \left(\prod_{k=1}^L \int \frac{d^d \ell_k}{i\pi^{d/2}} \right) \frac{1}{D_1^{s_1} \cdots D_N^{s_N}}$$

Example

$$\mathcal{I}(s_1, s_2; d) = \int \frac{d^d \ell}{i\pi^{d/2}} \frac{1}{(\ell^2)^{s_1} ((\ell + p)^2)^{s_2}}$$



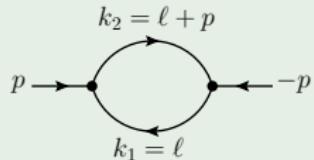
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Topic: relations between $\mathcal{I}(\vec{s} + \vec{n}, d + m)$ where $\vec{n} \in \mathbb{Z}^N$, $m \in 2\mathbb{Z}$

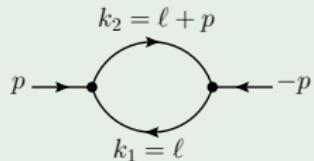
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Why?

$$-ie\Gamma^\mu = \text{Diagram} = \int \frac{d^4\ell}{(2\pi)^4} (-ie\gamma^\nu) \frac{i(\ell' + m)}{\ell'^2 - m^2 + i\epsilon} (-ie\gamma^\mu) \times \frac{i(\ell + m)}{\ell^2 - m^2 + i\epsilon} (-ie\gamma^\rho) \frac{-ig_{\mu\nu}}{(\ell - p)^2 + i\epsilon}$$

Form factors

$$\bar{u}(p')\Gamma^\mu u(p) = \gamma^\mu F_1(q^2) + \frac{[\not{q}, \gamma^\mu]}{4m} F_2(q^2)$$

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Reduction to scalar integrals

- Passarino–Veltman (contractions)
- Tarasov (dimension shifts)

$$F_2(q^2) = -\frac{e^2 m^2}{2\pi^2} \int \frac{d^8\ell}{i\pi^4} \frac{1}{\ell'^2 - m^2 + i\epsilon} \frac{1}{[\ell^2 - m^2 + i\epsilon]^2} \frac{1}{[(\ell - p)^2 + i\epsilon]^2}$$

Applications for recurrences & non-integer \vec{s} and d

- analytic regularization (e.g. $d = 4 - 2\epsilon$) [Speer]
- tensor reduction
- scalar reduction

$$\text{Diagram of a circle with a vertical chord} = \frac{1}{\epsilon} \left[\text{Diagram of a circle with a curved chord} + \text{Diagram of two circles connected by a horizontal line} \right]$$

- factorization of 1-scale subgraphs:

$$\text{Diagram of a complex graph} = \left(\text{Diagram of a circle} \right)^2 \text{Diagram of a complex graph with a scale factor } \epsilon$$

Applications for recurrences & non-integer \vec{s} and d

- analytic regularization (e.g. $d = 4 - 2\epsilon$) [Speer]
- tensor reduction
- scalar reduction

$$\text{Diagram of a 1-scale subgraph (a circle with a vertical chord) equals } \frac{1}{\epsilon} \left[\text{Diagram of a 1-scale subgraph (a circle with a curved self-loop)} + \text{Diagram of two circles connected by a horizontal line} \right]$$

- factorization of 1-scale subgraphs:

$$\text{Diagram of a 2-scale subgraph (a central node connected to four nodes, which are further connected) equals } \left(\text{Diagram of a 1-scale subgraph (a circle with a vertical chord)} \right)^2 \text{Diagram of a 1-scale subgraph with a vertical chord labeled } \epsilon \text{Diagram of a 1-scale subgraph with a vertical chord labeled } \epsilon \text{Diagram of a 1-scale subgraph with a vertical chord labeled } 3\epsilon$$

- differential equations
- recurrences in s_i [Laporta]
- recurrences in d [Lee]



$$\mathcal{I}(\vec{s}, d) := \left(\prod_{k=1}^L \int \frac{d^d \ell_k}{i\pi^{d/2}} \right) \frac{1}{D_1^{s_1} \cdots D_N^{s_N}}$$

Properties:

[Speer '69, Berkesch–Forsgård–Passare '14]

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Properties:

[Speer '69, Berkesch–Forsgård–Passare '14]

- ① converges in a non-empty, open domain
- ② unique, meromorphic extension to \mathbb{C}^{N+1}
- ③ poles are simple, located on hyperplanes

Remark: For $\vec{s} \in \mathbb{Z}^N$, poles coalesce.

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- ① converges in a non-empty, open domain
- ② unique, meromorphic extension to \mathbb{C}^{N+1}
- ③ poles are simple, located on hyperplanes
- ④ $\#\{\text{master integrals}\} = \mathcal{C} := \dim_{\mathbb{C}(\vec{s}, d)} \mathfrak{M} < \infty$ [Smirnov–Petukhov '11]
[Loeser–Sabbah '91]

$$\mathfrak{M} := \sum_{\vec{n} \in \mathbb{Z}^N} \mathbb{C}(\vec{s}, d) \cdot \mathcal{I}(\vec{s} + \vec{n}, d)$$

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$$\mathfrak{M} := \sum_{\vec{n} \in \mathbb{Z}^N} \mathbb{C}(\vec{s}, d) \cdot \mathcal{I}(\vec{s} + \vec{n}, d)$$

- ⑤ \mathfrak{M} is closed under $d \rightarrow d \pm 2$

[Tarasov '96, Baikov]

Remark: For $\vec{s} \in \mathbb{Z}^N$, poles coalesce.

Example

$$\mathcal{I}(s_1, s_2, d) = (-p^2)^{d/2-s_1-s_2} \frac{\Gamma(d/2 - s_1)\Gamma(d/2 - s_2)\Gamma(s_1 + s_2 - d/2)}{\Gamma(s_1)\Gamma(s_2)\Gamma(d - s_1 - s_2)}$$

- ① convergent for

$$\left\{ \operatorname{Re} s_1 < \operatorname{Re} \frac{d}{2} \right\} \cap \left\{ \operatorname{Re} s_2 < \operatorname{Re} \frac{d}{2} \right\} \cap \left\{ \operatorname{Re}(s_1 + s_2) > \operatorname{Re} \frac{d}{2} \right\}$$

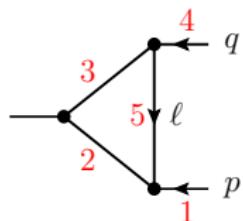
- ② poles on the families of hyperplanes ($k \in \mathbb{Z}_{\leq 0}$):

$$\{d/2 - s_1 = k\} \cup \{d/2 - s_2 = k\} \cup \{s_1 + s_2 - d/2 = k\}$$

- ③ $\mathfrak{M} = \mathbb{C}(s_1, s_2, d) \cdot \mathcal{I}(s_1, s_2, d)$ is 1-dimensional

Integration by parts

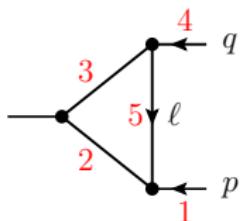
$$0 = \int d^d \ell \frac{\partial}{\partial \ell^\mu} [\ell^\mu f(\ell)]$$



$$f = \frac{1}{p^{2s_1} (\ell + p)^{2s_2} (\ell - q)^{2s_3} q^{2s_4} \ell^{2s_5}}$$

Integration by parts

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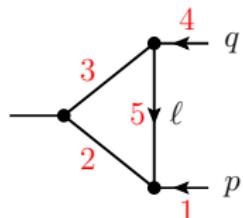
$$f = \frac{1}{p^{2s_1} (\ell + p)^{2s_2} (\ell - q)^{2s_3} q^{2s_4} \ell^{2s_5}}$$

Rewrite:

$$\ell^\mu \frac{\partial}{\partial \ell^\mu} \frac{1}{(\ell + p)^{2s_2}} = -s_2 \frac{2\ell(\ell + p)}{(\ell + p)^{2(s_2+1)}} = -\frac{s_2}{(\ell + p)^{2s_2}} \left(1 + \frac{\ell^2 - p^2}{(\ell + p)^2} \right)$$

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Triangle rule

$$(d - s_2 - s_3 - 2s_5) \mathcal{I} = A_2(B_5 - B_1) \mathcal{I} + A_3(B_5 - B_4) \mathcal{I}$$

$$(A_i \mathcal{I})(\vec{s}) := s_i \mathcal{I}(\vec{s} + \mathbf{e}_i)$$

$$(B_i \mathcal{I})(\vec{s}) := \mathcal{I}(\vec{s} - \mathbf{e}_i)$$

① Symbolic/analytic reductions:

- MINCER
- Gröbner bases
- S -bases
- LiteRed
- FORCER

[Larin et. al. '89,'91]

[Gerdt, Tarasov '04]

[Smirnov–Smirnov '06]

[Lee '12]

[Vermaseren et. al. '16]

② Laporta-like reductions

- solves linear equations on-the-fly
- systems grow with numer of shifts

③ Other methods

Lee-Pomeransky representation (Baikov similar):

- $\omega := s_1 + \dots + s_N - L \frac{d}{2}$
- $\mathcal{G} := \mathcal{U} + \mathcal{F}$

$$\mathcal{I}(\vec{s}) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \left(\prod_{i=1}^N \int_0^\infty \frac{x_i^{s_i-1} dx_i}{\Gamma(s_i)} \right) \mathcal{G}^{-d/2}$$

Example

$$\mathcal{I}(s_1, s_2) = \frac{\Gamma(\frac{d}{2})}{\Gamma(d - s_1 - s_2)} \int_0^\infty \frac{x_1^{s_1-1} dx_1}{\Gamma(s_1)} \int_0^\infty \frac{x_2^{s_2-1} dx_2}{\Gamma(s_2)} \left(\underbrace{x_1 + x_2}_{\mathcal{U}} \underbrace{- p^2 x_1 x_2}_{\mathcal{F}} \right)^{-\frac{d}{2}}$$

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The **Mellin transform** of a function $f: \mathbb{R}_+^N \rightarrow \mathbb{C}$ is

$$\mathcal{M}\{f\}(\vec{s}) := \left(\prod_{i=1}^N \int_0^\infty \frac{x_i^{s_i-1} dx_i}{\Gamma(s_i)} \right) f(x_1, \dots, x_N),$$

whenever this integral exists. Special case:

$$\mathcal{I}(\vec{s}) = \frac{\Gamma(d/2)}{\Gamma(d/2 - \omega)} \tilde{\mathcal{I}}(\vec{s}) \quad \text{for} \quad \tilde{\mathcal{I}}(\vec{s}) = \mathcal{M}\{\mathcal{G}^{-d/2}\}(\vec{s}).$$

Properties of the Mellin transform

① $\mathcal{M}\{\alpha f + \beta g\}(\vec{s}) = \alpha \mathcal{M}\{f\}(\vec{s}) + \beta \mathcal{M}\{g\}(\vec{s}) \quad (\alpha, \beta \in \mathbb{C})$

② $\mathcal{M}\{x_i f\}(\vec{s}) = s_i \mathcal{M}\{f\}(\vec{s} + \mathbf{e}_i)$

$$\int_0^\infty \frac{x_i^{s_i-1} dx_i}{\Gamma(s_i)} (x_i f) = \int_0^\infty \frac{s_i x_i^{s_i} dx_i}{s_i \Gamma(s_i)} f = \int_0^\infty \frac{s_i x_i^{(s_i+1)-1} dx_i}{\Gamma(s_i + 1)} f$$

③ $\mathcal{M}\{-\partial_i f\}(\vec{s}) = \mathcal{M}\{f\}(\vec{s} - \mathbf{e}_i)$

$$\int_0^\infty \frac{x_i^{s_i-1} dx_i}{\Gamma(s_i)} (-\partial_i f) = - \left[\frac{x_i^{s_i-1}}{\Gamma(s_i)} f \right]_{x_i=0}^\infty + \int_0^\infty \frac{x_i^{s_i-2} dx_i}{\Gamma(s_i - 1)} f$$

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$$\textcircled{2} \quad \mathcal{M}\{x_i f\}(\vec{s}) = s_i \mathcal{M}\{f\}(\vec{s} + \mathbf{e}_i) =: (A_i \mathcal{M}\{f\})(\vec{s})$$

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Shift operators:

$$(B_i F)(\vec{s}) := F(\vec{s} - \mathbf{e}_i) \quad (\mathbf{n}_i F)(\vec{s}) = s_i F(\vec{s}) \quad \text{for}$$

$$(A_i F)(\vec{s}) := s_i F(\vec{s} + \mathbf{e}_i) \quad \mathbf{n}_i := A_i B_i$$

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Given any differential operator $P \in A^N[d]$ in the Weyl algebra

$$A^N[d] := \mathbb{C}[d] \langle x_1, \dots, x_N, \partial_1, \dots, \partial_N \mid [\partial_i, x_j] = \delta_{i,j} \rangle$$

such that $P \bullet \mathcal{G}^{-d/2} = 0$ (**annihilator**), the substitutions

$$x_i \mapsto A_i, \quad \partial_i \mapsto -B_i, \quad x_i \partial_i \mapsto -\mathbf{n}_i$$

define a shift operator $\mathcal{M}\{P\} \in S^N[d]$ in the shift algebra

$$S^N[d] := \mathbb{C}[d] \langle A_1, \dots, A_N, B_1, \dots, B_N \mid [-B_j, A_i] = \delta_{i,j} \rangle$$

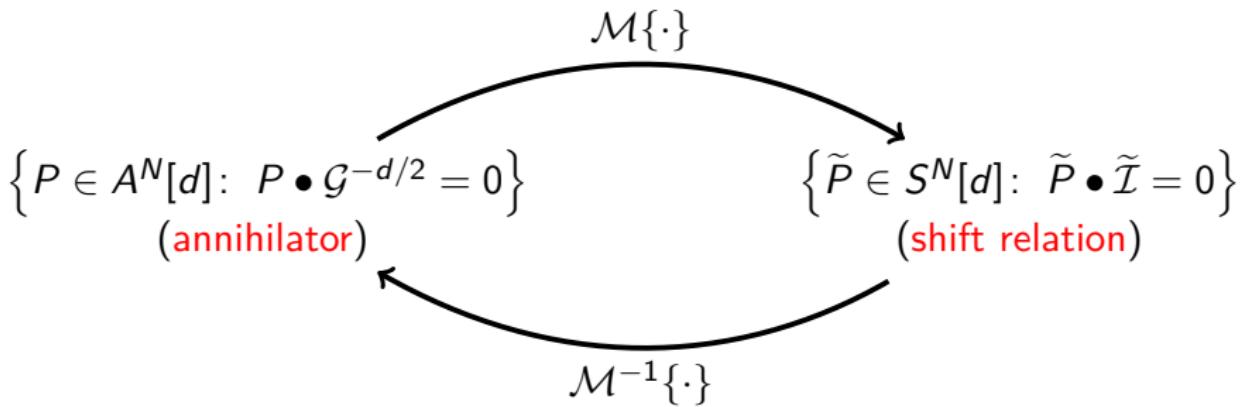
such that $\mathcal{M}\{P\} \bullet \mathcal{M}\{\mathcal{G}^{-d/2}\} = \mathcal{M}\{P \bullet \mathcal{G}^{-d/2}\} = 0$ (**relation**).

Example ($\mathcal{G} = x_1 + x_2 - p^2 x_1 x_2$)

$$\textcircled{1} \quad [(-p^2)(-d/2 - x_1 \partial_1 + 1)x_1 + (-d/2 - x_1 \partial_1 - x_2 \partial_2)] \bullet \mathcal{G}^{-d/2} = 0$$

$$\textcircled{2} \quad (-p^2)(-d/2 + \mathbf{n}_1 + 1)A_1 \tilde{\mathcal{I}} = -(-d/2 + \mathbf{n}_1 + \mathbf{n}_2) \tilde{\mathcal{I}}$$

$$\textcircled{3} \quad (-p^2)s_1 \tilde{\mathcal{I}}(s_1 + 1, s_2) = -\frac{-d/2 + s_1 + s_2}{-d/2 + s_1 + 1} \tilde{\mathcal{I}}(s_1, s_2)$$



The inverse Mellin transform of $f^*(\vec{s}) := \mathcal{M}\{f\}(\vec{s})$ is

$$f(x) = \mathcal{M}^{-1}\{f^*\}(x) = \left(\prod_{i=1}^N \int_{\sigma_i + i\mathbb{R}} \frac{\Gamma(s_i) ds_i}{(2\pi i) x_i^{s_i}} \right) f^*(\vec{s}).$$

Therefore, **every** shift relation comes from an annihilator.

Theorem (Loeser–Sabbah, Bitoun–Bogner–Klausen–Panzer)

$$(-1)^N \mathcal{C} = \chi \left(\mathbb{C}^N \setminus \{x_1 \cdots x_N \mathcal{G} = 0\} \right) = \chi \left((\mathbb{C}^*)^N \setminus \{\mathcal{G} = 0\} \right)$$

\Rightarrow implies finiteness [Smirnov & Petukhov]

Tools to compute the Euler characteristic $\chi(X) = \sum_i (-1)^i \dim H^i(X)$:

- $\chi(X) = \chi(X \setminus Z) + \chi(Z)$ [inclusion–exclusion]
- $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$ [products]
- $\chi(E) = \chi(B) \cdot \chi(F)$ [fibrations $F \rightarrow E \rightarrow B$]
- D -modules and Gröbner bases (e.g. SINGULAR) [Oaku & Takayama]
- CharacteristicClasses in Macaulay2 [Helmer]
- $\mathcal{C} = N! \cdot \text{Vol NP}(\mathcal{G})$ for non-degenerate \mathcal{G} [Kouchnirenko/Khovanskii]

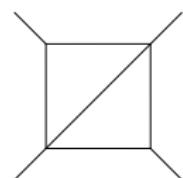
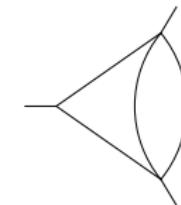
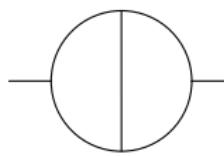
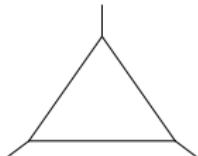
For some infinite families one can prove explicit formulas:

$$\mathcal{C} \left(\text{Diagram A} \right) = \mathcal{C} \left(\text{Diagram B} \right) = \frac{L(L+1)}{2}$$

$$\mathcal{C} \left(\text{Diagram C} \right) = 2^{L+1} - 1 \quad [\text{Kalmykov \& Kniehl}]$$

Further examples (results agree with AZURITE):

Graph G



$\mathcal{C}(G)$ massless
 $\mathcal{C}(G)$ massive

4

7

3

30

4

19

20

55

IBP certificates and relations by Ansatz

Suppose we are given $n = \mathcal{C}$ master integrands in a monomial basis:

$$\vec{B}^\top = (B_1, \dots, B_n) \quad \text{where} \quad B_i = \mathcal{M}\left\{x^{b_i}\right\} \quad \text{for} \quad b_i \in \mathbb{N}_0^N.$$

To reduce an integral $\mathcal{M}\left\{x^{b_0}\right\}$, we are looking for a non-zero solution of

$$\left(\sum_{i=0}^n P_i(d, \vec{\theta}) x^{b_i} \right) \cdot \mathcal{G}^{-d/2} = 0, \quad \text{where} \quad P_0, \dots, P_n \in \mathbb{C}[d, \vec{s}]$$

and $\vec{\theta} = (-x_1 \partial_1, \dots, -x_N \partial_N)$.

IBP certificates and relations by Ansatz

Suppose we are given $n = \mathcal{C}$ master integrands in a monomial basis:

$$\vec{B}^T = (B_1, \dots, B_n) \quad \text{where} \quad B_i = \mathcal{M}\left\{x^{b_i}\right\} \quad \text{for} \quad b_i \in \mathbb{N}_0^N.$$

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and $\vec{\theta} = (-x_1 \partial_1, \dots, -x_N \partial_N)$.

- An Ansatz for the P_i turns this into a sparse linear system.
- Can be used to compute relations between a given set of integrals.

integral reduction with contiguous relations

In a basis $\vec{B}^T = (\mathcal{M}\{x^{b_1}\}, \dots, \mathcal{M}\{x^{b_n}\})$, the shift operators are matrices

$$A_i \vec{B} = \mathbf{M}_i \vec{B} \quad \text{where} \quad \mathbf{M}_i(\vec{s}) \in \mathrm{GL}_n(\mathbb{C}(d, s_1, \dots, s_N)).$$

Once these matrices are computed, and $B_1 = \mathcal{M}\{1\}$, an arbitrary integral

$$\mathcal{M}\{x^s\} \propto \vec{e}_1^T \cdot \left(\prod_{k=1}^N \mathbf{M}_k^{s_k} \right) \cdot \vec{B}$$

can be reduced simply by multiplying and shifting matrices using

$$A_i \cdot \mathbf{M}_j(\vec{s}) = \mathbf{M}_j(\vec{s} + \vec{e}_i) \cdot A_i$$

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- works for non-integer \vec{s}
- straightforward parallelization
- compute \mathbf{M}_k once and for all
- first explicit examples (box and doublebox) [Tarasov, Panzer]

- The Mellin transform translates IBP relations to annihilators [Tkachov, Baikov, Lee, Pomeransky].
- The number of master integrals for generic \vec{s} is

$$\mathcal{C} = \left| \chi\left((\mathbb{C}^*)^N \setminus \{\mathcal{G} = 0\} \right) \right| < \infty.$$

- For generic \vec{s} , IBP relations can be obtained from annihilator Ansatz.
- Reduction simplifies drastically once **contiguous relations** are known.
- Goal: Extend IBP reduction from $\vec{s} \in \mathbb{Z}^N$ to generic \vec{s} .