

# Rapidity evolution of gluon TMD from low to moderate $x$

I. Balitsky

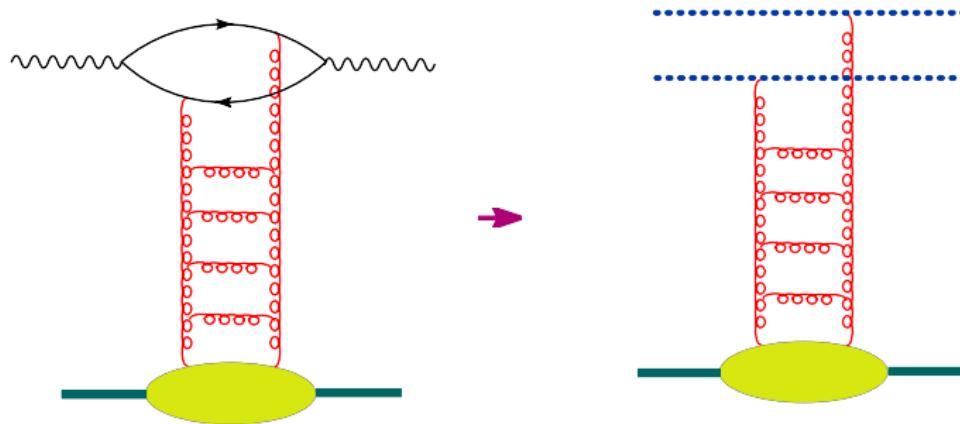
JLAB & ODU

QCD Evolution Workshop, 2 June 2016

- Reminder: rapidity factorization and evolution of color dipoles
- Definitions of small- $x$  and “moderate- $x$ ” gluon TMDs
- Method of calculation: shock-wave approach + light-cone expansion.
- One loop: real corrections and virtual corrections.
- One-loop evolution of gluon TMD
- DGLAP, Sudakov and BK limits of TMD evolution equation
- Gluon TMDs in particle production
- Conclusions and outlook

## DIS at high energy: Wilson lines and color dipoles

- At high energies, particles move along straight lines  $\Rightarrow$  the amplitude of  $\gamma^* A \rightarrow \gamma^* A$  scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



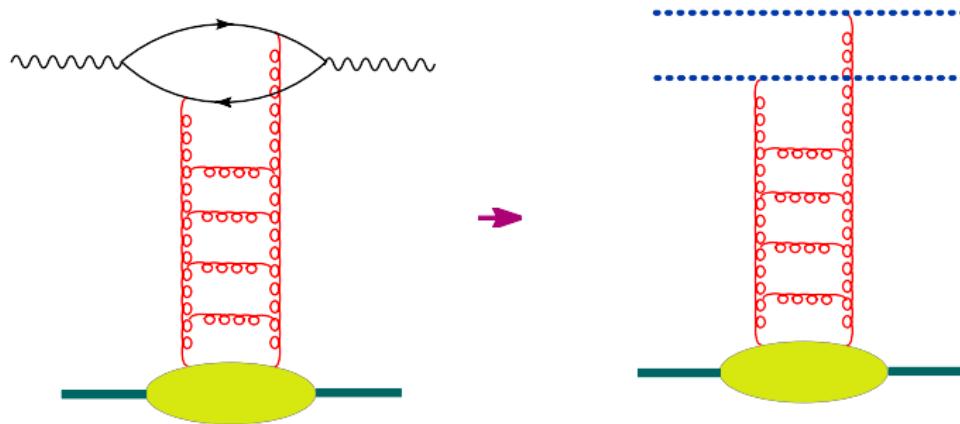
$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\} | B \rangle$$

$$U(x_\perp) = \text{Pexp} \left[ ig \int_{-\infty}^{\infty} du \; n^\mu A_\mu(un + x_\perp) \right]$$

Wilson line

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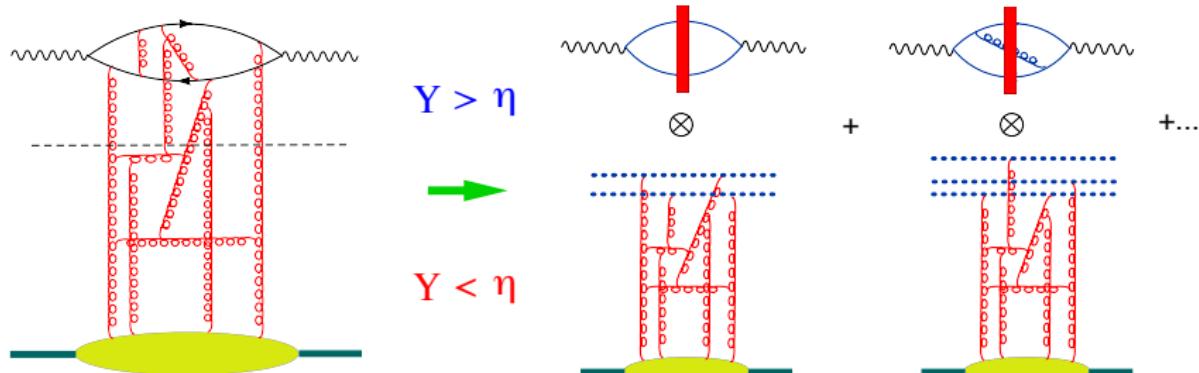


$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\} | B \rangle$$

$$U(x_\perp) = P \exp \left[ ig \int_{-\infty}^{\infty} du n^\mu A_\mu(un + x_\perp) \right] \quad \text{Wilson line}$$

Formally,  $\rightarrow$  means the operator expansion in Wilson lines

# Rapidity factorization: OPE in Wilson lines



$\eta$  - rapidity factorization scale

Rapidity  $Y > \eta$  - coefficient function ("impact factor")

Rapidity  $Y < \eta$  - matrix elements of (light-like) Wilson lines with rapidity divergence cut by  $\eta$

$$U_x^\eta = \text{Pexp} \left[ ig \int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp) \right], \quad A_\mu^\eta(x) = \int \frac{d^4 k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k)$$

## Spectator frame: propagation in the shock-wave background.



Each path is weighted with the gauge factor  $P e^{ig \int dx_\mu A^\mu}$ . Quarks and gluons do not have time to deviate in the transverse space  $\Rightarrow$  we can replace the gauge factor along the actual path with the one along the straight-line path.



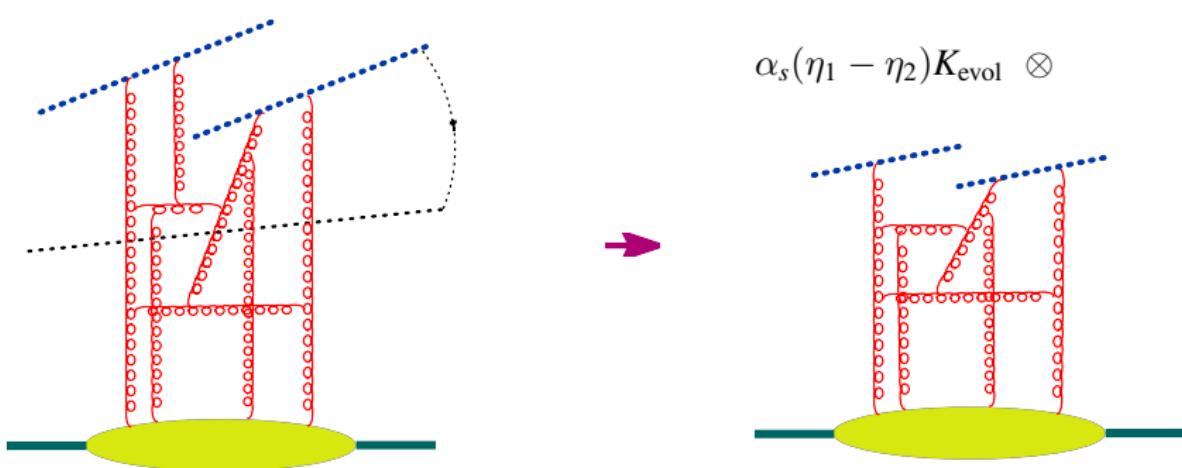
[ $x \rightarrow z$ : free propagation]  $\times$

[ $U^{ab}(z_\perp)$  - instantaneous interaction with the  $\eta < \eta_2$  shock wave]  $\times$

[ $z \rightarrow y$ : free propagation]

## Reminder: evolution of color dipoles at small $x$

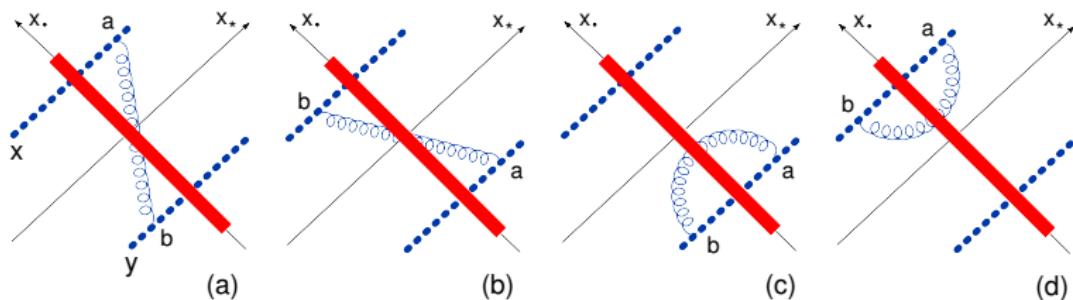
To get the evolution equation for color dipoles, consider the dipole with the rapidities up to  $\eta_1$  and integrate over the gluons with rapidities  $\eta_1 > \eta > \eta_2$ . This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to  $\eta_2$ ).



# Rapidity evolution of color dipoles in the leading order

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \dots \Rightarrow$$

$$\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}$$



$$U_z^{ab} = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

⇒ Evolution equation is non-linear

# Non linear evolution equation

$$\hat{\mathcal{U}}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_\perp) \hat{U}^\dagger(y_\perp)\}$$

## BK equation

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z}{(x-z)^2 (y-z)^2} \left\{ \hat{\mathcal{U}}(x, z) + \hat{\mathcal{U}}(z, y) - \hat{\mathcal{U}}(x, y) - \hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y) \right\}$$

I. B. (1996), Yu. Kovchegov (1999)  
Alternative approach: JIMWLK (1997-2000)

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(LLA:  $\alpha_s \ll 1, \alpha_s \eta \sim 1$ )

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LLA for DIS in pQCD  $\Rightarrow$  BFKL

(LLA:  $\alpha_s \ll 1, \alpha_s \eta \sim 1$ )

LLA for DIS in sQCD  $\Rightarrow$  BK eqn

(LLA:  $\alpha_s \ll 1, \alpha_s \eta \sim 1, \alpha_s A^{1/3} \sim 1$ )

(s for semiclassical)

NLO kernels for BK (and JIMWLK) are now known.

At small  $x$  - Weizsäcker-Williams unintegrated gluon distribution

$$\sum_X \text{tr} \langle p | U \partial^i U^\dagger(z_\perp) | X \rangle \langle X | U \partial_i U^\dagger(0_\perp) \} | p \rangle$$

Rapidity factorization: each gluon has rapidity  $\leq \ln x_B$ .

Rewrite (later  $n \equiv p_1$ )

$$\alpha_s \mathcal{D}(x_B, z_\perp) = -\frac{\alpha_s}{2\pi(p \cdot n)x_B} \int du \sum_X \langle p | \tilde{\mathcal{F}}_\xi^a(z_\perp + un) | X \rangle \langle X | \mathcal{F}^{a\xi}(0) | p \rangle$$

$$\mathcal{F}_\xi^a(z_\perp + un) \equiv [\infty n + z_\perp, un + z_\perp]^{am} n^\mu F_{\mu\xi}^m(un + z_\perp)$$

$$\tilde{\mathcal{F}}_\xi^a(z_\perp + un) \equiv n^\mu F_{\mu\xi}^m(un + z_\perp) [un + z_\perp, \infty n + z_\perp]^{ma}$$

and define the “WW unintegrated gluon distribution”

$$\mathcal{D}(x_B, k_\perp) = \int d^2 z_\perp e^{-i(k,z)_\perp} \mathcal{D}(x_B, z_\perp) \quad x_B s \gg k_\perp^2 \gg \Lambda_{\text{QCD}}^2$$

NB:  $\alpha_s \mathcal{D}(x_B, z_\perp)$  is renorm-invariant.

$$\begin{aligned}
 \mathcal{D}(x_B, k_\perp, \eta) &= \int d^2 z_\perp e^{-i(k,z)_\perp} \mathcal{D}(x_B, z_\perp, \eta), \\
 \alpha_s \mathcal{D}(x_B, z_\perp, \eta) \\
 &= \frac{-x_B^{-1} \alpha_s}{2\pi(p \cdot n)} \int du e^{-ix_B u(p \cdot n)} \sum_X \langle p | \tilde{\mathcal{F}}_\xi^a(z_\perp + un) | X \rangle \langle X | \mathcal{F}^{a\xi}(0) | p \rangle
 \end{aligned}$$

There are more involved definitions with the above TMD multiplied by some Wilson-line factors but we will discuss the “primordial” TMD.

$$\begin{aligned}\mathcal{D}(x_B, k_\perp, \eta) &= \int d^2 z_\perp e^{-i(k,z)_\perp} \mathcal{D}(x_B, z_\perp, \eta), \\ \alpha_s \mathcal{D}(x_B, z_\perp, \eta) &= \frac{-x_B^{-1} \alpha_s}{2\pi(p \cdot n)} \int du e^{-ix_B u(pn)} \sum_X \langle p | \tilde{\mathcal{F}}_\xi^a(z_\perp + un) | X \rangle \langle X | \mathcal{F}^{a\xi}(0) | p \rangle\end{aligned}$$

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The above TMD can have double-logarithmic contributions of the type  $(\alpha_s \eta \ln x_B)^n$  while the WW distribution has only single-log terms  $(\alpha_s \ln \eta)^n$  described by the BK evolution.

# Some definitions

Sudakov variables:

$$k = \alpha p_1 + \beta p_2 + k_{\perp}$$

Dimensionless light-cone coordinates

$$x_* \equiv x_\mu p_2^\mu = \sqrt{\frac{s}{2}} x_+, \quad x_\bullet \equiv x_\mu p_1^\mu = \sqrt{\frac{s}{2}} x_-$$

Gluon operators ( $x_B \equiv x_B$  for DIS and  $-x_B \equiv \frac{1}{z}$  for annihilation)

$$\begin{aligned} \mathcal{F}_i^a(k_{\perp}, x_B) &= \int d^2 z_{\perp} e^{-i(k,z)_{\perp}} \mathcal{F}_i^a(z_{\perp}, x_B), \\ \mathcal{F}_i^a(z_{\perp}, x_B) &\equiv \frac{2}{s} \int dz_* e^{ix_B z_*} [\infty, z_*]_z^{am} F_{\bullet i}^m(z_*, z_{\perp}) \end{aligned}$$

and similarly

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At first we study gluon TMDs with Wilson lines stretching to  $+\infty$  (like in SIDIS).

## Double fun. interval for cross sections

$$\begin{aligned} \langle p | \tilde{\mathcal{F}}_i^a(k'_\perp, x'_B) \mathcal{F}^{ai}(k_\perp, x_B) | p \rangle &\equiv \sum_X \langle p | \tilde{\mathcal{F}}_i^a(k'_\perp, x'_B) | X \rangle \langle X | \mathcal{F}^{ai}(k_\perp, x_B) | p \rangle \\ &= -2\pi\delta(x_B - x'_B)(2\pi)^2\delta^{(2)}(k_\perp - k'_\perp) 2\pi x_B \mathcal{D}(x_B = x_B, k_\perp, \eta) \end{aligned}$$

Short-hand notation

$$\langle p | \tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m \mathcal{O}_1 \dots \mathcal{O}_n | p \rangle \equiv \sum_X \langle p | \tilde{T}\{\tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m\} | X \rangle \langle X | T\{\mathcal{O}_1 \dots \mathcal{O}_n\} | p \rangle$$

This matrix element can be represented by a double functional integral

$$\begin{aligned} &\langle \tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m \mathcal{O}_1 \dots \mathcal{O}_n \rangle \\ &= \int D\tilde{A} D\tilde{\bar{\psi}} D\tilde{\psi} e^{-iS_{\text{QCD}}(\tilde{A}, \tilde{\psi})} \int DA D\bar{\psi} D\psi e^{iS_{\text{QCD}}(A, \psi)} \tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m \mathcal{O}_1 \dots \mathcal{O}_n \end{aligned}$$

The boundary condition  $\tilde{A}(\vec{x}, t = \infty) = A(\vec{x}, t = \infty)$  (and similarly for quark fields) reflects the sum over all intermediate states  $X$ .

## Gauge invariance

Due to the boundary condition  $\tilde{A}(\vec{x}, t = \infty) = A(\vec{x}, t = \infty)$  the matrix element

$$\begin{aligned} & \langle \tilde{\mathcal{F}}_i^a(z'_\perp, x'_B)[z'_\perp + \infty p_1, z_\perp + \infty p_1] \mathcal{F}^{ai}(z_\perp, x_B) \rangle \\ &= \int D\tilde{A} D\tilde{\psi} \bar{D}\tilde{\psi} e^{-iS_{\text{QCD}}(\tilde{A}, \tilde{\psi})} \int DA D\bar{\psi} \bar{D}\psi e^{iS_{\text{QCD}}(A, \psi)} \\ & \quad \tilde{\mathcal{F}}_i^a(z'_\perp, x'_B)[z'_\perp + \infty p_1, z_\perp + \infty p_1] \mathcal{F}^{ai}(z_\perp, x_B) \end{aligned}$$

is gauge invariant

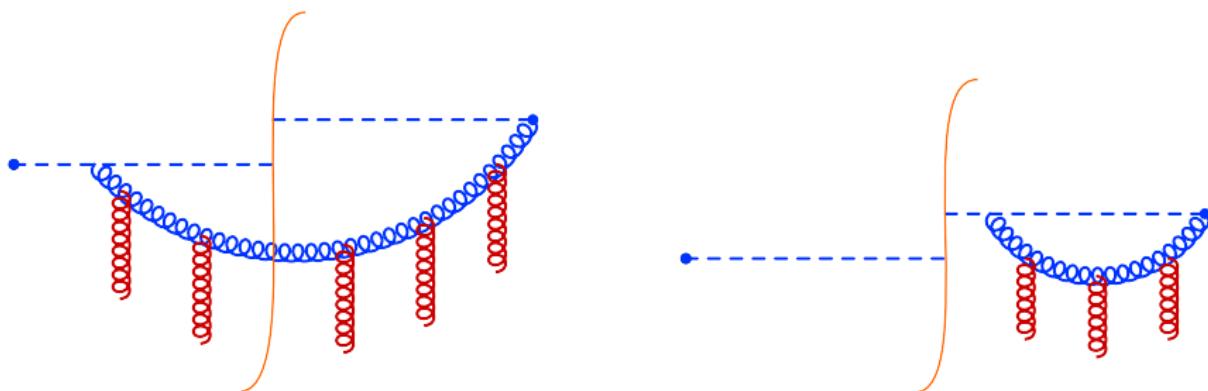
However, the gauge link  $[z'_\perp + \infty p_1, z_\perp + \infty p_1]$  does not contribute at least at the one-loop level ( $\gamma_{\text{cusp}}$  and self-energy diagrams vanish)

## Rapidity evolution: one loop

We study evolution of  $\tilde{\mathcal{F}}_i^{a\eta}(x_\perp, x_B) \mathcal{F}_j^{a\eta}(y_\perp, x_B)$  with respect to rapidity cutoff  $\eta$

$$A_\mu^\eta(x) = \int \frac{d^4 k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k)$$

Matrix element of  $\tilde{\mathcal{F}}_i^a(k'_\perp, x'_B) \mathcal{F}^{ai}(k_\perp, x_B)$  at one-loop accuracy:  
diagrams in the “external field” of gluons with rapidity  $< \eta$ .

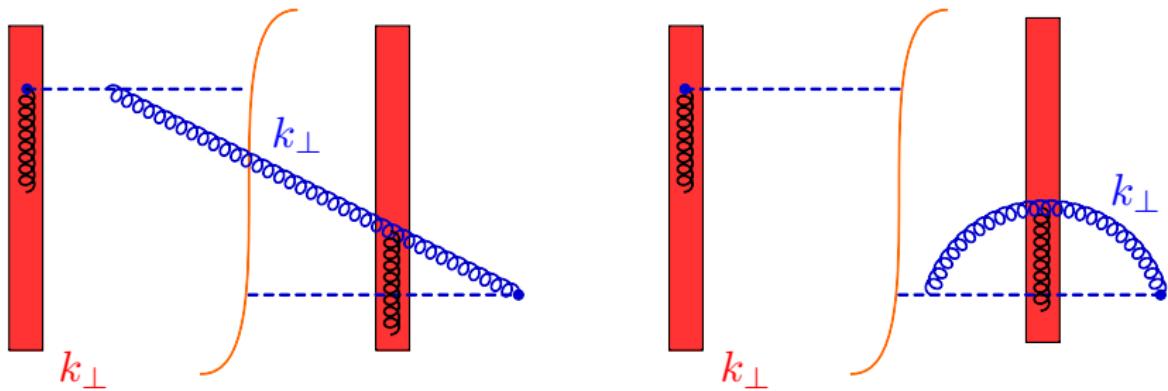


**Figure :** Typical diagrams for one-loop contributions to the evolution of gluon TMD.

(Fields  $\tilde{\mathcal{A}}$  to the left of the cut and  $\mathcal{A}$  to the right.)

# Shock-wave formalism and transverse momenta

$\alpha \gg \alpha$  and  $k_{\perp} \sim k_{\perp}$   $\Rightarrow$  shock-wave external field



Characteristic longitudinal scale of fast fields:  $x_* \sim \frac{1}{\beta}$ ,  $\beta \sim \frac{k_{\perp}^2}{\alpha s} \Rightarrow x_* \sim \frac{\alpha s}{k_{\perp}^2}$

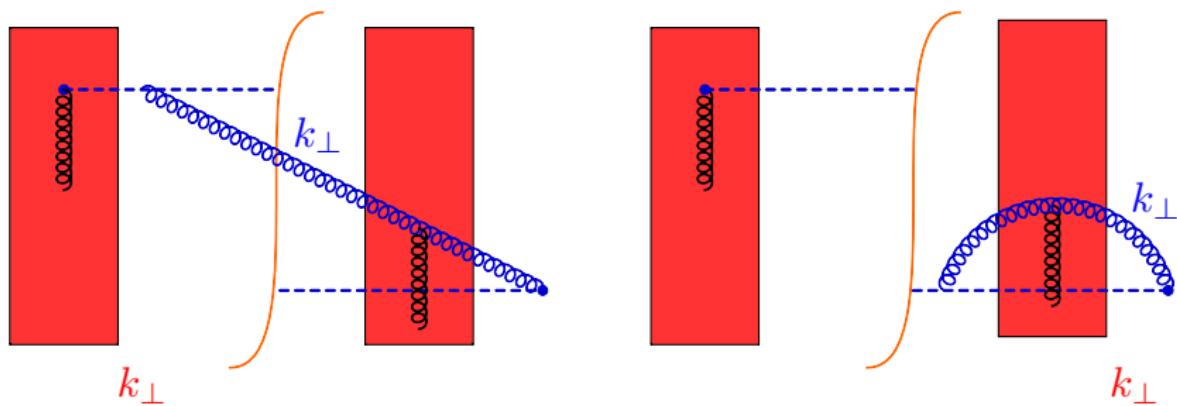
Characteristic longitudinal scale of slow fields:  $x_* \sim \frac{1}{\beta}$ ,  $\beta \sim \frac{k_{\perp}^2}{\alpha s} \Rightarrow x_* \sim \frac{\alpha s}{k_{\perp}^2}$

If  $\alpha \gg \alpha$  and  $k_{\perp}^2 \leq k_{\perp}^2 \Rightarrow x_* \gg x_*$

$\Rightarrow$  Diagrams in the shock-wave background at  $k_{\perp} \sim k_{\perp}$

## Problem: different transverse momenta

$\alpha \gg \alpha$  and  $k_{\perp} \gg k_{\perp}$   $\Rightarrow$  the external field may be wide



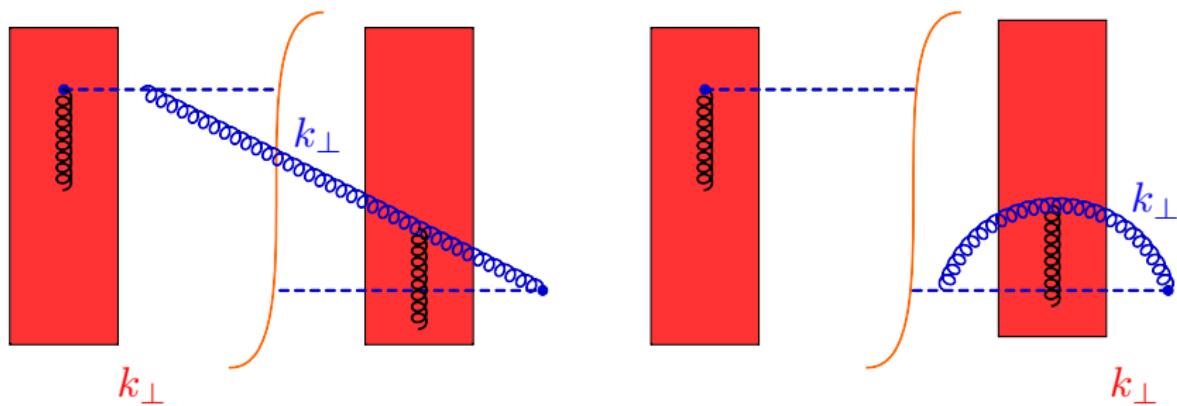
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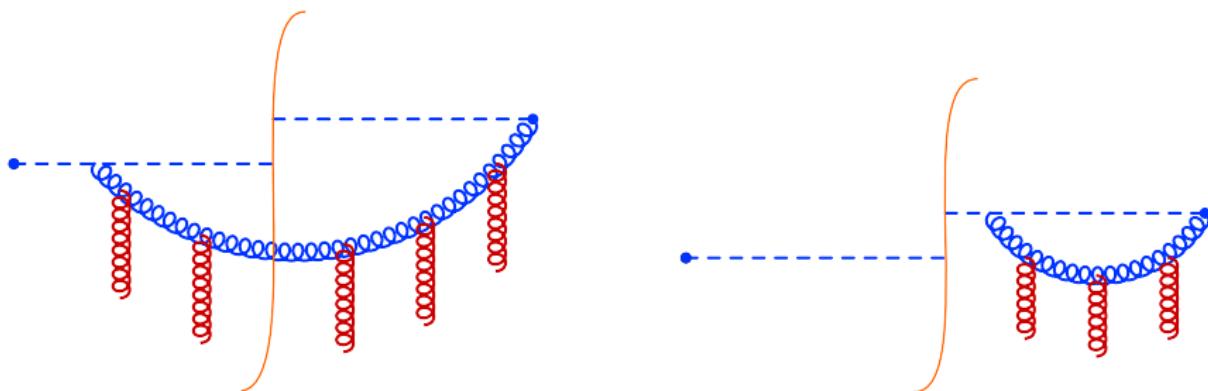
If  $\alpha \gg \alpha$  and  $k_\perp^2 \gg k_\perp^2 \Rightarrow x_* \sim x_*$   $\Rightarrow$  shock-wave approximation is invalid

Fortunately, at  $k_\perp^2 \gg k_\perp^2$  we can use another approximation

$\Rightarrow$  Light-cone expansion of propagators at  $k_\perp \gg k_\perp$

## Method of calculation

We calculate one-loop diagrams in the fast-field background



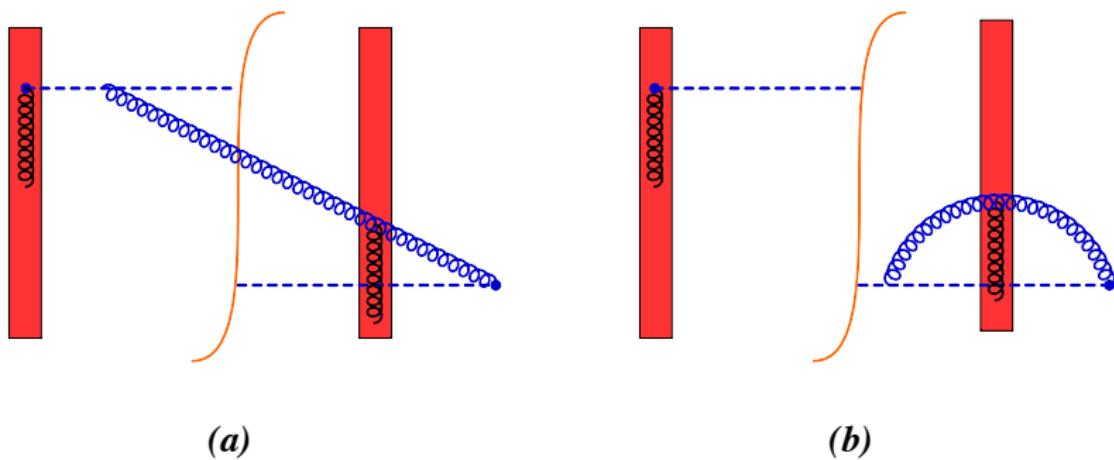
in following way:

if  $k_\perp \sim k_\perp$   $\Rightarrow$  propagators in the shock-wave background

if  $k_\perp \gg k_\perp$   $\Rightarrow$  light-cone expansion of propagators

We compute one-loop diagrams in these two cases and write down  
“interpolating” formulas correct both at  $k_\perp \sim k_\perp$  and  $k_\perp \gg k_\perp$

# One-loop corrections in the shock-wave background



**Figure :** Typical diagrams for one-loop evolution kernel. The shaded area denotes shock wave of background fast fields.

# Shock-wave calculation

Reminder:

$$\tilde{\mathcal{F}}_i^a(z_\perp, x_B) \equiv \frac{2}{s} \int dz_* e^{-i x_B z_*} F_{\bullet i}^m(z_*, z_\perp) [z_*, \infty]_z^{ma}$$

At  $x_B \sim 1$   $e^{-i x_B z_*}$  may be important even if shock wave is narrow.  
Indeed,  $x_* \sim \frac{\alpha s}{k_\perp^2} \ll x_* \sim \frac{\alpha s}{k_\perp^2} \Rightarrow$  shock-wave approximation is OK,  
but  $x_B \sigma_* \sim x_B \frac{\alpha s}{k_\perp^2} \sim \frac{\alpha s}{k_\perp^2} \geq 1 \Rightarrow$  we need to “look inside” the shock wave.

# Shock-wave calculation

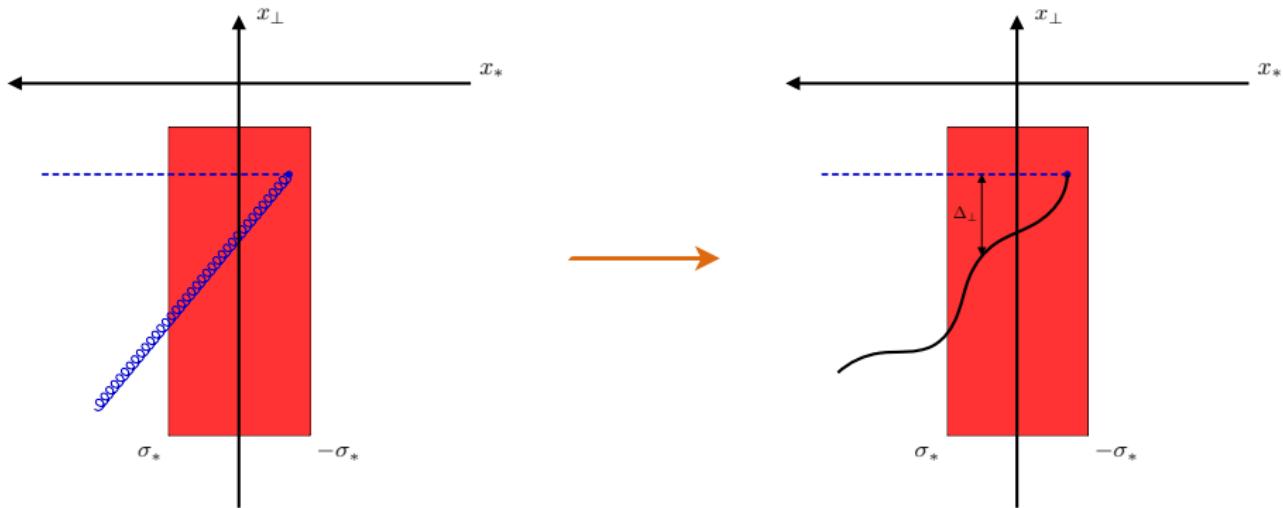
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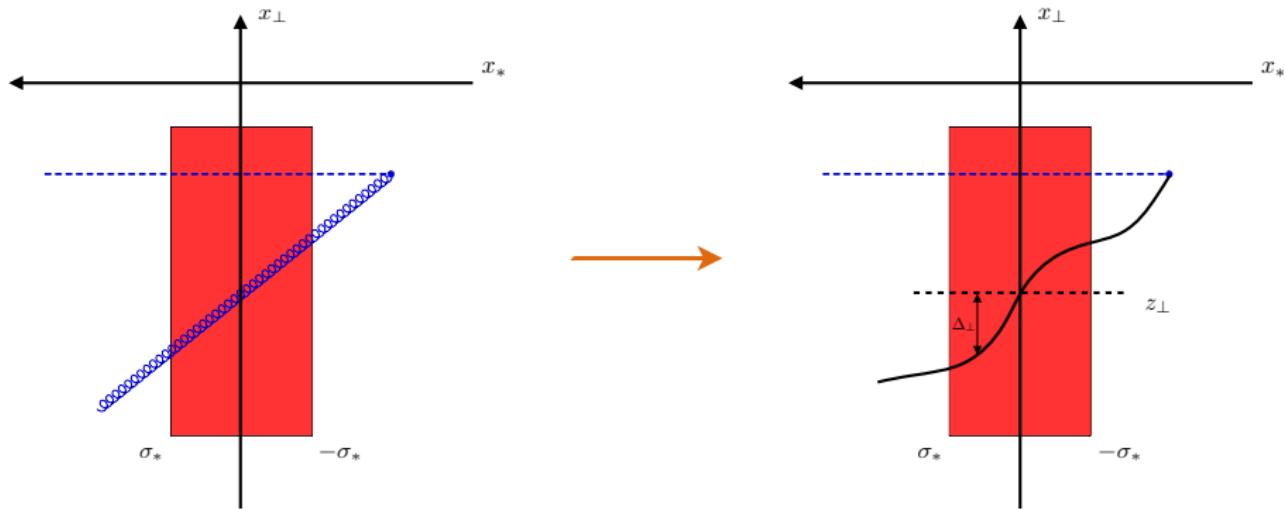
Technically, we consider small but finite shock wave:  
take the external field with the support in the interval  $[-\sigma_*, \sigma_*]$  (where  $\sigma_* \sim \frac{\alpha s}{k_\perp^2}$ ), calculate diagrams with points in and out of the shock wave, and  
check that the  $\sigma_*$ -dependence cancels in the sum of “in” and “out” contributions.

## Point(s) inside the shock wave: linear terms



$\Delta_\perp$  is small  $\Rightarrow$  expansion of  $P e^{ig \int dx_\mu A^\mu u}$  around  $y_\perp$   $\Rightarrow$  same operator  $\mathcal{F}(y_\perp, x_B)$   $\Rightarrow$  linear evolution.

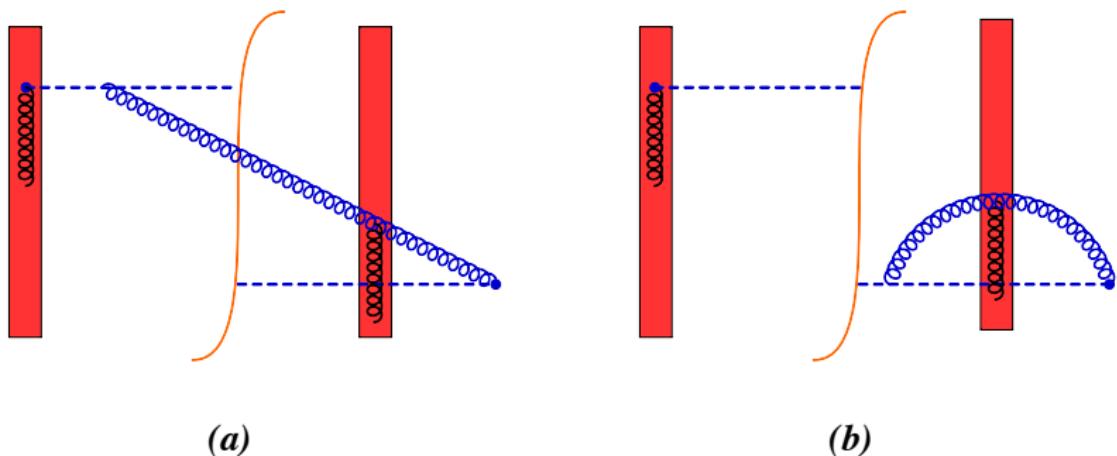
## Point(s) outside the shock wave: non-linear terms



$\Delta_\perp$  is small  $\Rightarrow$  expansion of  $P e^{ig \int dx_\mu A^m u}$  around  $z_\perp$

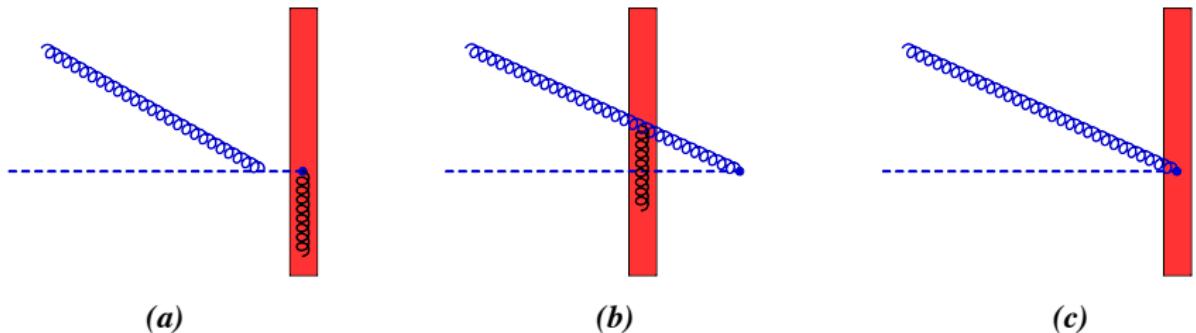
$\Rightarrow$  Wilson line  $U_z = [\infty_* p_1 + z_\perp, -\infty_* p_1 + z_\perp]$  in addition to  $U_y \Rightarrow$  non-linear terms in the evolution equation

# One-loop corrections in the shock-wave background



**Figure :** Typical diagrams for production (a) and virtual (b) contributions to the evolution kernel.

# Real corrections: square of “Lipatov vertex”



**Figure :** Lipatov vertex of gluon emission.

## Definition

$$L_{\mu i}^{ab}(k, y_\perp, x_B) = i \lim_{k^2 \rightarrow 0} k^2 \langle T\{A_\mu^a(k) \mathcal{F}_i^b(y_\perp, x_B)\} \rangle$$

Result of calculation (in the background-Feynman gauge)

$$\begin{aligned}
 L_{\mu i}^{ab}(k, y_\perp, x_B) &= 2g e^{-i(k,y)_\perp} \left( \frac{p_{2\mu}}{\alpha s} - \frac{\alpha p_{1\mu}}{k_\perp^2} \right) [\mathcal{F}_i(x_B, y_\perp) - U_i(y_\perp)]^{ab} \\
 &+ g(k_\perp | g_{\mu i} \left( \frac{\alpha x_B s}{\alpha x_B s + p_\perp^2} - U \frac{\alpha x_B s}{\alpha x_B s + p_\perp^2} U^\dagger \right) + 2\alpha p_{1\mu} \left( \frac{p_i}{\alpha x_B s + p_\perp^2} - U \frac{p_i}{\alpha x_B s + p_\perp^2} U^\dagger \right) \\
 &+ [2ix_B p_{2\mu} \partial_i U - 2i\partial_\mu^\perp U p_i + \frac{2p_{2\mu}}{\alpha s} \partial_\perp^2 U p_i] \frac{1}{\alpha x_B s + p_\perp^2} U^\dagger - \frac{2\alpha p_{1\mu}}{p_\perp^2} U_i | y_\perp )^{ab}
 \end{aligned}$$

$$U_i \equiv \mathcal{F}_i(0) = i(\partial_i U) U^\dagger.$$

Schwinger's notations  $(x_\perp | \mathcal{O}(\hat{p}_\perp, \hat{X}_\perp) | y_\perp) \equiv \int d^2 p \mathcal{O}(p_\perp, x_\perp) e^{-i(p,x-y)_\perp}$

# Lipatov vertex in the light-cone case

Result of calculation (in the background-Feynman gauge)

$$\begin{aligned} L_{\mu i}^{ab}(k, y_\perp, x_B) \rangle &= \frac{2g e^{-i(k,y)_\perp}}{\alpha x_B s + k_\perp^2} \mathcal{F}_l^{ab}(x_B + \frac{k_\perp^2}{\alpha s}, y_\perp) \\ &\times \left[ \frac{\alpha x_B s}{k_\perp^2} \left( \frac{k_\perp^2}{\alpha s} p_{2\mu} - \alpha p_{1\mu} \right) \delta_i^l - \delta_\mu^l k_i + \frac{\alpha x_B s g_{\mu i} k^l}{k_\perp^2 + \alpha x_B s} + \frac{2\alpha k_i k^l}{k_\perp^2 + \alpha x_B s} p_{1\mu} \right] \end{aligned}$$

NB:

$$k^\mu L_{\mu i}^{ab}(k, y_\perp, x_B) = 0$$

for both shock-wave and light-cone Lipatov vertices.

It is convenient to write Lipatov vertex in the light-like gauge  $p_2^\mu A_\mu = 0$  by replacement  $\alpha p_1^\mu \rightarrow \alpha p_1^\mu - k^\mu = -k_\perp^\mu - \frac{k_\perp^2}{\alpha s}$

$$\begin{aligned} L_{\mu i}^{ab}(k, y_\perp, x_B)^{\text{light-like}} &= 2g e^{-i(k,y)_\perp} \\ &\times \left[ \frac{k_\mu^\perp \delta_i^l}{k_\perp^2} - \frac{\delta_\mu^l k_i + \delta_i^l k_\mu^\perp - g_{\mu i} k^l}{\alpha x_B s + k_\perp^2} - \frac{k_\perp^2 g_{\mu i} k^l + 2k_\mu^\perp k_i k^l}{(\alpha x_B s + k_\perp^2)^2} \right] \mathcal{F}_l^{ab}(x_B + \frac{k_\perp^2}{\alpha s}, y_\perp) + O(p_{2\mu}) \end{aligned}$$

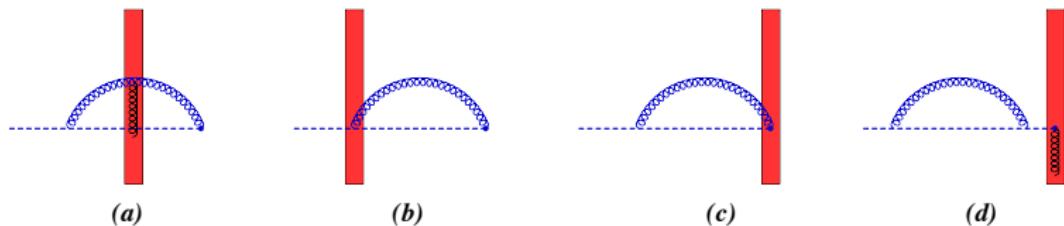
# Lipatov vertex at arbitrary momenta

“Interpolating formula” between the shock-wave and light-cone Lipatov vertices

$$\begin{aligned} L_{\mu i}^{ab}(k, y_\perp, x_B)^{\text{light-like}} \\ = g(k_\perp | \mathcal{F}^j(x_B + \frac{k_\perp^2}{\alpha s}) \left\{ \frac{\alpha x_B s g_{\mu i} - 2 k_\mu^\perp k_i}{\alpha x_B s + k_\perp^2} (k_j U + U p_j) \frac{1}{\alpha x_B s + p_\perp^2} U^\dagger \right. \\ \left. - 2 k_\mu^\perp U \frac{g_{ij}}{\alpha x_B s + p_\perp^2} U^\dagger - 2 g_{\mu j} U \frac{p_i}{\alpha x_B s + p_\perp^2} U^\dagger + \frac{2 k_\mu^\perp}{k_\perp^2} g_{ij} \right\} | y_\perp )^{ab} + O(p_{2\mu}) \end{aligned}$$

This formula is actually correct (within our accuracy  $\alpha_{\text{fast}} \ll \alpha_{\text{slow}}$ ) in the whole range of  $x_B$  and transverse momenta

## Virtual corrections: similar calculation



**Figure :** Virtual gluon corrections.

Result of the calculation (in light-like and background-Feynman gauges)

$$\begin{aligned} \langle \mathcal{F}_i^n(y_\perp, x_B) \rangle^{\text{Fig. 5}} &= -ig^2 f^{nkl} \int_{\sigma'}^\sigma \frac{d\alpha}{\alpha} (y_\perp | - \frac{p_\perp^j}{p_\perp^2} \mathcal{F}_k(x_B) (\vec{i} \partial_l + U_l) \\ &\quad \times (2\delta_j^k \delta_i^l - g_{ij} g^{kl}) U \frac{1}{\alpha x_B s + p_\perp^2} U^\dagger + \mathcal{F}_i(x_B) \frac{\alpha x_B s}{p_\perp^2 (\alpha x_B s + p_\perp^2)} |y_\perp)^{kl} \end{aligned}$$

NB: with  $\alpha < \sigma$  cutoff there is no UV divergence.

## Rapidity vs UV cutoff

Regularizing the IR divergence with a small gluon mass  $m^2$  we obtain

$$\int_0^\sigma \frac{d\alpha}{\alpha} \int d^2 p_\perp \frac{\alpha x_B s}{(p_\perp^2 + m^2)(\alpha x_B s + p_\perp^2 + m^2)} \simeq \frac{\pi}{2} \ln^2 \frac{\sigma x_B s + m^2}{m^2} \quad (1)$$

Simultaneous regularization of UV and rapidity divergence is a consequence of our specific choice of cutoff in rapidity.

For a different rapidity cutoff we may have the UV divergence in the remaining integrals which has to be regulated with suitable UV cutoff.

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Simultaneous regularization of UV and rapidity divergence is a consequence of our specific choice of cutoff in rapidity.

For a different rapidity cutoff we may have the UV divergence in the remaining integrals which has to be regulated with suitable UV cutoff.

We calculated

$$\int \frac{d\alpha d\beta d\beta' d^2 p_\perp}{(\beta - i\epsilon)(\beta' + x_B - i\epsilon)(\alpha\beta s - p_\perp^2 - m^2 + i\epsilon)(\alpha\beta' s - p_\perp^2 - m^2 + i\epsilon)}$$

by taking residues in the integrals over Sudakov variables  $\beta$  and  $\beta'$  and cutting the obtained integral over  $\alpha$  from above by the cutoff by  $\alpha < \sigma$

## Rapidity vs UV cutoff

Instead, let us take the residue over  $\alpha$ :

$$\begin{aligned} & ix_B \int \frac{d^2 p_\perp}{m^2 + p_\perp^2} \int d\beta d\beta' \frac{\theta(\beta)\theta(-\beta') - \theta(-\beta)\theta(\beta')}{(\beta' + x_B - i\epsilon)(\beta - i\epsilon)(\beta' - \beta)} \\ &= \int \frac{d^2 p_\perp}{m^2 + p_\perp^2} \int \frac{d\beta d\beta'}{\beta' + x_B - i\epsilon} \frac{ix_B \theta(\beta)}{(\beta - i\epsilon)(\beta' - \beta + i\epsilon)} = x_B \int \frac{d^2 p_\perp}{m^2 + p_\perp^2} \int_0^\infty \frac{d\beta}{\beta(\beta + x_B)} \end{aligned}$$

which is integral (1) with change of variable  $\beta = \frac{p_\perp^2}{\alpha s}$ .

## Rapidity vs UV cutoff

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which is integral (1) with change of variable  $\beta = \frac{p_\perp^2}{\alpha s}$ .

A conventional way of rewriting this integral in the framework of collinear factorization approach is

$$x_B \int \frac{d^2 p_\perp}{m^2 + p_\perp^2} \int_0^\infty \frac{d\beta}{\beta(\beta + x_B)} = \int \frac{d^2 p_\perp}{m^2 + p_\perp^2} \int_0^1 \frac{dz}{1-z}$$

where  $z = \frac{x_B}{x_B + \beta}$  is a fraction of momentum  $(x_B + \beta)p_2$  of “incoming gluon” (described by  $\mathcal{F}_i$  in our formalism) carried by the emitted “particle” with fraction  $x_B p_2$ .

If we cut the rapidity of the emitted gluon by cutoff in fraction of momentum  $z$ , we would still have the UV divergent expression which must be regulated by a suitable UV cutoff.

# Evolution equation for the gluon TMD operator

A. Tarasov and I.B.

$$\begin{aligned} & \frac{d}{d \ln \sigma} (\tilde{\mathcal{F}}_i^a(x_\perp, x_B) \mathcal{F}_j^a(y_\perp, x_B))^{\ln \sigma} \\ = & -\alpha_s \int d^2 k_\perp \operatorname{Tr} \{ \tilde{L}_i^\mu(k, x_\perp, x_B)^{\text{light-like}} L_{\mu j}(k, y_\perp, x_B)^{\text{light-like}} \} \\ & - \alpha_s \operatorname{Tr} \left\{ \tilde{\mathcal{F}}_i(x_\perp, x_B)(y_\perp | - \frac{p^m}{p_\perp^2} \mathcal{F}_k(x_B) (i \overleftrightarrow{\partial}_l + U_l) (2\delta_m^k \delta_j^l - g_{jm} g^{kl}) U \frac{1}{\sigma x_B s + p_\perp^2} U^\dagger \right. \\ & \quad \left. + \mathcal{F}_j(x_B) \frac{\sigma x_B s}{p_\perp^2 (\sigma x_B s + p_\perp^2)} | y_\perp ) \right. \\ & + (x_\perp | \tilde{U} \frac{1}{\sigma x_B s + p_\perp^2} \tilde{U}^\dagger (2\delta_i^k \delta_m^l - g_{im} g^{kl}) (i \partial_k - \tilde{U}_k) \tilde{\mathcal{F}}_l(x_B) \frac{p^m}{p_\perp^2} \right. \\ & \quad \left. + \tilde{\mathcal{F}}_i(x_B) \frac{\sigma x_B s}{p_\perp^2 (\sigma x_B s + p_\perp^2)} | x_\perp ) \mathcal{F}_j(y_\perp, x_B) \right\} + O(\alpha_s^2) \end{aligned}$$

This expression is UV and IR convergent.

It describes the rapidity evolution of gluon TMD operator in for any  $x_B$  and transverse momenta!

# Evolution equation for the gluon TMD

$$\begin{aligned} & \frac{d}{d \ln \sigma} \langle p | (\tilde{\mathcal{F}}_i^a(x_\perp, x_B) \mathcal{F}_j^a(y_\perp, x_B))^{\ln \sigma} | p \rangle \\ = & -\alpha_s \int d^2 k_\perp \langle p | \text{Tr} \{ \tilde{L}_i^\mu(k, x_\perp, x_B)^{\text{light-like}} \theta(1 - x_B - \frac{k_\perp^2}{\alpha s}) L_{\mu j}(k, y_\perp, x_B)^{\text{light-like}} \} | p \rangle \\ & - \alpha_s \langle p | \text{Tr} \left\{ \tilde{\mathcal{F}}_i(x_\perp, x_B)(y_\perp| - \frac{p_\perp^m}{p_\perp^2} \mathcal{F}_k(x_B)(i \overset{\leftarrow}{\partial}_l + U_l)(2\delta_m^k \delta_j^l - g_{jm}g^{kl})U \frac{1}{\sigma x_B s + p_\perp^2} U^\dagger \right. \\ & \quad + \mathcal{F}_j(x_B) \frac{\sigma x_B s}{p_\perp^2(\sigma x_B s + p_\perp^2)} | y_\perp) \\ & + (x_\perp | \tilde{U} \frac{1}{\sigma x_B s + p_\perp^2} \tilde{U}^\dagger (2\delta_i^k \delta_m^l - g_{im}g^{kl})(i\partial_k - \tilde{U}_k) \tilde{\mathcal{F}}_l(x_B) \frac{p_\perp^m}{p_\perp^2} \\ & \quad \left. + \tilde{\mathcal{F}}_i(x_B) \frac{\sigma x_B s}{p_\perp^2(\sigma x_B s + p_\perp^2)} | x_\perp) \mathcal{F}_j(y_\perp, x_B) \right\} | p \rangle + O(\alpha_s^2) \end{aligned}$$

The factor  $\theta(1 - x_B - \frac{k_\perp^2}{\alpha s})$  reflects kinematical restriction that the fraction of initial proton's momentum carried by produced gluon should be smaller than  $1 - x_B$

# Light-cone limit

$$\begin{aligned} \langle p | \tilde{\mathcal{F}}_i^n(x_B, x_\perp) \mathcal{F}^{in}(x_B, x_\perp) | p \rangle^{\ln \sigma} &= \frac{\alpha_s}{\pi} N_c \int_{\sigma'}^\sigma \frac{d\alpha}{\alpha} \int_0^\infty d\beta \left\{ \theta(1 - x_B - \beta) \right. \\ &\times \left[ \frac{1}{\beta} - \frac{2x_B}{(x_B + \beta)^2} + \frac{x_B^2}{(x_B + \beta)^3} - \frac{x_B^3}{(x_B + \beta)^4} \right] \langle p | \tilde{\mathcal{F}}_i^n(x_B + \beta, x_\perp) \right. \\ &\times \left. \mathcal{F}^{ni}(x_B + \beta, x_\perp) | p \rangle^{\ln \sigma'} - \frac{x_B}{\beta(x_B + \beta)} \langle p | \tilde{\mathcal{F}}_i^n(x_B, x_\perp) \mathcal{F}^{in}(x_B, x_\perp) | p \rangle^{\ln \sigma'} \right\} \end{aligned}$$

In the LLA the cutoff in  $\sigma \Leftrightarrow$  cutoff in transverse momenta

$$\langle p | \tilde{\mathcal{F}}_i^n(x_B, x_\perp) \mathcal{F}^{in}(x_B, x_\perp) | p \rangle^{k_\perp^2 < \mu^2} = \frac{\alpha_s}{\pi} N_c \int_0^\infty d\beta \int_{\frac{\mu'^2}{\beta s}}^{\frac{\mu^2}{\beta s}} \frac{d\alpha}{\alpha} \{ \text{same} \}$$

$\Rightarrow$  DGLAP equation  $\Rightarrow (z' \equiv \frac{x_B}{x_B + \beta})$

DGLAP kernel

$$\frac{d}{d\eta} \alpha_s \mathcal{D}(x_B, 0_\perp, \eta) = \frac{\alpha_s}{\pi} N_c \int_{x_B}^1 \frac{dz'}{z'} \left[ \left( \frac{1}{1 - z'} \right)_+ + \frac{1}{z'} - 2 + z'(1 - z') \right] \alpha_s \mathcal{D}\left(\frac{x_B}{z'}, 0_\perp, \eta\right)$$

## Low- $x$ case: BK evolution of the WW distribution

Low- $x$  regime:  $x_B = 0 + \text{characteristic transverse momenta } p_\perp^2 \sim (x-y)_\perp^{-2} \ll s$   
⇒ in the whole range of evolution ( $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ ) we have  $\frac{p_\perp^2}{\sigma s} \ll 1 \Rightarrow$  the kinematical constraint  $\theta(1 - \frac{k_\perp^2}{\alpha s})$  can be omitted  
⇒ non-linear evolution equation

$$\begin{aligned} & \frac{d}{d\eta} \tilde{U}_i^a(z_1) U_j^a(z_2) \\ &= -\frac{g^2}{8\pi^3} \text{Tr}\left\{ (-i\partial_i^{\zeta_1} + \tilde{U}_i^{\zeta_1}) \left[ \int d^2 z_3 (\tilde{U}_{z_1} \tilde{U}_{z_3}^\dagger - 1) \frac{z_{12}^2}{z_{13}^2 z_{23}^2} (U_{z_3} U_{z_2}^\dagger - 1) \right] (i \overset{\leftarrow}{\partial}_j^{\zeta_2} + U_j^{\zeta_2}) \right\} \end{aligned}$$

where  $\eta \equiv \ln \sigma$  and  $\frac{z_{12}^2}{z_{13}^2 z_{23}^2}$  is the BK kernel

This eqn holds true also at small  $x_B$  up to  $x_B \sim \frac{(x-y)_\perp^{-2}}{s}$  since in the whole range of evolution  $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$  one can neglect  $\sigma x_B s$  in comparison to  $p_\perp^2$  in the denominators  $(p_\perp^2 + \sigma x_B s) \Leftrightarrow$  effectively  $x_B = 0$ .

## Sudakov double logs

Sudakov limit:  $x_B \equiv x_B \sim 1$  and  $k_\perp^2 \sim (x - y)_\perp^{-2} \sim \text{few GeV}$ .

One can show that the non-linear terms are power suppressed  $\Rightarrow$

$$\begin{aligned} & \frac{d}{d \ln \sigma} \langle p | \tilde{\mathcal{F}}_i^a(x_B, x_\perp) \mathcal{F}_j^a(x_B, y_\perp) | p \rangle \\ &= 4\alpha_s N_c \int \frac{d^2 p_\perp}{p_\perp^2} \left[ e^{i(p, x-y)_\perp} \langle p | \tilde{\mathcal{F}}_i^a \left( x_B + \frac{p_\perp^2}{\sigma s}, x_\perp \right) \mathcal{F}_j^a \left( x_B + \frac{p_\perp^2}{\sigma s}, y_\perp \right) | p \rangle \right. \\ & \quad \left. - \frac{\sigma x_B s}{\sigma x_B s + p_\perp^2} \langle p | \tilde{\mathcal{F}}_i^a(x_B, x_\perp) \mathcal{F}_j^a(x_B, y_\perp) | p \rangle \right] \end{aligned}$$

Double-log region:  $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$  and  $\sigma x_B s \gg p_\perp^2 \gg (x - y)_\perp^{-2}$

$$\Rightarrow \frac{d}{d \ln \sigma} \mathcal{D}(x_B, z_\perp, \ln \sigma) = -\frac{\alpha_s N_c}{\pi^2} \mathcal{D}(x_B, z_\perp, \ln \sigma) \int \frac{d^2 p_\perp}{p_\perp^2} [1 - e^{i(p, z)_\perp}]$$

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Double-log region:  $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$  and  $\sigma x_B s \gg p_\perp^2 \gg (x - y)_\perp^{-2}$

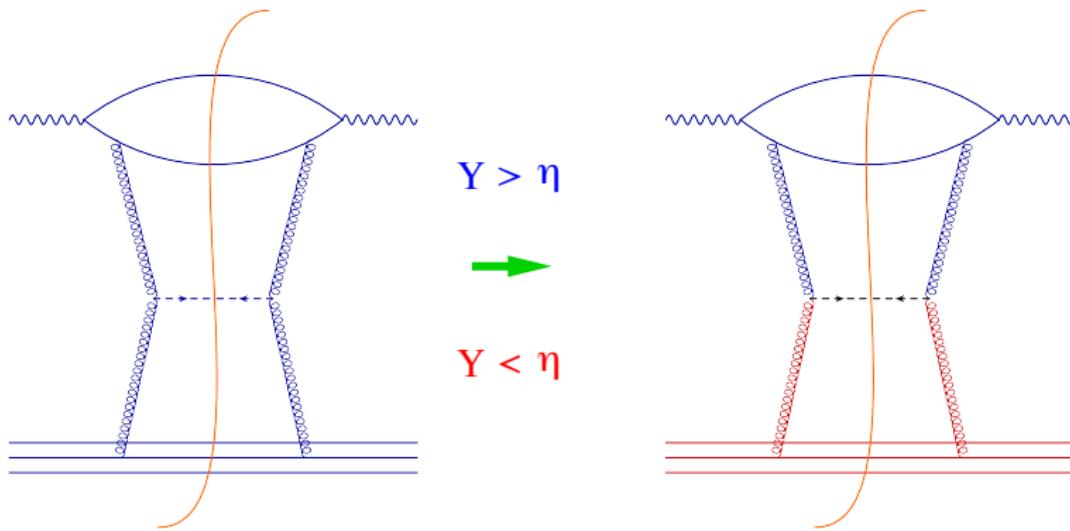
$$\Rightarrow \frac{d}{d \ln \sigma} \mathcal{D}(x_B, z_\perp, \ln \sigma) = -\frac{\alpha_s N_c}{\pi^2} \mathcal{D}(x_B, z_\perp, \ln \sigma) \int \frac{d^2 p_\perp}{p_\perp^2} [1 - e^{i(p, z)_\perp}]$$

$\Rightarrow$  Sudakov double logs

$$\mathcal{D}(x_B, k_\perp, \ln \sigma) \sim \exp \left\{ -\frac{\alpha_s N_c}{2\pi} \ln^2 \frac{\sigma s}{k_\perp^2} \right\} \mathcal{D}(x_B, k_\perp, \ln \frac{k_\perp^2}{s})$$

# Gluon TMD in particle production

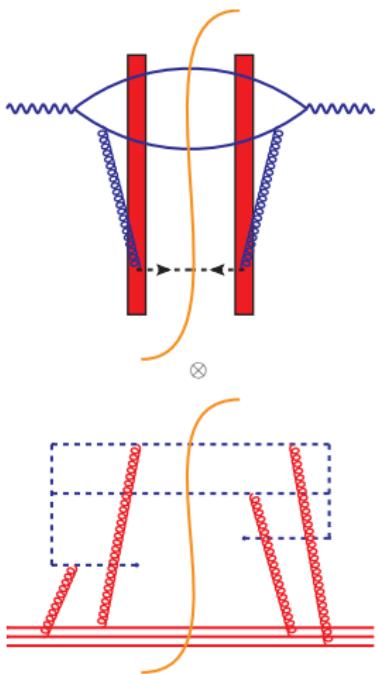
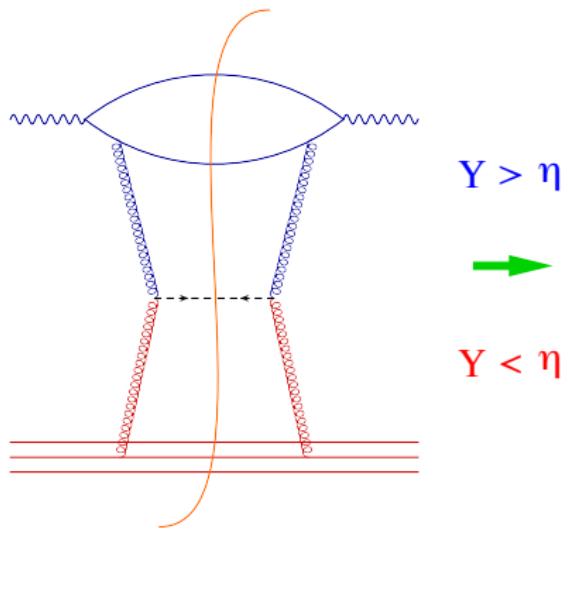
Suppose we produce a scalar particle (e.g. Higgs) in a gluon-gluon fusion. For simplicity, assume the vertex is local:  $\int dz F_{\mu\nu}^a F_a^{\mu\nu} \Phi(z)$ . Again, we integrate over rapidities  $Y > \eta$ :



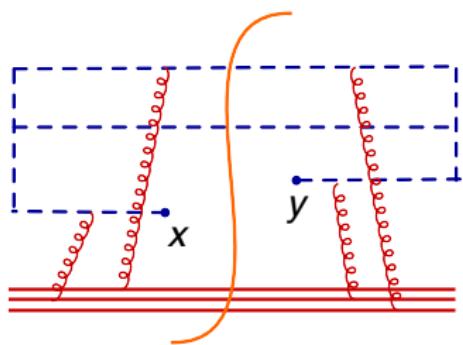
Gluons with rapidities  $Y < \eta$  shrink to a pancake

# Gluon TMD in particle production

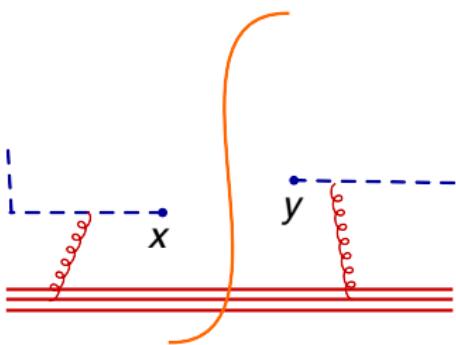
⇒ Rapidity factorization for particle production



# Matrix element between hadron states $\Rightarrow \sum_X = 1$



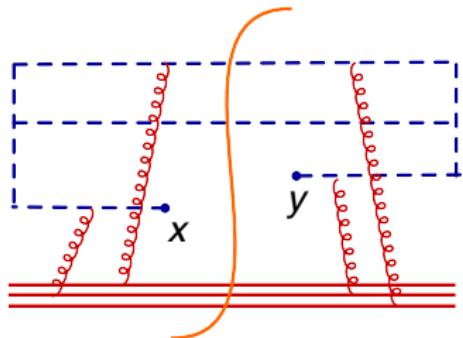
$$UU^+ = 1$$



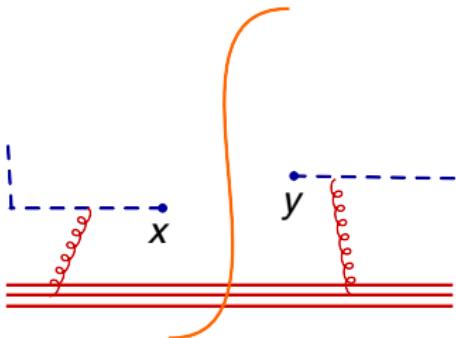
$\Rightarrow$

$$\sum_X \langle p | \boxed{\phantom{...}}_x | p \rangle = \langle p | \boxed{y} | p \rangle$$

# Gluon TMD with gauge links to $-\infty$



$$UU^t = 1$$



$$\sum_X \langle p | \begin{array}{c} \text{---} \\ | \end{array} \xrightarrow{x} \begin{array}{c} \text{---} \\ | \end{array} \xleftarrow{y} \text{---} | p \rangle = \langle p | \begin{array}{c} \text{---} \\ | \end{array} \xrightarrow{y} \text{---} | p \rangle$$

$$\Rightarrow \alpha_s \mathcal{D}(x_B, z_\perp) = -\frac{\alpha_s}{2\pi(p \cdot n)x_B} \int du e^{-ix_B u(pn)} \langle p | \tilde{\mathcal{F}}_\xi^a(z_\perp + un)[z_\perp, 0]_{-\infty} \mathcal{F}^{a\xi}(0) | p \rangle$$

$$\begin{aligned} \mathcal{F}_\xi^a(z_\perp + un) &\equiv [-\infty n + z_\perp, un + z_\perp]^{am} n^\mu F_{\mu\xi}^m(un + z_\perp) \\ \tilde{\mathcal{F}}_\xi^a(z_\perp + un) &\equiv n^\mu F_{\mu\xi}^m(un + z_\perp)[un + z_\perp, -\infty n + z_\perp]^{ma} \end{aligned}$$

$$L_{\mu i}^{ab}(k, y_\perp, \beta_B)_{-\infty}^{\text{light-like}} = g(k_\perp | U \mathcal{F}^j(\beta_B + \frac{k_\perp^2}{\alpha s})$$

$$\left[ \frac{\alpha \beta_B s g_{\mu i} - 2 k_\mu^\perp k_i}{\alpha \beta_B s + k_\perp^2} \frac{(p+k)_j}{\alpha \beta_B s + p_\perp^2} - \frac{2 k_\mu^\perp g_{ij} + 2 g_{\mu j} p_i}{\alpha \beta_B s + p_\perp^2} \right] + 2 U \frac{p_\mu^\perp}{p_\perp^2} \mathcal{F}_i(\beta_B + \frac{k_\perp^2}{\alpha s}) |y_\perp|^{ab}$$

## Lipatov vertex for gauge links to $-\infty$

$$L_{\mu i}^{ab}(k, y_\perp, \beta_B)_{-\infty}^{\text{light-like}} = g(k_\perp | U \mathcal{F}^j(\beta_B + \frac{k_\perp^2}{\alpha s}) \\ \left[ \frac{\alpha \beta_B s g_{\mu i} - 2k_\mu^\perp k_i}{\alpha \beta_B s + k_\perp^2} \frac{(p+k)_j}{\alpha \beta_B s + p_\perp^2} - \frac{2k_\mu^\perp g_{ij} + 2g_{\mu j} p_i}{\alpha \beta_B s + p_\perp^2} \right] + 2U \frac{p_\mu^\perp}{p_\perp^2} \mathcal{F}_i(\beta_B + \frac{k_\perp^2}{\alpha s}) | y_\perp )^{ab}$$

Compare to L. vertex for gauge links to  $+\infty$

$$L_{\mu i}^{ab}(k, y_\perp, \beta_B)_{+\infty}^{\text{light-like}} = g(k_\perp | \left\{ \frac{\alpha \beta_B s g_{\mu i} - 2k_\mu^\perp k_i}{\alpha \beta_B s + k_\perp^2} \mathcal{F}^j(\beta_B + \frac{k_\perp^2}{\alpha s}) U \frac{(p+k)_j}{\alpha \beta_B s + p_\perp^2} U^\dagger \right. \\ \left. - 2\mathcal{F}^j(\beta_B + \frac{k_\perp^2}{\alpha s}) U \frac{g_{ij} k_\mu^\perp + g_{\mu j} p_i}{\alpha \beta_B s + p_\perp^2} U^\dagger + \frac{2k_\mu^\perp}{k_\perp^2} \mathcal{F}_i(\beta_B + \frac{k_\perp^2}{\alpha s}) \right\} | y_\perp )^{ab}$$

# Evolution equation for the gluon TMD

Replace

$\infty n \rightarrow -\infty n$  everywhere

and

$x_B \rightarrow -x_B$  in the virtual correction:

$$\begin{aligned} & \frac{d}{d \ln \sigma} \langle p | (\mathcal{F}_i^a(x_\perp, x_B) \mathcal{F}_j^a(y_\perp, x_B))^{\ln \sigma} | p \rangle \\ = & -\alpha_s \int d^2 k_\perp \langle p | \text{Tr} \{ L_i^\mu(k, x_\perp, x_B)^{\text{light-like}} \theta(1 - x_B - \frac{k_\perp^2}{\alpha s}) L_{\mu j}(k, y_\perp, x_B)^{\text{light-like}} \} | p \rangle \\ & - \alpha_s \langle p | \text{Tr} \left\{ \mathcal{F}_i(x_\perp, x_B)(y_\perp) U^\dagger \frac{1}{\sigma x_B s - p_\perp^2 + i\epsilon} U (2\delta_m^k \delta_j^l - g_{jm} g^{kl}) (i\partial_l + U_l) \mathcal{F}_k(x_B) \frac{p^m}{p_\perp^2} \right. \\ & \quad + \mathcal{F}_j(x_B) \frac{\sigma x_B s}{p_\perp^2 (\sigma x_B s - p_\perp^2 + i\epsilon)} | y_\perp \rangle \\ & + (x_\perp | \frac{p^m}{p_\perp^2} \mathcal{F}_l(x_B) (i \overset{\leftarrow}{\partial}_k + U_k) (2\delta_i^k \delta_m^l - g_{im} g^{kl}) U^\dagger \frac{1}{\sigma x_B s - p_\perp^2 - i\epsilon} U \\ & \quad + \mathcal{F}_i(x_B) \frac{\sigma x_B s}{p_\perp^2 (\sigma x_B s - p_\perp^2 - i\epsilon)} | x_\perp \rangle \mathcal{F}_j(y_\perp, x_B) \} | p \rangle + O(\alpha_s^2) \end{aligned}$$

## 1 Conclusions

- The evolution equation for gluon TMD at any  $x_B$  and transverse momenta.
- Interpolates between linear DGLAP and Sudakov limits and the non-linear low- $x$  BK regime

## 2 Outlook

- Conformal invariance (for N=4 SYM)?
- Transition between collinear factorization and  $k_T$  factorization.

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Thank you for attention!