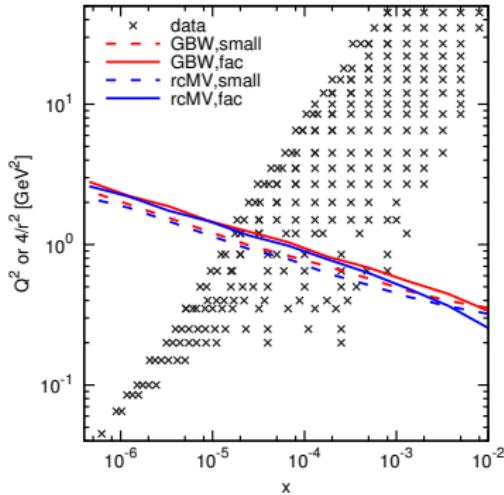
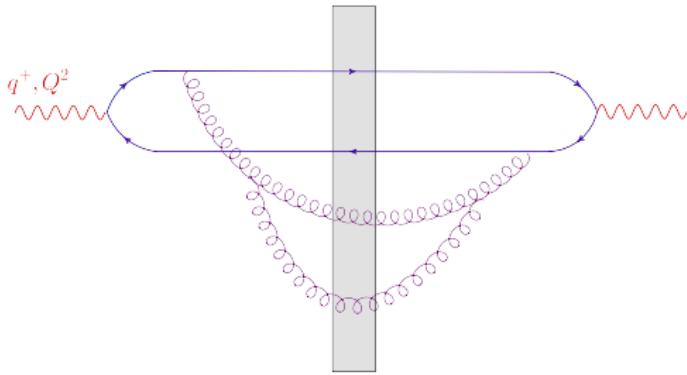


# Resumming large radiative corrections in the non-linear evolution in QCD at high energy

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w/ J.D. Madrigal, A.H. Mueller, G.Soyez, and D.N. Triantafyllopoulos



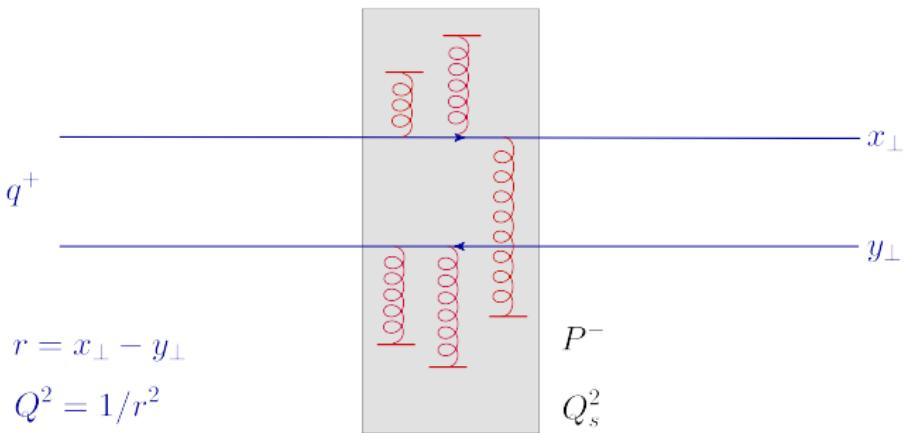
# Introduction

- Wilson lines : the right d.o.f. for QCD scattering at high energy
  - multiple scattering in the eikonal approximation
- High-energy evolution described by the Balitsky-JIMWLK equations
  - infinite hierarchy, non-linear generalization of BFKL
  - large number of colors  $\Rightarrow$  a closed equation: Balitsky-Kovchegov
- This non-linear evolution has recently been promoted to NLO
  - essential for a realistic phenomenology  
*(Balitsky, Chirilli, 2008, 2013; Kovner, Lublinsky, Mulian, 2013)*
- Large corrections enhanced by double or single collinear logarithms
  - lack of convergence, unstable evolution at NLO
- Similar problems encountered & solved for NLO BFKL
  - collinear resummations, formulated in Mellin space  
*(Salam, Ciafaloni, Colferai, Stasto, 98-03; Altarelli, Ball, Forte, 00-03)*

# Introduction

- Mellin representation is not suitable beyond the linear approximation
- Non-linear effects (multiple scattering) most naturally discussed in terms of **transverse coordinates**
  - eikonal approximation, Wilson lines
- Alternative resummation, formulated in **transverse coordinates**  
*(E.I., J. Madrigal, A. Mueller, G. Soyez, and D. Triantafyllopoulos, 2015)*
  - collinearly improved version of the BK equation  
*(arXiv:1502.05642, Phys.Lett. B744 (2015) 293)*
  - promising phenomenology (so far, only DIS)  
*(arXiv:1507.03651, Phys.Lett. B750 (2015) 643)*
  - extension to full JIMWLK (finite  $N_c$ ) : under way  
*(Y. Hatta and E.I., to appear)*
  - further applications to  $pp$ ,  $pA$ ,  $AA$  (initial conditions)

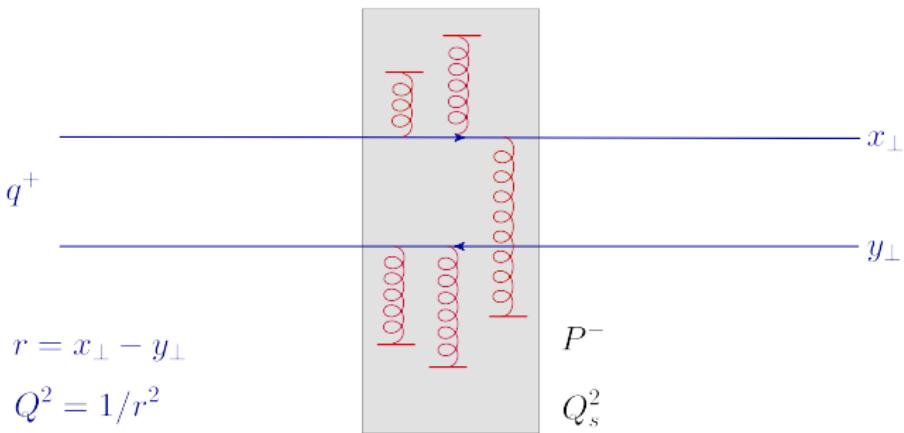
# Dipole–hadron scattering ( $\gamma^* p$ , $\gamma^* A$ , $pA$ , ...)



- Dipole ('projectile'): large  $q^+$ , transverse resolution  $Q^2 = 1/r^2$
- Hadron ('target'): large  $P^-$ , saturation momentum  $Q_s^2 \gg \Lambda_{\text{QCD}}^2$
- Scattering is weak ( $T \ll 1$ ) if  $Q^2 \gg Q_s^2$  and strong ( $T \sim 1$ ) otherwise
- Eikonal approximation : Wilson lines for the quark and the antiquark

$$V(\mathbf{x}) = P \exp \left\{ ig \int dx^+ A_a^-(x^+, \mathbf{x}) t^a \right\}$$

# Dipole–hadron scattering ( $\gamma^* p$ , $\gamma^* A$ , $pA$ , ...)

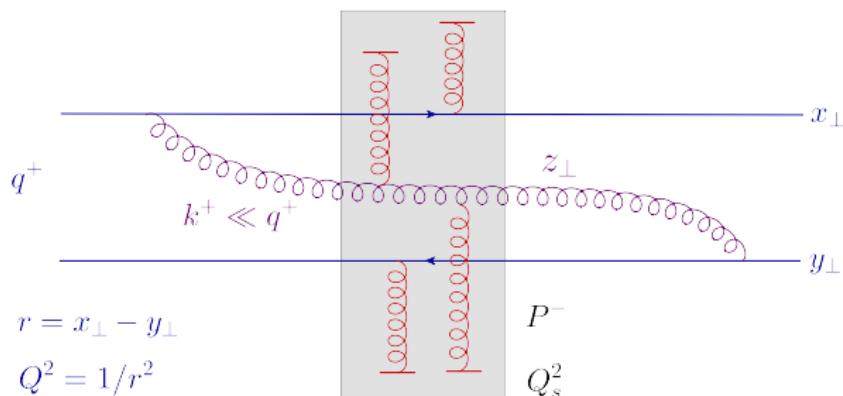


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- Scattering is weak ( $T \ll 1$ ) if  $Q^2 \gg Q_s^2$  and strong ( $T \sim 1$ ) otherwise
- Dipole *S*-matrix : F.T. of unintegrated gluon distribution in the target

$$S_{xy} = \frac{1}{N_c} \text{tr}(V_x V_y^\dagger) = 1 - T_{xy} \quad (\text{average over the target: CGC})$$

# High energy evolution

- Probability  $\sim \alpha_s \ln \frac{1}{x}$  to radiate a soft gluon with  $x \equiv \frac{k^+}{q^+} \ll 1$

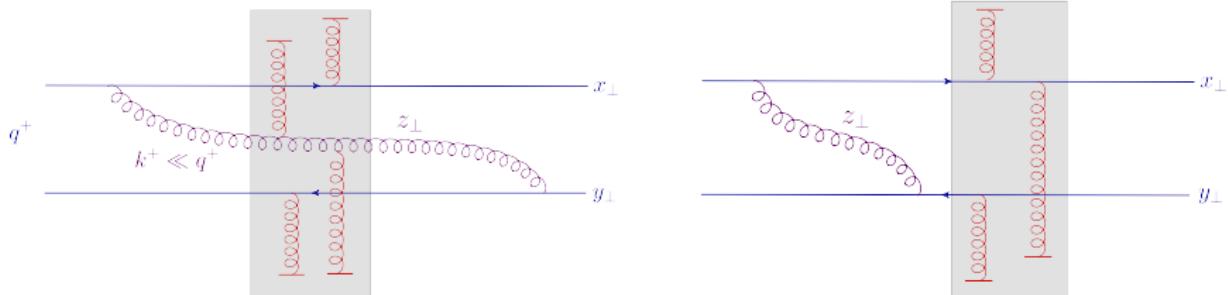


$$xq^+ \sim \frac{Q_s^2}{2P^-} \implies x \simeq \frac{Q_s^2}{2q^+P^-} \ll 1$$

- When  $\alpha_s \ln \frac{1}{x} \sim 1$ , need for resummation:  $(\alpha_s Y)^n$  with  $Y \equiv \ln \frac{1}{x}$ 
  - BFKL evolution of the dipole in the background of the dense target
  - multiple scattering  $\implies$  non-linear evolution  $\implies$  Balitsky-JIMWLK

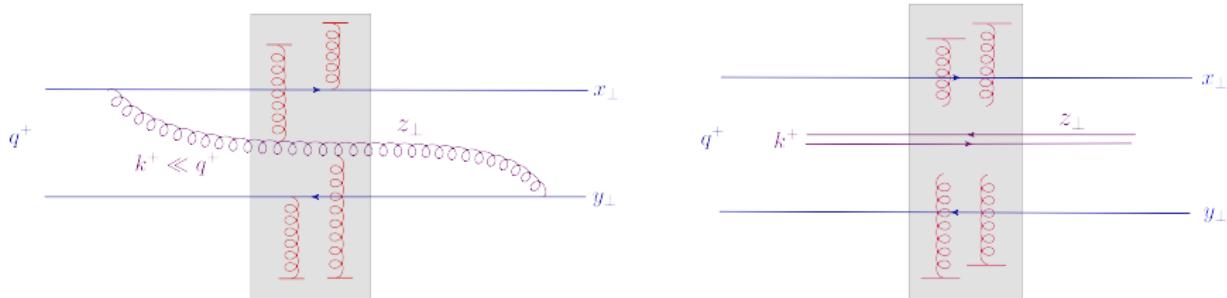
# The BK equation (Balitsky, '96; Kovchegov, '99)

- Both 'real' graphs (the soft gluon crosses the target) and 'virtual'



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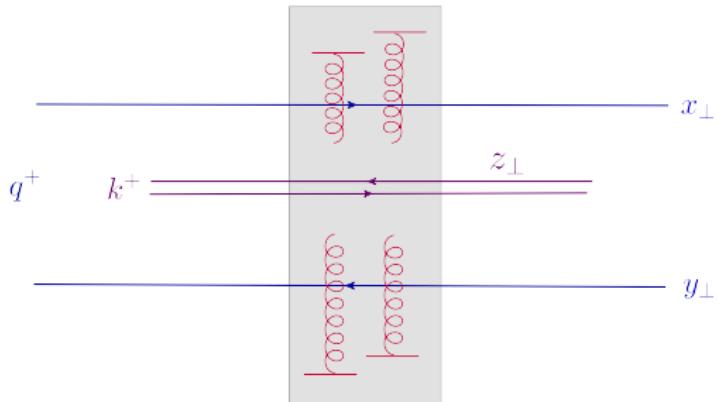
- Large  $N_c$  : the original dipole splits into two new dipoles
- Evolution equation for the dipole  $S$ -matrix  $S_{xy}(Y)$

$$\frac{\partial S_{xy}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int d^2 z \frac{(x-y)^2}{(x-z)^2(y-z)^2} [S_{xz}S_{zy} - S_{xy}]$$

- 'dipole kernel' : probability for the dipole to split (Al Mueller, 1990)
- Mean field approximation to the Balitsky-JIMWLK hierarchy

# Deconstructing the BK equation

- Non-linear equation for the scattering amplitude  $T_{xy} \equiv 1 - S_{xy}$

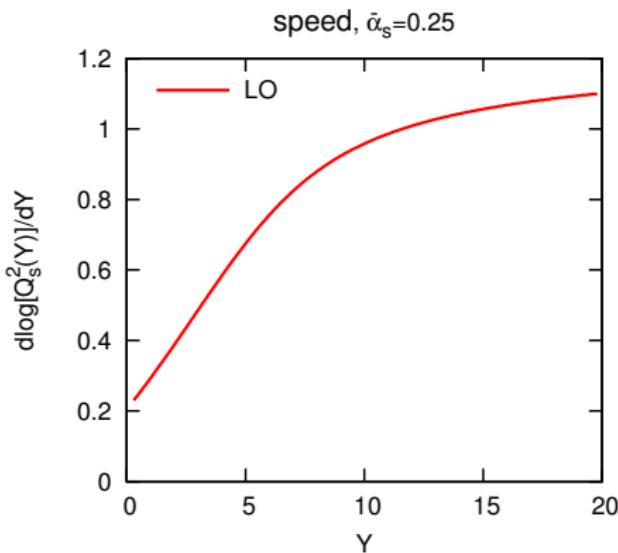
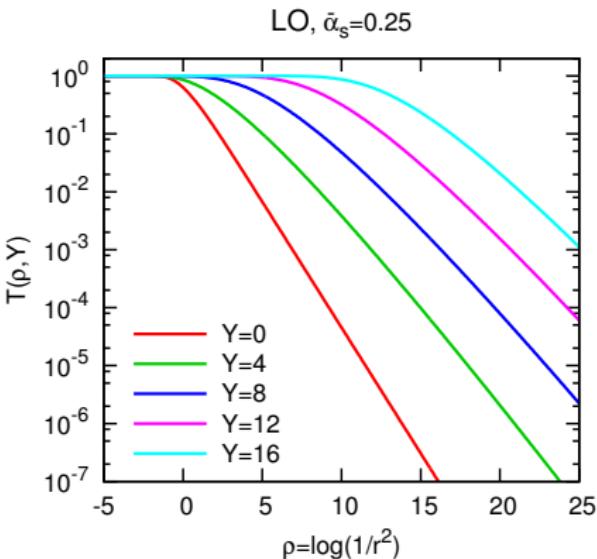


$$\frac{\partial T_{xy}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2} [T_{xz} + T_{zy} - T_{xy} - T_{xz}T_{zy}]$$

- weak scattering (dilute target):  $T(r, Y) \ll 1 \Rightarrow$  BFKL equation
- non-linear term enforces unitarity bound:  $T(r, Y) \leq 1$
- saturation momentum  $Q_s(Y)$ :  $T(r, Y) = 0.5$  when  $r = 1/Q_s(Y)$

# The saturation front

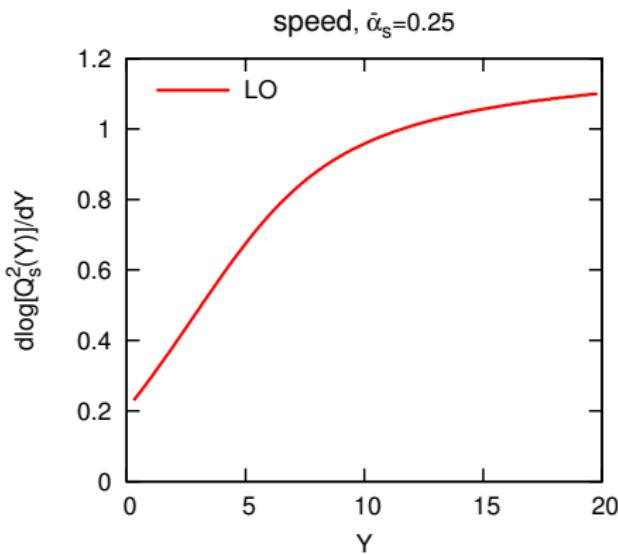
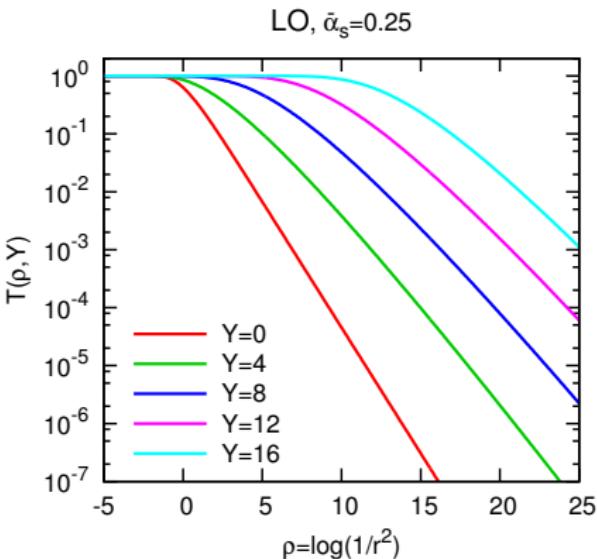
- $T(\rho, Y)$  as a function of  $\rho = \ln(1/r^2 Q_0^2)$  with  $Q_0 \equiv Q_s(Y=0)$



- color transparency at large  $\rho$  (small  $r$ ) :  $T \propto r^2 = e^{-\rho}$
- unitarization at small  $\rho$  (large  $r$ ) :  $T = 1$  (black disk limit)
- transition point  $\rho_s \equiv \ln[Q_s^2(Y)/Q_0^2]$  increases with  $Y$

# The saturation front

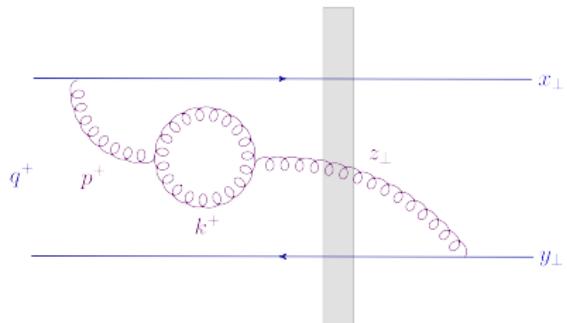
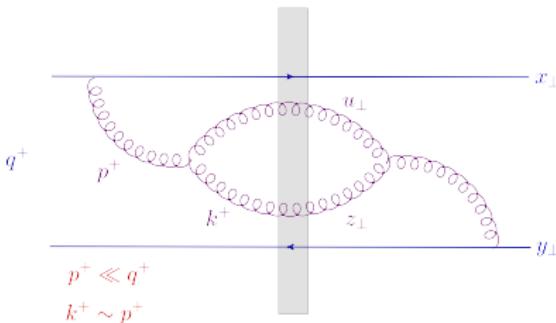
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- unitarization at small  $\rho$  (large  $r$ ) :  $T = 1$  (black disk limit)
- saturation exponent:  $\lambda_s \equiv d\rho_s/dY \simeq 1$  for  $Y \gtrsim 10$  : way too large

# Next-to-leading order

- Any effect of  $\mathcal{O}(\bar{\alpha}_s^2 Y) \implies \mathcal{O}(\bar{\alpha}_s)$  correction to the BFKL kernel



- The prototype: two successive emissions, one **soft** and one **non-soft**
- The maximal correction thus expected:  $\mathcal{O}(\bar{\alpha}_s \rho)$  with  $\rho \equiv \ln(Q^2/Q_s^2)$
- But one finds an even larger effect:  $\mathcal{O}(\bar{\alpha}_s \rho^2)$  ('double collinear log')
- Originally found as a NLO correction to the BFKL kernel  
*(Fadin, Lipatov, Camici, Ciafaloni ... 95-98; Balitsky, Chirilli, 07)*

# BK equation at NLO *Balitsky, Chirilli (arXiv:0710.4330)*

- “Reasonably simple” (= it fits into one slide)
- Note however:  $N_f = 0$ , large  $N_c$ , tiny fonts

$$\begin{aligned} \frac{\partial S_{\mathbf{x}\mathbf{y}}}{\partial Y} &= \frac{\bar{\alpha}_s}{2\pi} \int d^2 z \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2} (S_{\mathbf{x}\mathbf{z}} S_{\mathbf{z}\mathbf{y}} - S_{\mathbf{x}\mathbf{y}}) \left\{ 1 + \right. \\ &\quad + \bar{\alpha}_s \left[ \bar{b} \ln(\mathbf{x}-\mathbf{y})^2 \mu^2 - \bar{b} \frac{(\mathbf{x}-\mathbf{z})^2 - (\mathbf{y}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2} \ln \frac{(\mathbf{x}-\mathbf{z})^2}{(\mathbf{y}-\mathbf{z})^2} \right. \\ &\quad \left. \left. + \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2} \ln \frac{(\mathbf{x}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2} \ln \frac{(\mathbf{y}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2} \right] \right\} \\ &+ \frac{\bar{\alpha}_s^2}{8\pi^2} \int \frac{d^2 u \, d^2 z}{(\mathbf{u}-\mathbf{z})^4} (S_{\mathbf{x}\mathbf{u}} S_{\mathbf{u}\mathbf{z}} S_{\mathbf{z}\mathbf{y}} - S_{\mathbf{x}\mathbf{u}} S_{\mathbf{u}\mathbf{y}}) \\ &\quad \left\{ -2 + \frac{(\mathbf{x}-\mathbf{u})^2(\mathbf{y}-\mathbf{z})^2 + (\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{u})^2 - 4(\mathbf{x}-\mathbf{y})^2(\mathbf{u}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{u})^2(\mathbf{y}-\mathbf{z})^2 - (\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{u})^2} \ln \frac{(\mathbf{x}-\mathbf{u})^2(\mathbf{y}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{u})^2} \right. \\ &\quad \left. + \frac{(\mathbf{x}-\mathbf{y})^2(\mathbf{u}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{u})^2(\mathbf{y}-\mathbf{z})^2} \left[ 1 + \frac{(\mathbf{x}-\mathbf{y})^2(\mathbf{u}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{u})^2(\mathbf{y}-\mathbf{z})^2 - (\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{u})^2} \right] \ln \frac{(\mathbf{x}-\mathbf{u})^2(\mathbf{y}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{u})^2} \right\} \end{aligned}$$

# Deconstructing NLO BK

$$\begin{aligned} \frac{\partial S_{\mathbf{x}\mathbf{y}}}{\partial Y} = & \frac{\bar{\alpha}_s}{2\pi} \int d^2 \mathbf{z} \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2 (\mathbf{y}-\mathbf{z})^2} (S_{\mathbf{xz}} S_{\mathbf{zy}} - S_{\mathbf{x}\mathbf{y}}) \left\{ 1 + \right. \\ & + \bar{\alpha}_s \left[ \bar{b} \ln(\mathbf{x}-\mathbf{y})^2 \mu^2 - \bar{b} \frac{(\mathbf{x}-\mathbf{z})^2 - (\mathbf{y}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2} \ln \frac{(\mathbf{x}-\mathbf{z})^2}{(\mathbf{y}-\mathbf{z})^2} \right. \\ & \left. \left. + \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2} \ln \frac{(\mathbf{x}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2} \ln \frac{(\mathbf{y}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2} \right] \right\} \\ & + \frac{\bar{\alpha}_s^2}{8\pi^2} \int \frac{d^2 \mathbf{u} d^2 \mathbf{z}}{(\mathbf{u}-\mathbf{z})^4} (S_{\mathbf{xu}} S_{\mathbf{uz}} S_{\mathbf{zy}} - S_{\mathbf{xu}} S_{\mathbf{uy}}) \\ & \left\{ -2 + \frac{(\mathbf{x}-\mathbf{u})^2 (\mathbf{y}-\mathbf{z})^2 + (\mathbf{x}-\mathbf{z})^2 (\mathbf{y}-\mathbf{u})^2 - 4(\mathbf{x}-\mathbf{y})^2 (\mathbf{u}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{u})^2 (\mathbf{y}-\mathbf{z})^2 - (\mathbf{x}-\mathbf{z})^2 (\mathbf{y}-\mathbf{u})^2} \ln \frac{(\mathbf{x}-\mathbf{u})^2 (\mathbf{y}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{z})^2 (\mathbf{y}-\mathbf{u})^2} \right. \\ & \left. + \frac{(\mathbf{x}-\mathbf{y})^2 (\mathbf{u}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{u})^2 (\mathbf{y}-\mathbf{z})^2} \left[ 1 + \frac{(\mathbf{x}-\mathbf{y})^2 (\mathbf{u}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{u})^2 (\mathbf{y}-\mathbf{z})^2 - (\mathbf{x}-\mathbf{z})^2 (\mathbf{y}-\mathbf{u})^2} \right] \ln \frac{(\mathbf{x}-\mathbf{u})^2 (\mathbf{y}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{z})^2 (\mathbf{y}-\mathbf{u})^2} \right\} \end{aligned}$$

- blue : leading-order (LO) terms
- red : NLO terms enhanced by (double or single) transverse logarithms
- black : pure  $\bar{\alpha}_s$  corrections (no logarithms)

# Collinear logarithms

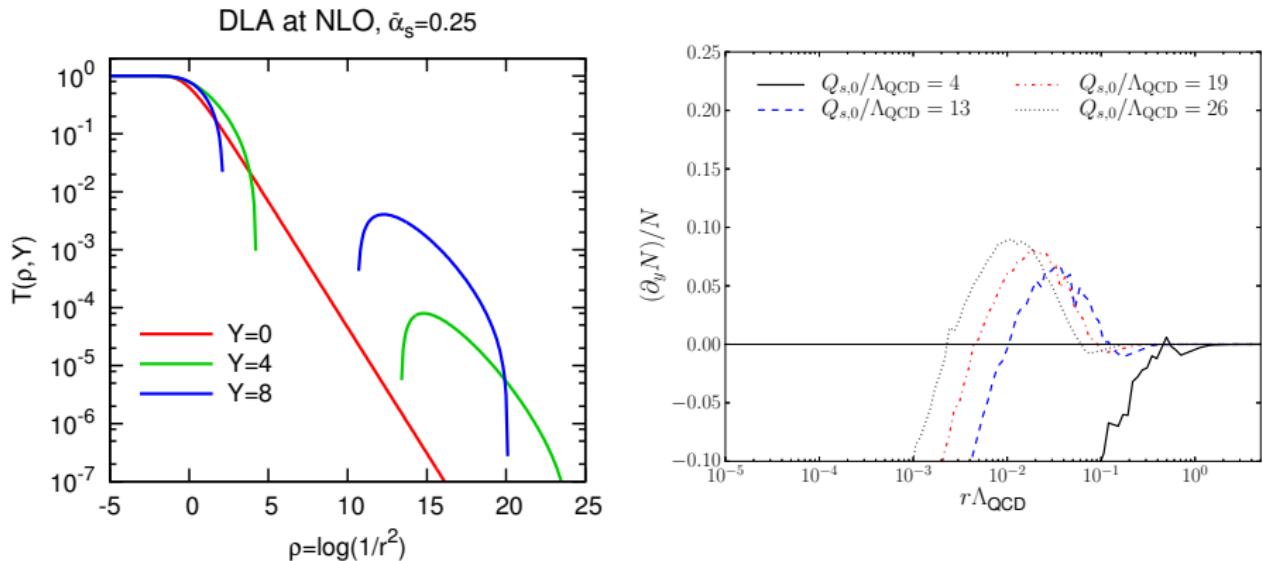
- Ratios of widely separated dipole sizes: requires  $Q^2 = 1/r^2 \gg Q_s^2$
- The **double** logarithm is already manifest:

$$-\frac{1}{2} \ln \frac{(\mathbf{x}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2} \ln \frac{(\mathbf{y}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2} \simeq -\frac{1}{2} \ln^2 \frac{(\mathbf{x}-\mathbf{z})^2}{r^2} \quad \text{if} \quad |\mathbf{z}-\mathbf{x}| \simeq |\mathbf{z}-\mathbf{y}| \gg r$$

- The **single logs** are still hidden: needs to perform the integral over  $\mathbf{u}$   
 $1/Q_s \gg |\mathbf{z}-\mathbf{x}| \simeq |\mathbf{z}-\mathbf{y}| \simeq |\mathbf{z}-\mathbf{u}| \gg |\mathbf{u}-\mathbf{x}| \simeq |\mathbf{u}-\mathbf{y}| \gg r$ 
  - all dipoles are relatively small ( $\ll 1/Q_s$ ): **weak scattering**
  - logarithmic phase-space for the intermediate gluon at  $\mathbf{u}$
- Keeping just the collinear logarithms ( $|\mathbf{z}-\mathbf{x}| \rightarrow z$ ):

$$\frac{\partial T(Y, r)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} dz^2 \frac{r^2}{z^4} \left\{ 1 - \bar{\alpha}_s \left( \frac{1}{2} \ln^2 \frac{z^2}{r^2} + \frac{11}{12} \ln \frac{z^2}{r^2} \right) \right\} T(Y, z)$$

# Unstable numerical solution

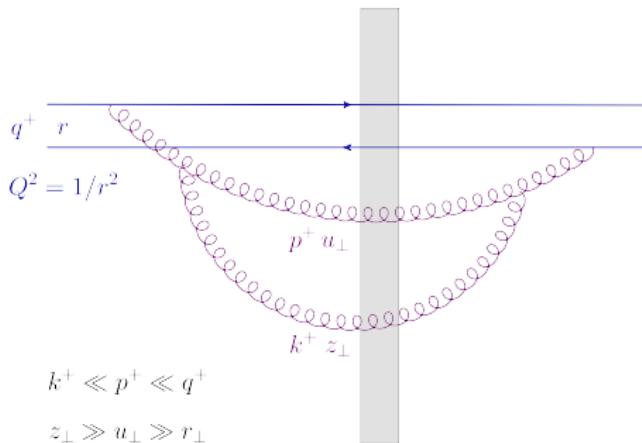


- Left: LO BK + the double collinear logarithm
- Right: full NLO BK (*Lappi, Mäntysaari, arXiv:1502.02400*)
- The main source of instability: the double collinear logarithm

# Time ordering & Double collinear logs

- Strong ordering in both longitudinal and transverse momenta

$$q^+ \gg p^+ \gg k^+ \quad \& \quad r^2 \ll u_\perp^2 \ll z_\perp^2 \ll 1/Q_s^2$$



- lifetime  $\approx$  energy denominator

$$\Delta t \simeq \frac{1}{\Delta E} \simeq \frac{1}{p^- + k^-}$$

- light-cone energies

$$p^- = \frac{p_\perp^2}{2p^+} \simeq \frac{1}{p^+ u_\perp^2}$$

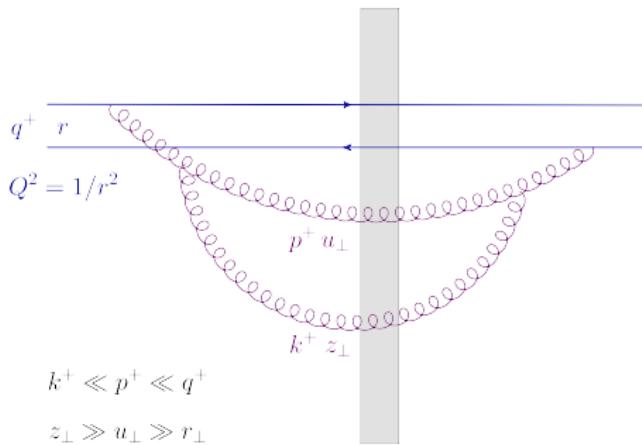
- Integrate out the harder gluon ( $p^+, u_\perp$ ) to double-log accuracy:

$$\bar{\alpha}_s \int_{r^2}^{z^2} \frac{du^2}{u^2} \int_{k^+}^{p^+} \frac{dp^+}{p^+} \frac{p^+ u^2}{p^+ u^2 + k^+ z^2}$$

# Time ordering & Double collinear logs

- Strong ordering in both longitudinal and transverse momenta

$$q^+ \gg p^+ \gg k^+ \quad \& \quad r^2 \ll u_\perp^2 \ll z_\perp^2 \ll 1/Q_s^2$$



- lifetime  $\approx$  energy denominator

$$\Delta t \simeq \frac{1}{\Delta E} \simeq \frac{1}{p^- + k^-}$$

- to have double logs, one needs

$$\tau_p \simeq p^+ u_\perp^2 \gg \tau_k \simeq k^+ z_\perp^2$$

- Integrate out the harder gluon  $(p^+, u_\perp)$  to double-log accuracy:

$$\bar{\alpha}_s \int_{r^2}^{z^2} \frac{du^2}{u^2} \int_{k^+}^{q^+} \frac{dp^+}{p^+} \Theta(p^+ u^2 - k^+ z^2) = \bar{\alpha}_s Y \ln \frac{z^2}{r^2} - \frac{\bar{\alpha}_s}{2} \ln^2 \frac{z^2}{r^2}$$

# Resummation of double logs in DLA

- Double-collinear logs can be systematically resummed to all orders by enforcing time-ordering within naive DLA

$$\frac{\partial T(Y, r)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_0^2} \frac{dz^2}{z^2} \frac{r^2}{z^2} T(Y, z)$$

# Resummation of double logs in DLA

- Double-collinear logs can be systematically resummed to all orders by enforcing time-ordering within naive DLA  $\implies$  equation non-local in  $Y$

$$\frac{\partial T(Y, r)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} \frac{dz^2}{z^2} \frac{r^2}{z^2} \Theta\left(Y - \ln \frac{z^2}{r^2}\right) T\left(Y - \ln \frac{z^2}{r^2}, z\right)$$

- The importance of time-ordering in BFKL had already been recognized

Ciafaloni (88), CCFM (90), Lund group (Andersson et al, 96),  
Kwiecinski et al (96), Salam (98) ... Motyka, Stasto (09), G. Beuf (14)

- 'coherence effects', 'kinematical constraint', 'choice of energy scale', 'choice of the high-energy factorization scheme'
- N.B. Non-local in  $Y$  but local in the variable  $\eta \equiv Y - \rho$ : target rapidity

$$\eta \equiv Y - \ln \frac{Q^2}{Q_s^2} = \ln \frac{1}{x_{Bj}} \quad \text{with} \quad x_{Bj} \equiv \frac{Q^2}{2q^+ P^-} = \frac{k^-}{P^-}$$

- $k^-$  is not the natural evolution variable for the right-moving projectile

# Getting local

- So far, no change in the kernel: double-logs come from **non-locality**
- Equivalently: a **local** equation but with a **resummed kernel**

$$\frac{\partial T(Y, r)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} \frac{dz^2}{z^2} \frac{r^2}{z^2} \mathcal{K}_{\text{DLA}} \left( \ln \frac{z^2}{r^2} \right) T(Y, z)$$

- ... and the **resummed initial condition** (impact factor)

$$T(0, r) = \bar{\alpha}_s r^2 Q_0^2 \ln \frac{1}{r^2 Q_0^2} \mathcal{K}_{\text{DLA}} \left( \ln \frac{1}{r^2 Q_0^2} \right)$$

- ... where  $\mathcal{K}_{\text{DLA}}(\rho)$  resums **powers of  $\bar{\alpha}_s \rho^2$**  to all orders:

$$\mathcal{K}_{\text{DLA}}(\rho) \equiv \frac{J_1(2\sqrt{\bar{\alpha}_s \rho^2})}{\sqrt{\bar{\alpha}_s \rho^2}} = 1 - \frac{\bar{\alpha}_s \rho^2}{2} + \frac{(\bar{\alpha}_s \rho^2)^2}{12} + \dots$$

- $\mathcal{K}_{\text{DLA}}(\rho)$ : Jacobian for the change of factorization scheme  $\eta \rightarrow Y$

# Resummation of double logs in BK

- The local equation can be easily promoted to include BK physics

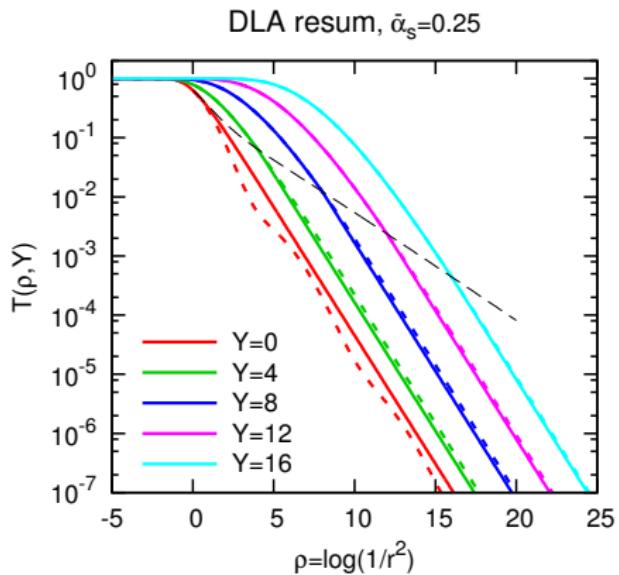
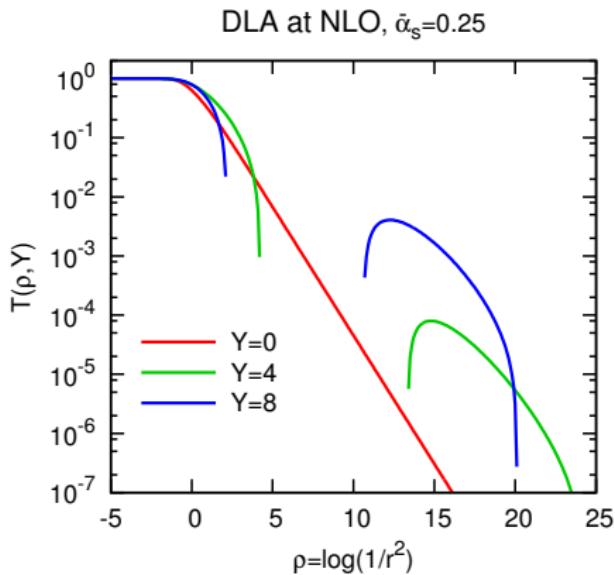
$$\frac{\partial S_{\mathbf{x}\mathbf{y}}}{\partial Y} = \bar{\alpha}_s \int \frac{d^2 z}{2\pi} \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} \mathcal{K}_{\text{DLA}}(\rho(\mathbf{x}, \mathbf{y}, \mathbf{z})) (S_{\mathbf{x}\mathbf{z}} S_{\mathbf{z}\mathbf{y}} - S_{\mathbf{x}\mathbf{y}})$$

- $\rho(\mathbf{x}, \mathbf{y}, \mathbf{z})$  : the symmetrized version of the collinear double-log:

$$\rho^2(\mathbf{x}, \mathbf{y}, \mathbf{z}) \equiv \ln \frac{(\mathbf{x}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2} \ln \frac{(\mathbf{y}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2}$$

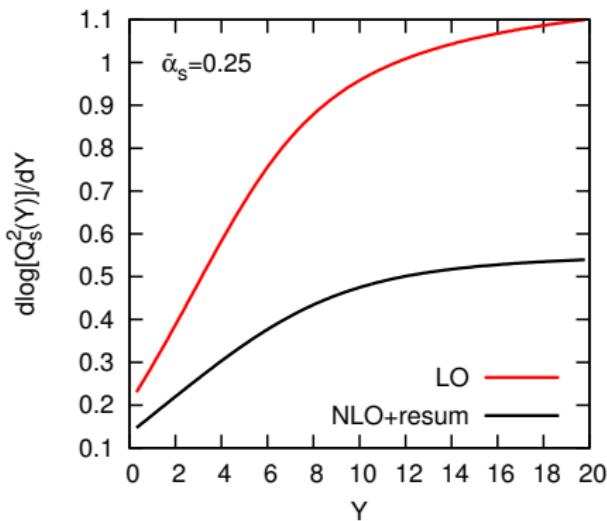
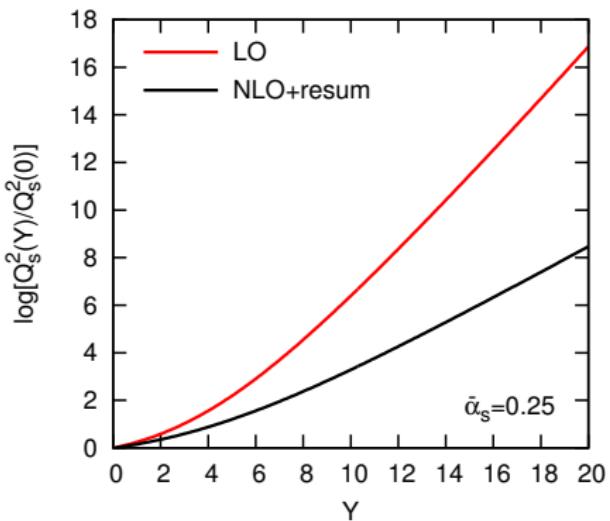
- The first correction, of  $\mathcal{O}(\bar{\alpha}_s \rho^2)$ , coincides with the NLO double-log
- Exactly resums double log terms to all orders
- The resummation stabilizes and slows down the evolution

# Numerical solutions: saturation front



- Fixed coupling  $\bar{\alpha}_s = 0.25$ 
  - left:  $\mathcal{K}_{\text{DLA}}$  expanded to NLO
  - right: full kernel, double collinear logs resummed to all orders

# Numerical solutions: saturation momentum

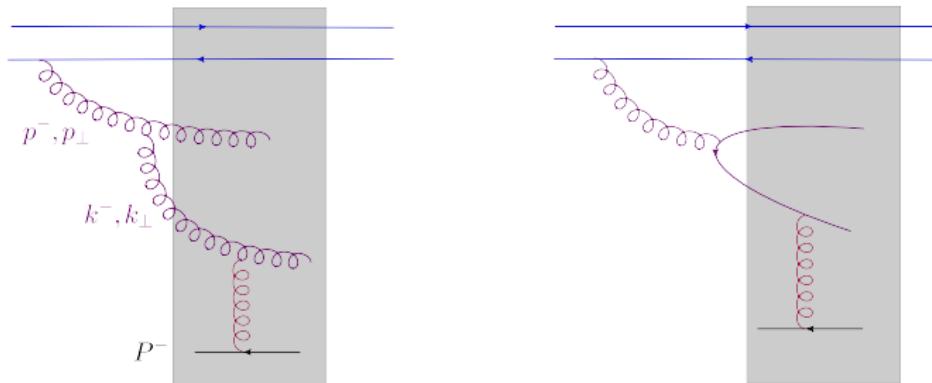


- Fixed coupling  $\bar{\alpha}_s = 0.25$ 
  - left: saturation momentum  $\rho_s(Y) \equiv \ln[Q_s^2(Y)/Q_0^2]$
  - right: saturation exponent  $\lambda_s \equiv d\rho_s/dY$
  - $Y \gtrsim 10 : \lambda_s|_{\text{NLO}} \simeq 0.5 \simeq \text{half of } \lambda_s|_{\text{LO}} \dots \text{but still too large}$

# The single logs are important too ...

$$\frac{\partial T(r)}{\partial Y} = \bar{\alpha}_s \int dz^2 \frac{r^2}{z^4} \left\{ 1 - \bar{\alpha}_s \left( \frac{1}{2} \ln^2 \frac{z^2}{r^2} + A_1 \ln \frac{z^2}{r^2} \right) \right\} T(z), \quad A_1 = \frac{11}{12} + \frac{N_f}{6N_c^3}$$

- “one BFKL gluon (small  $x$ )  $\times$  one DGLAP emission (large  $x$  parton)”



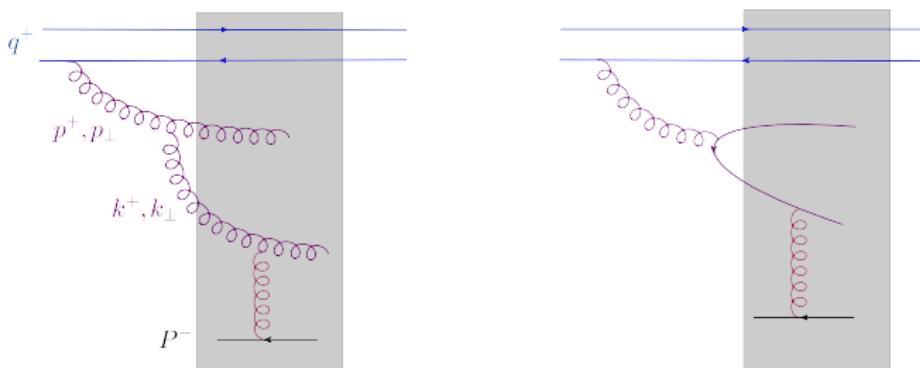
- This however refers to the evolution of the **target** (“soft to hard”)

$$x \equiv \frac{k^-}{P^-}, \quad p^- < k^- < P^-, \quad Q^2 \gg p_{\perp}^2 \gg k_{\perp}^2$$

# The single logs are important too ...

$$\frac{\partial T(r)}{\partial Y} = \bar{\alpha}_s \int dz^2 \frac{r^2}{z^4} \left\{ 1 - \bar{\alpha}_s \left( \frac{1}{2} \ln^2 \frac{z^2}{r^2} + A_1 \ln \frac{z^2}{r^2} \right) \right\} T(z), \quad A_1 = \frac{11}{12} + \frac{N_f}{6N_c^3}$$

- “one BFKL gluon (small  $x$ )  $\times$  one DGLAP emission (large  $x$  parton)”



- **Projectile evolution:**  $q^+ \gg p^+ \gg k^+$  &  $Q^2 \gg p_\perp^2 \gg k_\perp^2$ 
  - subleading contribution from strong double ordering
  - lifetimes are now comparable:  $\tau_p \sim \tau_k$

# Guidance from DGLAP evolution

- The **collinear limit of BFKL** is properly reproduced by **DGLAP**
  - target evolution  $\implies$  rapidity  $\eta = Y - \rho$

$$T(Y, r) \propto xG\left(\eta, \rho = \ln \frac{1}{r^2 Q_0^2}\right) = \int \frac{d\omega}{2\pi i} e^{\omega(Y-\rho)+\bar{\alpha}_s \gamma(\omega)\rho}$$

- The 'most gluonic' anomalous dimension (largest eigenvalue)

$$\gamma(\omega) = \int_0^1 dz z^\omega \left[ P_{GG}(z) + \frac{C_F}{N_c} P_{qG}(z) \right] = \frac{1}{\omega} - A_1 + A_2 \omega + \mathcal{O}(\omega^2)$$

# Guidance from DGLAP evolution

- The **collinear limit of BFKL** is properly reproduced by **DGLAP**
  - target evolution  $\implies$  rapidity  $\eta = Y - \rho$

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- The ‘most gluonic’ anomalous dimension (largest eigenvalue)

$$\gamma(\omega) = \int_0^1 dz z^\omega \left[ P_{GG}(z) + \frac{C_F}{N_c} P_{qG}(z) \right] \simeq \frac{1}{\omega} - A_1$$

- Keep only the first 2 terms  $\implies A_1$  exponentiates
- Expanding the above generates the collinear limit of **NLO BFKL**:

$$\frac{\partial T(r)}{\partial Y} = \bar{\alpha}_s \int dz^2 \frac{r^2}{z^4} \left\{ 1 - \bar{\alpha}_s \left( \frac{1}{2} \ln^2 \frac{z^2}{r^2} + A_1 \ln \frac{z^2}{r^2} \right) \right\} T(z)$$

# Guidance from DGLAP evolution

- The collinear limit of BFKL is properly reproduced by DGLAP

- target evolution  $\Rightarrow$  rapidity  $\eta = Y - \rho$

$$T(Y, r) \propto xG \left( \eta, \rho = \ln \frac{1}{r^2 Q_0^2} \right) = e^{-\bar{\alpha}_s A_1 \rho} \int \frac{d\omega}{2\pi i} e^{\omega(Y-\rho) + \frac{\bar{\alpha}_s}{\omega} \rho}$$

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- W/o expansion: the collinearly improved BFKL at DLA accuracy

$$\frac{\partial T(Y, r)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} \frac{dz^2}{z^2} \left( \frac{r^2}{z^2} \right)^{1+\bar{\alpha}_s A_1} \mathcal{K}_{\text{DLA}} \left( \ln \frac{z^2}{r^2} \right) T(Y, z)$$

- double logs: rapidity shift  $Y \rightarrow \eta = Y - \rho$
- single logs: one BFKL emission  $\times$  arbitrary many DGLAP emissions

# Guidance from DGLAP evolution

- The **collinear limit of BFKL** is properly reproduced by **DGLAP**
  - target evolution  $\implies$  rapidity  $\eta = Y - \rho$

$$T(Y, r) \propto xG \left( \eta, \rho = \ln \frac{1}{r^2 Q_0^2} \right) = e^{-\bar{\alpha}_s A_1 \rho} \int \frac{d\omega}{2\pi i} e^{\omega(Y-\rho) + \frac{\bar{\alpha}_s}{\omega} \rho}$$

- The 'most gluonic' anomalous dimension (largest eigenvalue)

$$\gamma(\omega) = \int_0^1 dz z^\omega \left[ P_{GG}(z) + \frac{C_F}{N_c} P_{qG}(z) \right] \simeq \frac{1}{\omega} - A_1$$

- With one-loop running coupling:  $\bar{\alpha}_s(Q) = 1/(\bar{b} \ln(Q^2/\Lambda^2))$

$$\frac{\partial T(Y, r)}{\partial Y} = \bar{\alpha}_s(r) \int_{r^2}^{1/Q_s^2} \frac{dz^2}{z^2} \frac{r^2}{z^2} \left[ \frac{\bar{\alpha}_s(r)}{\bar{\alpha}_s(z)} \right]^{A_1/\bar{b}} \mathcal{K}_{\text{DLA}} \left( \ln \frac{z^2}{r^2} \right) T(Y, z)$$

- $\bar{\alpha}_s(r) \equiv \bar{\alpha}_s(Q = 2/r)$

# A collinearly improved BK equation

- Extension to full BK physics is now straightforward:

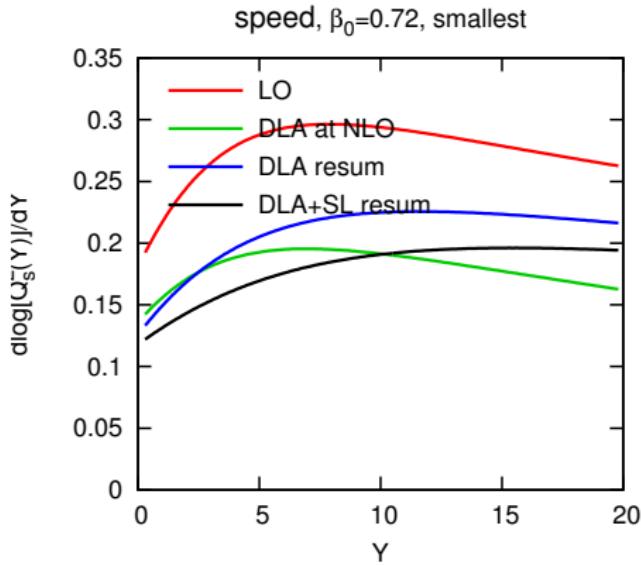
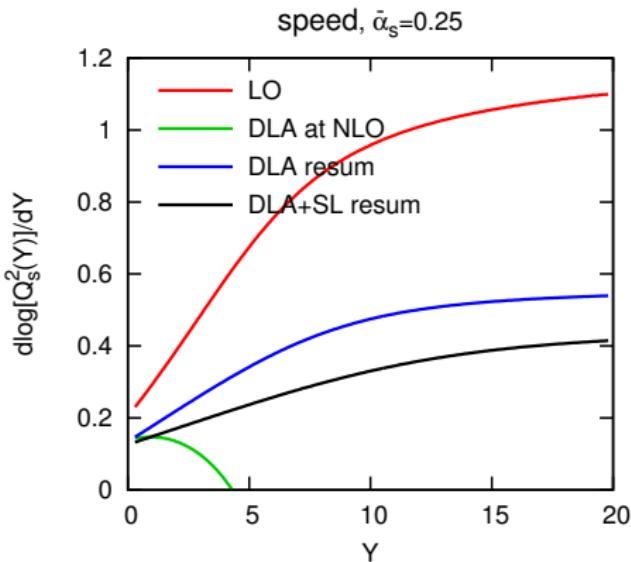
$$\frac{\partial S_{\mathbf{x}\mathbf{y}}}{\partial Y} = \int \frac{d^2 z}{2\pi} \bar{\alpha}_s(r_{\min}) \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} \left[ \frac{\bar{\alpha}_s(r)}{\bar{\alpha}_s(z_<)} \right]^{\pm A_1/\bar{b}} \mathcal{K}_{\text{DLA}}(\rho(\mathbf{x}, \mathbf{y}, \mathbf{z})) \\ \times (S_{\mathbf{x}\mathbf{z}} S_{\mathbf{z}\mathbf{y}} - S_{\mathbf{x}\mathbf{y}})$$

- $r_{\min} \equiv \min \{|\mathbf{x}-\mathbf{y}|, |\mathbf{x}-\mathbf{z}|, |\mathbf{y}-\mathbf{z}|\}$  : the smallest of the 3 dipoles
  - to cancel the potentially large logs proportional to  $\bar{b}$  at NLO
- $z_< \equiv \min \{|\mathbf{x}-\mathbf{z}|, |\mathbf{y}-\mathbf{z}|\}$ , + sign when  $r = |\mathbf{x}-\mathbf{y}| < z_<$

$$\mathcal{K}_{\text{DLA}}(\rho) \equiv \frac{J_1(2\sqrt{\bar{\alpha}_s\rho^2})}{\sqrt{\bar{\alpha}_s\rho^2}} = 1 - \frac{\bar{\alpha}_s\rho^2}{2} + \frac{(\bar{\alpha}_s\rho^2)^2}{12} + \dots$$

$$\rho^2(\mathbf{x}, \mathbf{y}, \mathbf{z}) \equiv \ln \frac{(\mathbf{x}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2} \ln \frac{(\mathbf{y}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2}, \quad A_1 \equiv \frac{11}{12} + \frac{N_f}{6N_c^3}$$

# Saturation exponent $\lambda_s \equiv d \ln Q_s^2 / dY$



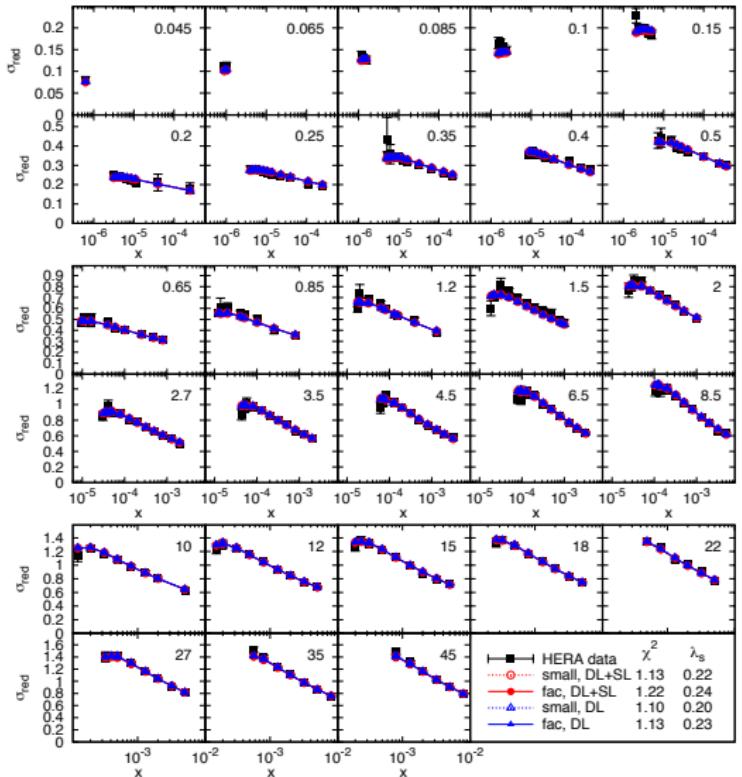
- Fixed coupling
  - LO:  $\lambda_s \simeq 4.88\bar{\alpha}_s \simeq 1$
  - resummed DL:  $\lambda_s \simeq 0.5$
  - DL + SL:  $\lambda_s \simeq 0.4$
- Running coupling
  - LO:  $\lambda_s = 0.25 \div 0.30$
  - DL + SL:  $\lambda_s \simeq 0.2$
  - better convergence

# Fitting the HERA data

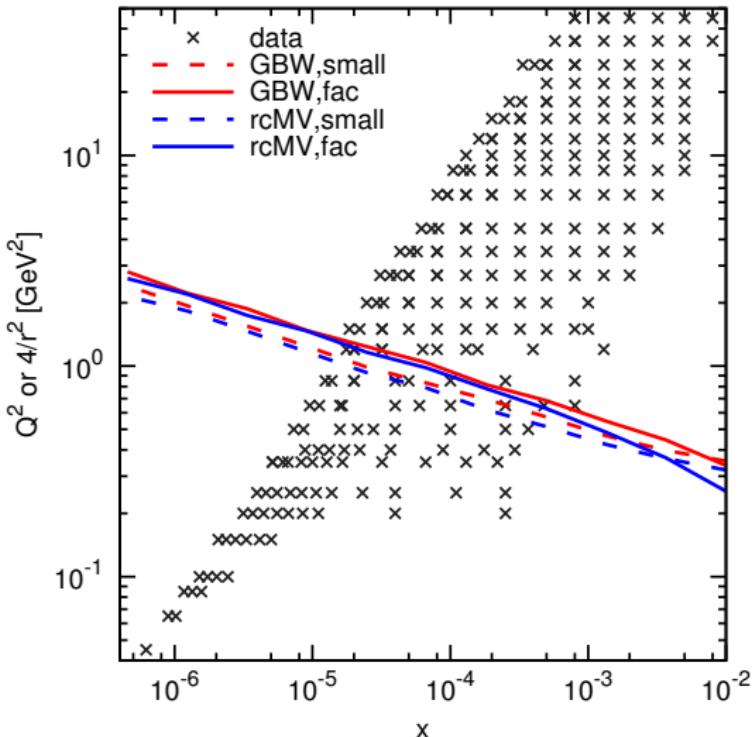
(E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1507.03651)

- Numerical solutions to the **collinearly-improved BK equation** using **initial conditions** (at  $x_0 = 0.01$ ) which involve **4 free parameters**
  - a similar strategy as for the DGLAP fits
- Combined analysis by ZEUS and H1 (2009): **small error bars**
  - Bjorken'  $x \leq 0.01$
  - $Q^2 < Q_{\max}^2$  with  $Q_{\max}^2 = 50 \div 400 \text{ GeV}^2$
- 3 light quarks + charm quark, all treated on the same footing
  - good quality fits for  $m_{u,d,s} = 0 \div 140 \text{ MeV}$  and  $m_c = 1.3 \text{ or } 1.4 \text{ GeV}$
- $\chi^2$  per point around 1.1-1.2
- **Very discriminatory:** the fits favor
  - initial condition: MV model with running coupling
  - smallest-dipole prescription for the running

# The HERA fit: rcMV initial condition



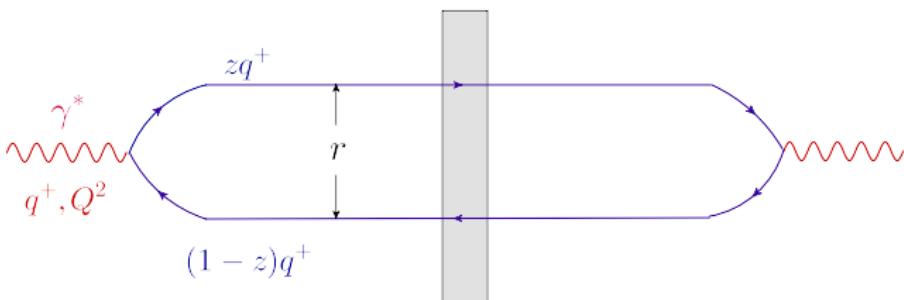
# The HERA fit: rcMV initial condition



- Saturation line  $Q_s^2(x)$  superposed over the data points
  - saturation exponent:  $\lambda_s = 0.20 \div 0.24$

**THANK YOU !**

# Dipole factorization for DIS at small $x$



$$\sigma_{\gamma^* p}(Q^2, x) = 2\pi R_p^2 \sum_f \int d^2 r \int_0^1 dz |\Psi_f(r, z; Q^2)|^2 T(r, x)$$

$$x = \frac{Q^2}{2P \cdot q} \simeq \frac{Q^2}{s} \ll 1 \quad (\text{Bjorken' } x)$$

- $T(r, x)$  : scattering amplitude for a  $q\bar{q}$  color dipole with transverse size  $r$ 
  - $r^2 \sim 1/Q^2$  : the resolution of the dipole in the transverse plane
  - $x$  : longitudinal fraction of a gluon from the target that scatters

# Fitting the HERA data: initial conditions

- Various choices for the **initial condition** at  $x_0 = 0.01$  :

$$\text{GBW : } T(Y_0, r) = \left\{ 1 - \exp \left[ - \left( \frac{r^2 Q_0^2}{4} \right)^p \right] \right\}^{1/p}$$

$$\text{rcMV : } T(Y_0, r) = \left\{ 1 - \exp \left[ - \left( \frac{r^2 Q_0^2}{4} \bar{\alpha}_s(r) \left[ 1 + \ln \left( \frac{\bar{\alpha}_{\text{sat}}}{\bar{\alpha}_s(r)} \right) \right] \right)^p \right] \right\}^{1/p}$$

- One loop **running coupling** with scale  $\mu = 2C_\alpha/r$  :

$$\bar{\alpha}_s(r) = \frac{1}{b_0 \ln [4C_\alpha^2 / (r^2 \Lambda^2)]}$$

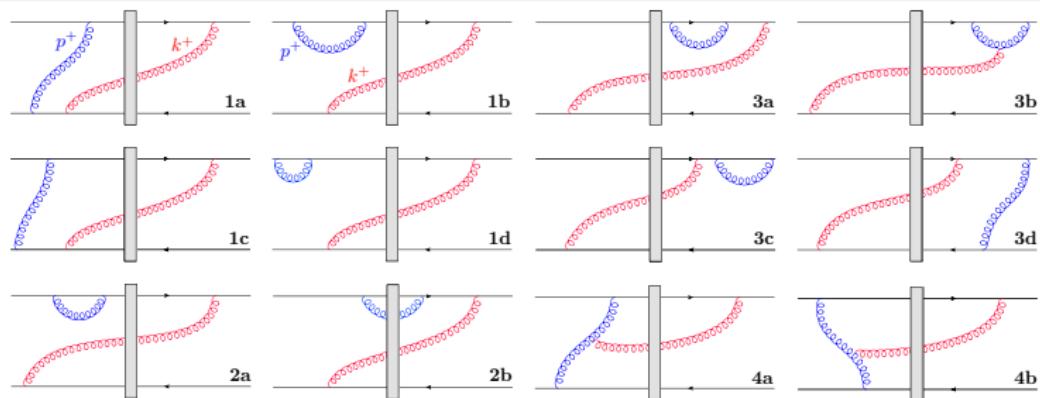
- **4 free parameters:**  $R_p$  (proton radius),  $Q_0$ ,  $p$ ,  $C_\alpha$

# The HERA fit in tables

init cdt.	RC schm	sing. logs	$\chi^2$ per data point			parameters				
			$\sigma_{\text{red}}$	$\sigma_{\text{red}}^{cc}$	$F_L$	$R_p$ [fm]	$Q_0$ [GeV]	$C_\alpha$	$p$	$C_{\text{MV}}$
GBW	small	yes	1.135	0.552	0.596	0.699	0.428	2.358	2.802	-
GBW	fac	yes	1.262	0.626	0.602	0.671	0.460	0.479	1.148	-
rcMV	small	yes	1.126	0.578	0.592	0.711	0.530	2.714	0.456	0.896
rcMV	fac	yes	1.222	0.658	0.595	0.681	0.566	0.517	0.535	1.550
GBW	small	no	1.121	0.597	0.597	0.716	0.414	6.428	4.000	-
GBW	fac	no	1.164	0.609	0.594	0.697	0.429	1.195	4.000	-
rcMV	small	no	1.097	0.557	0.593	0.723	0.497	7.393	0.477	0.816
rcMV	fac	no	1.128	0.573	0.591	0.703	0.526	1.386	0.502	1.015

init cdt.	RC schm	sing. logs	$\chi^2/\text{npts}$ for $Q_{\text{max}}^2$			
			50	100	200	400
GBW	small	yes	1.135	1.172	1.355	1.537
GBW	fac	yes	1.262	1.360	1.654	1.899
rcMV	small	yes	1.126	1.172	1.167	1.158
rcMV	fac	yes	1.222	1.299	1.321	1.317
GBW	small	no	1.121	1.131	1.317	1.487
GBW	fac	no	1.164	1.203	1.421	1.622
rcMV	small	no	1.097	1.128	1.095	1.078
rcMV	fac	no	1.128	1.177	1.150	1.131

# The Anti-Time-Ordered graphs



$$\frac{\tau_p}{\tau_p + \tau_k} \rightarrow \frac{\tau_k}{\tau_p + \tau_k} \simeq \Theta(\tau_k - \tau_p)$$

- At DLA, the softer gluon  $k^+$  lives longer than the harder one  $p^+$

$$\bar{\alpha}_s \int_{r^2}^{z^2} \frac{du^2}{u^2} \int_{k^+}^{q^+} \frac{dp^+}{p^+} \Theta(k^+ z^2 - p^+ u^2) = \frac{\bar{\alpha}_s \rho^2}{2}$$

- The DLA terms exactly cancel in the sum of all the ATO graphs