Noncommutative geometry and particle physics: Pati–Salam unification

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January 16, 2020

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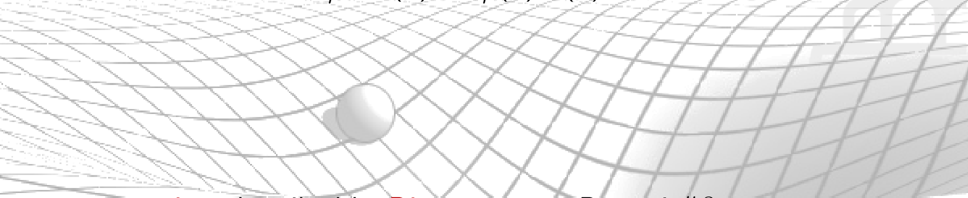
A fermion in spacetime

Minimal ingredients to describe a free fermion:

- coordinates on spacetime M:

$$x_\mu \cdot x_\nu(p) = x_\mu(p) x_\nu(p), etc.,$$



- propagation, described by Dirac operator $D_M = i\gamma^\mu \partial_\mu$

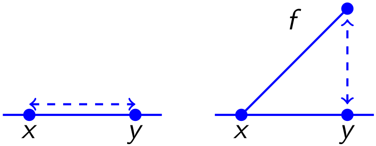
Noncommutative geometry

- This combination of coordinate algebra and operators is central to the spectral, or noncommutative approach [Connes 1994].

$$(\mathcal{A}, \mathcal{H}, D)$$

- Reconstruction of M in the commutative case [Connes 1989]: $(C^\infty(M), L^2(\mathcal{S}_M), D_M)$:

$$d(x,y) = \max_f \{|f(x) - f(y)| : \text{ gradient } f \leq 1\}$$



- The gradient of f is given by the commutator $[D_M, f] = D_M f - f D_M$ $(e.g.\ [D_{\mathbb{S}^1}, f] = -i\frac{df}{dt})$

Emerging bosons

Our fermionic starting point induces a bosonic theory:

- "Inner perturbations" by the coordinates [C 1996, CCS 2013]:

$$D_M \rightsquigarrow D_M + \sum_j a_j [D_M, a_j']$$

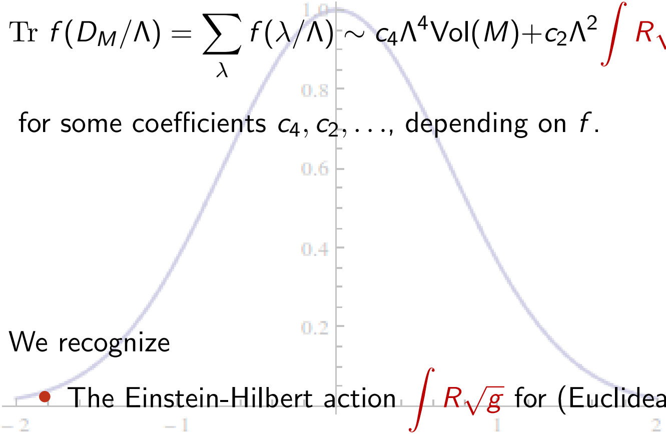
  for functions a_j, a_j' depending on the coordinates x_μ.

- Then,

$$\sum_j a_j [D_M, a_j'] = A^\nu \gamma^\mu (\partial_\mu x^\nu) = A^\mu \gamma_\mu$$

  where A^μ is the electromagnetic 4-potential describing the photon.

Moreover, it is possible to derive a bosonic action from the (Euclidean) Dirac operator via the spectral action [Chamseddine–Connes 1996]:



$$\mathrm{Tr}\, f(D_M/\Lambda) = \sum_\lambda f(\lambda/\Lambda) \sim c_4 \Lambda^4 \mathrm{Vol}(M) + c_2 \Lambda^2 \int R\sqrt{g} + c_0 \int (\partial_{[\mu} A_{\nu]})^2 + \cdots$$

for some coefficients $c_4, c_2, \ldots$, depending on f.

We recognize

- The Einstein-Hilbert action $\int R\sqrt{g}$ for (Euclidean) gravity
- The Lagrangian $\int (\partial_{[\mu} A_{\nu]})^2$ for the electromagnetic field

Noncommutative fine structure of spacetime

Replace spacetime by
spacetime $\times$ **noncommutative space**: $M \times F$

- F is considered as finite internal space (Kaluza–Klein like)
- F is described by noncommutative matrices, that play the role of coordinates, just as spacetime is described by $x_\mu(p)$.
- These matrices geometrically encode non-abelian gauge symmetries.
- ‘Propagation’ of particles in F is described by a 'Dirac operator' D_F which is actually simply a hermitian matrix.

Note that the spectral approach is now the only way to describe the geometry of F.

Finite **commutative** spaces

- Finite space F

$$F = \quad {}_1\bullet \qquad {}_2\bullet \qquad \cdots\cdots \qquad {}_N\bullet$$

- Coordinate functions on F are given by N-tuples in $\mathbb{C}^N$, and the corresponding algebra $C^\infty(F)$ corresponds to diagonal matrices

$$\begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & f(N) \end{pmatrix}$$

- The finite Dirac operator is an arbitrary hermitian matrix D_F, giving rise to a distance function on F as

$$d(p,q) = \max_{f \in C^\infty(F)} \{ |f(p) - f(q)| : \| [D_F, f] \| \leq 1 \}$$

Example: two-point space

$$F = {}_1\bullet \qquad {}_2\bullet$$

- Then the algebra of smooth functions

$$C^\infty(F) := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

- A finite Dirac operator is given by

$$D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}; \qquad (c \in \mathbb{C})$$

- The distance formula then becomes

$$d(1,2) = \max \left\{ |\lambda_1 - \lambda_2| : \left\| \begin{pmatrix} 0 & \overline{c}(\lambda_2 - \lambda_1) \\ c(\lambda_1 - \lambda_2) & 0 \end{pmatrix} \right\| \leq 1 \right\} = \frac{1}{|c|}$$

Finite **noncommutative** spaces

The geometry of F gets much more interesting if we allow for a *noncommutative* structure at each point of F.

- Instead of diagonal matrices, we consider block diagonal matrices

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_N \end{pmatrix},$$

  where the $a_1, a_2, \dots, a_N$ are square matrices of size $n_1, n_2, \dots, n_N$.

- Hence we will consider the matrix algebra

$$A_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C})$$

  where $\mathbb{C}$ can be replaced by $\mathbb{R}$ or $\mathbb{H}$.

- A finite Dirac operator is still given by a hermitian matrix.

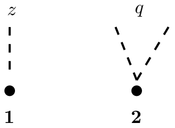
Example: noncommutative two-point space

Coordinates on F are elements in $\mathbb{C} \oplus \mathbb{H}$

- A complex number z
- A quaternion $q = q_0 + i q_k \sigma^k$; in terms of Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It describes a two-point space, with internal structure:



- 'Dirac operator'

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

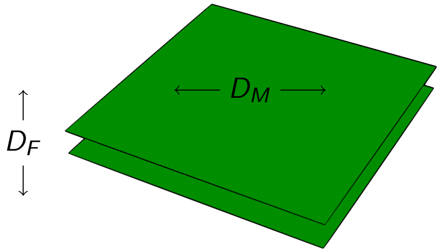
- Inner perturbations:

$$D_F \rightsquigarrow D_F + \sum_j a_j [D_F, a_j'] = \begin{pmatrix} 0 & \overline{c}\overline{\phi_1} & \overline{c}\overline{\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$

- Distance between the two points is now $1/\sqrt{|c\phi_1|^2 + |c\phi_2|^2}$.
- We may call ϕ_1 and ϕ_2 the Higgs field.
- Indeed, the group of unitary block diagonal matrices is now $U(1) \times SU(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Almost-commutative spacetimes



We now combine mild matrix noncommutativity with spacetime:

- coordinates of the almost-commutative spacetime $M \times F$:

$$\boxed{\hat{x}^\mu(p) = (z^\mu(p), q^\mu(p))}$$

  as elements in $\mathbb{C} \oplus \mathbb{H}$ (for each μ and each point p of M)

- The combined Dirac operator becomes

$$\boxed{D_{M\times F} = D_M + \gamma_5 D_F}$$

  Note that $D_{M\times F}^2 = D_M^2 + D_F^2$, which will be useful later on.

Inner perturbations on $M \times F$

So, we describe $M \times F$ by:

$$\hat{x}^\mu = (z^\mu, q^\mu)\,; \qquad D_{M\times F} = D_M + \gamma_5 D_F$$

As before, we consider inner perturbations of $D_{M\times F}$ by $\hat{x}^\mu(p)$:

- The inner perturbations of D_F become scalar fields ϕ_1, ϕ_2.
- The inner perturbations of D_M become matrix-valued:

$$\sum_j a_j [D_M, a_j'] = a_\nu \gamma^\mu (\partial_\mu \hat{x}^\nu) =: A_\mu \gamma^\mu$$

with A_μ taking values in $\mathbb{C} \oplus \mathbb{H}$:

$$A_\mu = \begin{pmatrix} B_\mu & 0 & 0 \\ 0 & W_\mu^3 & W_\mu^+ \\ 0 & W_\mu^- & -W_\mu^3 \end{pmatrix}$$

corresponding to hypercharge and the W-bosons.

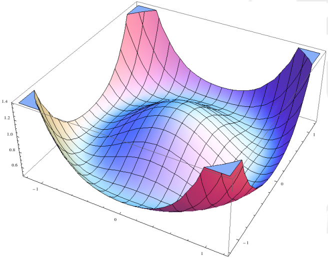
Action functional: electroweak theory

Use $D^2_{M\times F} = D^2_M + D^2_F$ to compute the spectral action (in 4d)

$$\mathrm{Tr}\ f(D_{M\times F}/\Lambda)$$
$$\sim c_4\Lambda^4\mathrm{Vol}(M) + c_2\Lambda^2\int R\sqrt{g} + c_0\int F_{\mu\nu}F^{\mu\nu} - c_2'\frac{|\phi|^2}{\Lambda^2} + c_0'\frac{|\phi|^4}{2\Lambda^4} + \cdots$$

We now recognize in terms of the field-strength $F_{\mu\nu}$ for A_μ:

- The Yang–Mills term $F_{\mu\nu}F^{\mu\nu}$ for hypercharge and W-boson
- The Higgs potential $-\mu^2|\phi|^2 + \lambda|\phi|^4$



Standard Model as an almost-commutative spacetime

Describe $M \times F_{SM}$ by [CCM 2007]

- Coordinates: $\hat{x}^\mu(p) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (with unimodular unitaries $U(1)_Y \times SU(2)_L \times SU(3)$).
- Dirac operator $D_{M\times F} = D_M + \gamma_5 D_F$ where

$$D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

is a 96×96-dimensional hermitian matrix where 96 is:

$$3 \times 2 \times (\underline{2} \otimes \underline{1} + \underline{1} \otimes \underline{1} + \underline{1} \otimes \underline{1} + \underline{2} \otimes \underline{3} + \underline{1} \otimes \underline{3} + \underline{1} \otimes \underline{3})$$

- 3: families
- 2: anti-particles
- $\underline{2} \otimes \underline{1}$: (ν_L, e_L)
- $\underline{1} \otimes \underline{1}$: ν_R
- $\underline{1} \otimes \underline{1}$: e_R
- $\underline{2} \otimes \underline{3}$: (u_L, d_L)
- $\underline{1} \otimes \underline{3}$: u_R
- $\underline{1} \otimes \underline{3}$: d_R

The Dirac operator on F_{SM}

$$D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

- The operator S is given by

$$S_l := \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ Y_\nu^* & 0 & 0 & 0 \\ 0 & Y_e^* & 0 & 0 \end{pmatrix}, \qquad S_q \otimes 1_3 = \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ Y_u^* & 0 & 0 & 0 \\ 0 & Y_d^* & 0 & 0 \end{pmatrix} \otimes 1_3,$$

  where Y_ν, Y_e, Y_u and Y_d are 3×3 mass matrices acting on the three generations.
- The symmetric operator T only acts on the right-handed (anti)neutrinos, $T\nu_R = Y_R\overline{\nu_R}$ for a 3×3 symmetric Majorana mass matrix Y_R, and $Tf = 0$ for all other fermions $f \neq \nu_R$.

Inner perturbations

Just as before, we find

- Inner perturbations of D_M give a matrix

$$A_\mu = \begin{pmatrix} B_\mu & 0 & 0 & 0 \\ 0 & W_\mu^3 & W_\mu^+ & 0 \\ 0 & W_\mu^- & -W_\mu^3 & 0 \\ 0 & 0 & 0 & (G_\mu^a) \end{pmatrix}$$

corresponding to hypercharge, weak and strong interaction.

- Inner perturbations of D_F give

$$\begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \rightsquigarrow \begin{pmatrix} Y_\nu \phi_1 & -Y_e \overline{\phi}_2 \\ Y_\nu \phi_2 & Y_e \overline{\phi}_1 \end{pmatrix}$$

corresponding to SM-Higgs field. Similarly for Y_u, Y_d.

Dynamics and interactions

If we reconsider the spectral action:

$$\mathrm{Tr}\ f(D_{M\times F}/\Lambda) \sim c_4\Lambda^4 \mathrm{Vol}(M) + c_0 \int F_{\mu\nu}F^{\mu\nu} - c_2' \frac{|\phi|^2}{\Lambda^2} + c_0' \frac{|\phi|^4}{2\Lambda^4} + \cdots$$

we observe [CCM 2007]:

- The coupling constants of hypercharge, weak and strong interaction are expressed in terms of the single constant c_0 which implies

  $$\boxed{g_3^2 = g_2^2 = \frac{5}{3} g_1^2}$$

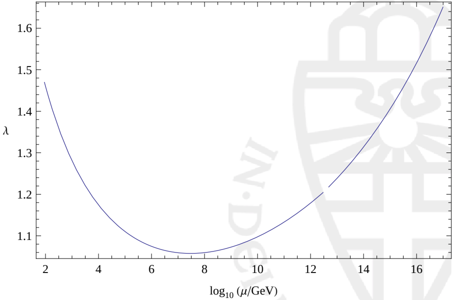
  In other words, there should be grand unification.
- Moreover, the quartic Higgs coupling λ is related via

  $$\lambda \approx 24 \frac{3+\rho^4}{(3+\rho^2)^2} g_2^2; \qquad \rho = \frac{m_\nu}{m_{\mathrm{top}}}$$

This has been used to derive predictions as follows:

- Interpret the spectral action as an effective field theory at $\Lambda_{\text{GUT}} \approx 10^{13} - 10^{16}$ GeV.
- Run the quartic coupling constant λ to SM-energies to predict

$$m_h^2 = \frac{4\lambda M_W^2}{3g_2^2}$$

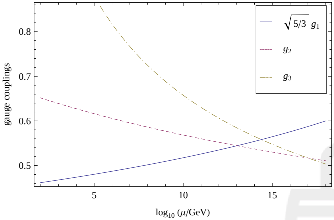


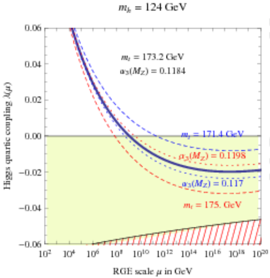
This gives [CCM 2007]

$$\boxed{167 \text{ GeV} \leq m_h \leq 176 \text{ GeV}}$$

Three problems

1. This prediction is falsified by the now measured value.
2. In the Standard Model there is not the presumed grand unification.
3. There is a problem with the low value of m_h, making the Higgs vacuum un/metastable [Elias-Miro et al. 2011].





Beyond the SM with noncommutative geometry
[Chamseddine–Connes–vS, 2013, 2015]

- The matrix coordinates of the Standard Model arise naturally as a restriction of the following coordinates

$$\hat{x}^\mu(p) = \left(q_R^\mu(p), q_L^\mu(p), m^\mu(p)\right) \in \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

corresponding to a Pati–Salam unification:

$$U(1)_Y \times SU(2)_L \times SU(3) \to SU(2)_R \times SU(2)_L \times SU(4)$$

- The 96 fermionic degrees of freedom are structured as

$$\left(\begin{array}{cc|cc} \nu_R & u_{iR} & \nu_L & u_{iL} \\ e_R & d_{iR} & e_L & d_{iL} \end{array}\right) \qquad (i = 1, 2, 3)$$

- Again the finite Dirac operator is a 96×96-dimensional matrix (details in [CCS 2013]).

Inner perturbations

- Inner perturbations of D_M now give three gauge bosons:

$$W_R^\mu, \qquad W_L^\mu, \qquad V^\mu$$

corresponding to $SU(2)_R \times SU(2)_L \times SU(4)$.

- For the inner perturbations of D_F we distinguish two cases, depending on the initial form of D_F:

  I The Standard Model $D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$

  II A more general D_F with zero $\overline{f}_L - f_L$-interactions.

Scalar sector of the spectral Pati–Salam model

I For a SM D_F, the resulting scalar fields are composite fields, expressed in scalar fields whose representations are:

	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\phi_{\dot{a}}^{b}$	2	2	1
$\Delta_{\dot{a}I}$	2	1	4
Σ_{J}^{I}	1	1	15

II For a more general finite Dirac operator, we have fundamental scalar fields:

particle	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\Sigma_{\dot{a}J}^{bJ}$	2	2	$1+15$
$H_{\dot{a}I\dot{b}J}$ {	3	1	10
	1	1	6

Action functional

As for the Standard Model, we can compute the spectral action which describes the usual Pati–Salam model with

- unification of the gauge couplings

$$g_R = g_L = g.$$

- A rather involved, fixed scalar potential, still subject to further study

Phenomenology of the spectral Pati–Salam model

However, independently from the scalar potential we can analyze the running at one loop of the gauge couplings [CCS 2015, AMST 2015]:

1. At Λ there is a unification of the Pati–Salam gauge couplings

$$g_R = g_L = g$$

   This is where the spectral action is valid as an effective theory.

2. Run the Pati–Salam gauge couplings down to a presumed PS $\to$ SM symmetry breaking scale m_R

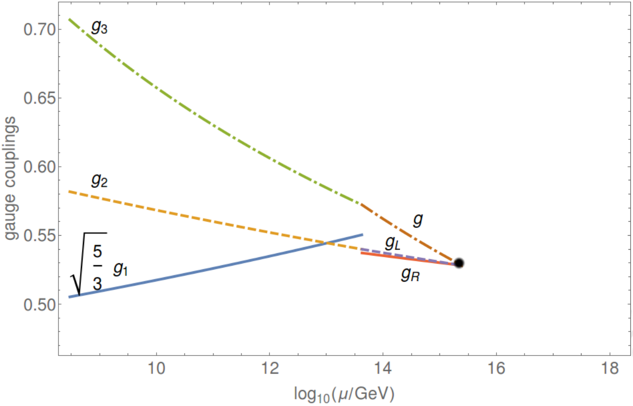
3. Relate their values to the Standard Model gauge couplings at this scale via

$$\frac{1}{g_1^2} = \frac{2}{3}\frac{1}{g^2} + \frac{1}{g_R^2}, \qquad \frac{1}{g_2^2} = \frac{1}{g_L^2}, \qquad \frac{1}{g_3^2} = \frac{1}{g^2},$$

   and run couplings down to scale M_Z.

4. Look for values of m_R and Λ that yield the measured values of g_1, g_2, g_3 at energy scale M_Z.

For the Standard Model Dirac operator, we have found that with $m_R \approx 4.25 \times 10^{13}$ GeV there can be unification at $\Lambda \approx 2.5 \times 10^{15}$ GeV:



In this case, we can also say something about the scalar particles that remain after SSB:

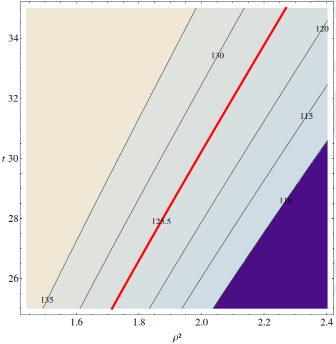
	$U(1)_Y$	$SU(2)_L$	$SU(3)$
$\begin{pmatrix} \phi_1^0 \\ \phi_1^+ \end{pmatrix} = \begin{pmatrix} \phi_{\dot{1}}^{1} \\ \phi_{\dot{1}}^{2} \end{pmatrix}$	1	2	1
$\begin{pmatrix} \phi_2^- \\ \phi_2^0 \end{pmatrix} = \begin{pmatrix} \phi_{\dot{2}}^{1} \\ \phi_{\dot{2}}^{2} \end{pmatrix}$	-1	2	1
σ	0	1	1
η	$-\frac{2}{3}$	1	3

- It turns out that these scalar fields have a little influence on the running of the SM-gauge couplings (at one loop).
- However, this sector contains the real scalar singlet σ of [CC 2012].

- Suppose that the real scalar singlet σ is coupled to the Higgs sector in the following way:

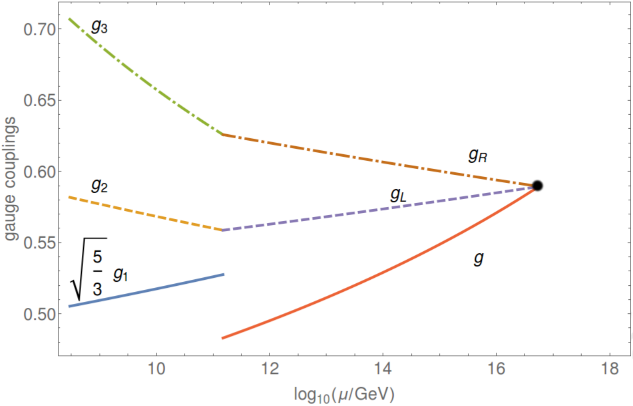
$$V(\sigma, h) = -\frac{4g_2^2}{\pi^2} f_2 \Lambda^2 (h^2 + \sigma^2) + \frac{1}{24}\lambda_h h^4 + \frac{1}{2} h^2 \sigma^2 + \frac{1}{4}\lambda_\sigma \sigma^4$$

- Instead of the notorious Higgs mass prediction from the nc Standard Model, this real scalar singlet gives a Higgs mass varying with $\rho = m_{\text{top}}/m_\nu$ and the unification scale $t = \log(\Lambda_{\text{GUT}}/M_Z)$



- This allows for $m_h = 125.5\text{GeV}$ and $m_\sigma \sim 10^{12}\text{GeV}$.

For the more general case, we have found that with $m_R \approx 1.5 \times 10^{11}$ GeV there can be unification at $\Lambda \approx 6.3 \times 10^{16}$ GeV:



Significance of the spectral action

- Spectral action gives effective field theory at GUT-scale: it is the classical starting point for subsequent (conventional) RG-flow analysis
- Basic obstacle for getting the quantum field theory is that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is a one-particle description of geometry
- So, the question is whether we can extend the framework of noncommutative geometry to multi-particle states?

- The first step is to replace Hilbert space $\mathcal{H}$ by fermionic Fock space

$$\bigwedge \mathcal{H} = \bigoplus_{n \geq 0} \bigwedge^n \mathcal{H}$$

- The one-particle Dirac–Hamiltonian $|D|$ on $\mathcal{H}$ is second-quantized to give

$$\bigwedge (e^{-\beta|D|}) = e^{-\beta H}$$

where H is the second-quantized Hamiltonian:

$$H = |D| \otimes 1 \cdots \otimes 1 + 1 \otimes |D| \otimes \cdots \otimes 1 + 1 \otimes \cdots \otimes 1 \otimes |D|$$

Gibbs states and entropy

- Consider the density matrix for this second-quantized Hamiltonian:

$$\rho_\beta = \mathcal{N}^{-1} \cdot \bigwedge(e^{-\beta|D|})$$

- Of course, this is the Gibbs state for a Fermi gas on the (noncommutative) space that is described by $(\mathcal{A}, \mathcal{H}, D)$.

Theorem (Chamseddine–Connes–vS, 2018)

The (von Neumann) entropy,

$$S(\rho_\beta) = -\mathrm{Tr}\ \rho_\beta \log \rho_\beta,$$

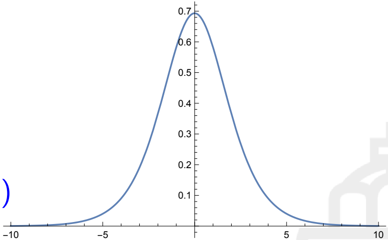
of the above Gibbs state ρ_β is given by a spectral action $\mathrm{Tr}\ h(\beta D)$ for the function $h(x) = \mathcal{E}(e^{-x})$ where $\mathcal{E}(y)$ is the entropy of a partition of unit interval in two intervals with size of ratio y.

Analysis of the function h

We have

$$\mathcal{E}(y) = \log(y+1) - \frac{y \log y}{y+1}$$

$$h(x) = \mathcal{E}(e^{-x}) = \frac{x}{1+e^x} + \log(1+e^{-x})$$



Proposition

The function h is a Laplace transform:

$$h(x) = \int_0^\infty g(t) e^{-tx^2}$$

with

$$g(t) = \frac{-1}{8\sqrt{\pi}\, t^{5/2}} \sum_{n \in \mathbb{Z}} (-1)^n n^2 q^{n^2}; \qquad q = e^{-1/4t}.$$

This allows us to use heat asymptotics of $e^{-t\beta^2 D^2}$.

Asymptotic expansion of entropy

If $\mathrm{Tr}\ e^{-tD^2} \sim \sum_k t^k b_k$ then

$$S(\rho_\beta) = \mathrm{Tr}\ h(\beta D) \sim \sum_k \beta^{2k} \gamma(k) b_k \qquad \gamma(k) = \frac{1-2^{-2k}}{k} \pi^{-k} \xi(2k)$$

in terms of the Riemann ξ-function :

$$\xi(s) := \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$$

$\gamma(-1)$	$\gamma(-1/2)$	$\gamma(0)$	$\gamma(1/2)$	$\gamma(1)$	$\gamma(3/2)$
$\frac{9\zeta(3)}{2}$	$\frac{\pi^{3/2}}{3}$	$\log 2$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{8}$	$\frac{7\zeta(3)}{8\pi^{5/2}}$

Further reading

A. Chamseddine, A. Connes, WvS.

Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification. *JHEP* 11 (2013) 132. [arXiv:1304.8050]

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Noncommutative Geometry and Particle Physics. Mathematical Physics Studies, Springer, 2015.

and also: `http://www.waltervansuijlekom.nl`