Heavy flavour corrections to polarised and unpolarised deep-inelastic scattering at 3-loop order

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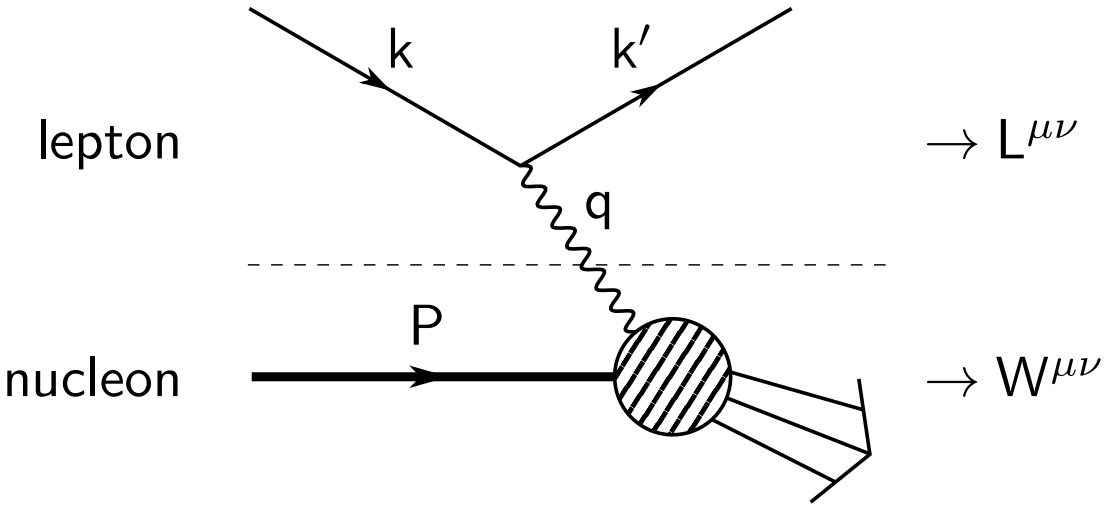
Heavy flavour contributions to deep-inelastic scattering



Kinematic variables: $Q^2 = -q^2$, $x = \frac{Q^2}{2P.q}$

Cross section: $$\frac{d\sigma}{dx dQ^2} \propto L_{\mu\nu} W^{\mu\nu}$$

Hadronic tensor: $$W_{\mu\nu} = (\dots)_{\mu\nu} F_L(x, Q^2) + (\dots)_{\mu\nu} F_2(x, Q^2)$$

Structure functions contain light and heavy quark contributions.

Motivation for NNLO heavy flavour corrections

- Precision of DIS world data: $\sim 1\%$ for F_2
  $\rightarrow$ requires $\mathcal{O}(\alpha_s^3)$ description
- Heavy quarks yield essential contributions to structure functions
  $\sim 20 - 30\%$ in the small x region
- Heavy quark contributions to the scaling violations
  have different shape than massless contributions

$\Rightarrow$ NNLO heavy quark contributions are important for
precise measurement of the strong coupling constant

$$\delta\alpha_s(M_Z) \approx 1\%$$

and heavy quark masses [Alekhin et al. '12 (and updates)]

$$m_c(m_c) = 1.25 \pm 0.02(\text{exp})^{+0.03}_{-0.02}(\text{scale})^{+0.00}_{-0.07}(\text{thy})\text{GeV}$$

$$m_b(m_b) = 3.91 \pm 0.14(\text{exp})^{+0.00}_{-0.11}(\text{thy})\text{GeV} \quad \text{(preliminary)}$$

($\overline{\text{MS}}$ scheme)

Heavy flavour contributions to deep-inelastic scattering

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Structure functions:

$$\boxed{F_2(x, Q^2, m^2) \;=\; x \sum_j \mathbb{C}_{2,j}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes f_j(x, \mu^2)}$$

Wilson coefficients (perturbative)

PDFs (non-perturbative)

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PDFs (non-perturbative)

x- and N-space are connected by a Mellin transformation:

$$M[f(x)](N) = \int_0^1 \mathrm{d}x\, x^{N-1} f(x)$$

Representation simplifies in Mellin space.

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massless Wilson coefficients

NNLO: [Moch, Vermaseren, Vogt '05]

heavy-flavor Wilson coefficients

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For F_2 and $Q^2/m^2 \gtrsim 10$ the heavy flavor Wilson coefficients factorise:
[Buza, Matiounine, Smith, Migneron, van Neerven '96]

Heavy flavor Wilson coefficients: $\boxed{H_{2,j}(N) = \sum_i A_{ij}(N)\, C_{2,i}(N)}$

massive operator matrix elements (OMEs)

LO: [Witten '76; Babcock, Sievers '78; Shifman, Vainshtein, Zakharov '78; Leveille, Weiler '79; Glück, Reya '79; Glück, Hoffmann, Reya '82]
NLO: [Laenen, van Neerven, Riemersma, Smith '93; Buza, Matiounine, Smith, Migneron, van Neerven '96; Bierenbaum, Blümlein, Klein '07a, '07b, '08, '09a]

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Heavy flavor Wilson coefficients: $H_{2,j}(N) = \sum_i A_{ij}(N) C_{2,i}(N)$

OMEs A_{ij} also essential to define the variable flavor number scheme
$\rightarrow$ describe transition $N_F \rightarrow N_F + 1$ massless quarks
$\rightarrow$ transitions relevant for the PDFs at the LHC

Massive operator matrix elements

Definition of the OMEs A_{ij}

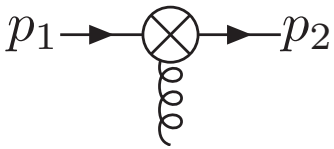
$$A_{ij} := \langle j | O_i | j \rangle$$

$|j\rangle$: partonic states (massless, on-shell)
O_i: local light-cone operators
Example:

$$O^{\mathrm{NS}}_{q,a;\mu_1,\ldots,\mu_N} = i^{N-1} S[\overline{\Psi}\gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N} \frac{\lambda^a}{2} \Psi] - \text{trace terms}$$

Feynman rules for operators

$\propto (\Delta.p)^{N-1}$



$\propto \sum_{j=0}^{N-2} (\Delta.p_1)^j (\Delta.p_2)^{N-2-j}$

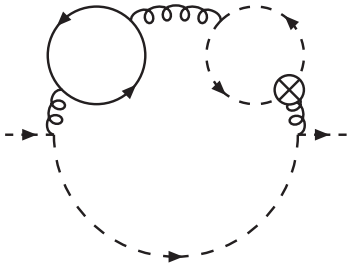
$\vdots$

Depend on integer variable N (Mellin variable)

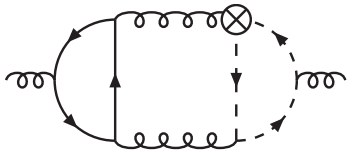
Massive operator matrix elements at NNLO

Fixed moments for OMEs: $N = 2 \ldots 10(14)$ ✓ [Bierenbaum, Blümlein, Klein, '09b]

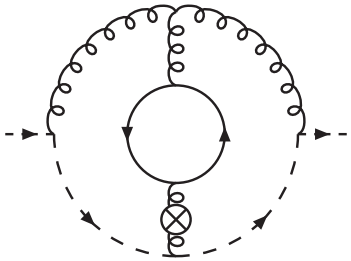
All logarithmic terms from renormalisation ✓ [Behring et al. '14]



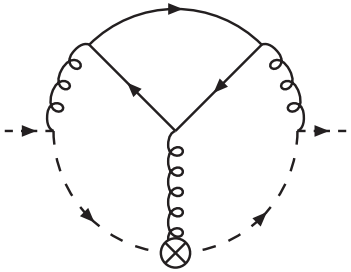
$A^{\mathrm{PS}}_{qq,Q}$
8 diagrams
✓ [Ablinger et al. '10]



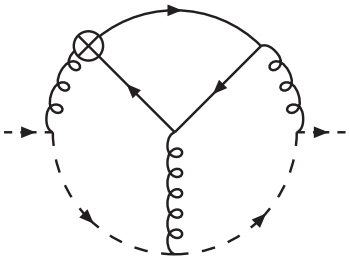
$A_{qg,Q}$
132 diagrams
✓ [Ablinger et al. '10]



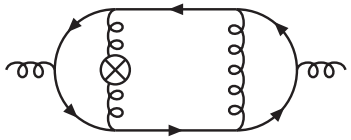
$A_{gq,Q}$
89 diagrams
✓ [Ablinger et al. '14a]



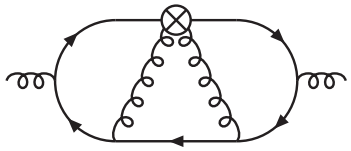
$A^{\mathrm{NS}}_{qq,Q}$ & $A^{\mathrm{TR}}_{qq,Q}$
112 diagrams
✓ [Ablinger et al. '14b]



A^{PS}_{Qq}
125 diagrams
✓ [Ablinger et al. '14c]



$A_{gg,Q}$
642 diagrams
✓



A_{Qg}
1233 diagrams
in progress
(1003 diags. done)

Feynman diagrams drawn using `axodraw` [Vermaseren '94]

Wilson coefficients at $Q^2 \gg m^2$ in terms of OMEs

- Factorisation into massive OMEs and massless Wilson coefficients
  Example: [Buza, Matiounine, Smith, Migneron, van Neerven '96] [Bierenbaum, Blümlein, Klein, '09b]

$$
\begin{aligned}
H_{q,2}^{\mathrm{PS}}(N_F+1) = a_s^2 \Bigg[& A_{Qq}^{\mathrm{PS},(2)}(N_F+1) + \frac{C_{q,2}^{\mathrm{PS},(2)}(N_F+1)}{N_F+1} \Bigg] \\
+ a_s^3 \Bigg[& A_{Qq}^{\mathrm{PS},(3)}(N_F+1) + \frac{C_{q,2}^{\mathrm{PS},(3)}(N_F+1)}{N_F+1} \\
& + A_{gq,Q}^{(2)}(N_F+1) \frac{C_{g,2}^{(1)}(N_F+1)}{N_F+1} \\
& + A_{Qq}^{\mathrm{PS},(2)}(N_F+1) C_{q,2}^{\mathrm{NS},(1)}(N_F+1) \Bigg]
\end{aligned}
$$

- Similar relations for other coefficients
- Status of Wilson coefficients:

$L_{q,2}^{\mathrm{PS}}$	$(\propto A_{qq,Q}^{\mathrm{PS},(3)})$	✓ [Ablinger et al. '10] [Behring et al. '14]		$H_{q,2}^{\mathrm{PS}}$	$(\propto A_{Qq}^{\mathrm{PS},(3)})$	✓ [Ablinger et al. '14c]
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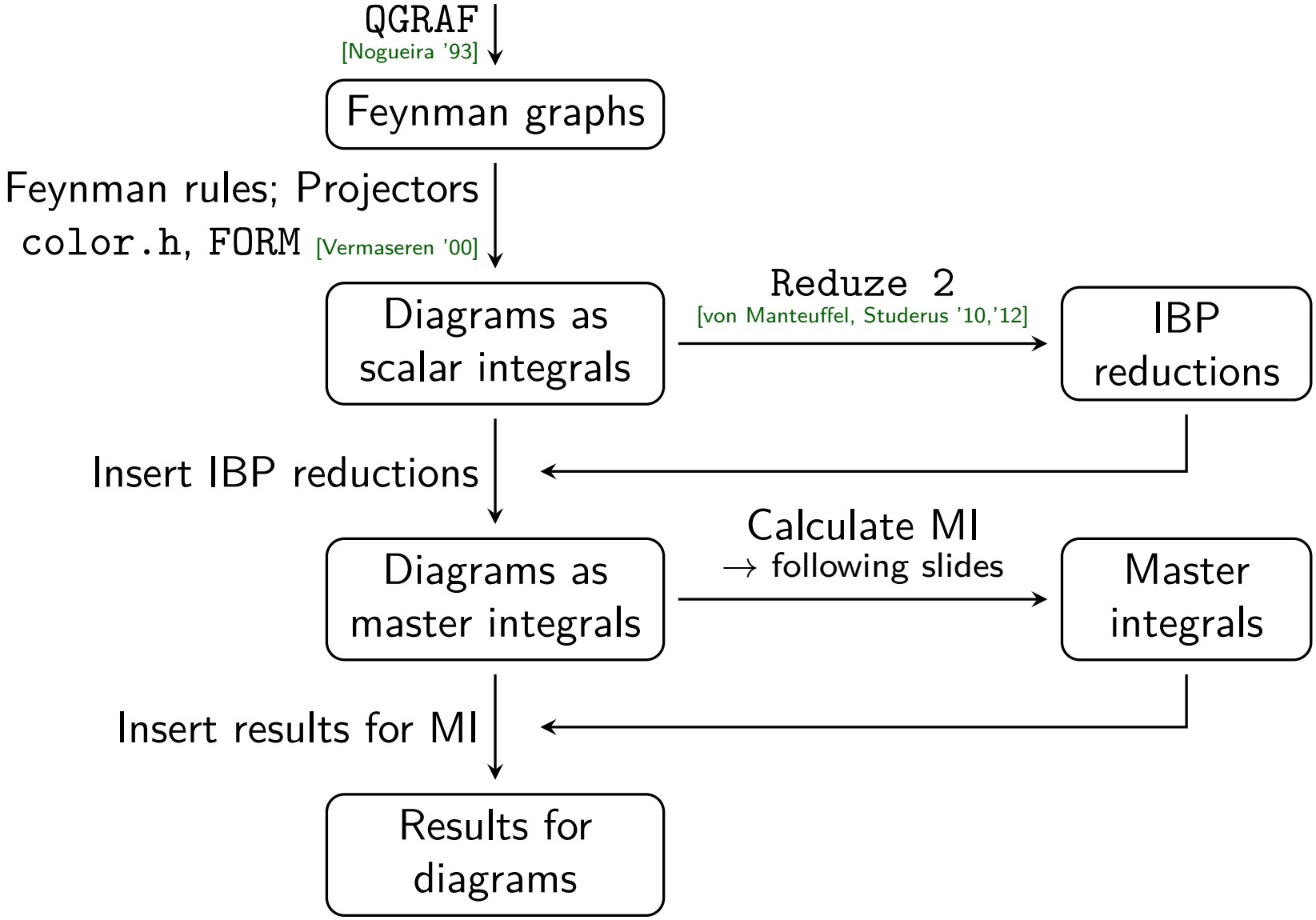
$L_{q,2}^{\mathrm{NS}}$	$(\propto A_{qq,Q}^{\mathrm{NS},(3)})$	✓ [Ablinger et al. '14b]				

Outline of the calculation



Calculation of master integrals

Master integrals are calculated using a range of techniques:

- Hypergeometric function techniques
- Mellin-Barnes representations

$\Rightarrow$ Yields multi-sum representations

$\Rightarrow$ Simplify using summation algorithms based on $\Sigma\Pi$ fields/rings implemented in `Sigma` [Schneider '01-], `EvaluateMultiSums` and `SumProduction` [Ablinger, Blümlein, Hasselhuhn, Schneider'10-] and special function tools from `HarmonicSums` [Ablinger, Blümlein, Schneider '10,'13]

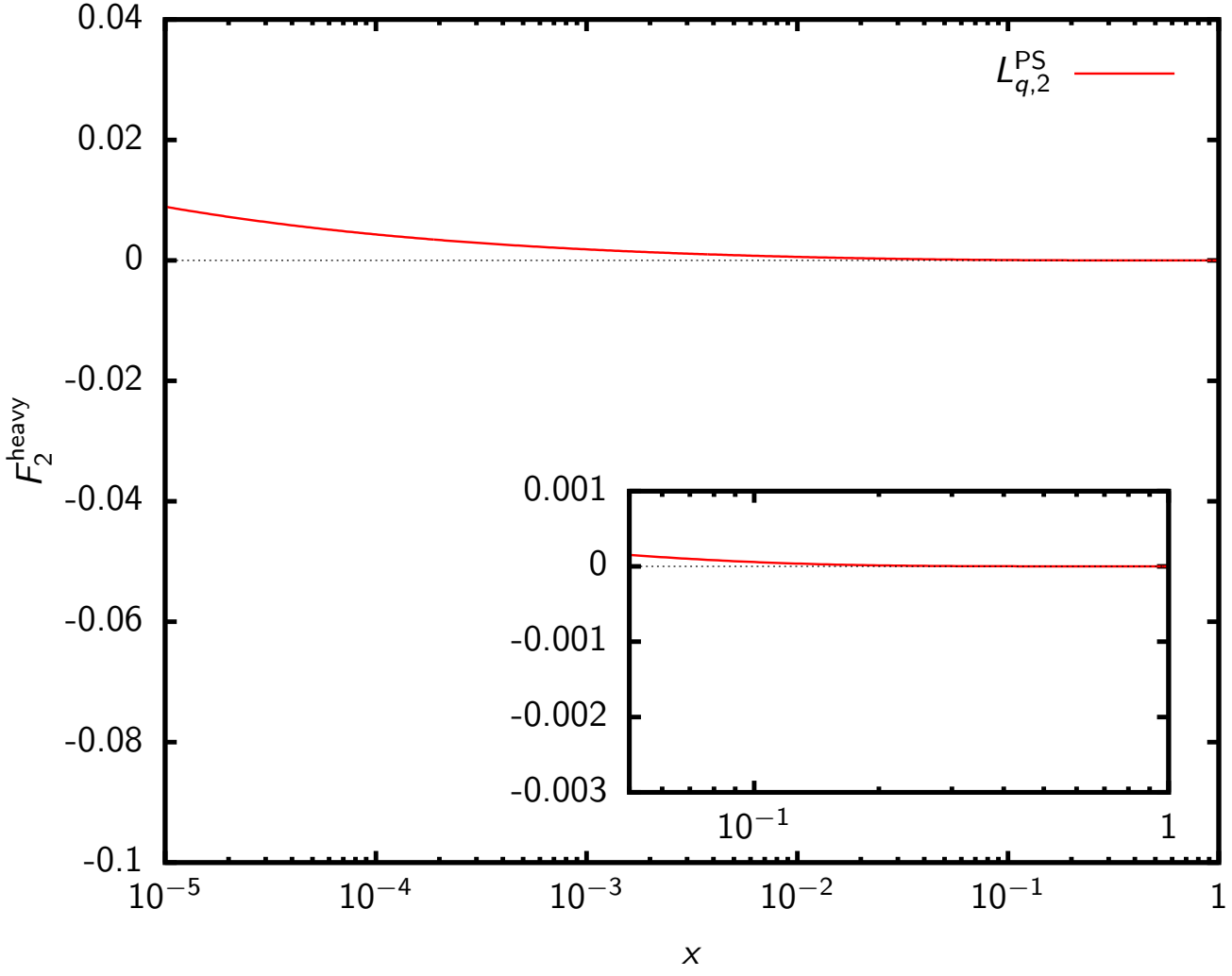
Moreover, we use

- Coupled systems of differential equations/difference equations [Ablinger et al. '15]
  `SolveCoupledSystem`
- Almkvist-Zeilberger algorithm [Almkvist, Zeilberger '90; Apagodu, Zeilberger '06]
  $\rightarrow$ `MultiIntegrate` [Ablinger '12]

$\Rightarrow$ Yields scalar recurrences for the integrals

$\Rightarrow$ Solve using the packages listed above

Contributions to the structure function F_2



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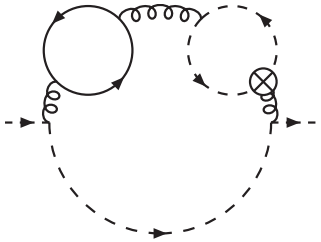

$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\mathrm{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\mathrm{pole}} = 1.59\,\mathrm{GeV}$; ABM13 $N_F = 3$ PDFs

Renormalisation: α_s in $\overline{\mathrm{MS}}$ scheme, m_c in on-shell scheme

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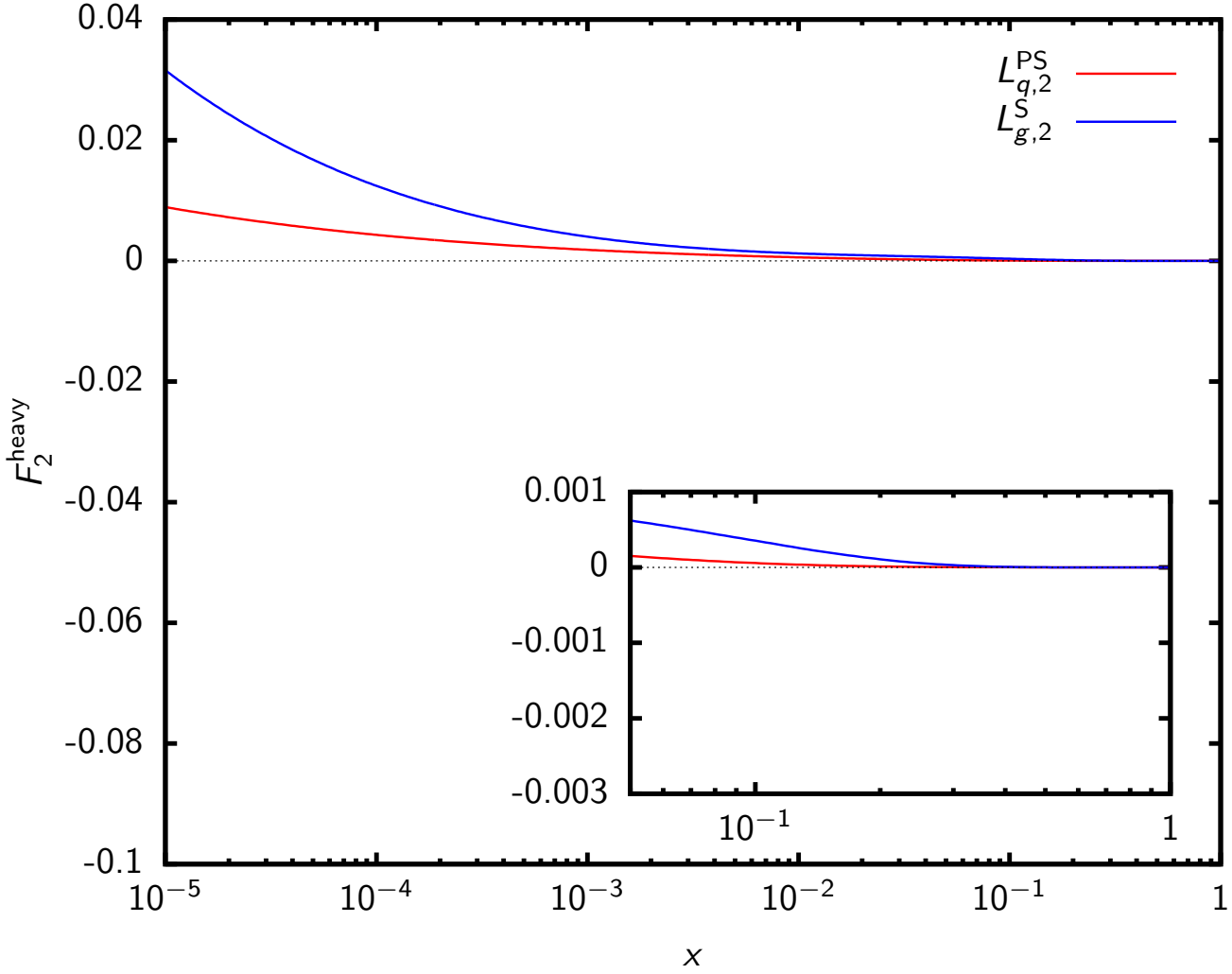

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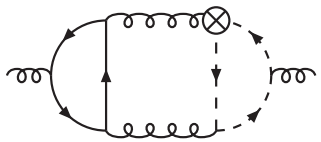

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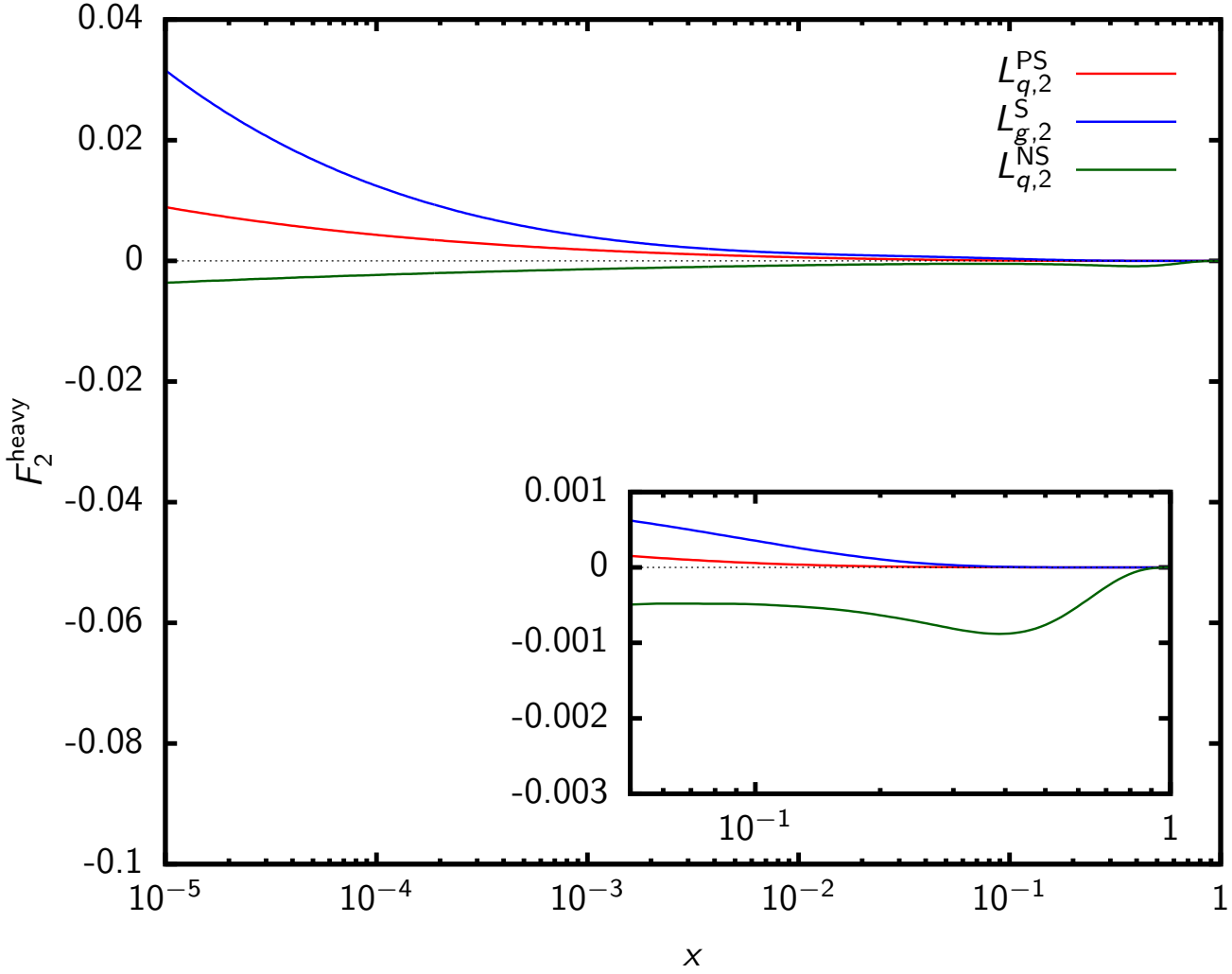

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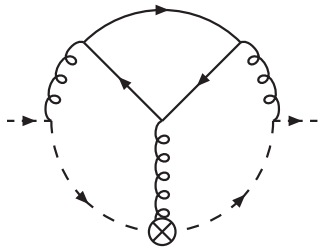

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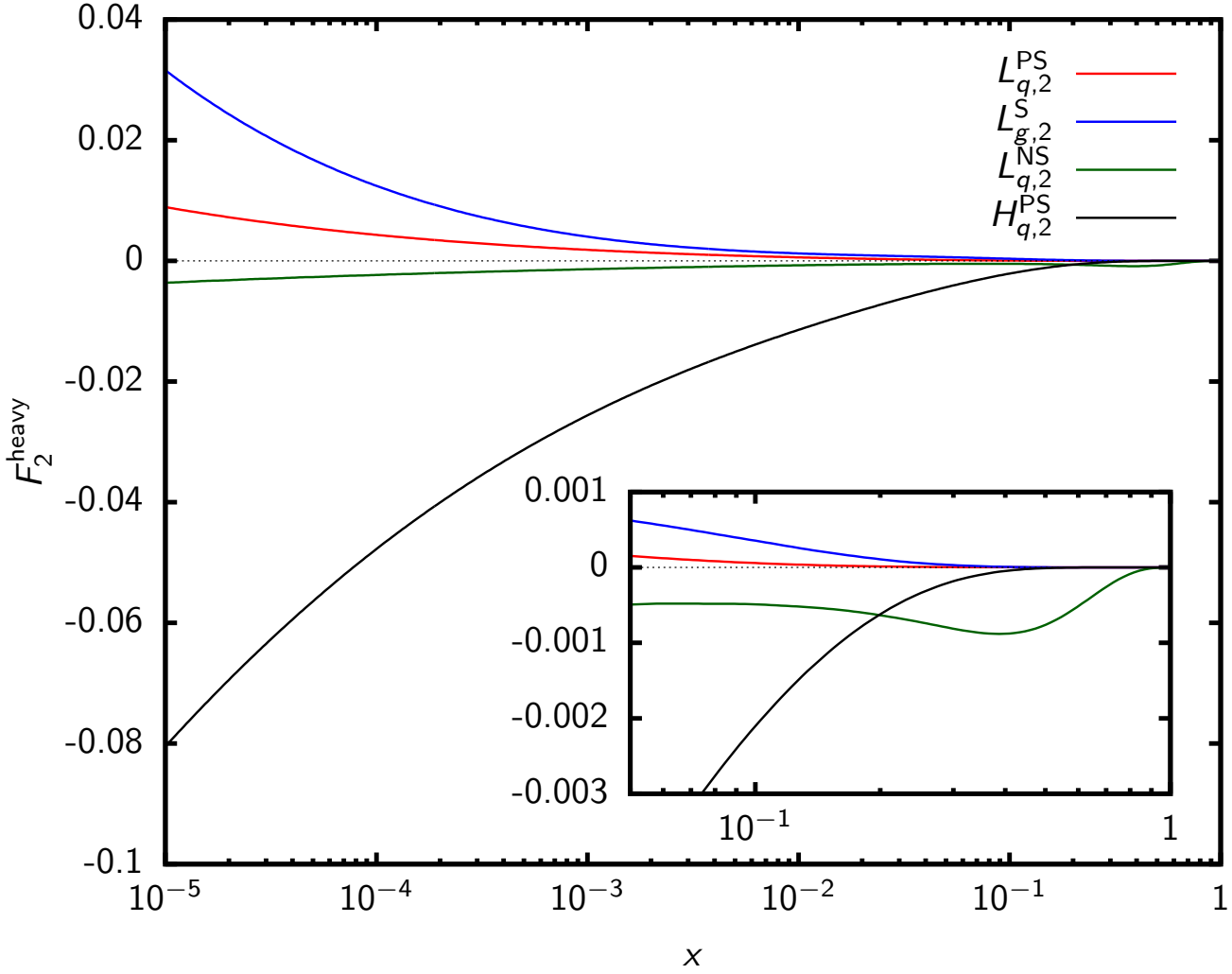

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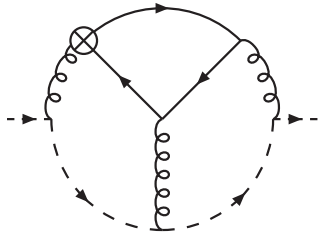
$H_{q,2}^{\text{PS}}$ [Ablinger et al. '14c]



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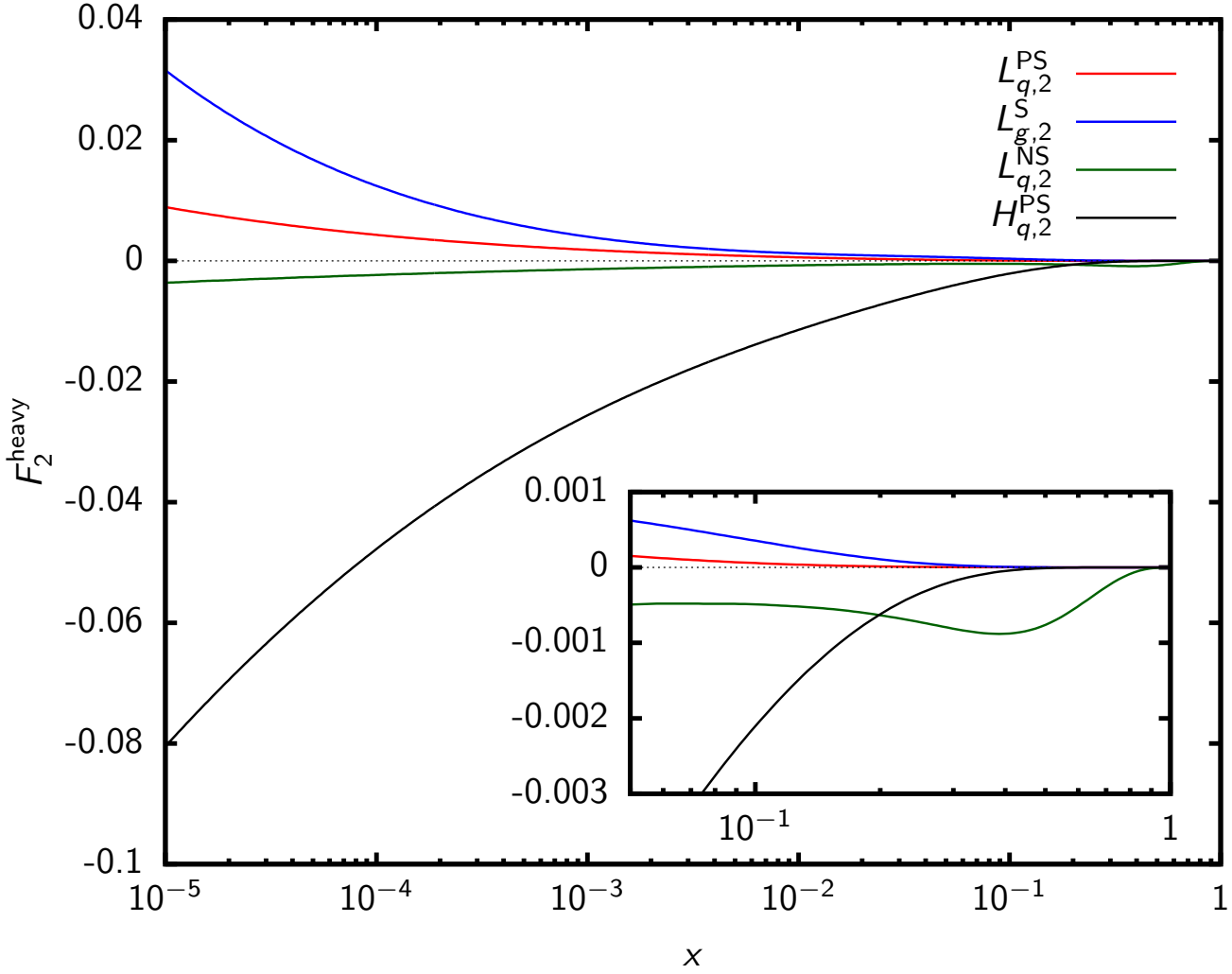

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$H_{q,2}^{\text{PS}}$ [Ablinger et al. '14c]

$H_{g,2}^{\text{S}}$ not yet known at $\mathcal{O}(\alpha_s^3)$ $\rightarrow$ work in progress

$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs
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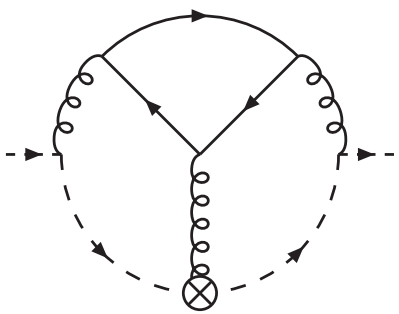
Transversity

- Tensor operator ($\rightarrow$ transversity h_1):

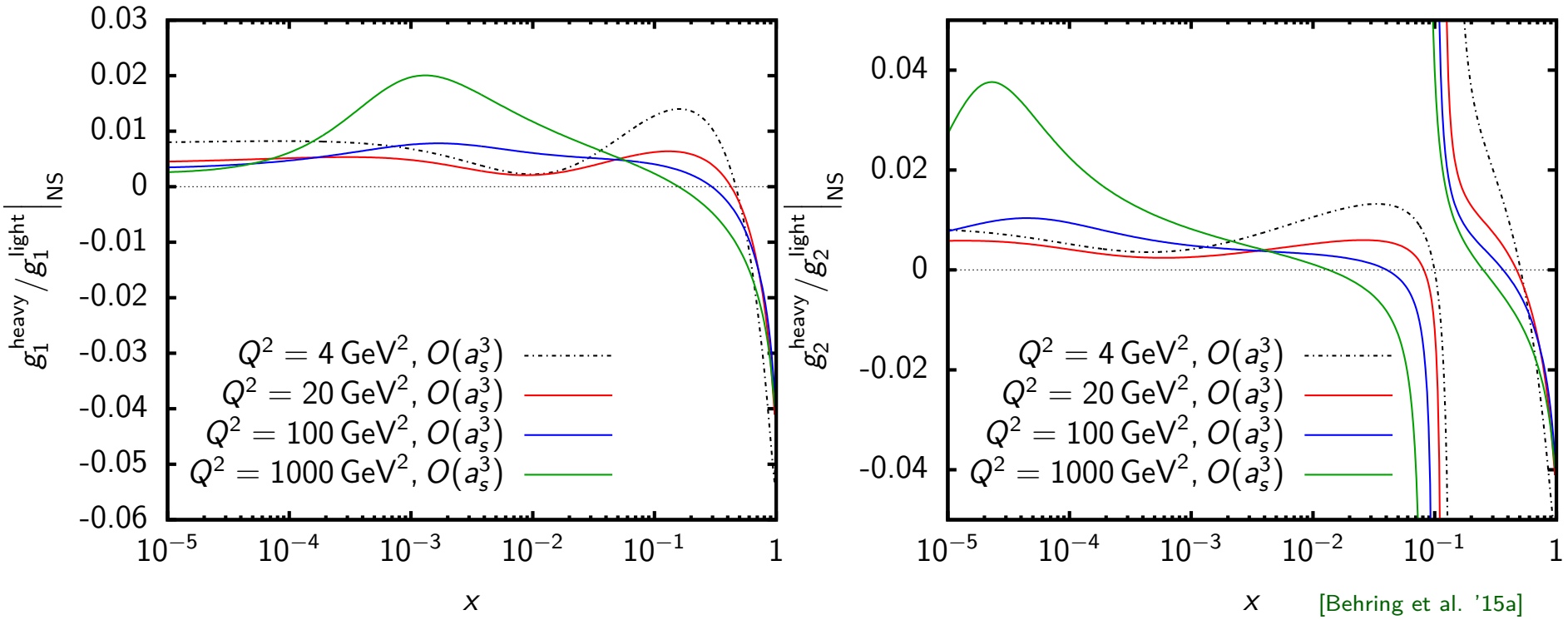
$$O_{q,r}^{\mathrm{TR,NS},\mu\mu_1\ldots\mu_N} = \frac{1}{2} i^{N-1} S\left[\bar{\psi}\sigma^{\mu\mu_1} D^{\mu_2}\ldots D^{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{trace terms}$$

- We calculated its massive operator matrix element $A_{qq,Q}^{\mathrm{TR,NS}}$
  [Ablinger et al. '14b]



- Results for transversity:
  - N_F-dependent parts of the 3-loop anomalous dimension $\gamma_{qq}^{\mathrm{TR,NS}}$
  - 3-loop massive operator matrix element $A_{qq,Q}^{\mathrm{TR,NS},(3)}$
  - Once the corresponding massless Wilson coefficients are known, also the asymptotic heavy flavour Wilson coefficients for transversity can be constructed using our results

Non-singlet part of polarised structure functions g_1 & g_2

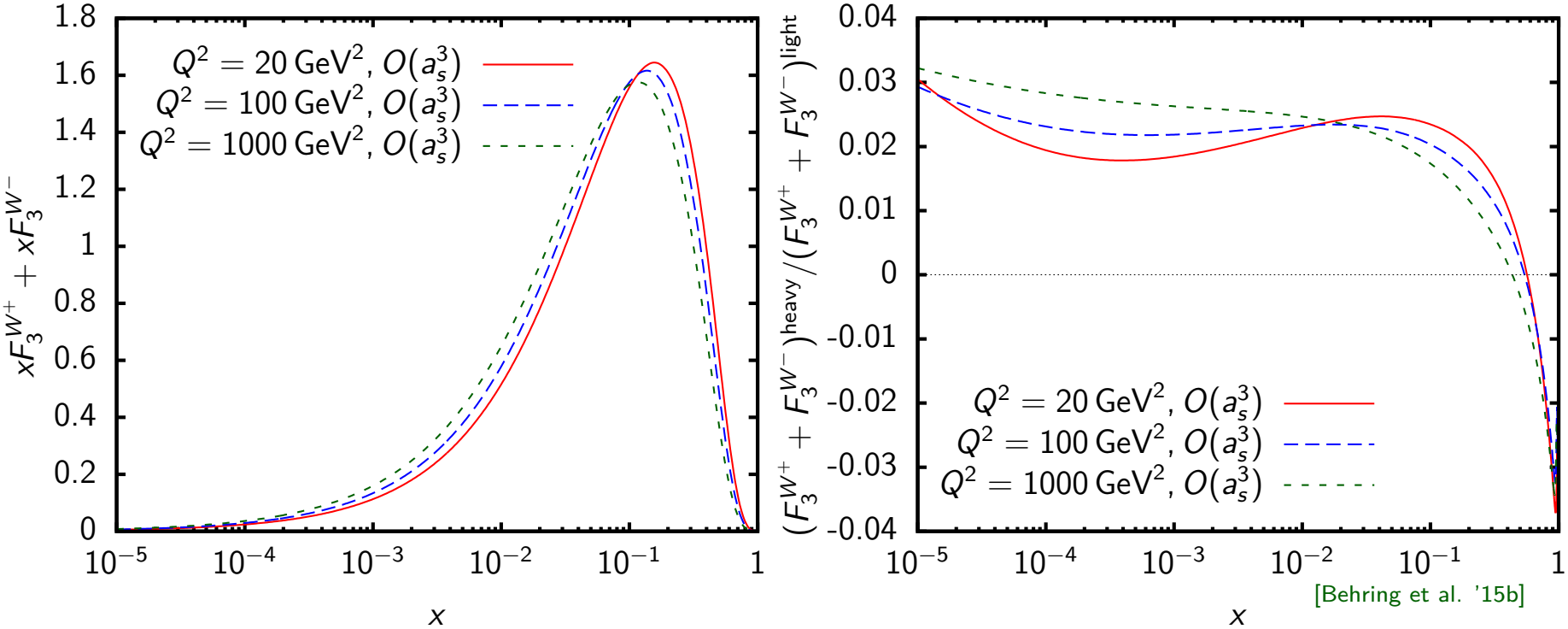


[Behring et al. '15a]

- Odd moments of $A^{\text{NS}}_{qq,Q}$ calculated as well [Ablinger et al. '14b]
- They enter the non-singlet contribution to g_1
- Twist-2 part of g_2 determined via Wandzura-Wilczek relation:

$$g_2(x, Q^2) = -g_1(x, Q^2) + \int_x^1 \frac{\mathrm{d}y}{y} g_1(y, Q^2)$$

Charged current function xF_3



[Behring et al. '15b]

- Odd moments of $A_{qq,Q}^{\mathrm{NS}}$ enter also $xF_3^{W^+} + xF_3^{W^-}$
- Two non-singlet Wilson coefficients:
  - $L_{q,3}^{\mathrm{NS}}$: W couples to light quarks ($u \to d, \ldots$)
  - $H_{q,3}^{\mathrm{NS}}$: W couples to heavy quark ($s \to c, \ldots$)

Sum rules

Polarised Bjorken sum rule [Bjorken '70]

$$\int_0^1 \mathrm{d}x \left[g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2) \right] = \frac{1}{6} \left| \frac{g_A}{g_V} \right| C_{\mathrm{pBj}}(\hat{a}_s)$$

Gross-Llewellyn-Smith sum rule [Gross, Llewellyn-Smith '69]

$$\int_0^1 \mathrm{d}x \left[F_3^{\bar{\nu} p}(x, Q^2) + F_3^{\nu p}(x, Q^2) \right] = 6 C_{\mathrm{GLS}}(\hat{a}_s)$$

- QCD corrections: Coefficients C_{pBj} and C_{GLS} follow from moment $N = 1$ of the Wilson coefficients
- Massless corrections known to $\mathcal{O}(\alpha_s^4)$
- $A_{qq,Q}^{NS}(N = 1) = 0$ due to fermion number conservation

$\Rightarrow$ Corrections from heavy quarks at $Q^2 \gg m^2$ reduce to shift $N_F \to N_F + 1$ in massless coefficient
[Behring et al. '15a '15b]

Conclusions

- Heavy quark corrections yield important contributions to DIS
  $\rightarrow$ essential for precision measurements
  of α_s (1%) and m_c (3%). [Alekhin et al. '12]
- New mathematical and computer-algebraic methods required for analytic calculation of the 3-loop corrections
  $\rightarrow$ includes new classes of higher transcendental functions
  and function spaces
- Completed massive OMEs and Wilson coefficients:
  - $A_{qq,Q}^{\rm PS}$, $A_{qg,Q}$, $A_{qq,Q}^{\rm NS}$, $A_{qq,Q}^{\rm TR}$, $A_{Qq}^{\rm PS}$, $A_{gq,Q}$, $A_{gg,Q}$,
  - $L_{q,2}^{\rm PS}$, $L_{g,2}^{\rm S}$, $L_{q,2}^{\rm NS}$, $H_{q,2}^{\rm PS}$, $L_{q,g_1}^{\rm NS}$, $L_{q,3}^{\rm NS}$
- Allows also applications to polarised and charged-current structure functions and sum rules.
- Calculation of the remaining massive OME A_{Qg} and Wilson coefficient $H_{g,2}^{\rm S}$ is in progress.

Anomalous dimension $\gamma_{qq}^{TR,NS}$

$$\gamma_{\pm}^{\mathrm{NS}}(N) = \sum_{k=1}^{\infty} a_s^k \left[\gamma_{qq}^{\mathrm{NS},(k-1)}(N) + (-1)^N \gamma_{q\bar{q}}^{\mathrm{NS},(k-1)}(N)\right]$$

$$\begin{aligned}
\gamma_{qq}^{\mathrm{NS,TR},(2)}(N) = {} & C_F^2 T_F N_F \Bigg\{ \frac{256}{3} S_{3,1} + \left[-\frac{512}{3} S_{-2,1} + \frac{1280}{9} S_2 - \frac{512}{3} S_3 - \frac{440}{3}\right] S_1 - \frac{2560}{9} S_{-2,1} \\
& - \frac{256}{3} S_{-2,2} + \frac{1024}{3} S_{-2,1,1} + \frac{4\left(207N^3 + 414N^2 + 311N + 56\right)}{9N(N+1)^2} - \frac{128}{3} S_2^2 \\
& - \frac{80}{3} S_2 + \left[\frac{1280}{9} - \frac{256}{3} S_1\right] S_{-3} + \left[\frac{2560}{9} S_1 - \frac{256}{3} S_2\right] S_{-2} + \frac{1856}{9} S_3 \\
& - \frac{512}{3} S_4 - \frac{256}{3} S_{-4} + \left[128 S_1 - 96\right] \zeta_3 \Bigg\} \\
& + C_F C_A T_F N_F \Bigg\{ \left[\frac{256}{3} S_{-2,1} - \frac{16\left(209N^2 + 209N - 9\right)}{27N(N+1)} + 64 S_3\right] S_1 - \frac{256}{3} S_{3,1} \\
& + \frac{1280}{9} S_{-2,1} + \frac{128}{3} S_{-2,2} - \frac{512}{3} S_{-2,1,1} - \frac{16\left(15N^3 + 30N^2 + 12N - 5\right)}{3N(N+1)^2} \\
& + \left[\frac{128}{3} S_1 - \frac{640}{9}\right] S_{-3} + \frac{5344}{27} S_2 + \left[\frac{128}{3} S_2 - \frac{1280}{9} S_1\right] S_{-2} - \frac{448}{3} S_3 + \frac{320}{3} S_4 \\
& + \frac{128}{3} S_{-4} + \left[96 - 128 S_1\right] \zeta_3 \Bigg\} \\
& + C_F T_F^2 N_F^2 \left\{ \frac{8\left(17N^2 + 17N - 8\right)}{9N(N+1)} - \frac{128}{27} S_1 - \frac{640}{27} S_2 + \frac{128}{9} S_3 \right\}
\end{aligned}$$

$$\gamma_{q\bar{q}}^{\mathrm{NS,TR},(2)}(N) = \frac{32}{3} C_F T_F N_F \left(\frac{C_A}{2} - C_F\right) \left[\frac{13N + 7}{3N(N+1)^2} - \frac{2}{N(N+1)} S_1\right]$$