Double parton scattering for perturbative transverse momenta

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QCD evolution workshop 2016

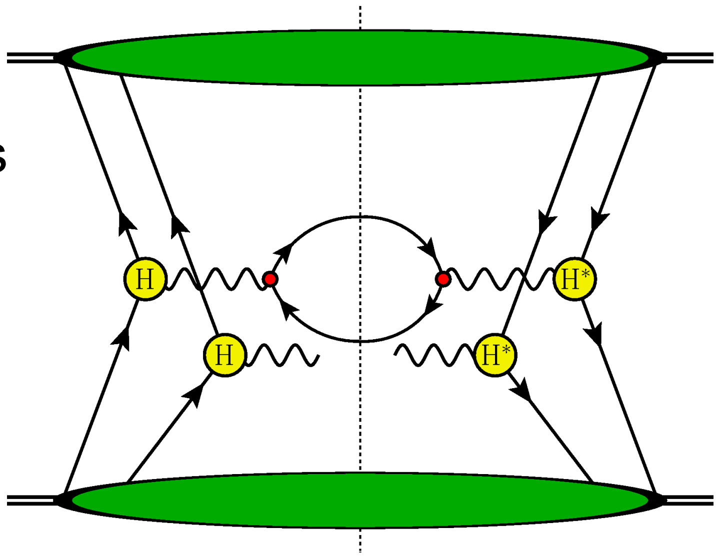
May 31, 2016

Content - outline

- Brief motivation/introduction
- Soft factors
- Color for DPDFs/DTMDs
- Evolution equations
  - Writing them down for DTMDs
  - Solving them
- Matching: cross section contributions for large **y**
- Conclusions

Motivation

- DPDs: double parton distribution functions
- Factorization: stick to singlets in final states
  - Double Drell-Yan
  - Higgs + W/Z
- For perturbative $q_T \rightarrow$ significant predictive results



- Motivation and goals
  - Formulate description to handle soft factors
  - Write down evolution equations
  - Solve evolution equations
  - Matching equations for DPDFs/DTMDs

Diehl, Ostermeier, Schäfer, JHEP 1203 (2012) 089

Short-distance expansion

- Differences compared to TMDs
  - Two hard processes involved
  - Two coefficient functions per DTMD
  - Positions $\mathbf{z}_1$ and $\mathbf{z}_2$ (compare with $\mathbf{b}_T$ for the TMD case)
  - Additional distance $\mathbf{y}$

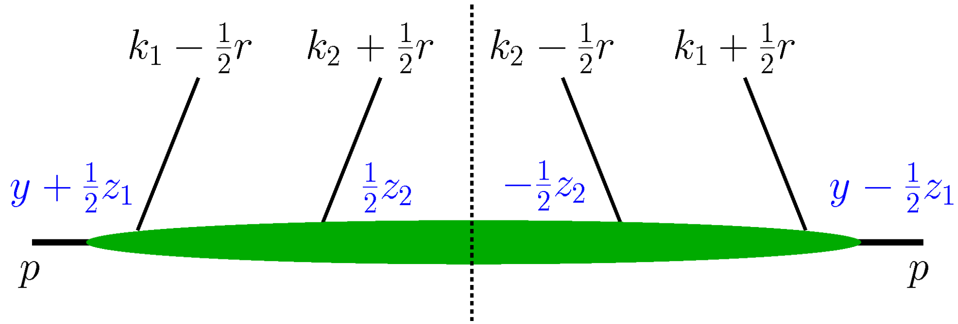


Figure: modified from Diehl, Ostermeier, Schäfer, JHEP03 (2012) 089

- Consider the limit
  - $|\mathbf{z}_1|, |\mathbf{z}_2|$ much smaller than $1/\Lambda$
  - $|\mathbf{z}_1|, |\mathbf{z}_2| \ll \mathbf{y}$, with $\mathbf{y}$ fixed

- Gives separate matching factors

$$F_{\mathrm{us}}(x_i, \mathbf{z}_i, \mathbf{y}) = C_f(x'_1, \mathbf{z}_1) \underset{x_1}{\otimes} C_f(x'_2, \mathbf{z}_2) \underset{x_2}{\otimes} F_{\mathrm{us}}(x'_i, \mathbf{y})$$

with

$$C(x') \underset{x}{\otimes} F(x') = \int_x^1 \frac{dx'}{x'} C(x') F\left(\frac{x}{x'}\right)$$

Soft factors

- Wilson line structure from factorization formula.
- Nontrivial color complications. Collinear and soft factors carry color indices.
- Wilson line self-interactions drop out in cross section.

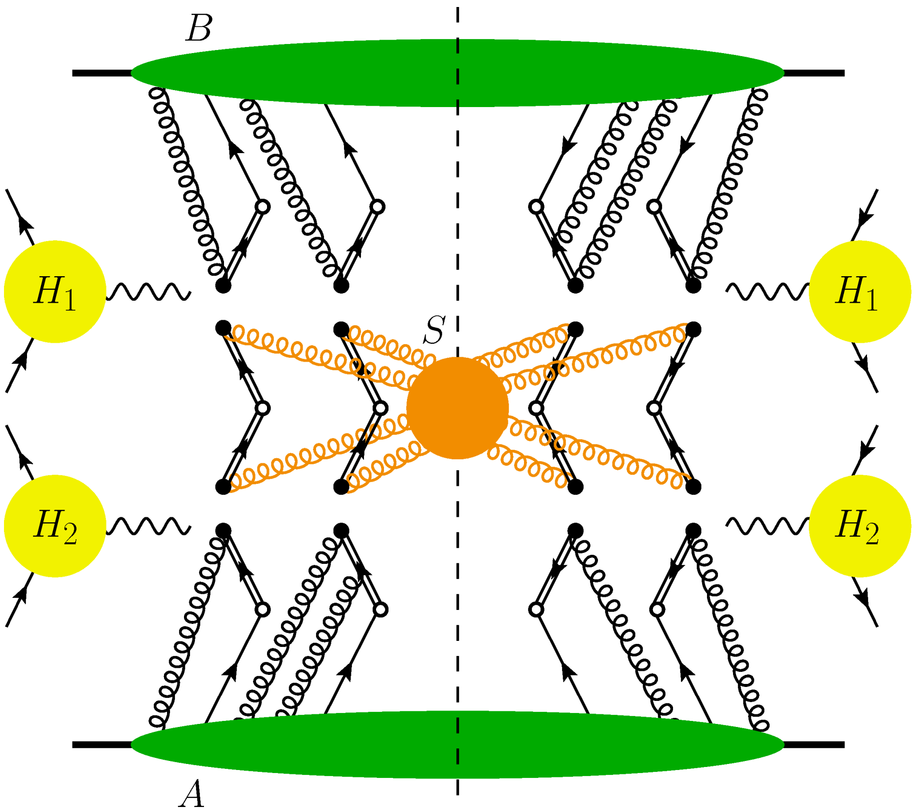


Figure: Diehl, Gaunt, Ostermeier, Plößl, Schäfer, JHEP01 (2016) 076

Collins, *Foundations of perturbative QCD*, (2011); Aybat, Rogers, PRD 83 (2011) 114042; Diehl, Gaunt, Ostermeier, Plößl, Schäfer, JHEP01 (2016) 076

Soft factors

- TMDs
  - Soft functions for the single TMD related to *K* through

$$K(\mathbf{z};\mu) = \frac{1}{2}\left[\frac{\partial}{\partial y_A}\log S(\mathbf{z}; y_A, -\infty) - \frac{\partial}{\partial y_B}\log S(\mathbf{z}; +\infty, y_B)\right]$$

  - Soft function not matrix valued
  - Square root construction for TMD (see Collins' book)

- DTMDs
  - For DPDs: matrix valued functions (working hypothesis)

$$S(\mathbf{z}_1, \mathbf{z}_2, \mathbf{y}, y_A, y_B) = \exp\left[(y_A - y_B)K(\mathbf{z}_1, \mathbf{z}_2, \mathbf{y})\right]$$

  - Soft function matrix valued
  - Square root construction extended to matrix expressions

Collins, *Foundations of perturbative QCD*, (2011); Aybat, Rogers, PRD 83 (2011) 114042

Soft factors
(technical details)

- Subtracted DPD distributions are defined as

$$F_{qq}(v_c) = \lim_{v_L^2 \to 0} S_{qq}^{-1}(v_L, v_C) F_{us,qq}(v_L)$$

with F_{us} vector in color space and S a matrix.

- Matrix equivalent of square root construction

$$S^{-1}(v_L, v_C) = S^{1/2}(-v_C, v_R) S^{-1/2}(v_L, v_R) S^{-1/2}(v_L, v_C)$$

using composition law

$$S(v_A, -v_B) S(v_B, v_C) = S(v_A, v_C)$$

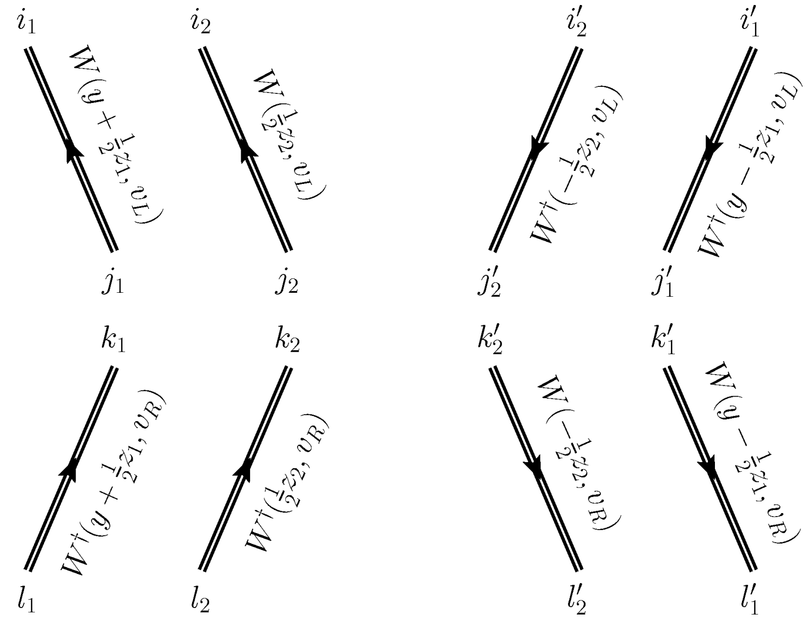
and a similar expression for left moving particles.

- Wilson line self-interactions drop out in *F*.

Collins, *Foundations of perturbative QCD*, (2011); Aybat, Rogers, PRD 83 (2011) 114042

Soft factors

- Wilson line structure for double Drell-Yan



with Wilson lines

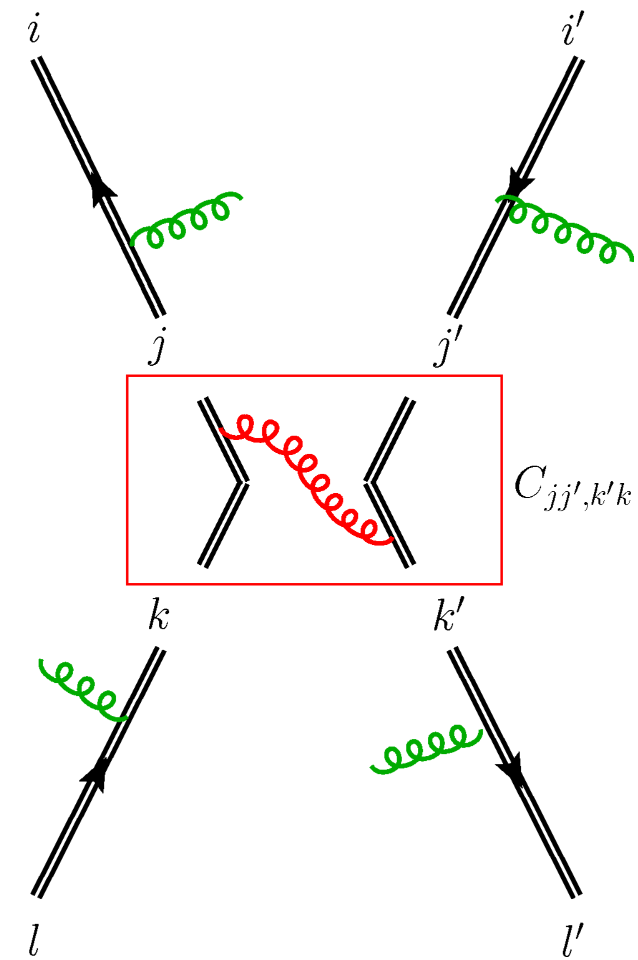
$$W_{ij}(\mathbf{z}, v) = \mathcal{P}\exp\left[-igt^a \int_{-\infty}^{0} d\lambda v A^a(z+\lambda v)\right]_{ij}^{z^+=z^-=0}$$

and similarly for the adjoint representation.

- We will need uncontracted color indices in the middle.

Soft factors

- Uncontracted indices in the middle
- Soft factor for DTMDs factorizes in small-distance expansion as
  $$S(\mathbf{z_1}, \mathbf{z_2}, \mathbf{y}) = C_s(\mathbf{z_1})\, C_s(\mathbf{z_2})\, S(\mathbf{y})$$
- Wilson lines in S(**y**) pairwise at the same transverse position.
- We require a simplification of the color indices.



Color structure

- Recall full Wilson line structure

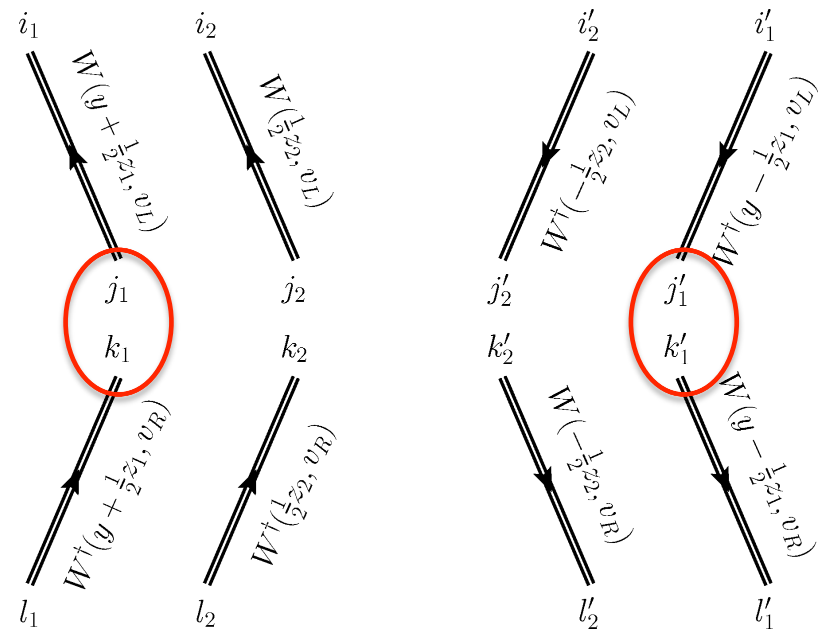
- Hard scattering couples four parton lines, insert color projectors

- Examples of color projectors
  - Quarks:

$$p_1^{j_1 j_1' \, k_1 k_1'} = \frac{1}{N_c} \delta_{j_1 j_1'} \delta_{k_1 k_1'}$$

$$p_8^{j_1 j_1' \, k_1 k_1'} = 2 t^a_{j_1 j_1'} t^a_{k_1 k_1'}$$

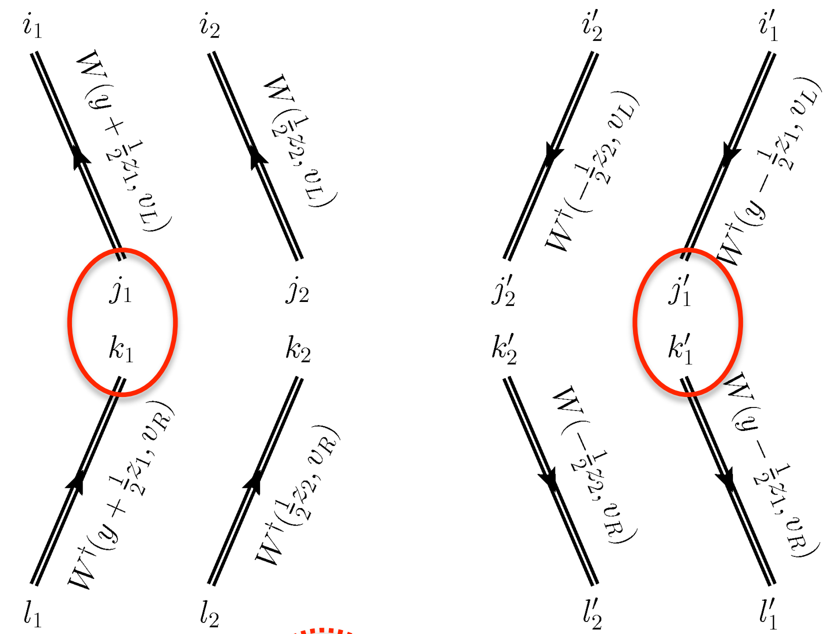
  - For gluons: more possibilities
  - Mixed quark-gluon projectors also exist



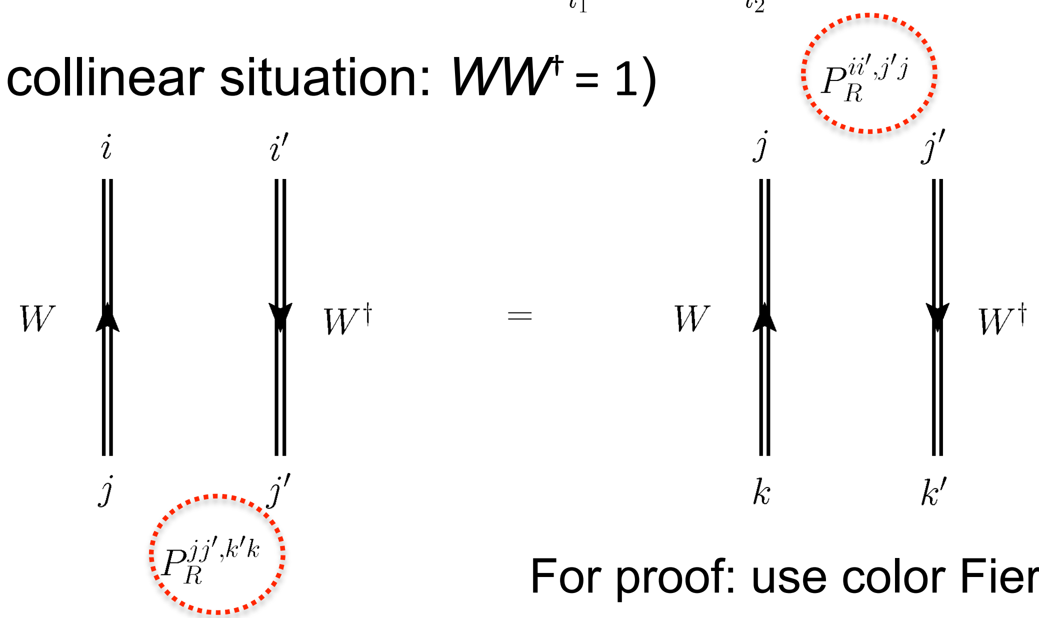
- Highly nontrivial whether color structure can be factorized.

Color structure

- Recall full Wilson line structure
- Hard scattering couples four parton lines, insert color projectors



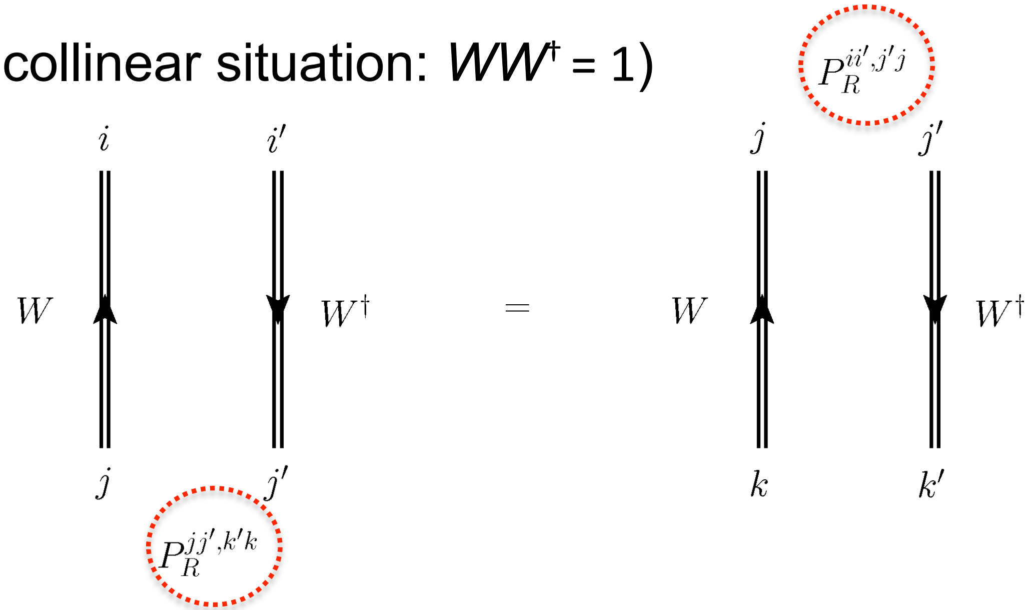
- Color trick (in collinear situation: $WW^\dagger = 1$)



For proof: use color Fierz identity

Color structure

- Color trick (in collinear situation: $WW^{\dagger} = 1$)



- For proof: use color Fierz identity:

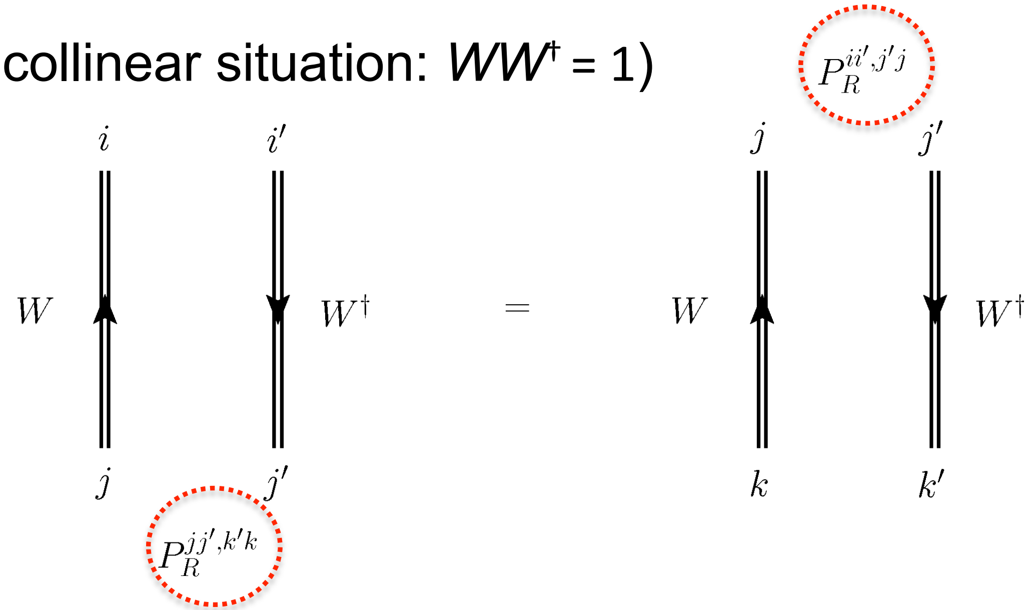
$$2t^a_{ii'}t^a_{jj'} = \delta_{ij'}\delta_{i'j} - \frac{1}{N_c}\delta_{ii'}\delta_{jj'}$$

- Trick also works for adjoint Wilson lines. Use color Fierz identity and

$$W^{ab} = 2\,\mathrm{Tr}\Big(t^a W t^b W^\dagger\Big)$$

Color structure

- Color trick (in collinear situation: $WW^{\dagger} = 1$)

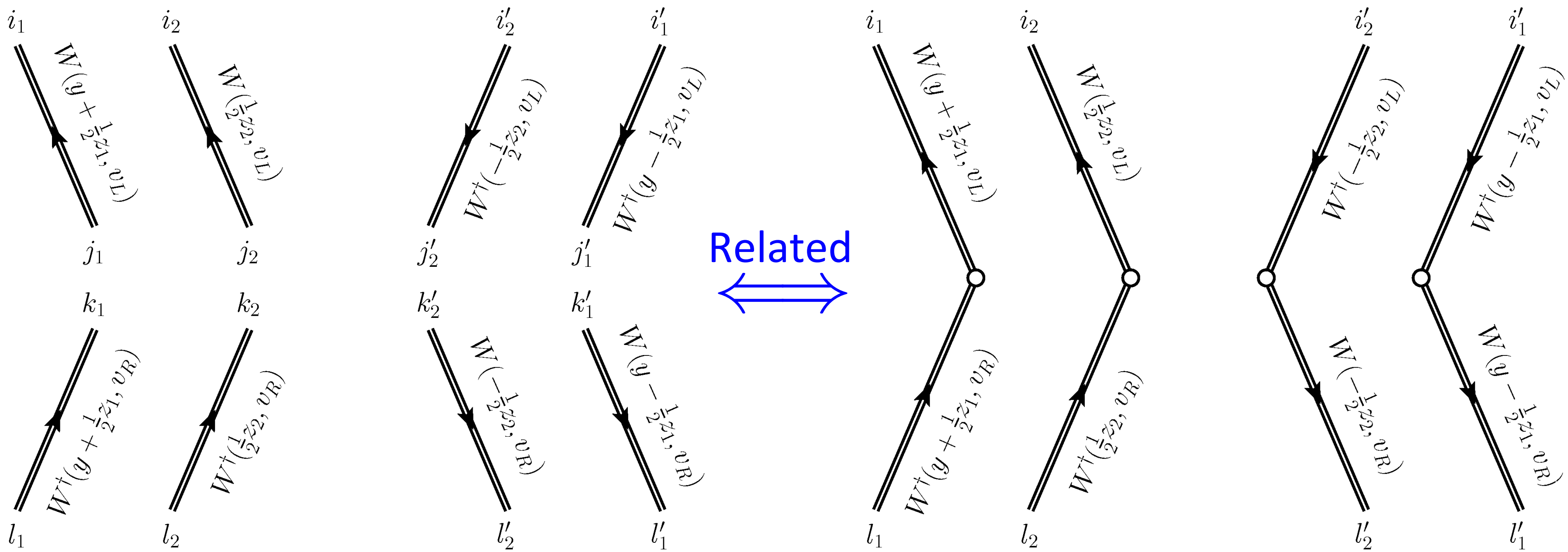


- Dynamical and not just some color algebra

- With same trick show that S(**y**) is color diagonal.

Implications for soft factor

- Color projection of fields at infinity rather than $\xi^+ = \xi^- = 0$.



- Allows for relating most general soft function with open indices in the middle with soft function with contracted indices in the middle.

- For collinear factorization case only!

Renormalization and rapidity evolution

- Short-distance expansion
  - The two hard processes are separated
- Evolution equations for DTMDs
  - Two renormalization scales: μ_1 and μ_2
- Soft factor recap
  - Working hypothesis

$$S(\mathbf{z}_1, \mathbf{z}_2, \mathbf{y}, y_A, y_B) = \exp\left[(y_A - y_B) K(\mathbf{z}_1, \mathbf{z}_2, \mathbf{y})\right]$$

  - Soft factor becomes

$${}^{RR'}S(\mathbf{z_1}, \mathbf{z_2}, \mathbf{y}) = {}^{R}C_s(\mathbf{z_1})\, {}^{R}C_s(\mathbf{z_2})\, {}^{RR}S(\mathbf{y})\, \delta_{RR'}$$

- For phenomenology: only four independent collinear soft functions

Renormalization and rapidity evolution

- TMDs

$$\frac{\partial}{\partial \log \mu} F(x, \mathbf{z}; \mu, \zeta) = \gamma_F(\mu, \zeta) F(x, \mathbf{z}; \mu, \zeta)$$

$$\frac{\partial}{\partial \log \zeta} F(x, \mathbf{z}, \mu, \zeta) = \frac{1}{2} K(\mathbf{z}; \mu) F(x, \mathbf{z}, \mu, \zeta)$$

- DTMDs

$$\frac{\partial}{\partial \log \mu_1} {}^{R}F(x_i, \mathbf{z}_i, \mathbf{y}; \mu_i, \zeta) = \gamma_F(\mu_1, x_1\zeta/x_2)\, {}^{R}F(x_i, \mathbf{z}_i, \mathbf{y}; \mu_i, \zeta)$$

$$\frac{\partial}{\partial \log \mu_2} {}^{R}F(x_i, \mathbf{z}_i, \mathbf{y}; \mu_i, \zeta) = \gamma_F(\mu_2, x_2\zeta/x_1)\, {}^{R}F(x_i, \mathbf{z}_i, \mathbf{y}; \mu_i, \zeta)$$

$$\frac{\partial}{\partial \log \zeta} {}^{R}F(x_i, \mathbf{z}_i, \mathbf{y}, \mu_i, \zeta) = \frac{1}{2} \sum_{R'} {}^{RR'}K(\mathbf{z}_i, \mathbf{y}; \mu_i)\, {}^{R'}F(x_i, \mathbf{y}, \mu_i, \zeta)$$

- DTMD renormalizations are independent, since they are separated.

Collins, *Foundations of perturbative QCD*, (2011); Aybat, Rogers, PRD 83 (2011) 114042

Evolution: TMDs vs DTMDs

PDF/TMDs

- Soft function <u>not</u> matrix valued
- Just the position of one parton
- Renormalization scale μ
- Rapidity evolution scale ζ
- One coefficient function per TMD

DPDF/DTMDs

- Soft function matrix valued
- Positions of two partons <u>and</u> the distance y
- Renormalization scales μ_1, μ_2
- Rapidity evolution scale ζ
  - ζ dependence also for collinear distribution if $R \neq 1$.
- Two coefficient functions per DTMD

DTMD evolution

- The evolution of DTMDs is in the short-distance matching given by

$$
\begin{aligned}
&{}^{R}F(x_i,\mathbf{z}_i,\mathbf{y};\mu_1,\mu_2,\zeta) \\
&\quad = \exp\Bigg\{\int_{\mu_{01}}^{\mu_1}\frac{d\mu}{\mu}\left[\gamma_F(\mu,\mu^2)-\gamma_K(\mu)\log\frac{\sqrt{x_1\zeta/x_2}}{\mu}\right]+{}^{R}K(\mathbf{z}_1,\mu_{01})\log\frac{\sqrt{x_1\zeta/x_2}}{\mu_{01}} \\
&\qquad\quad +\int_{\mu_{02}}^{\mu_2}\frac{d\mu}{\mu}\left[\gamma_F(\mu,\mu^2)-\gamma_K(\mu)\log\frac{\sqrt{x_2\zeta/x_1}}{\mu}\right]+{}^{R}K(\mathbf{z}_2,\mu_{02})\log\frac{\sqrt{x_2\zeta/x_1}}{\mu_{02}} \\
&\qquad\quad +{}^{R}J(\mathbf{y},\mu_{01},\mu_{02})\log\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}\Bigg\} \\
&\qquad \times {}^{R}C(x_1',\mathbf{z}_1;\mu_{01},\mu_{01}^2)\underset{x_1}{\otimes}{}^{R}C(x_2',\mathbf{z}_2;\mu_{02},\mu_{02}^2)\underset{x_2}{\otimes}{}^{R}F(x_i',\mathbf{y};\mu_{01},\mu_{02},\zeta_0)
\end{aligned}
$$

- From additive structure of the Collins-Soper evolution kernel we have the sum for the two contributions for the μ_1 and μ_2 dependences.

- $K(\mathbf{z}_1,\mathbf{z}_2,\mathbf{y})$-kernel splits in three separate contributions: $K(\mathbf{z}_1,\mu_{01})$, $K(\mathbf{z}_2,\mu_{02})$ and $J(\mathbf{y},\mu_{01},\mu_{02})$ when collinear soft function becomes diagonal.

Cross section contribution

- Cross section contribution given by

$$
\begin{aligned}
W_{\text{large }\mathbf{y}} = \sum_R \exp\Bigg\{ & \int_{\mu_{01}}^{\mu_1} \frac{d\mu}{\mu}\left[\gamma_F(\mu,\mu^2) - \gamma_K(\mu)\log\frac{Q_1^2}{\mu^2}\right] + {}^R K(\mathbf{z}_1,\mu_{01})\log\frac{Q_1^2}{\mu_{01}^2} \\
& + \int_{\mu_{02}}^{\mu_2} \frac{d\mu}{\mu}\left[\gamma_F(\mu,\mu^2) - \gamma_K(\mu)\log\frac{Q_2^2}{\mu^2}\right] + {}^R K(\mathbf{z}_2,\mu_{02})\log\frac{Q_2^2}{\mu_{02}^2} \Bigg\} \\
& \times {}^R C(\overline{x}_1', \mathbf{z}_1; \mu_{01}, \mu_{01}^2) \underset{\overline{x}_1}{\otimes} {}^R C(\overline{x}_2', \mathbf{z}_2; \mu_{02}, \mu_{02}^2) \underset{\overline{x}_2}{\otimes} \\
& \times {}^R C(x_1', \mathbf{z}_1; \mu_{01}, \mu_{01}^2) \underset{x_1}{\otimes} {}^R C(x_2', \mathbf{z}_2; \mu_{02}, \mu_{02}^2) \underset{x_2}{\otimes} \\
& \times \left[\Phi(\nu\mathbf{y})\right]^2 \exp\left[{}^R J(\mathbf{y},\mu_{0i})\log\frac{\sqrt{Q_1^2 Q_2^2}}{\zeta_0}\right] {}^R F(\overline{x}_i, \mathbf{y}; \mu_{0i}, \zeta_0)\, {}^R F(x_i, \mathbf{y}; \mu_{0i}, \zeta_0)
\end{aligned}
$$

- The $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{y}$ contributions nicely factorize.

Cross section contribution

- Cross section contribution given by

$$
\begin{aligned}
W_{\text{large }\mathbf{y}} = \sum_R \exp\Bigg\{ & \int_{\mu_{01}}^{\mu_1} \frac{d\mu}{\mu}\left[\gamma_F(\mu,\mu^2) - \gamma_K(\mu)\log\frac{Q_1^2}{\mu^2}\right] + {}^R K(\mathbf{z}_1,\mu_{01})\log\frac{Q_1^2}{\mu_{01}^2} \\
& + \int_{\mu_{02}}^{\mu_2} \frac{d\mu}{\mu}\left[\gamma_F(\mu,\mu^2) - \gamma_K(\mu)\log\frac{Q_2^2}{\mu^2}\right] + {}^R K(\mathbf{z}_2,\mu_{02})\log\frac{Q_2^2}{\mu_{02}^2} \Bigg\} \\
& \times {}^R C(\overline{x}_1', \mathbf{z}_1; \mu_{01}, \mu_{01}^2) \underset{\overline{x}_1}{\otimes} {}^R C(\overline{x}_2', \mathbf{z}_2; \mu_{02}, \mu_{02}^2) \underset{\overline{x}_2}{\otimes} \\
& \times {}^R C(x_1', \mathbf{z}_1; \mu_{01}, \mu_{01}^2) \underset{x_1}{\otimes} {}^R C(x_2', \mathbf{z}_2; \mu_{02}, \mu_{02}^2) \underset{x_2}{\otimes} \\
& \times \left[\Phi(\nu\mathbf{y})\right]^2 \exp\left[{}^R J(\mathbf{y},\mu_{0i})\log\frac{\sqrt{Q_1^2Q_2^2}}{\zeta_0}\right] {}^R F(\overline{x}_i,\mathbf{y};\mu_{0i},\zeta_0)\, {}^R F(x_i,\mathbf{y};\mu_{0i},\zeta_0)
\end{aligned}
$$

- There is ζ – dependence for color non-singlet DPDFs.

Polarizations (work in progress)

- Including parton labels in equations for DTMDs and cross section. E.g.

$$
\begin{aligned}
&{}^{R}F_{a_1 a_2}(x_i, \mathbf{z}_i, \mathbf{y}; \mu_1, \mu_2, \zeta) \\
&= \sum_{b_1 b_2} \exp\Bigg\{ \int_{\mu_{01}}^{\mu_1} \frac{d\mu}{\mu} \left[\gamma_{F,a_1}(\mu, \mu^2) - \gamma_{K,a_1}(\mu) \log \frac{\sqrt{x_1\zeta/x_2}}{\mu} \right] + {}^{R}K_{a_1}(\mathbf{z}_1, \mu_{01}) \log \frac{\sqrt{x_1\zeta/x_2}}{\mu_{01}} \\
&\qquad + \int_{\mu_{02}}^{\mu_2} \frac{d\mu}{\mu} \left[\gamma_{F,a_2}(\mu, \mu^2) - \gamma_{K,a_2}(\mu) \log \frac{\sqrt{x_2\zeta/x_1}}{\mu} \right] + {}^{R}K_{a_2}(\mathbf{z}_2, \mu_{02}) \log \frac{\sqrt{x_2\zeta/x_1}}{\mu_{02}} \\
&\qquad + {}^{R}J(\mathbf{y}, \mu_{01}, \mu_{02}) \log \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} \Bigg\} \\
&\quad \times {}^{R}C_{a_1 b_1}(x_1', \mathbf{z}_1; \mu_{01}, \mu_{01}^2) \underset{x_1}{\otimes} {}^{R}C_{a_2 b_2}(x_2', \mathbf{z}_2; \mu_{02}, \mu_{02}^2) \underset{x_2}{\otimes} {}^{R}F_{b_1 b_2}(x_i', \mathbf{y}; \mu_{01}, \mu_{02}, \zeta_0)
\end{aligned}
$$

- Parton labels like a_1 not only q, $\bar{q}$ and g, but also δq, Δq, δg, Δg, etc.
- Splitting kernels from PDF/TMDs can largely be recycled

$$C_{q/g}(x', \mathbf{z}; \mu_0, \mu_0^2) \qquad C_{g/g}(x', \mathbf{z}; \mu_0, \mu_0^2) \qquad C_{g/\delta g}(x', \mathbf{z}; \mu_0, \mu_0^2) \quad \text{etc.}$$

Collins, *Foundations of perturbative QCD*, (2011); Aybat, Rogers, PRD 83 (2011) 114042; Bacchetta, Prokudin, NPB 875 (2013) 536; Echevarría, Kasemets, Mulders, Pisano, JHEP 1507 (2015) 158; MGAB, Diehl, Kasemets, *work in progress*.

Conclusions

- We use short-distance expansion
  - $|\mathbf{z}_1|$, $|\mathbf{z}_2|$ much smaller than $1/\Lambda$
  - $|\mathbf{z}_1|$, $|\mathbf{z}_2| \ll \mathbf{y}$

  although part of our results are also valid outside this region.

- Description for soft function
  - Separation in a $\mathbf{y}$-dependent contribution and two pieces depending on either $\mathbf{z}_1$ or $\mathbf{z}_2$.
  - We have shown the correct way to deal with color.

- Matching equations for DPDs
  - Evolution equations for DTMDs.
  - Expression for matching at level of individual DTMDs/DPDFs and cross section.

- Work in progress: explicit expressions for matching of all polarization-modes.

Backup slides

Properties of the DTMD soft factor

- Parity and boost
  $S_{a_1a_2}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{y},v_A,v_B) = S_{a_1a_2}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{y},v_B,v_A)$
- Parity and time reversal
  $S_{a_1a_2}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{y},v_A,v_B) = S_{a_1a_2}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{y},-v_A,-v_B)$
- Hermitian conjugation
  $S_{a_1a_2}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{y},v_A,v_B) = S^{\dagger}_{a_1a_2}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{y},v_A,v_B)$
- Charge conjugation
  $S^*_{qq} = S_{qq}$ and $S^*_{gg} = S_{gg}$
- Composition law
  $S_{a_1a_2}(v_A,-v_B)S_{a_1a_2}(v_B,v_C) = S_{a_1a_2}(v_A,v_C)$

- Independent collinear soft matrix elements (singlet configurations are 1)
  $^{88}S = {}^{88}S_{qq} = {}^{AA}S_{qg} = {}^{SS}S_{qg} = {}^{AA}S_{gq} = {}^{SS}S_{gq} = {}^{AA}S_{gg} = {}^{SS}S_{gg}$
  $^{DD}S = {}^{DD}S_{gg}$
  $^{27\,27}S = {}^{27\,27}S_{gg}$

Solving evolution equations for DTMDs

DTMDs: μ_1 and μ_2 scale evolution

- μ_1 scale evolution governed by an equation of the form

$$\frac{\partial}{\partial \log \mu_1} {}^R F(x_i, \mathbf{z}_i, \mathbf{y}; \mu_i, \zeta) = \gamma_F(\mu_1, x_1\zeta/x_2)\, {}^R F(x_i, \mathbf{z}_i, \mathbf{y}; \mu_i, \zeta)$$

  and similarly for μ_2.

- For the starting values:
  - Starting scales μ_{10} and μ_{20} for μ_1 and μ_2.
  - We define the ζ value as the geometric mean

- We get the result

$$\begin{aligned} {}^R F(x_i, \mathbf{z}_i, \mathbf{y}; \mu_i, \zeta) &= {}^R F(x_i, \mathbf{z}_i, \mathbf{y}; \mu_{0i}, \zeta) \\ &\times \exp\left\{ \int_{\mu_{01}}^{\mu_1} \frac{d\mu}{\mu} \gamma_F(\mu, x_1\zeta/x_2) + \int_{\mu_{02}}^{\mu_2} \frac{d\mu}{\mu} \gamma_F(\mu, x_2\zeta/x_1) \right\} \end{aligned}$$

- Note the additive structure

Solving evolution equations for DTMDs

DTMDs: μ_1 and μ_2 scale evolution *and* ζ evolution

- ζ evolution governed by

$$\frac{\partial}{\partial \log \zeta} {}^{R}F(x_i, \mathbf{z}_i, \mathbf{y}, \mu_i, \zeta) = \frac{1}{2} \sum_{R'} {}^{RR'}K(\mathbf{z}_i, \mathbf{y}; \mu_i) {}^{R'}F(x_i, \mathbf{y}, \mu_i, \zeta)$$

- Solving for rapidity dependence, we then get the result

$$\begin{aligned} {}^{R}F(x_i, \mathbf{z}_i, \mathbf{y}; \mu_i, \zeta) = {}^{R}F(x_i, \mathbf{z}_i, \mathbf{y}; \mu_{0i}, \zeta) & \\ \times \exp\Bigg\{ \int_{\mu_{01}}^{\mu_1} \frac{d\mu}{\mu} \Bigg[\gamma_F(\mu, x_1\zeta/x_2) - \gamma_K(\mu) \log \frac{\sqrt{x_1\zeta/x_2}}{\mu} \Bigg] & \\ + \int_{\mu_{02}}^{\mu_2} \frac{d\mu}{\mu} \Bigg[\gamma_F(\mu, x_2\zeta/x_1) - \gamma_K(\mu) \log \frac{\sqrt{x_2\zeta/x_1}}{\mu} \Bigg] & \\ + \Big[{}^{R}K(\mathbf{z}_1, \mu_{01}) + {}^{R}K(\mathbf{z}_2, \mu_{02}) + {}^{R}J(\mathbf{y}, \mu_{0i}) \Big] \log \frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} \Bigg\} & \end{aligned}$$

- The *K*-kernel splitting in three separate contributions is crucial, but only true in limit where we can do the DTMD → DPDF matching.

Coefficient functions

- Consider the limit
  - $|\mathbf{z}_1|$, $|\mathbf{z}_2|$ much smaller than $1/\Lambda$
  - $|\mathbf{z}_1|$, $|\mathbf{z}_2| \ll$ **y**, with **y** fixed

- We calculate the coefficient functions for a value of **z** at $\mathrm{O}(\alpha_s)$.
  - Collinear contribution given by
    $$ {}^{R}C_{f,ab}(x,\mathbf{z},v_L) = \delta_{ab}\delta(1-x) + \alpha_s\, {}^{R}C^{(1)}_{f,ab}(x,\mathbf{z},v_L) + \mathcal{O}(\alpha_s^2) $$
  - Soft function contribution given by
    $$ {}^{R}C_{s,a}(\mathbf{z},v_L,v_C) = 1 + \alpha_s\, {}^{R}C^{(1)}_{s,a}(\mathbf{z},v_L,v_C) + \mathcal{O}(\alpha_s^2) $$

- The expression for the coefficient function at order at $\mathrm{O}(\alpha_s)$ is then given by

$$ {}^{R}C_{ab}(x,\mathbf{z},\ldots/\zeta\ldots) = \delta_{ab}\delta(1-x) $$
$$ +\alpha_s \lim_{v_L^2\to 0}\left[{}^{R}C^{(1)}_{f,ab}(x,\mathbf{z},v_L) - \delta_{ab}\delta(1-x)\,{}^{R}C^{(1)}_{s,a}(\mathbf{z},v_L,v_C)\right] + \mathcal{O}(\alpha_s^2) $$