Theoretical derivation of diffusion-tensor coefficients for the transport of charged particles in magnetic fields

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i. Formal expression of diffusion-tensor coefficients

Diffusion-tensor coefficients

- Spatial diffusion tensor:

$$D_{ij} = \frac{c^2}{3\nu_{\parallel}}\widehat{b}_i\widehat{b}_j + \frac{c^2}{3\nu_{\perp}}(\delta_{ij} - \widehat{b}_i\widehat{b}_j) + \frac{c^2}{3\nu_A}\epsilon_{ijk}\widehat{b}_k$$

$\left(\langle\omega_i\rangle = \langle\omega\rangle\widehat{b}_i,\ \nu_{\parallel} = \nu,\ \nu_{\perp} = (\nu^2 + \langle\omega\rangle^2)/\nu, \text{ and } \nu_A = (\nu^2 + \langle\omega\rangle^2)/\langle\omega\rangle\right)$

- Coefficients related to the velocity correlation function through a time integration (Kubo, 1957):

$$D_{ij}(t) = \int_0^t \mathrm{d}t' \, \langle v_{0i} v_j(t')\rangle$$

- $\langle\cdot\rangle$: averages taken over several space and time correlation scales of the turbulent field
- Ergodic fluctuations $\rightarrow \langle\cdot\rangle$ = averaging over an ensemble of systems
- Aim: relate $\langle v_i(t)\rangle$ to the statistical properties of the turbulence

Solution as a Dyson series

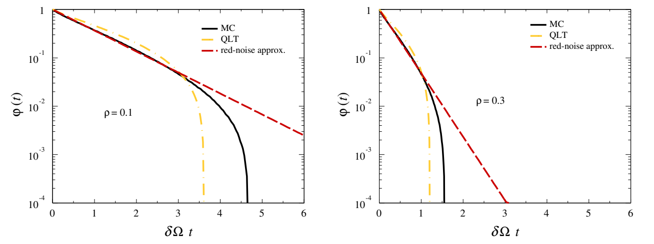
- Equation of motion: $\mathrm{d}v_i(t)/\mathrm{d}t = \delta\Omega\, \epsilon_{ijk} v_j(t)\delta b_k(t)$, with $\delta\Omega = c^2 Z|e|\delta B/E$ the gyrofrequency and $\delta b_k(t) \equiv \delta b_k(\mathbf{x}(t))$ the k-th component of the magnetic field, expressed in units of δB, at the spatial coordinate $\mathbf{x}(t)$ of the particle at time t
- First iterative solution: $\langle v_i(t)\rangle = v_{i0} + \delta\Omega\, \epsilon_{ijk} \int_0^t \mathrm{d}t' \langle v_j(t')\delta b_k(t')\rangle$
- Dyson series:

$$\langle v_{i_0}(t)\rangle = v_{0i_0} + \sum_{n=1}^{\infty} \delta\Omega^n\, \epsilon_{i_0 i_1 j_1}\, \epsilon_{i_1 i_2 j_2} \ldots \epsilon_{i_{n-1} i_n j_n} v_{0 i_n} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \langle \delta b_{j_1}(t_1) \ldots \delta b_{j_n}(t_n)\rangle$$

- Use of the Wick theorem in the Gaussian approximation to express the expectation value in the integrand in terms of all possible permutations of products of contractions of pairs of $\langle \delta b_{i_1}(t_{j_1})\delta b_{i_2}(t_{j_2})\rangle$

ii. Diffusion in isotropic magnetic turbulence

2-pt correlation function of the experienced magnetic field



- Formal expression:

$$\langle \delta b_i(t)\delta b_j(0)\rangle = \iint \mathrm{d}\mathbf{k}\mathrm{d}\mathbf{k}' \, \langle \delta b_i(\mathbf{k})\delta b_j(\mathbf{k}')e^{i\mathbf{k}\cdot\mathbf{x}(t)}\rangle$$

- Ansatz in the case of isotropic turbulence:

$$\langle \delta b_{i_1}(t_{j_1})\delta b_{i_2}(t_{j_2})\rangle = \frac{\delta_{i_1 i_2}}{3}\varphi(t_{j_1} - t_{j_2})$$

- Red-noise approximation for $\varphi(t)$ with an exponential function,

$\varphi(t) \simeq \exp\left(-t/\tau\right)$, $\tau \simeq \frac{1}{c}\frac{\int_{k_\star}^{k_{\max}} \mathrm{d}k k^{-1}\,\mathcal{E}(k)}{\int_{k_\star}^{k_{\max}} \mathrm{d}k\,\mathcal{E}(k)}$

Diagrammatic representation

- Expression for the "propagator" $\langle v_i(t) \rangle = u(t) v_{0i}$:

$$u(t) = 1 + \sum_{n=1}^{\infty} \left(\frac{-2\,\delta\Omega^2}{3} \right)^n \times \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{2n-1}} \mathrm{d}t_{2n} \sum_{\{i<j\}} \prod \varphi(t_i - t_j),$$

  with $\sum_{\{i<j\}} \prod \varphi(t_i - t_j)$ the $(2n-1)!!$ permutations of products of contractions of pairs

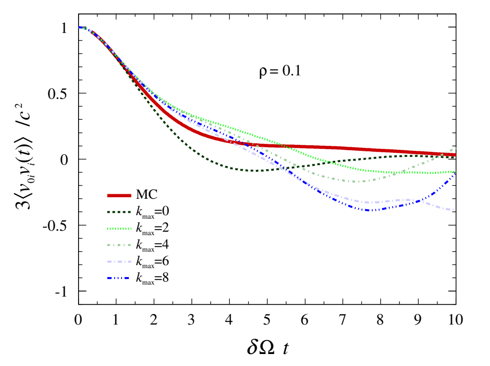
- Representation of the various iterations contributing to $u(t)$ in the form of diagrams with integrations over the ordered times for dashed lines crossing the continuous ones:

- For instance (with $0 \leq t_6 \leq t_5 \leq t_4 \leq t_3 \leq t_2 \leq t_1 \leq t$):

=

$$\left(\frac{-2\delta\Omega^2}{3} \right)^3 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \int_0^{t_3} \mathrm{d}t_4 \int_0^{t_4} \mathrm{d}t_5 \int_0^{t_5} \mathrm{d}t_6 \; \varphi(t_1 - t_3)\varphi(t_2 - t_5)\varphi(t_4 - t_6)$$

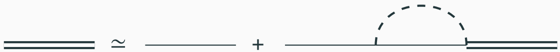
Truncation of the series?



- Summation of all terms up to some order k_{max}?
- Series absolutely convergent for all t, very many terms required for $t > 3/(2\delta\Omega^2\tau)$
- However, combinatorics rapidly non-tractable...

Partial summation

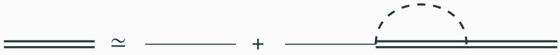
- Simplest scheme: Bourret propagator
- $\rightarrow$ substitute the "mass operator" in the Dyson equation for unconnected contributions:



- $\rightarrow$ sum of unconnected diagrams
- NB: exact solution in the case of white-noise process (cancellation of nested/crossed diagrams)
- Decoupling in Laplace-transform space using the change of variables
- $\rightarrow$ nonphysical oscillations around 0 for $t > 3/(2\delta\Omega^2\tau)$

Partial summation

- Next scheme: Kraichnan propagator
- $\to$ substitute the free propagator inside the dotted loop for the Bourret propagator:

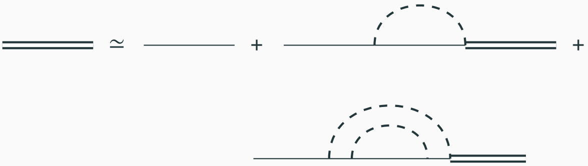


- $\to$ sum of unconnected/nested diagrams
- Series in Laplace-transform space:

$$\left[U^{(2)}(p)\right]^{-1} = p - \sum_{n\geq 1} \frac{\left(-2\,\delta\Omega^2/3\right)^n}{(p+\tau^{-1})^n(p+2\tau^{-1})^{n-1}}$$
$$- \sum_{n\geq 3} \frac{\left(-2\,\delta\Omega^2/3\right)^n}{(p+\tau^{-1})^2(p+2\tau^{-1})^{n-1}(p+3\tau^{-1})^{n-2}}$$
$$- \sum_{n\geq 4} \frac{\left(-2\,\delta\Omega^2/3\right)^n}{(p+\tau^{-1})^2(p+2\tau^{-1})^2(p+3\tau^{-1})^{n-2}(p+4\tau^{-1})^{n-3}}$$
$$- \cdots$$

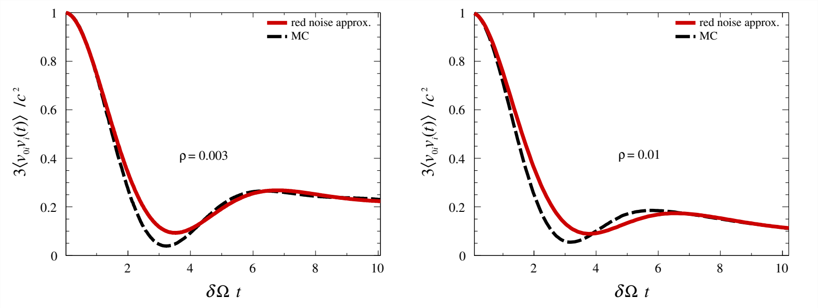
Partial summation

- n formally sent to infinity, but physical solution with a truncation to $n \leq 2$:

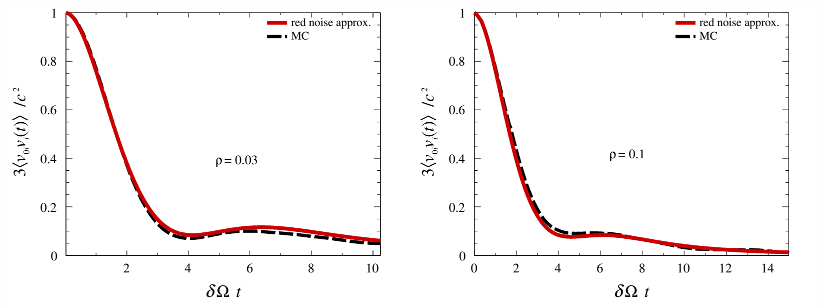


- Applicable in both the "high-rigidity" and "gyro-resonant" regimes (white-noise limit applicable to the high-rigidity regime only)

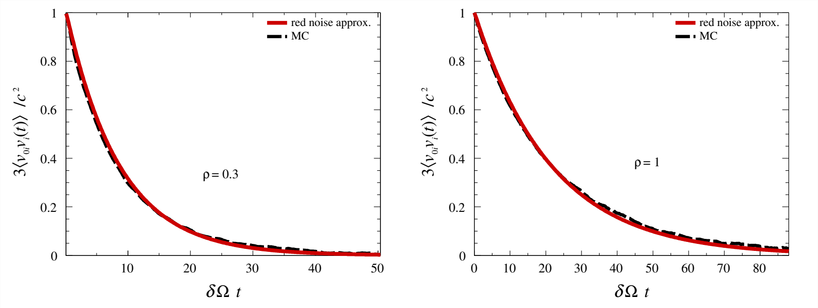
Velocity decorrelations



Velocity decorrelations



Velocity decorrelations



iii. Hints in presence of a mean field

2-pt correlation function of the experienced magnetic field

- $\mathrm{d}v_i(t)/\mathrm{d}t = \delta\Omega\ \epsilon_{ijk} v_j(t)\delta b_k(t) + \Omega_0\ \epsilon_{ijk} v_j(t) b_{0k}$
- Auxiliary variable $\mathbf{v}(t) = \exp{(t\widehat{\Omega}_0)}\mathbf{w}(t)$ (Plotnikov+, A&A 2011)
- $\mathrm{d}\mathbf{w}(t)/\mathrm{d}t = \exp{(t\widehat{\Omega}_0)}^{-1}\widehat{\delta\Omega}(t)\exp{(t\widehat{\Omega}_0)}\mathbf{w}(t)$
- Presence of mean field: turbulence becoming effectively anisotropic:

$$\langle \delta b_{i_1}(t_{j_1})\delta b_{i_2}(t_{j_2})\rangle = \frac{1}{3}\left(\varphi_{\|}(t_{j_1}-t_{j_2})\pi^{\|}_{i_1 i_2} + \varphi_{\perp}(t_{j_1}-t_{j_2})\pi^{\perp}_{i_1 i_2} + \varphi_A(t_{j_1}-t_{j_2})\epsilon_{ijk}\langle\widehat{b}_k\rangle\right)$$

- Longer time-scale memory $\tau_{\|}$ compared to $\tau_{\perp} = \tau_A$ in the red-noise approximation

Outlook

- Red-noise approximation valid for both the gyro-resonant and the high-rigidity regimes
- Regime of Larmor radius smaller than the smallest scale of the turbulence: summation schemes (contribution of crossed diagrams?)
- Better modelings of the $\langle \delta b_i(t) \delta b_j(0) \rangle$ functions...