

Evolution equations for light-cone distribution amplitudes of heavy-light hadrons

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QCD evolution 2016

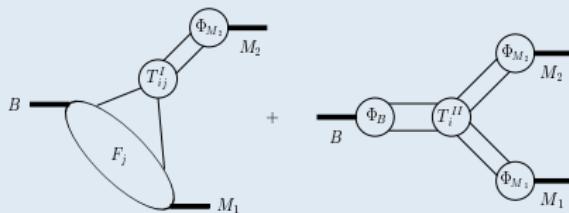


Exclusive B-Decays

- Heavy quark expansion methods ($m_b \gg \Lambda_{\text{QCD}}$)
- Soft-collinear factorization (final state particle energies $\gg \Lambda_{\text{QCD}}$)

BBNS approach:

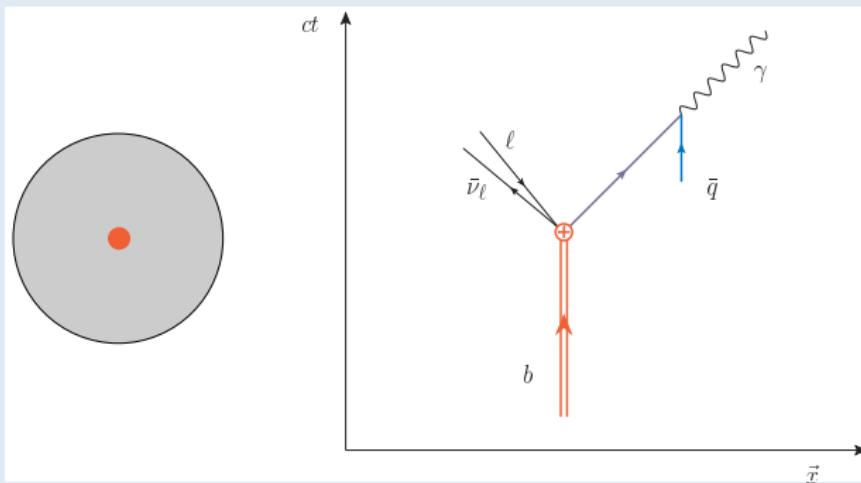
$$\langle M_1 M_2 | O_i | B \rangle = F^{B \rightarrow M_1}(0) \int_0^1 du T^{(1)}(u) \Phi_{M_2}(u) + \int_0^\infty d\omega \int_0^1 du dv T^{(2)}(\omega, u, v) \Phi_+(\omega) \Phi_{M_1}(u) \Phi_{M_2}(v)$$



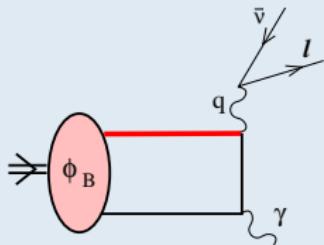
u, v — momentum fractions
 ω — light quark energy
 in B-meson
 $\Phi_{M,B}$ — distribution amplitudes



Simplest exclusive B decay: $B \rightarrow \ell\nu\gamma$



- Decay rate depends on the probability amplitude to find a light antiquark at a given light-like distance from the heavy quark — the B-meson distribution amplitude



$$= \int_0^\infty \frac{d\omega}{\omega} T(\omega, E_\gamma; \mu_F) \Phi_+(\omega, \mu_F)$$



B-Meson Distribution Amplitude

Definition

Grozin, Neubert '97

$$\langle 0 | \left[\bar{q}(zn) \not{n}[zn, 0] \gamma_5 h_v(0) \right]_R | \bar{B}(v) \rangle = i f_B^{\text{stat}}(\mu) \Phi_+(z, \mu)$$

- v_μ is the heavy quark velocity
- n_μ is the light-like vector, $n^2 = 0$, such that $n \cdot v = 1$
- $\Phi_+(z - i0, \mu)$ is analytic function of z in the lower half-plane

In momentum space

$$\Phi_+(\omega, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz e^{i\omega z} \Phi_+(z - i0, \mu)$$

- $\omega > 0$ is the $(2\times)$ light quark energy in the b-quark rest frame

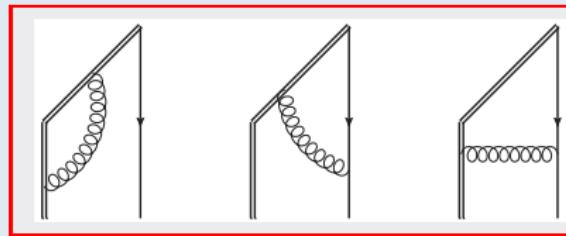


Lange-Neubert evolution equation

light quark with attached cusped Wilson line

Lange, Neubert '03

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \frac{\alpha_s C_F}{\pi} H_{LN} \right) \Phi_+(\omega, \mu) = 0$$



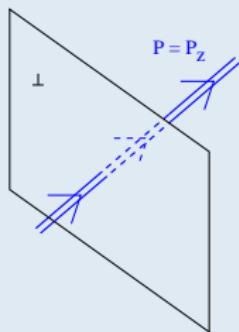
$$[H_{LN}f](\omega) = - \int_0^\infty d\omega' \left[\frac{\omega}{\omega'} \frac{\theta(\omega' - \omega)}{\omega' - \omega} + \frac{\theta(\omega - \omega')}{\omega - \omega'} \right]_+ + \left[\ln \frac{\mu}{\omega} - \frac{5}{4} \right] f(\omega)$$

- Explicit solution found only recently

G. Bell et. al., '13



Collinear conformal transformations



- Special conformal transformation

$$x_- \rightarrow x'_- = \frac{x_-}{1 + 2ax_-}$$

- translations $x_- \rightarrow x'_- = x_- + c$

- dilatations $x_- \rightarrow x'_- = \lambda x_-$

form the so-called **collinear subgroup** $SL(2, R)$

$$\alpha \rightarrow \alpha' = \frac{a\alpha + b}{c\alpha + d}, \quad ad - bc = 1$$

$$\Phi(\alpha) \rightarrow \Phi'(\alpha) = (c\alpha + d)^{-2j} \Phi \left(\frac{a\alpha + b}{c\alpha + d} \right)$$

$$\begin{aligned} p_+ &= \frac{1}{\sqrt{2}}(p_0 + p_z) \rightarrow \infty \\ p_- &= \frac{1}{\sqrt{2}}(p_0 - p_z) \rightarrow 0 \\ px &\rightarrow p_+ x_- \end{aligned}$$

where $\Phi(x) \rightarrow \Phi(x_-) = \Phi(\alpha n_-)$ is the quantum field with scaling dimension ℓ and spin projection s "living" on the light-ray

Conformal spin:

$$j = (l + s)/2$$



Collinear conformal transformations — *continued*

generators $j = 1$ for quarks

$$\begin{aligned} S_+ &= z^2 \partial_z + 2jz \\ S_0 &= z \partial_z + j \\ S_- &= -\partial_z \end{aligned}$$

$SL(2)$ algebra

$$\begin{aligned} [S_+, S_-] &= 2S_0 \\ [S_0, S_+] &= S_+ \\ [S_0, S_-] &= -S_- \end{aligned}$$

- one-loop evolution equations for light quarks (and gluons) are $SL(2)$ invariant

⇒ evolution kernels can be written in terms of the two-particle quadratic Casimir operator

$$S_{12}^2 = S_+^{(12)} S_-^{(12)} + S_0^{(12)} (S_0^{(12)} - 1) \quad S_+^{(12)} = S_+^{(1)} + S_+^{(2)} \dots$$

$$[\mathcal{H}_{light}, S_{\pm,0}] = 0 \quad \Rightarrow \mathcal{H}_{light} = h(S_{12}^2)$$

Bukhvostov, Frolov, Kuraev, Lipatov '85

- Not true for heavy-light evolution kernels



Conformal symmetry of the LN equation

Translation invariance is broken, however

Knödlseder, Offen '11

$$[S_+, \mathcal{H}_{LN}] = 0 \quad [S_0, \mathcal{H}_{LN}] = 1$$

S_+ : generator of special conformal transformations, S_0 : generator of dilatations

- These commutation relations fix \mathcal{H}_{LN} up to a constant

VB, Manashov '14

$$\mathcal{H}_{LN} = \ln(i\mu S^+) - \psi(1) - \frac{5}{4}$$

in position space

$$S_+ = z^2 \partial_z + 2z$$

in momentum space

$$\mathcal{S}_+ = i[\omega \partial_\omega^2 + 2\partial_\omega]$$



Solution of the LN equation

\mathcal{H}_{LN} and S_+ share the same eigenfunctions; in position space

$$iS_+ Q_s(z) = s Q_s(z) \Rightarrow Q_s(z) = -\frac{1}{z^2} \exp\left\{\frac{is}{z}\right\}$$

Inversion: $JS_- = S_+ J$

$$\begin{aligned} Jf(z) &= \frac{1}{z^2} f\left(-\frac{1}{z}\right) \\ Q_s(z) &= -Je^{-isz} \end{aligned}$$

eigenvalues \equiv anomalous dimensions

$$\mathcal{H}_{LN} Q_s(z) = \left[\ln(i\mu S^+) - \psi(1) - \frac{5}{4} \right] Q_s(z) = \left[\ln(\mu s) - \psi(1) - \frac{5}{4} \right] Q_s(z)$$

solution in momentum space

$$\Phi_+(\omega, \mu) = \int_0^\infty ds \sqrt{\omega s} J_1(2\sqrt{\omega s}) \eta(s, \mu) \quad s_0 = e^{5/4 - \gamma_E}$$

$$\eta(s, \mu) = \eta(s, \mu_0) \left(\frac{\mu}{\mu_0}\right)^{-\frac{8}{3\beta_0}} \left(\frac{\mu_0 s}{s_0}\right)^{\frac{8}{3\beta_0} \ln L} L^{-\frac{16\pi}{3\beta_0^2 \alpha_s(\mu_0)}} \quad L = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$$

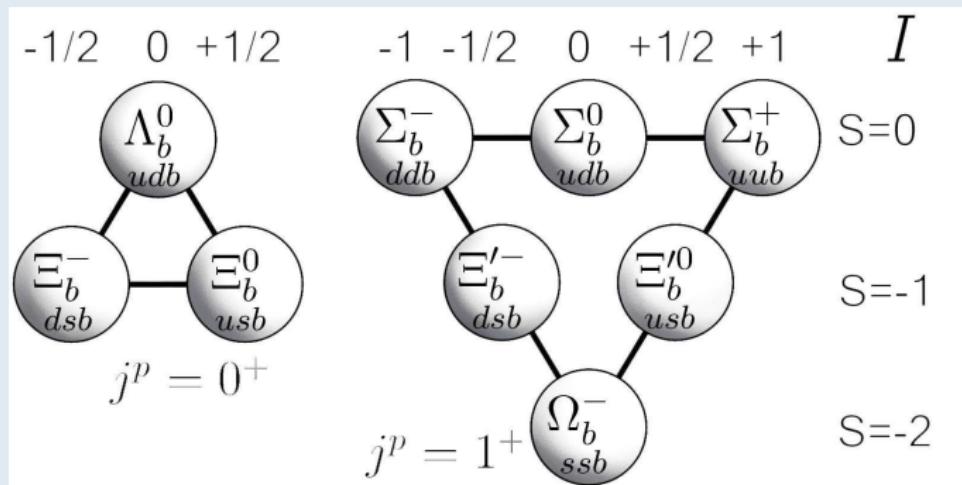
- The same result obtained in

Bell, Feldmann, Y.-M. Wang, M. W. Y. Yip, JHEP 1311 (2013) 191



Heavy-light baryons

can be classified according to spin and parity of the light diquark



$\Sigma_b, \Xi_b, \Omega_b$ and $\Sigma_b^*, \Xi_b^*, \Omega_b^*$ sextets are degenerate in the heavy b -quark limit



Evolution equations

*Ball, Braun, Gardi, '08
Ali, Hambrock, Parkhomenko, Wang, '13*

- RG equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} + \frac{2\alpha_s}{3\pi} \mathbb{H} \right) \Psi_B(z_1, z_2, \mu) = 0.$$

- Kernels

$$q^\uparrow q^\uparrow \quad \mathbb{H}_\perp = \ln(i\mu S_+^{(1)}) + \ln(i\mu S_+^{(2)}) + 2\psi(J_{12}) - 4\psi(2)$$

$$q^\uparrow q^\downarrow \quad \mathbb{H}_\Lambda = \ln(i\mu S_+^{(1)}) + \ln(i\mu S_+^{(2)}) + 2\psi(J_{12}) - 4\psi(2) - \frac{1}{J_{12}(J_{12}-1)}$$

where $S_{12}^2 = J_{12}(J_{12}-1)$ is the SL(2) quadratic Casimir operator



$q^\uparrow q^\uparrow$: Complete integrability

Braun, Derkachov, Manashov, [arXiv:1406.0664]

- Two conserved charges:

$$\mathbb{Q}_1 = i(S_+^{(1)} + S_+^{(2)}) , \quad \mathbb{Q}_2 = S_0^{(1)} S_+^{(2)} - S_0^{(2)} S_+^{(1)}$$

first/second order differential operators in position space

$$\mathbb{Q}_1 = i(\partial_{z_1} z_1^2 + \partial_{z_2} z_2^2) , \quad \mathbb{Q}_2 = \partial_{z_1} \partial_{z_2} z_1 z_2 (z_1 - z_2)$$

- Commute with each other and with evolution kernel

$$[\mathbb{Q}_1, \mathbb{H}_\perp] = [\mathbb{Q}_2, \mathbb{H}_\perp] = [\mathbb{Q}_1, \mathbb{Q}_2] = 0$$

$\implies \mathbb{H}_\perp$ is diagonal on eigenfunctions of $\mathbb{Q}_1, \mathbb{Q}_2$



$q^\uparrow q^\uparrow$: Exact solution

Eigenfunctions of “conserved charges” $\mathbb{Q}_1, \mathbb{Q}_2$ *Derkachov, Korchemsky, Manashov, 2003*

$$Q_{s,x}(z_1, z_2) = \frac{s}{z_1^2 z_2^2} \int_0^1 d\alpha \left(\frac{\alpha}{\bar{\alpha}}\right)^{ix} \exp\left[is(\bar{\alpha}/z_1 + \alpha/z_2)\right]$$

Anomalous dimensions

Braun, Derkachov, Manashov, [arXiv:1406.0664]

$$\mathbb{H}_\perp Q_{s,x}(z_1, z_2) = \gamma(s, x; \mu) Q_{s,x}(z_1, z_2)$$

$$\gamma(s, x; \mu) = 2 \ln(\mu s / s_0) + \mathcal{E}(x), \quad \mathcal{E}(x) = \psi(1 + ix) + \psi(1 - ix) + 2\gamma_E$$

Continuous spectrum characterized by two real parameters (quasimomenta) $s, x \in \mathbb{R}$, $s > 0$



$q^\uparrow q^\uparrow$: Exact solution — *continued*

$$\begin{aligned}\widetilde{\Psi}_\perp(\omega, u; \mu) &= \omega^2 u \bar{u} \int_{-\infty}^{\infty} \frac{dx}{2\pi} \int_0^{\infty} s ds \widetilde{Q}_{s,x}(\omega, u) \eta_\perp(s, x; \mu) \\ \eta_\perp(s, x; \mu) &= \eta_\perp(s, x; \mu_0) \left(\frac{\mu}{\mu_0} \right)^{-\frac{8}{3\beta_0}} \left(\frac{\mu_0 s}{s_0} \right)^{\frac{8}{3\beta_0} \ln L} L^{\frac{4}{3\beta_0}} \left[\mathcal{E}(x) - \frac{4\pi}{\beta_0 \alpha_s(\mu_0)} \right]\end{aligned}$$

where

$$\widetilde{Q}_{s,x}(\omega, u) = \frac{1}{\omega} \sum_{n=0}^{\infty} i^n \varkappa_n^{-1} C_n^{3/2} (1-2u) H_n(x) \frac{1}{\sqrt{s\omega}} J_{2n+3}(2\sqrt{s\omega})$$

$H_n(x)$: Hahn polynomials

$$\varkappa_n = \frac{(n+1)(n+2)}{4(2n+3)}$$



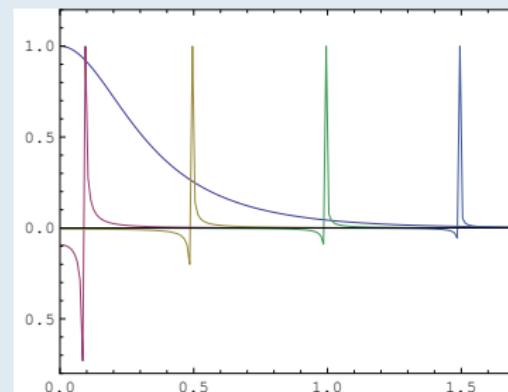
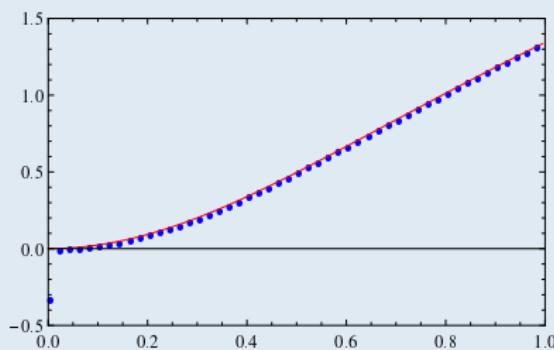
$q^\uparrow q^\downarrow$: e.g. Λ_b

Additional attractive interaction: gluon exchange between light quarks of opposite helicity

$$\mathbb{H}_\Lambda = \mathbb{H}_\perp - \delta\mathbb{H},$$

$$\delta\mathbb{H} = 1/J_{12}(J_{12} - 1)$$

→ will treat as perturbation



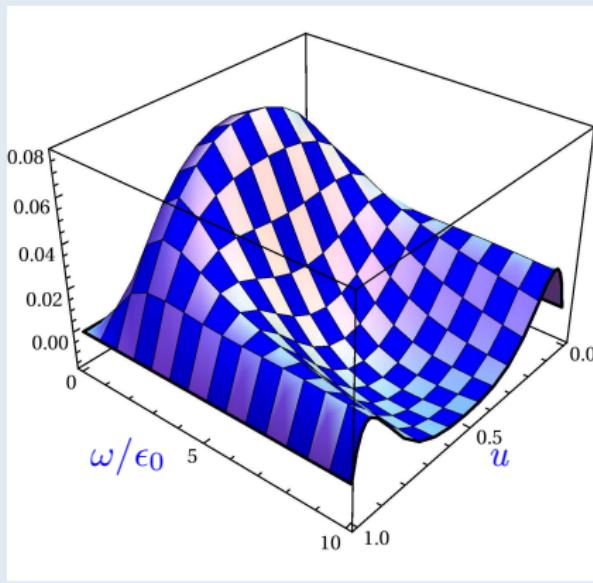
Continuum spectrum is intact; dispersion relation is not affected

Lowest state becomes separated by a finite gap — bound state

$$\Delta E = E_0 \simeq -0.3214$$



Asymptotic Λ_b distribution amplitude



ω

total energy carried by the light quarks

$u \leftrightarrow 1 - u$: energy fractions carried by the first and the second quark



B-meson, higher-twist

Braun, Manashov, Offen, Phys. Rev. D 92 (2015) 074044

twist-two

$$\langle 0 | \bar{q}(nz) \not{v} \gamma_5 h_v(0) | \bar{B}(v) \rangle = iF(\mu) \Phi_+(z, \mu)$$

twist-three

$$\langle 0 | \bar{q}(nz) \not{v} \gamma_5 h_v(0) | \bar{B}(v) \rangle = iF(\mu) \Phi_-(z, \mu)$$

$$\langle 0 | \bar{q}(nz_1) g G_{\mu\nu}(nz_2) n^\nu \sigma^{\mu\rho} n_\rho \gamma_5 h_v(0) | \bar{B}(v) \rangle = -2iF(\mu) \Phi_3(z_1, z_2, \mu)$$

Equations of motion

Beneke, Feldmann, '01

$$(1 + z\partial_z) \Phi_-(z) = \Phi_+(z) + 2 \int_0^z w dw \Phi_3(z, w)$$



Twist-three evolution equation in the $N_c \rightarrow \infty$ limit

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \frac{\alpha_s}{2\pi} \mathcal{H} \right) F(\mu) \Phi_3(z_1, z_2, \mu) = 0$$

where

$$\mathcal{H} = N_c \left[\psi(J_{qg} + 3/2) + \psi(J_{qg} - 3/2) - 3\psi(1) - 5/4 + \ln(i\mu S_g^+) \right] + \mathcal{O}(1/N_c)$$

is completely integrable:

$$[\mathbb{Q}_1, \mathbb{Q}_2] = [\mathbb{Q}_1, \mathbb{H}] = [\mathbb{Q}_2, \mathbb{H}] = 0$$

with

$$\mathbb{Q}_1 = i(S_q^+ + S_g^+)$$

$$\mathbb{Q}_2 = \frac{9}{4}iS_q^+ - iS_g^+ (S_g^+ S_q^- + S_g^0 S_q^0) - iS_g^0 (S_q^0 S_g^+ - S_g^0 S_q^+)$$

(open spin chain)



Solution

Eigenfunctions of “conserved charges” $\mathbb{Q}_1, \mathbb{Q}_2$ *Derkachov, Korchemsky, Manashov, 2003*

$$Y_{s,x}(z_1, z_2) = \frac{is^2}{z_1^2 z_2^3} \int_0^1 du u \bar{u} e^{is(u/z_1 + \bar{u}/z_2)} {}_2F_1\left(-\frac{1}{2} - ix, -\frac{1}{2} + ix \middle| 2\right) - \frac{u}{\bar{u}}$$

+ special solution:

$$Y_s^{(0)}(z_1, z_2) = \frac{is^2}{z_1^2 z_2^3} \int_0^1 du u \bar{u} e^{is(u/z_1 + \bar{u}/z_2)}$$

Anomalous dimensions: one discrete level + continuum spectrum

$$\mathbb{E}(s, x) = \ln(\mu s) + \psi(3/2 + ix) + \psi(3/2 - ix) - 3\psi(1) - 5/4$$

+ discrete level:

$$\mathbb{E}_0 \equiv \mathbb{E}(s, x = i/2) = \ln(\mu s) - \psi(1) - 1/4$$



Solution — *continued*

- main result:
- The two-particle DA $\phi_-(\omega, \mu)$ decouples from quark-gluon correlations (continuum spectrum)

$$\begin{aligned}\phi_+(\omega, \mu) &= \int_0^\infty ds \tilde{\phi}_+(s, \mu) \sqrt{\omega s} J_1(2\sqrt{\omega s}) , \\ \phi_-(\omega, \mu) &= \int_0^\infty ds \left[\tilde{\phi}_+(s, \mu) + \eta_0(s, \mu) \right] J_0(2\sqrt{\omega s}) .\end{aligned}$$

The scale-dependence of the coefficients $\tilde{\phi}_+(s, \mu)$ and $\eta_0(s, \mu)$ differs by a simple overall factor

$$\begin{aligned}\tilde{\phi}_+(s, \mu) &= R(s; \mu, \mu_0) \tilde{\phi}_+(s, \mu_0) \\ \eta_0(s, \mu) &= L^{N_c/\beta_0} R(s; \mu, \mu_0) \eta_0(s, \mu_0) \quad L = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\end{aligned}$$

$$R(s; \mu, \mu_0) = \exp \left[- \int_{\mu_0}^{\mu} \frac{d\tau}{\tau} \Gamma_{cusp}(\alpha_s(\tau)) \ln(\tau s/s_0) \right]$$

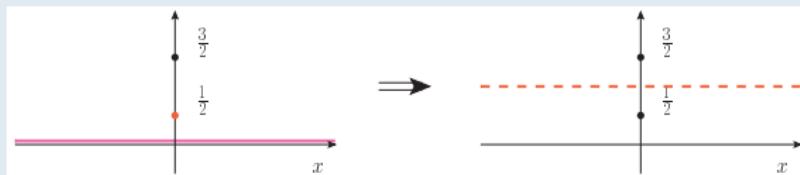


Large energy limit

- Hierarchy of contributions with rising anomalous dimensions breaks down at $\omega > \mu$

$$\Phi_3(\omega_1, \omega_2, \mu) = \int_0^\infty ds \left[\eta_0(s, \mu) \tilde{Y}_s^{(0)}(\omega_1, \omega_2) + \frac{1}{2} \int_{-\infty}^\infty dx \eta(s, x, \mu) \tilde{Y}_{s,x}(\omega_1, \omega_2) \right]$$

One finds $\tilde{Y}_s^{(0)}(\omega_1, \omega_2) \sim \text{const}$, $\tilde{Y}_{s,x}(\omega_1, \omega_2) \sim \omega_2^{1/2 \pm ix}$ for $\omega_2 \rightarrow \infty$, but ...

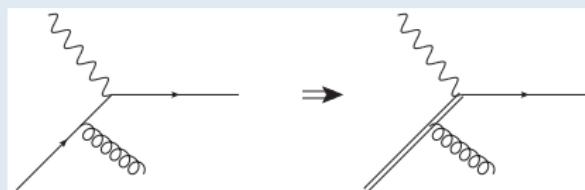


... $\Phi_3(\omega_1, \omega_2, \mu) \sim 1/\omega_2$ in the sum (integral) over all terms, unless the initial condition at $\mu = \mu_0$ is more singular



Effective field theory for $x \rightarrow 1$

For soft gluon emission



... and renormalization of HQET operators matches large- N limit of light-quark ones

- well known for leading twist *Korchemsky, '89*
- true for non-leading twist, for the lowest part of the spectrum

- Gaps in the spectrum of heavy-light operators exactly coincide with the gaps for light-light operators at $N \rightarrow \infty$
- New: Analytic expressions for the eigenfunctions (multiplicatively renormalized higher-twist operators) at $N \rightarrow \infty$



Summary

- Heavy-light RG kernel is given in terms of the generator of conformal transformations

Conjecture:

$$\Gamma_{\text{cusp}}(\alpha_s) \ln(i\mu S_+) + \gamma(\alpha_s)$$

- Baryons with light quarks with aligned helicities:

Complete integrability; anomalous dimensions form a continuum spectrum

- Baryons with light quarks with anti-aligned helicities:

Attractive interaction between the light quarks creates a bound state — asymptotic DA.

- B-meson higher-twist DAs:

Complete integrability in the $N_c \rightarrow \infty$ limit; gap in the spectrum;
 $\phi_-(\omega)$ decouples from quark-gluon operators

- Matching to light quark (gluon) operators for a large number of derivatives



Supplementary slides



Quantum Inverse Scattering Method

Faddeev, Sklyanin, Takhtajan, Kulish,...

Monodromy matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = L_1(u)L_2(u)$$

where

$$L_k(u) = u + i \begin{pmatrix} S_0^{(k)} & S_-^{(k)} \\ S_+^{(k)} & -S_0^{(k)} \end{pmatrix}$$

$$C(u) = u \mathbb{Q}_1 + \mathbb{Q}_2, \quad [C(u), C(v)] = [C(u), \mathbb{H}_\perp] = 0$$

