Transverse momentum dependent (TMD) distributions at NNLO

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in collaboration with I.Scimemi & M.Echevarria
based on[1604.07869]

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TMD is involved function of several variables

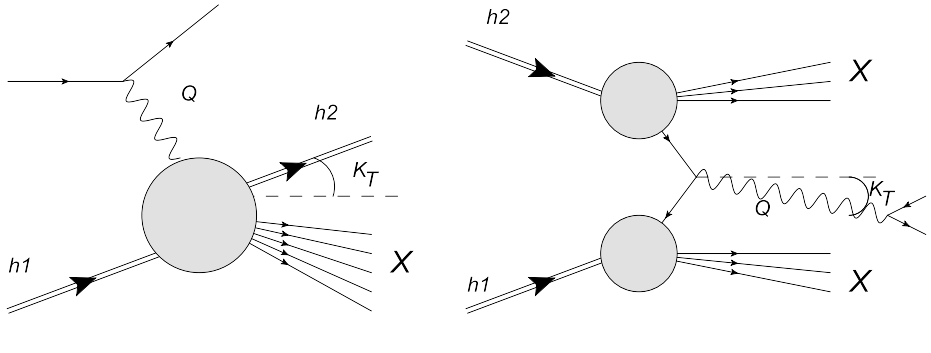
- Non-Pertrubative part is most interesting (but theoretically less accessible)
- Pertubative part plays lesser role

Object: to obtain as much pertrubative information about TMD as possible

- Anomalous dimensions and kernels (all known up to 3-loops)
- Coefficient and matching functions:
  - TMD PDFs are known up to NNLO [Catani et al,12][Gehrmann et al,14]
  - TMD FF known up NLO only (gluon part unknown even at this level)

Outline

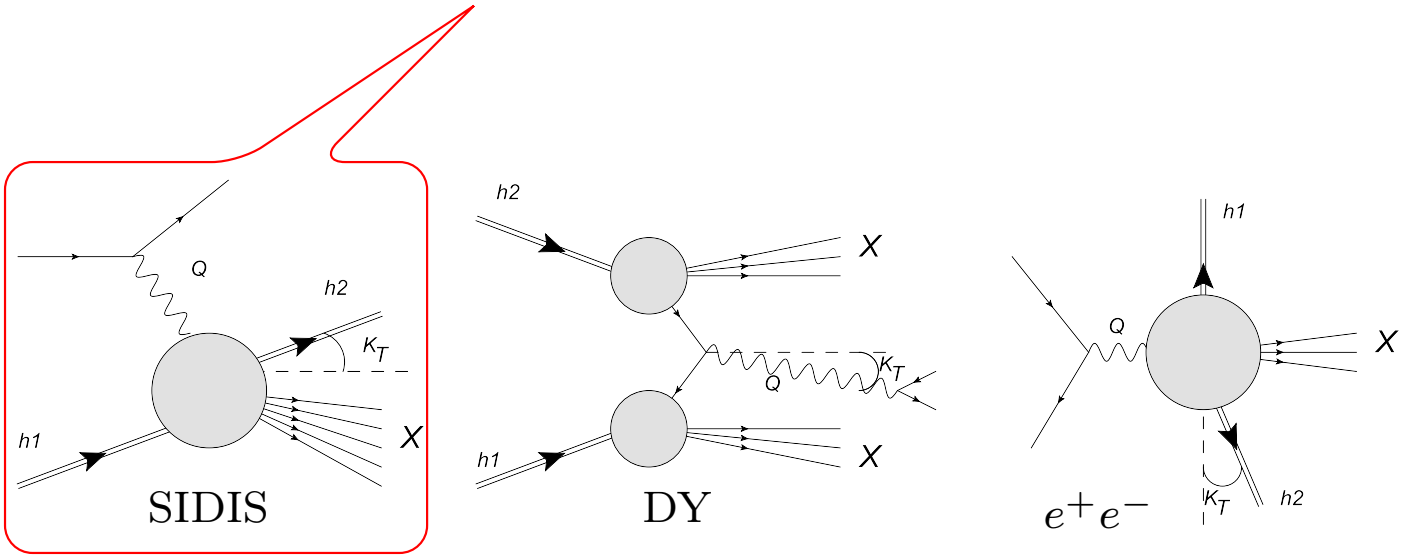
The evaluation reveals many small details that were needed to fix to complete the TMD definition.
I discuss these details and present NNLO evaluation of TMDs.



TMD factorization

Hadronic tensor is alike for all processes. We consider SIDIS

$$d\sigma \sim \int d^4x\, e^{iqx} \sum_X \langle h_1|J^\mu(x)|X;h_2\rangle\langle X;h_2|J^\nu(0)|h_1\rangle$$



TMD factorization

Applying TMD factorization ($Q^2 \gg q_T^2$) we factorize the cross-section

$$d\sigma \sim \int d^4x\, e^{iqx} \sum_X \langle h_1|J^\mu(x)|X;h_2\rangle\langle X;h_2|J^\nu(0)|h_1\rangle$$

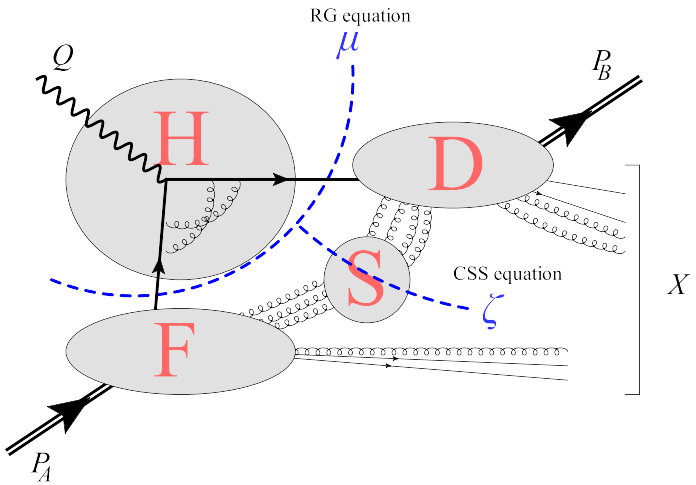
TMD factorization

$$d\sigma \sim \int d^2b_T\, e^{-i(qb)_T} H(Q^2)\ \ \Phi_{h1}(z_1,b_T)S(b_T)\Delta_{h_2}(z_2,b_T) + Y$$

TMD soft factor (very singular)

TMD FF (singular)

TMD PDF (singular)



It is not complete factorization
due to Soft Factor
that mixes singularities
of PDF and FF

TMD factorization

At $Q^2 \gg q_T^2 \gg \Lambda^2_{QCD}$, collinear factorization allows to recombine singularities

$$d\sigma \sim \int d^2 b_T \, e^{-i(qb)_T} H(Q^2) \;\; \Phi_{h1}(z_1, b_T) S(b_T) \Delta_{h_2}(z_2, b_T) + Y$$

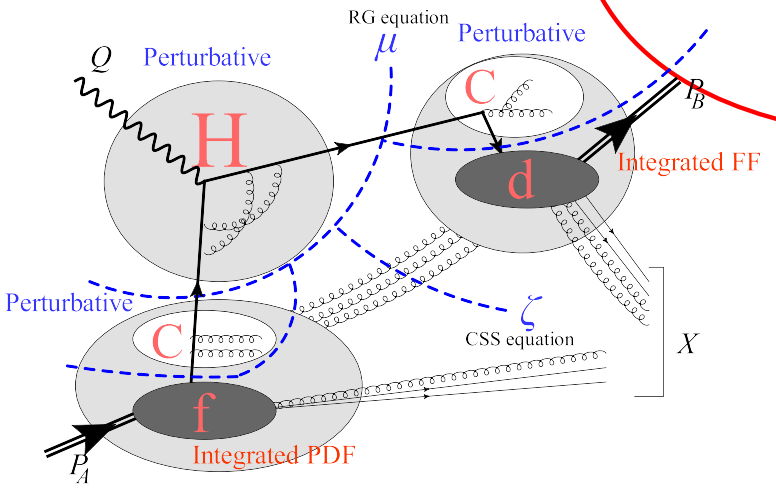
collinear factorization

$$d\sigma \sim \int d^2 b_T \, e^{-i(qb)_T} H(Q^2) \Big[q_{h1} \otimes C(b_T) \Big](z_1) \Big[d_{h2} \otimes \mathbb{C}(b_T) \Big](z_2)$$

integrated FF

TMD matching coef. (regular)

integrated PDF



TMD factorization

Splitting rapidity singularities individual TMD can be defined

$$d\sigma \sim \int d^2 b_T \, e^{-i(qb)_T} H(Q^2) \;\; \Phi_{h1}(z_1, b_T) S(b_T) \Delta_{h_2}(z_2, b_T) + Y$$

spliting rapidity singularities
$S(b_T) \to \sqrt{S(b_T;\zeta)}\sqrt{S(b_T;\zeta^{-1})}$

$$d\sigma \sim \int d^2 b_T \, e^{-i(qb)_T} H(Q^2) \; F(z_1, b_T) \; D(z_2, b_T)$$

TMD PDF (regular) TMD FF (regular)

TMD distributions are non-perturbative objects
defined on the whole range of b_T.
At $b_T \to 0$ they can be expressed via integrated distributions

$$F(z, b_T) = \Big[C(b_T) \otimes q \Big](z)$$

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TMD factorization

In this way we come to the previous result

$$\int H(Q^2)\ \Phi_{h1}(z_1, b_T) S(b_T) \Delta_{h_2}(z_2, b_T) + Y$$

collinear factorization

spliting rapidity singularities

$$\int H(Q^2) \big[q \otimes C\big](z_1) \big[d \otimes \mathbb{C}\big](z_2)$$

$$\int H(Q^2)\ F(z_1, b_T)\ D(z_2, b_T)$$

small-b_T OPE

$$D(z, b_T) = \Big[\mathbb{C}(b_T) \otimes d\Big](z)$$
$$F(x, b_T) = \Big[C(b_T) \otimes q\Big](x)$$

TMD factorization

In this way we come to the previous result

$$\int H(Q^2)\ \Phi_{h1}(z_1, b_T) S(b_T) \Delta_{h_2}(z_2, b_T) + Y$$

collinear factorization

$$\int H(Q^2) \big[q \otimes C\big](z_1) \big[d \otimes \mathbb{C}\big](z_2)$$

spliting rapidity singularities

$$\int H(Q^2)\ F(z_1, b_T)\ D(z_2, b_T)$$

small-b_T OPE

$$D(z, b_T) = \Big[\mathbb{C}(b_T) \otimes d\Big](z)$$
$$F(x, b_T) = \Big[C(b_T) \otimes q\Big](x)$$

$$F(x, b_T) = \underbrace{e^{\int \frac{d\zeta}{\zeta} \mathcal{D}}}_{\text{evolution}} \underbrace{C(x, b_T; \mu_b) \otimes q(x, \mu_b)}_{\text{matching}} \underbrace{e^{-g_K^{NP}(b_T)} f_{NP}(x, b_T)}_{\text{non-pertrubative}}$$

Many pertrubative results are universal for all TMDs and independent on states. It is natural to formulate the the statement in terms of TMD operators.

Formal definition of bare TMD operator

Operator for (unpolarized) TMD PDF

$$O_q^{bare}(x, b_T) = \frac{1}{2}\sum_X \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \left\{ T\left[\bar{q}_i\, \tilde{W}_n^T\right]_a \left(\frac{\xi}{2}\right) \; |X\rangle \gamma_{ij}^+ \langle X| \; \bar{T}\left[\tilde{W}_n^{T\dagger} q_j\right]_a \left(-\frac{\xi}{2}\right) \right\},$$

Operator for (unpolarized) TMD FF

$$\mathbb{O}_q^{bare}(z, b_T) = \frac{1}{4zN_c} \int \frac{d\xi^-}{2\pi} e^{-ip^+\xi^-/z}$$
$$\langle 0| T\left[\tilde{W}_n^{T\dagger} q_j\right]_a \left(\frac{\xi}{2}\right) \sum_X |X, \frac{\delta}{\delta J}\rangle \gamma_{ij}^+ \langle X, \frac{\delta}{\delta J}| \bar{T}\left[\bar{q}_i\, \tilde{W}_n^T\right]_a \left(-\frac{\xi}{2}\right) |0\rangle,$$

$$\xi = [0, \xi^-, \xi_T]$$

- W_n is Wilson line along n ($n^2 = 0$).
- Gluon operators are similar $O_g \sim T[F_{+\mu} W](\xi/2)\bar{T}[W^\dagger F_{+\mu}](-\xi/2)$.

Formal definition of TMD operator

Applying these operators to the hadron states we obtain unsubtracted TMDs

$$\begin{aligned}\Phi_{q\leftarrow h}(x,b_T) &= \langle h|O_q^{bare}(x,b_T)|h\rangle \\ \Delta_{q\to h}(z,b_T) &= \langle h|\mathbb{O}_q^{bare}(z,b_T)|h\rangle\end{aligned}$$

To define individual TMD we have to take into account rapidity divergences, UV divergences and overlap regions

$$\begin{aligned}F_{q\leftarrow h}(x,b_T;\zeta,\mu) &= \sqrt{S(b_T;\zeta)}\langle h|Z_q(\mu)O_q^{bare}(x,b_T)|h\rangle\Big|_{zero-bin} \\ D_{q\to h}(x,b_T;\zeta,\mu) &= \sqrt{S(b_T;\zeta)}\langle h|Z_q(\mu)\mathbb{O}_q^{bare}(x,b_T)|h\rangle\Big|_{zero-bin}\end{aligned}$$

- μ is scale of UV renormalization.
- ζ is scale of rapidity-divergences separation.

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Formal definition of the TMD operator

In this way we come to the definition of TMD operator

$$O_q(x, b_T, \mu, \zeta) = Z_q(\zeta, \mu) R_q(\zeta, \mu) O_q^{bare}(x, b_T)$$
$$\mathbb{O}_q(z, b_T, \mu, \zeta) = Z_q(\zeta, \mu) R_q(\zeta, \mu) \mathbb{O}_q^{bare}(z, b_T),$$

Universal UV and rapidity renormalization constants

$R_q(\zeta, \mu) = \frac{\sqrt{S(b_T)}}{\text{zero-bin}}$
contains all IR divergences of operator

Z_q is UV renormalization const.

Similarly, one defines the gluon TMD operators

$$O_g(x, b_T, \mu, \zeta) = Z_g(\zeta, \mu) R_g(\zeta, \mu) O_g^{bare}(x, b_T),$$
$$\mathbb{O}_g(z, b_T, \mu, \zeta) = Z_g(\zeta, \mu) R_g(\zeta, \mu) \mathbb{O}_g^{bare}(z, b_T).$$

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Unlike usual operators, TMD operator has IR divergences, that cured by the multiplier R

$$R(\zeta,\mu) = \frac{\sqrt{S(b_T)}}{\text{zero-bin}}$$

Form of R is dependent on the rapidity regularization

Tilted WL's [Collins]

$$R_q(\zeta,\mu) = \frac{\sqrt{S(b_T;+\infty,y_s)}}{\sqrt{S(b_T;+\infty,-\infty)S(b_T;y_s,-\infty)}}$$

$$\zeta \sim m^2 e^{-2y_s}$$

δ-regularization [EIS]

zero-bin coincides with soft-factor

$$R_q(\zeta,\mu) = \frac{1}{\sqrt{S(b_T;\alpha\delta^+,\delta^+)}}$$

$$\zeta \sim \alpha Q^2$$

Universality of R

- R is universal for different processes (thus, definition of TMD operator is process independent)
- R obeys Casimir scaling $\frac{R_q}{R_g} = \sqrt{\frac{C_A}{C_F}}$

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Modified δ-regularization scheme

"Old-fashion" δ-regularization

$$\delta - \text{regularization} \qquad \frac{1}{k^+ + i0} \to \frac{1}{k^+ + i\delta}$$

Such regularization does not suite the demands at higher pert.orders:

- Violates non-Abelian exponentiation
- Zero-bin $\neq$ soft factor

Both occur at NNLO.

Modified δ-regularization

Collinear WL's

$$TMDPDF: \quad P\exp\left(-ig\int_0^\infty d\sigma(n\cdot A)(n\sigma)\right) \to P\exp\left(-ig\int_0^\infty d\sigma(n\cdot A)(n\sigma)e^{-\delta\sigma x}\right)$$

$$TMDFF: \quad P\exp\left(-ig\int_0^\infty d\sigma(n\cdot A)(n\sigma)\right) \to P\exp\left(-ig\int_0^\infty d\sigma(n\cdot A)(n\sigma)e^{-\delta\sigma/z}\right)$$

Soft WL's

$$SF: \quad P\exp\left(-ig\int_0^\infty d\sigma(n\cdot A)(n\sigma)\right) \to P\exp\left(-ig\int_0^\infty d\sigma(n\cdot A)(n\sigma)e^{-\delta\sigma}\right)$$

Modified δ-regularization

$$\frac{1}{(k_1^+ + i0)(k_1^+ + k_2^+ + +i0)...(k_1^+ + ... + k_n^+ + i0)} \to$$

$$\frac{1}{(k_1^+ + i\delta)(k_1^+ + k_2^+ + 2i\delta)...(k_1^+ + ... + k_n^+ + ni\delta)}$$

Proc.

- Non-Abelian exponentiation is satisfied at all orders [AV,1501.03316].
- Factors x, z makes zero-bin be equal to soft-factor (explicitly checked at NNLO)
- At NLO there is no difference between usual and modified δ-regularization.

Cons.

- δ-regularization violates gauge properties of WL by power-suppressed in δ terms.

Only calculation at $\delta \to 0$ is legitimate.
Note: Be aware of power divergent integrals!

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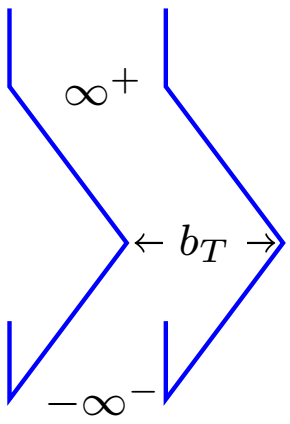
Soft factor

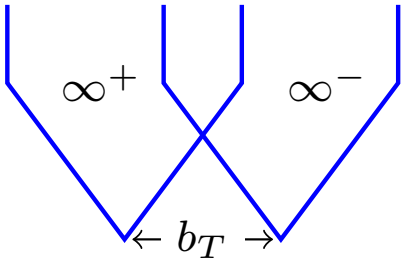


SIDIS

$$S(b_T) = \langle 0|T\Big[W_{\bar{n}}(-\infty, b_T)W_n(b_T, \infty)\Big]\bar{T}\Big[W_n^\dagger(\infty, 0)W_{\bar{n}}^\dagger(0, -\infty)\Big]|0\rangle$$

Soft factor is function of $\delta^+\delta^- = \boldsymbol{\delta}$ and $b_T^2 = \boldsymbol{B}$:

$$S(b_T) = \exp\left(C_K\left(a_s S^{[1]} + a_s^2 S^{[2]} + ...\right)\right)$$



DY

Singularities are presented in SF as

- $\frac{1}{\epsilon}$ from UV singularities and UV part of rapidity singularities
- $(\boldsymbol{\delta})^{-\epsilon}$ from collinear and rapidity singularities
- $\ln(\boldsymbol{\delta B})$ from IR part of rapidity singularities

The most important property of SF is that its logarithm is linear in $\ln(\delta^+\delta^-)$

$$S(b_T) = \exp\left(A(b_T,\epsilon)\ln(\delta^+\delta^-) + B(b_T,\epsilon)\right)$$

It allows to split rapidity divergences and define individual TMDs.

Linearity in $\ln(\boldsymbol{\delta})$

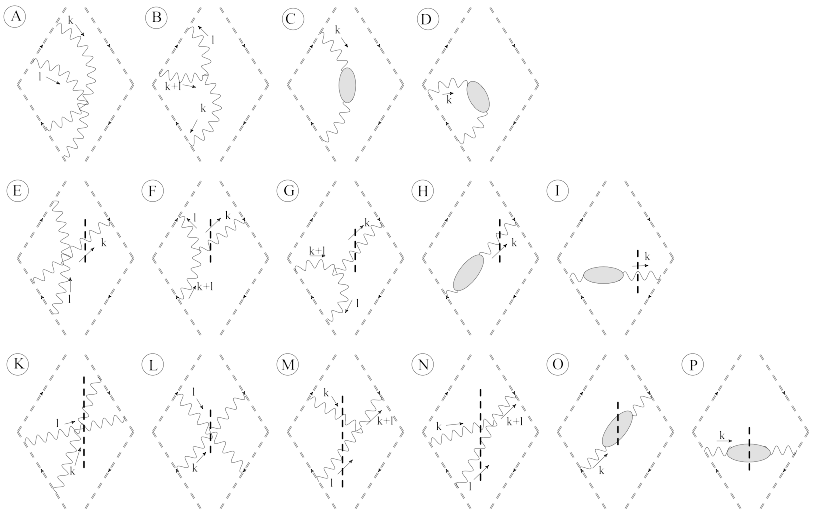
Generally (say at NNLO) one expects the following form (finite ϵ)

$$S^{[2]} = \underbrace{A_1\boldsymbol{\delta}^{-2\epsilon} + A_2\boldsymbol{\delta}^{-\epsilon}\boldsymbol{B}^{\epsilon} + \boldsymbol{B}^{2\epsilon}\Big(A_3\ln^2(\boldsymbol{\delta}B)}_{\text{cancel in sum of diagram}} + A_4\ln(\boldsymbol{\delta}B) + A_5\Big)$$

Proof

- A_1 should cancel since $\lim_{b_T\to 0} S^{[2]} = 0$ (modified δ-regularization supports!)
- A_2 should cancel since $\lim_{b_T\to 0} S^{[2]} = 0$ at $\delta = \delta b_T$ (mod. δ-regularization supports!)
- A_3 cancels due to Ward identity (alike leading UV pole for cusp)

These arguments work at all orders.



Result for Soft Factor [Echevarria,Scimemi,AV,1511.05590]

- Soft factor has been evaluated at NNLO at fixed(positive) ϵ
- All cancellations shown explicitly
- Depends only on $|\boldsymbol{\delta}|$, process independent.

$$
\begin{aligned}
S^{[2]} \quad = \quad & \Bigg[d^{(2,2)} \left(\frac{3}{\epsilon^3} + \frac{2\mathbf{l}_\delta}{\epsilon^2} + \frac{\pi^2}{6\epsilon} + \frac{4}{3}\mathbf{L}_\mu^3 - 2\mathbf{L}_\mu^2 \mathbf{l}_\delta + \frac{2\pi^2}{3}\mathbf{L}_\mu + \frac{14}{3}\zeta_3 \right) - \\
& d^{(2,1)} \left(\frac{1}{2\epsilon^2} + \frac{\mathbf{l}_\delta}{\epsilon} - \mathbf{L}_\mu^2 + 2\mathbf{L}_\mu \mathbf{l}_\delta - \frac{\pi^2}{4} \right) - d^{(2,0)} \left(\frac{1}{\epsilon} + 2\mathbf{l}_\delta \right) + \\
& C_A \left(\frac{\pi^2}{3} + 4 \ln 2 \right) \left(\frac{1}{\epsilon^2} + \frac{2\mathbf{L}_\mu}{\epsilon} + 2\mathbf{L}_\mu^2 + \frac{\pi^2}{6} \right) + C_A \left(8 \ln 2 - 9\zeta_3 \right) \left(\frac{1}{\epsilon} + 2\mathbf{L}_\mu \right) + \frac{656}{81} T_R N_f + \\
& C_A \left(-\frac{2428}{81} + 16 \ln 2 - \frac{7\pi^4}{18} - 28 \ln 2\, \zeta_3 + \frac{4}{3}\pi^2 \ln^2 2 - \frac{4}{3} \ln^4 2 - 32 \mathrm{Li}_4 \left(\frac{1}{2} \right) \right) + \mathcal{O}(\epsilon) \Bigg],
\end{aligned}
\tag{1}
$$

RGE for operators

$$\mu^2 \frac{d}{d\mu^2} O_f(x, b_T) = \frac{1}{2}\gamma_D^f(\mu, \zeta) O_f(x, b_T), \qquad \zeta \frac{d}{d\zeta} O_f(x, b_T) = -\mathcal{D}^f(\mu, b_T) O_f(x, b_T),$$

TMD anomalous dimensions obtained from renormalization factors

$$O_f(x, b_T) = \underbrace{Z_f(\mu, \zeta; \epsilon)}_{\to \gamma_V} \underbrace{R_f(\zeta; \epsilon, \delta)}_{\to \mathcal{D}} O^{bare}(x, b_T)$$

$$\gamma^f = 2\widehat{AD}(z_f - Z_f), \qquad \mathcal{D}^f = -\frac{\ln R_f}{\ln \zeta}\bigg|_{f.p.}$$

- Anomalous dimension same for TMDPDF and TMDFF
- Independent on regularization procedure
- Singular parts of R and Z are related to each other, e.g.

$$\frac{d \ln R_f}{d \ln \zeta}\bigg|_{s.p.} = \frac{d \ln Z_f}{d \ln \mu^2}$$

Small-b_T OPE

One can consider "transverse"-twist expansion of TMD at small-b_T

Coef. function (matching coef.)

$$O_q(x, b_T) = \sum_{n=0}^{\infty} \left(\frac{b_T^2}{B^2}\right)^n C^n_{q\to f}(x, \mathbf{L}_\mu; \mu, \zeta) \otimes O_{n,f}(x)$$

$$\int e^{ixp\xi} T[\bar{q}W^\dagger](\xi, b_T)\bar{T}[Wq](0)$$

$$\int e^{ixp\xi} T[\bar{q}W^\dagger](\xi)(\overleftrightarrow{\partial}_T B)^n \bar{T}[Wq](0)$$

Some unknown parameter
(character size)

Small-b_T OPE

One can consider "transverse"-twist expansion of TMD at small-b_T

Coef. function (matching coef.)

$$O_q(x, b_T) = \sum_{n=0}^{\infty} \left(\frac{b_T^2}{B^2}\right)^n C^n_{q\to f}(x, \mathbf{L}_\mu; \mu, \zeta) \otimes O_{n,f}(x)$$

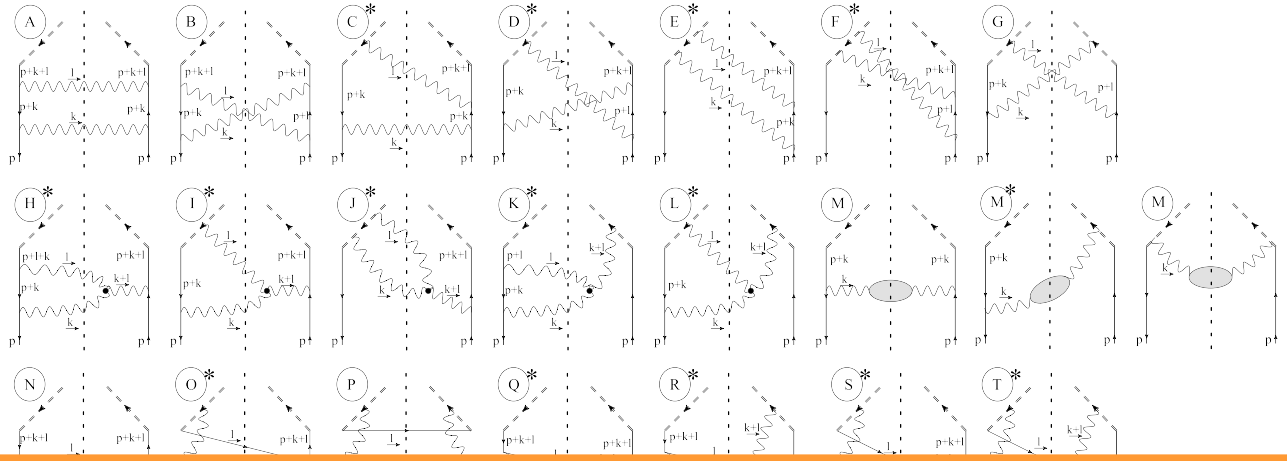
Matrix elemnent over hadron states

TMD PDF

PDF (of higher twists)

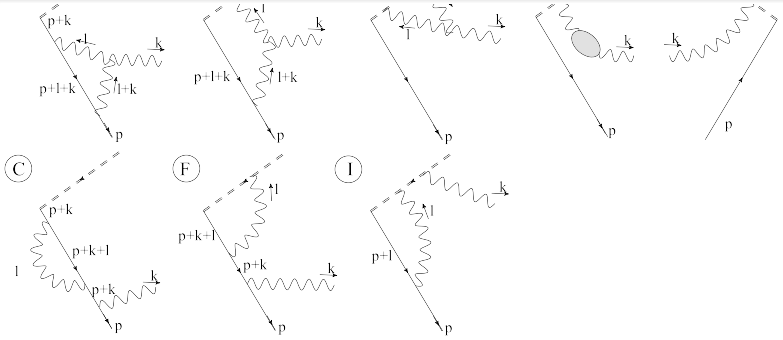
$$F_{q\leftarrow h}(x, b_T; \mu, \zeta) = \sum_{n=0}^{\infty} \left(\frac{b_T^2}{B^2}\right)^n C^n_{q\leftarrow f}(x, \mathbf{L}_\mu; \mu, \zeta) \otimes f^n_{f\leftarrow h}(x)$$

- At $n = 0$ f^0 is usual integrated PDF
- FF kinematics is analogous, but with overall factor $z^{-2+2\epsilon}$ (Collins normalization)



We have evaluated all flavor-channels TMD PDF and TMD FF at NLO and NNLO.

- $\gtrsim 100$ non-zero diagrams
- ~ 20 basic integrals (all taken as exact function of ϵ)
- Algebra done by *Mathematica*
- Multiple checks performed (cancellation of IR divergences by topologies, crossing, Ward identities, RGEs)
- Anomalous dimensions, operator renormalization constants found



$$\left[\bar{\eta} - i\delta\right]_\delta \quad \Gamma(1+a+2\epsilon) \quad \Gamma(1-2\epsilon)$$

$$\left[\frac{F^{(a0)}_{0101}}{\bar{\eta}+i\delta}\right]_+ = \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(-2\epsilon)}\frac{x^a}{1-x}\left(\ln(-i\delta) - \ln\bar{x} + \psi_{-2\epsilon} + \gamma_E\right) \tag{8.19}$$

$$\left[\frac{F^{(a0)}_{0101}}{\bar{\eta}+i\delta}\right]_\delta = \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(-2\epsilon)}\frac{1}{6}\Big[-3(\psi(1+a)+\psi(-2\epsilon))(\psi(1+a)+\psi(-2\epsilon)+2\ln(-i\delta)+4\gamma_E)$$
$$-3\ln(-i\delta)(4\gamma_E+\ln(i\delta)) + 3\psi'(1+a) + 3\psi'(-2\epsilon) - 12\gamma_E^2 - 2\pi^2\Big]$$

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Evaluation of coefficient coefficient

- Leading order are δ-function $\Longrightarrow$ coefficient functions from straightforward matching.
- **LO:** $C^{[0]}_{f\leftarrow f'} = \delta_{ff'}\delta(1-x)$, $\qquad \mathbb{C}^{[0]}_{f'\to f} = \delta_{ff'}\delta(1-z)$.
- **NLO:** $C^{[1]}_{f\leftarrow f'} = F^{[1]}_{f\leftarrow f'} - f^{[1]}_{f\leftarrow f'}$, $\qquad \mathbb{C}^{[1]}_{f\to f'} = D^{[1]}_{f'\to f} - \frac{d^{[1]}_{f'\to f}}{z^{2-2\epsilon}}$.
- **NNLO:**

$$C^{[2]}_{f\leftarrow f'} = F^{[2]}_{f\leftarrow f'} - \sum_r C^{[1]}_{f\leftarrow r}\otimes f^{[1]}_{r\leftarrow f'} - f^{[2]}_{f\leftarrow f'},$$

$$\mathbb{C}^{[2]}_{f'\to f} = D^{[2]}_{f'\to f} - \sum_r \mathbb{C}^{[1]}_{f\to r}\otimes \frac{d^{[1]}_{r\to f'}}{z^{2-2\epsilon}} - \frac{d^{[2]}_{f'\to f}}{z^{2-2\epsilon}}.$$

Note: f and d are zero in our scheme. Thus, only UV counter remains

$$f^{[1]}_{f\leftarrow f'} = \frac{-1}{\epsilon}P^{(1)}_{f\leftarrow f'}(x), \qquad f^{[2]}_{f\leftarrow f'} = \frac{-1}{2\epsilon}\left(\frac{P^{(1)}_{f\leftarrow r}\otimes P^{(1)}_{r\leftarrow f'}(x) + \beta_0 P^{(1)}_{f\leftarrow f'}(x)}{\epsilon} + P^{(1)}_{f\leftarrow f'}(x)\right)$$

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Screenshot

7.1 TMD parton distribution functions

The NNLO matching coefficients are

$$
\begin{aligned}
C^{(2,0)}_{q\leftarrow q}(x) = C_F^2 \Bigg\{ & p_{qq}(x) \Bigg[-20\mathrm{Li}_3(x) + 4\mathrm{Li}_3(\bar{x}) - 12\ln x \mathrm{Li}_2(\bar{x}) - 4\ln\bar{x}\mathrm{Li}_2(x) - 10\ln^2 x \ln\bar{x} \\
& + 2\ln^2 x \ln x + \frac{3}{2}\ln^2 x + (8+2\pi^2)\ln x + 20\zeta_3 \Bigg] + 8\bar{x}\mathrm{Li}_2(\bar{x}) + \frac{1+x}{3}\ln^3 x - 4\bar{x}\ln x \ln\bar{x} \\
& + \frac{7x+3}{2}\ln^2 x - 2x\ln\bar{x} + 2(1-12x)\ln x - \bar{x}\left(22 + \frac{\pi^2}{3}\right) + \frac{\pi^4}{72}\delta(\bar{x}) \Bigg\} \\
& + C_F C_A \Bigg\{ p_{qq}(x) \Bigg[8\mathrm{Li}_3(x) - 4\mathrm{Li}_3(\bar{x}) + 4\ln\bar{x}\mathrm{Li}_2(\bar{x}) - 4\ln x \mathrm{Li}_2(x) - \frac{\ln^3 x}{3} \\
& - \frac{11}{6}\ln^2 x - \frac{76}{9}\ln x + 6\zeta_3 - \frac{404}{27} \Bigg] - 4\bar{x}\mathrm{Li}_2(\bar{x}) - 2x\ln^2 x + 2x\ln\bar{x} \\
& + (10x+2)\ln x + \frac{44-\pi^2}{3}\bar{x} + \delta(\bar{x})\left(\frac{1214}{81} - \frac{67\pi^2}{36} - \frac{77}{9}\zeta_3 + \frac{\pi^4}{18}\right) \Bigg\} \\
& + C_F T_r N_f \Bigg\{ p_{qq}(x) \left[\frac{2}{3}\ln^2 x + \frac{20}{9}\ln x + \frac{112}{27} \right] - \frac{4}{3}\bar{x} + \delta(\bar{x})\left(-\frac{328}{81} + \frac{5\pi^2}{9} + \frac{28}{9}\zeta_3\right) \Bigg\} ;
\end{aligned}
\tag{7.1}
$$

$$
\begin{aligned}
C^{(2,0)}_{g\leftarrow q}(x) = C_A T_r \Bigg\{ & p_{qg}(x) \Bigg[4\mathrm{Li}_3(\bar{x}) - 4\ln\bar{x}\mathrm{Li}_2(\bar{x}) - 12\mathrm{Li}_2(x) + \frac{2}{3}\ln^3\bar{x} + 2\ln^2\bar{x} - \frac{3}{2}\ln^2 x - 12\ln\bar{x}\ln x \\
& + 8\ln x - \frac{3}{2}\ln\bar{x} - 6\zeta_3 + \frac{5\pi^2}{3} - \frac{113}{6} \Bigg] \\
& + p_{qg}(-x) \Bigg[8\mathrm{Li}_3(-x) - 4\mathrm{Li}_3\left(\frac{x}{1+x}\right) + 4\mathrm{Li}_3\left(\frac{1}{1+x}\right) - 4\ln x \mathrm{Li}_2(-x) + 2\mathrm{Li}_2(x^2) \\
& + 2\ln^2 x \ln(1+x) - 2\ln x \ln^2(1+x) + 4\ln x \ln(1+x) + \frac{\ln^2 x}{2} + 4\ln\bar{x}\ln x + \frac{3}{2}\ln\bar{x} \\
& + \frac{4}{3}\ln x - \frac{\pi^2}{3} - 2\zeta_3 - \frac{9}{2} \Bigg] + 32x\mathrm{Li}_3(x) + 16x\ln x \mathrm{Li}_2(\bar{x}) - 4\mathrm{Li}_2(-x) \\
& + \frac{8(5x^3-2)}{3x}\mathrm{Li}_2(\bar{x}) + \frac{2(2x+1)}{3}\ln^3 x + 16x\ln^2 x \ln\bar{x} - 2\ln^2\bar{x} - \frac{38}{3}x^2\ln^2 x \\
& - 4\ln x \ln(1+x) - 8x^2\ln x + \frac{8(13x-3\pi^2)x}{9}\ln x + \frac{8(83x^3+43)}{9x} \Bigg\}
\end{aligned}
$$

Screenshot

$$
\begin{aligned}
C^{(2,0)}_{q\leftarrow g}(x) = C_F C_A \Bigg\{ & p_{gq}(x) \Bigg[2\ln^2 x \ln\bar{x} + 2\ln x \ln^2\bar{x} + \frac{\ln^3 x}{3} - \frac{2}{3}\ln^3\bar{x} + 8\ln x \mathrm{Li}_2(x) - 12\mathrm{Li}_3(x) + \frac{44}{3}\mathrm{Li}_2(\bar{x}) \\
& - \frac{11}{3}\ln^2\bar{x} + 26\zeta_3 - \frac{1580}{27} \Bigg] \\
& + p_{gq}(-x) \Bigg[2\ln^2 x \ln(1+x) - 2\ln x \ln^2(1+x) + \frac{\ln^3 x}{3} - 2\ln x \mathrm{Li}_2(x^2) + 2\mathrm{Li}_3(x^2) \\
& + 4\mathrm{Li}_3\left(\frac{1}{1+x}\right) - 4\mathrm{Li}_3\left(\frac{x}{1+x}\right) + \frac{4x}{3}\mathrm{Li}_2(\bar{x}) + \frac{152}{9}\ln\bar{x} - 2\zeta_3 \Bigg] \\
& + 2x\mathrm{Li}_2(x^2) - 4x^2\mathrm{Li}_2(x) - \frac{2x}{3}\ln^3 x + 4x\ln x \ln(1+x) + \left(12 + 3x + \frac{8x^2}{3}\right)\ln^2 x \\
& + 2x\ln^2\bar{x} + \frac{608+66x}{9}\ln\bar{x} - \frac{498-12x+176x^2}{9}\ln x + \frac{4+508x+608x^2}{27} - \pi^2 x \Bigg\} \\
& + C_F^2 \Bigg\{ p_{gq}(x) \left[\frac{2}{3}\ln^3\bar{x} + 3\ln^2\bar{x} + 16\ln\bar{x} \right] - 2x\ln^2\bar{x} - 6x\ln\bar{x} + \frac{2-x}{3}\ln^3 x - \frac{4+3x}{2}\ln^2 x \\
& + 5(x-3)\ln x + 10 - x \Bigg\} + C_F T_r N_f \Bigg\{ p_{gq}(x) \left(\frac{4}{3}\ln^2\bar{x} + \frac{40}{9}\ln\bar{x} + \frac{224}{27} \right) - \frac{8x}{3}\ln\bar{x} - \frac{40x}{9} \Bigg\} ;
\end{aligned}
\tag{7.3}
$$

$$
\begin{aligned}
C^{(2,0)}_{g\leftarrow g}(x) = C_A^2 \Bigg\{ & p_{gg}(x) \Bigg[4\ln^2 x \ln\bar{x} + 4\ln x \ln^2\bar{x} - \frac{2}{3}\ln^3 x + 16\ln x \mathrm{Li}_2(x) - 24\mathrm{Li}_3(x) + 52\zeta_3 - \frac{808}{27} \Bigg] \\
& + p_{gg}(-x) \Bigg[4\ln^2 x \ln(1+x) - \frac{2}{3}\ln^3 x - \frac{8}{3}\ln^3(1+x) + \frac{4\pi^2}{3}\ln(1+x) - 4\ln x \mathrm{Li}_2(x^2) \\
& + 4\mathrm{Li}_3(x^2) + 8\mathrm{Li}_3(-x) + 16\mathrm{Li}_3\left(\frac{1}{1+x}\right) - 12\zeta_3 \Bigg] + \frac{8}{3}\bar{x}\left(\frac{11}{x} - 1 + 11x\right)\mathrm{Li}_2(\bar{x}) \\
& + \frac{2x}{3}\ln\bar{x} - \frac{8}{3}(1+x)\ln^3 x + \frac{44x^2 - 11x + 25}{3}\ln^2 x - \frac{536x^2 + 149x + 701}{9}\ln x \\
& + \frac{844x^3 - 744x^2 + 696x - 784}{9x} + \delta(\bar{x})\left(\frac{1214}{81} - \frac{67\pi^2}{36} - \frac{77}{9}\zeta_3 + \frac{5\pi^4}{72}\right) \Bigg\} \\
& + C_A T_r N_f \Bigg\{ \frac{224}{27}p_{gg}(x) - \frac{4}{3}x\ln\bar{x} + \frac{4}{3}(x+1)\ln^2 x + \frac{4}{9}(10x+13)\ln x - 8\bar{x}
\end{aligned}
$$

We have evaluated all flavor-channels TMD PDF and TMD FF and the their small-b_T matching coefficients at NLO and NNLO.

[Echevarria,Scimemi,AV,1604.07869]

see also [Echevarria,Scimemi,AV,1509.06392]

$$
\begin{aligned}
z^2 C^{(2,0)}_{q\to q}(z) = C_F \Bigg\{ & p_{qq}(z) \Bigg[104\mathrm{Li}_3(z) - 4\mathrm{Li}_3(\bar{z}) + 4\ln\bar{z}\mathrm{Li}_2(z) - 48\ln z \mathrm{Li}_2(z) - 8\ln^2 z \\
& + 2\ln^2 z \ln\bar{z} - 2\ln^2\bar{z}\ln z - \frac{9}{2}\ln^2 z - \left(8 + \frac{20}{3}\pi^2\right)\ln z - 104\zeta_3 \Bigg] \\
& + \bar{z}\left(24\mathrm{Li}_2(z) + 28\ln z \ln\bar{z} + 42 - \frac{13}{3}\pi^2\right) + (1+z)\ln^3 z \\
& - \frac{7z+43}{2}\ln^2 z + 2\ln\bar{z} + (62z - 22)\ln z + \frac{\pi^4}{72}\delta(\bar{z}) \Bigg\} \\
& + C_F C_A \Bigg\{ p_{qq}(z) \Bigg[4\mathrm{Li}_3(\bar{z}) + 12\mathrm{Li}_3(z) - 4\ln\bar{z}\mathrm{Li}_2(\bar{z}) - 8\ln z \mathrm{Li}_2(z) + 3\ln^3 z - 4\ln\bar{z}\ln^2 z \\
& - \frac{11}{6}\ln^2 z + \left(\frac{70}{3} - 2\pi^2\right)\ln z + 2\zeta_3 - \frac{404}{27} \Bigg] + 4\bar{z}\mathrm{Li}_2(\bar{z}) + 2(4+z)\ln^2 z - 2\ln\bar{z} \\
& + \frac{116-74z}{3}\ln z + \frac{44-\pi^2}{3}\bar{z} + \delta(\bar{z})\left(\frac{1214}{81} - \frac{67\pi^2}{36} - \frac{77}{9}\zeta_3 + \frac{13\pi^4}{18}\right) \Bigg\} \\
& + C_F T_r N_f \Bigg\{ p_{qq}(z) \left[\frac{2}{3}\ln^2 z - \frac{20}{9}\ln z + \frac{112}{27} \right] - \frac{16}{3}\bar{z}\ln z - \frac{4}{3}\bar{z} + \delta(\bar{z})\left(-\frac{328}{81} + \frac{5\pi^2}{9} + \frac{28}{9}\zeta_3\right) \Bigg\} ;
\end{aligned}
\tag{7.7}
$$

$$
\begin{aligned}
& + \left(\frac{49}{2}z - 28\right)\ln^2 z + 2z\ln^2\bar{z} + 8z\ln \\
& + C_F C_A \Bigg\{ p_{gq}(-z) \Bigg[8\mathrm{Li}_3\left(\frac{1}{1+z}\right) - \\
& - \frac{4}{3}\ln^3(1+z) + \frac{2\pi^2}{3}\ln(1+z) \Bigg] + p_g \\
& + 4\ln z \mathrm{Li}_2(z) + \frac{2}{3}\ln^3\bar{z} - 10\ln z \ln^2\bar{z} + \\
& + 32\mathrm{Li}_3(z) + 16\ln z \mathrm{Li}_2(\bar{z}) + 4z\mathrm{Li}_2(- \\
& - \left(\frac{16}{z} + 4 + 10z\right)\ln^3 z + 16\ln\bar{z}\ln^2 z \\
& - 2z\ln^2\bar{z} - 4z\ln\bar{z}\ln z + 4z\ln(1+z)\ln \\
& + \left(\frac{1180}{3z} + 826 - 8\pi^2 + 266z + \frac{176}{3}\right. \\
& + \left(\frac{2684}{9} + \frac{8\pi^2}{3}\right)z - \frac{464}{9}z^2 \Bigg\} + C_F
\end{aligned}
$$

$$
\begin{aligned}
& + \left(-\frac{7}{2} - 26z + 34z^2\right)\ln^2 z + (-8 + 73z - 76z^2)\ln z + 63 - 101z + 56z^2 \Bigg\} \\
& + T_r C_A \Bigg\{ p_{qq}(-z) \Bigg[8\mathrm{Li}_3\left(\frac{1}{1+z}\right) - 4\mathrm{Li}_3(-z) - 8\mathrm{Li}_3(z) - 4\ln z \mathrm{Li}_2(-z) + 4\ln z \mathrm{Li}_2(z) \\
& - \frac{4}{3}\ln^3(1+z) - 6\ln^2 z \ln(1+z) + \frac{2\pi^2}{3}\ln(1+z) - 2\zeta_3 \Bigg] + p_{qq}(z) \Bigg[20\mathrm{Li}_3(z) \\
& - 24\ln z \mathrm{Li}_2(z) - \frac{2}{3}\ln^3\bar{z} - 4\ln z \ln\bar{z} + 4\ln^2\bar{z}\ln z - \frac{11}{3}\ln^2\bar{z} + 30\ln\bar{z}\ln z \\
& - \frac{152}{9}\ln\bar{z} + 2\pi^2\ln\bar{z} - 4\pi^2\ln z - 6\zeta_3 \Bigg] + z(1+z)\left[8\mathrm{Li}_2(-z) + 8\ln(1+z)\ln z \right] \\
& + z\bar{z}\left[4\ln^2\bar{z} + \frac{20}{3}\ln\bar{z} \right] - \frac{8\bar{z}}{3z}(2 - z + 11z^2)\mathrm{Li}_2(z) + \frac{2}{3}(11 + 62z)\ln^2 z \\
& - \frac{8(2+z^3)}{3z}\ln\bar{z}\ln z + \frac{2(16 - 22z + 35z^2 - 59z^3)\ln^2 z}{3z} + \frac{2(24 - 165z - 699z^2 + 38z^3)\ln z}{9z} \\
& + \frac{-296 - 2446z + 278z^2 + 1548z^3}{27z} + \frac{\pi^2}{9z}(8 + 21z - 60z^2 + 76z^3) \Bigg\} \\
& + T_r^2 N_f \Bigg\{ \frac{4}{3}p_{qq}(z) \left[\ln^2 z + \ln^2\bar{z} - 6\ln\bar{z}\ln z - 10\ln z + \frac{10}{3}\ln\bar{z} - \pi^2 + \frac{56}{9} \right] \\
& - \frac{16}{3}z\bar{z}\left[\ln z + \ln\bar{z} + \frac{2}{3} \right] \Bigg\} ;
\end{aligned}
\tag{7.9}
$$

$$
\begin{aligned}
& + C_F T_r N_f \Bigg\{ \frac{44}{3}(1+z)\ln^3 z + \frac{2(16z^3 + 15z^2 + 21z +}{3z} \\
& - \frac{8(82z^3 + 81z^2 + 135z - 6)}{9z}\ln z + \frac{8(301z^3 - 108z^2}{27z}
\end{aligned}
$$

$$
\begin{aligned}
z^2 \mathbb{C}^{(2,0)}_{q\to q}(z) = T_r C_F \Bigg\{ & \frac{8}{3}\frac{\bar{z}}{z}(2 - z + 2z^2)\mathrm{Li}_2(\bar{z}) + 2(1+z)\ln^3 z \\
& - \frac{4}{9z}(40 + 282z + 159z^2 + 32z^3)\ln z - \frac{2}{3}\frac{\bar{z}}{z}(56 +
\end{aligned}
$$

$$
\begin{aligned}
z^2 \mathbb{C}^{(2,0)}_{g\to q}(z) = \left(C_F^2 - \frac{C_F C_A}{2}\right) \Bigg\{ & p_{qq}(-z) \Bigg[8\mathrm{Li}_3\left(\frac{1}{1+z}\right) - 8\mathrm{Li}_3 \\
& + 8\ln z \mathrm{Li}_2(z) - 8\ln z \mathrm{Li}_2(-z) - 4\ln z \ln^2(1+z) - 12\ln \\
& + 8\bar{z}\mathrm{Li}_2(\bar{z}) - 8(1+z)\mathrm{Li}_2(-z) - 8(1+z)\ln z \ln(1+ \\
& + (38 - 10z)\ln z - \frac{2\pi^2}{3}(1+z) + 30\bar{z} \Bigg\} .
\end{aligned}
$$

The result for quark sector were first presented by us in [1]. The mi
are presented here for the first time. Moreover, to our best know
gluon TMDFFs are also presented for the first time.

Crossing symmetry for TMD

TMD PDF vs. TMD FF operators

On level of unsubtracted TMDs the exact relation holds (at any order of pert.theory)

$$D_{f\to f'}(z) \quad = \quad -\frac{\mathcal{N}_{f,f'}}{z}\, F_{f\leftarrow f'}(z^{-1})$$

$$\mathcal{N}_{f,f'} = \frac{\#\text{physical states}_f}{\#\text{physical states}_{f'}}$$

- Violated diagram-by-diagram, due to IR divergences. Presented in the sum of diagrams.
- In fact, one can evaluate only PDF (FF) and obtain FF (PDF).
- We evaluate these kinematics independently and check the results.

This nice relation is significantly violated for matching coefficients

- ϵ-expansion and renormalization (choice of brunch for logs, factors of ζ_n)
- Extra factor from integrated FF normalization $\mathbb{O}(z, b_T) = z^{2-2\epsilon}\mathbb{O}(z)$ (while for TMD PDF $O(x, b_T) = O(x)$)

Finally: There are very little (numerically) traces of crossing between FF and PDF

Large-x behaviour of coefficient function

- At $x, z \to 1$ TMD behave as $\frac{\ln^{n-1}\bar{x}}{(1-x)_+}$
- In the matching function all non-trivial log's cancels

$$C^{[2]} = \overbrace{16 C_K^2 \boldsymbol{L}_\mu^2 \left(\frac{\ln \bar{x}}{1-x}\right)_+}^{P\otimes P} \underbrace{- \frac{2C_K}{(1-x)_+}\left(2C_K \boldsymbol{L}_\mu^3 + d^{(2,2)}\boldsymbol{L}_\mu^2 + \left(d^{(2,1)} - C_K\frac{\pi^2}{3}\right)\boldsymbol{L}_\mu\right)}_{\text{RGE predicted}} - \frac{2C_K d^{(2,0)}}{(1-x)_+}$$

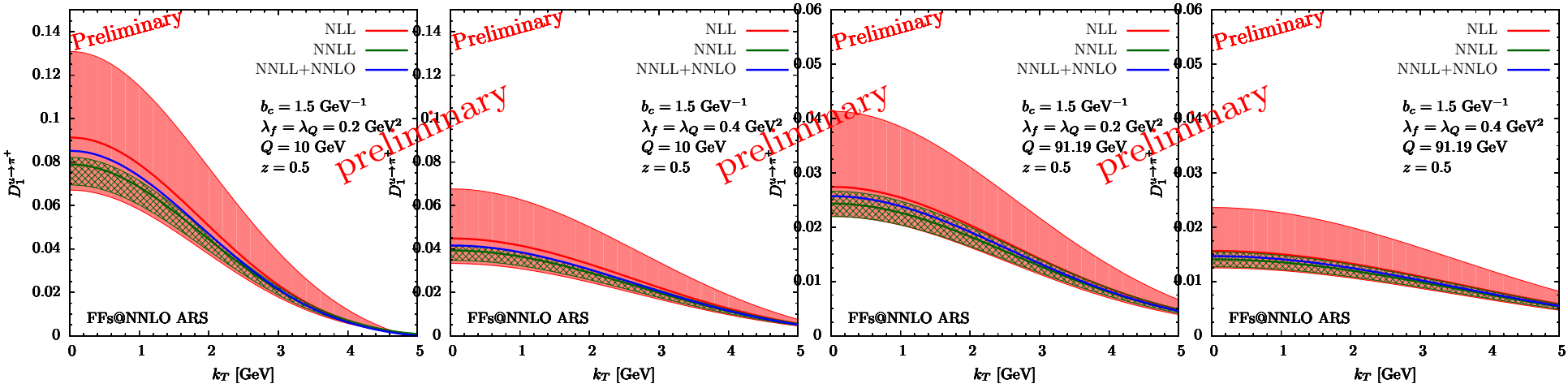
$$\mathcal{D} = C_K \sum_{n=0}^{\infty} a_s^n \sum_{k=0}^{n} \boldsymbol{L}_\mu^k d^{(n,k)}$$

In general one can show that at large x, z

$$C^{[n]} \sim (\text{RGE part}) - \frac{2C_K d^{(n,0)}}{(1-x)_+} + \mathcal{O}\left(\delta(\bar{x})\right)$$

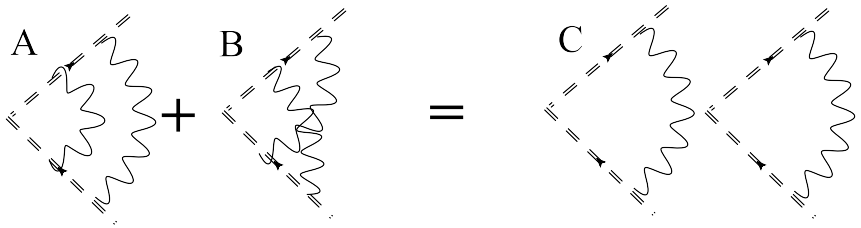
confirmed in [Lustermans,Waalewjin,Zeune,1605.02740]

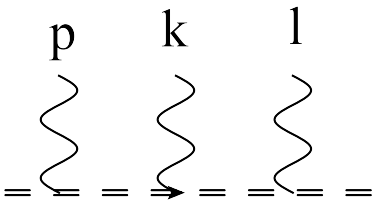
Conclusion



- Definition of TMD operators elaborated for PDF and FF kinematics
- UV and rapidity renormalization constants evaluated at NNLO (in modified δ-reg.scheme)
- Partonic TMD PDF and FF are evaluated at NNLO
- All matching coefficients are found at NNLO (for PDF coincide with [Catani at al,Gehrmann at al], for q/q TMD FF [Echevarria,Scememi,AV;1509.06392])
- Gluon TMD FF is considered for the first time
- Various properties and relations are discussed

Violation of exponentiation in δ-regularization



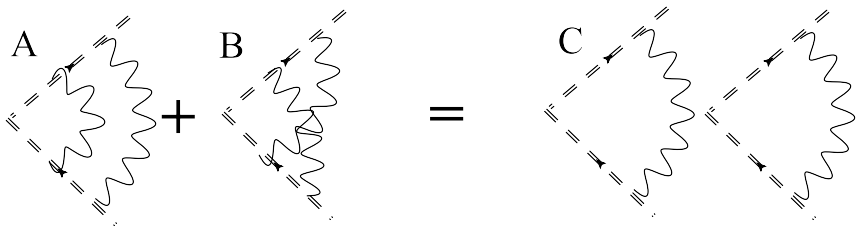


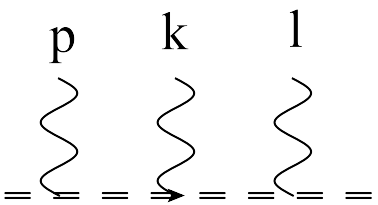
$$= \frac{1}{(p^+ + i\delta)(p^+ + k^+ + i\delta)(p^+ + k^+ + l^+ + i\delta)}$$

Within original δ-regularization, the exponentiation is broken

$$\mathrm{Diag}_A + \mathrm{Diag}_B = \frac{\mathrm{Diag}_C^2}{2} + \delta^+ \underbrace{\int \frac{d^dk}{k^2}\frac{d^dl}{l^2}\frac{1}{(k^+ + l^+)k^+l^+(k^- + l^-)k^-}}_{\frac{1}{\delta^+}\ \text{divergent}}$$

- That can result to artificial singularities in δ
- To incomplete cancellation of $\ln\delta$, that will cause problems at higher loops.





$$= \frac{1}{(p^+ + i\delta)(p^+ + k^+ + 2i\delta)(p^+ + k^+ + l^+ + 3i\delta)}$$

δ-regularization preserving exponentiation

The regularization should be implemented on the level of operator

$$P\exp\Big[-ig\int_0^\infty d\sigma A_\pm(\sigma n)\Big] \longrightarrow P\exp\Big[-ig\int_0^\infty d\sigma A_\pm(\sigma n)e^{-\delta^\pm|\sigma|}\Big]$$

Then exponentiation is exact

$$\mathrm{Diag}_A + \mathrm{Diag}_B = \frac{\mathrm{Diag}_C^2}{2}$$

In any form, δ-regularization violate gauge-invariance linearly, beware of linearly divergent integrals.

- Is there any regularization <u>with scale</u> for light-like half-infinite Wilson lines without any problem?

Structure of anomalous dimensions

$$O_f(x,b_T) = \underbrace{Z_f(\mu,\zeta;\epsilon)}_{\to\gamma_V}\underbrace{R_f(\zeta;\epsilon,\delta)}_{\to\mathcal{D}}O^{bare}(x,b_T)$$

Anomalous dimension for CSS evolution

$$\boxed{\mathcal{D}^f = \frac{1}{2}\frac{dS}{d\mathbf{l}_\zeta} - \underbrace{\frac{dZ_f}{d\ln\zeta}}_{\sim\frac{1}{\epsilon}} = \frac{1}{2}\frac{dS}{d\mathbf{l}_\zeta}\Big|_{finite}}$$

$$\begin{aligned} S^{[2]} = & \Bigg[d^{(2,2)}\left(\frac{3}{\epsilon^3} + \frac{2\mathbf{l}_\delta}{\epsilon^2} + \frac{\pi^2}{6\epsilon} + \frac{4}{3}\mathbf{L}_\mu^3 - 2\mathbf{L}_\mu^2\mathbf{l}_\delta + \frac{2\pi^2}{3}\mathbf{L}_\mu + \frac{14}{3}\zeta_3\right) - \\ & d^{(2,1)}\left(\frac{1}{2\epsilon^2} + \frac{\mathbf{l}_\delta}{\epsilon} - \mathbf{L}_\mu^2 + 2\mathbf{L}_\mu\mathbf{l}_\delta - \frac{\pi^2}{4}\right) - d^{(2,0)}\left(\frac{1}{\epsilon} + 2\mathbf{l}_\delta\right) + \ldots. \end{aligned}$$

$$\Longrightarrow \mathcal{D}^{[2]} = d^{(2,2)}\ln^2\left(\frac{b_T^2\mu^2}{4e^{-2\gamma_E}}\right) + d^{(2,1)}\ln\left(\frac{b_T^2\mu^2}{4e^{-2\gamma_E}}\right) + d^{(2,0)}$$

$$d^{(2,2)} = \frac{\Gamma^{(0)}\beta_0}{4}, \qquad d^{(2,1)} = \frac{\Gamma^{(1)}}{2}, \qquad d^{(2,0)} = C_K\left(\left(\frac{404}{27} - 14\zeta_3\right)C + A - \frac{112}{27}T_rN_f\right)$$

$$O_f(x,b_T) = \underbrace{Z_f(\mu,\zeta;\epsilon)}_{\to\gamma_V}\underbrace{R_f(\zeta;\epsilon,\delta)}_{\to\mathcal{D}}O^{bare}(x,b_T)$$

TMD anomalous dimension

$$Z_f^{[1]} = \frac{-\Gamma^{[1]}}{2\epsilon^2}\left(1+\epsilon\mathbf{l}_\zeta\right) + Z_f^{[1]} + \frac{\gamma_V^{[1]f}}{2\epsilon}$$

$$Z_f^{[2]} = \frac{\Gamma^{[2]^2}}{8\epsilon^4}\left(1+2\epsilon\mathbf{l}_\zeta+\epsilon^2\mathbf{l}_\zeta^2\right) + \ldots + Z_f^{[2]} + \frac{\gamma_V^{[2]f}}{4\epsilon}$$

$$Z_q^{[2]} = \frac{2C_F^2}{\epsilon^4} + \ldots.. + \frac{C_F}{\epsilon}\left[C_F\left(\pi^2-12\zeta_3\right) + C_A\left(-\frac{355}{27}-\frac{11\pi^2}{12}+13\zeta_3+\left(-\frac{67}{9}+\frac{\pi^2}{3}\right)\mathbf{l}_\zeta\right) + T_rN_f\left(\frac{92}{27}+\frac{\pi^2}{3}+\frac{20}{9}\mathbf{l}_\zeta\right)\right],$$

$$Z_g^{[2]} = \frac{2C_A^2}{\epsilon^4} + \ldots.. + \frac{C_A}{\epsilon}\left[C_A\left(-\frac{2147}{216}+\frac{11\pi^2}{36}+\zeta_3+\left(-\frac{67}{9}+\frac{\pi^2}{3}\right)\mathbf{l}_\zeta\right) + T_rN_f\left(\frac{121}{54}-\frac{\pi^2}{9}+\frac{20}{9}\mathbf{l}_\zeta\right)\right].$$

$$\Longrightarrow \gamma_V^{q(2)} = C_F^2\left(-3+4\pi^2-48\zeta_3\right) + C_FC_A\left(-\frac{961}{27}-\frac{11\pi^2}{3}+52\zeta_3\right) + C_FT_rN_f\left(\frac{260}{27}+\frac{4\pi^2}{3}\right),$$

$$\Longrightarrow \gamma_V^{g(2)} = C_A^2\left(-\frac{1384}{27}+\frac{11\pi^2}{9}+4\zeta_3\right) + C_AT_rN_f\left(\frac{512}{27}-\frac{4\pi^2}{9}\right) + 8C_FT_rN_f.$$

RGE for TMD and coefficient functions

RGE for operators

$$\mu^2 \frac{d}{d\mu^2} O_f(x, b_T) = \frac{1}{2}\gamma_D^f(\mu,\zeta) O_f(x, b_T), \qquad \mu^2 \frac{d}{d\mu^2} \mathbb{O}_f(z, b_T) = \frac{1}{2}\gamma_D^f(\mu,\zeta) \mathbb{O}_f(z, b_T).$$

$$\zeta \frac{d}{d\zeta} O_f(x, b_T) = -\mathcal{D}^f(\mu, b_T) O_f(x, b_T), \qquad \zeta \frac{d}{d\zeta} \mathbb{O}_f(z, b_T) = -\mathcal{D}^f(\mu, b_T) \mathbb{O}_f(z, b_T).$$

RGE for coefficient functions

The ζ-dependance can be solved out from the functions

$$\begin{aligned} C_{f\leftarrow f'}(x, b_T; \mu, \zeta) &= \exp\left(-\mathcal{D}^f(\mu, b_T)\mathbf{L}_{\sqrt{\zeta}}\right) \hat{C}_{f\leftarrow f'}(x, \mathbf{L}_\mu) \\ \mathbb{C}_{f\to f'}(x, b_T; \mu, \zeta) &= \exp\left(-\mathcal{D}^f(\mu, b_T)\mathbf{L}_{\sqrt{\zeta}}\right) \hat{\mathbb{C}}_{f\to f'}(z, \mathbf{L}_\mu). \end{aligned}$$

RGE for operators

$$\mu^2 \frac{d}{d\mu^2} O_f(x, b_T) = \frac{1}{2}\gamma_D^f(\mu,\zeta) O_f(x, b_T), \qquad \mu^2 \frac{d}{d\mu^2} \mathbb{O}_f(z, b_T) = \frac{1}{2}\gamma_D^f(\mu,\zeta) \mathbb{O}_f(z, b_T).$$

$$\zeta \frac{d}{d\zeta} O_f(x, b_T) = -\mathcal{D}^f(\mu, b_T) O_f(x, b_T), \qquad \zeta \frac{d}{d\zeta} \mathbb{O}_f(z, b_T) = -\mathcal{D}^f(\mu, b_T) \mathbb{O}_f(z, b_T).$$

RGE for coefficient functions

The μ-dependence is given by equation

$$\mu^2 \frac{d}{d\mu^2} \hat{C}_{f\leftarrow f'}(x, \mathbf{L}_\mu) = \sum_r \hat{C}_{f\to r}(x, \mathbf{L}_\mu) \otimes K^f_{r\leftarrow f'}(x, \mathbf{L}_\mu),$$

$$\mu^2 \frac{d}{d\mu^2} \hat{\mathbb{C}}_{f\to f'}(z, \mathbf{L}_\mu) = \sum_r \hat{\mathbb{C}}_{f\to r}(z, \mathbf{L}_\mu) \otimes \mathbb{K}^f_{r\to f'}(z, \mathbf{L}_\mu).$$

The kernels $\mathbf{K}$ and $\mathbb{K}$ are

$$\begin{aligned} K^f_{r\leftarrow f'}(x, \mathbf{L}_\mu) &= \frac{\delta_{rf'}}{2}\left(\Gamma^f_{cusp}\mathbf{L}_\mu - \gamma^f_V\right) - P_{r\leftarrow f'}(x), \\ \mathbb{K}^f_{r\to f'}(z, \mathbf{L}_\mu) &= \frac{\delta_{rf'}}{2}\left(\Gamma^f_{cusp}\mathbf{L}_\mu - \gamma^f_V\right) - \frac{\mathbb{P}_{r\to f'}(z)}{z^2}. \end{aligned}$$

Evaluation of partonic TMD

Evaluation of partonic TMD

- Simplest way to find (leading) coefficient function is to calculate partonic matrix element.
- For $n = 0$ we can set parton on-mass-schell $p^2 = 0$.

$$
\begin{aligned}
D^{[0]} &= \Delta^{[0]}, \\
D^{[1]} &= \Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2} + \left(Z_q^{[1]} - Z_2^{[1]}\right)\Delta^{[0]}, \\
D^{[2]} &= \Delta^{[2]} - \frac{S^{[1]}\Delta^{[1]}}{2} + \frac{3S^{[1]}S^{[1]}\Delta^{[0]}}{8} - \frac{S^{[2]}\Delta^{[0]}}{2} + \left(Z_D^{[1]} - Z_2^{[1]}\right)\left(\Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2}\right) \\
&\qquad + \left(Z_D^{[2]} - Z_2^{[2]} - Z_2^{[1]}Z_D^{[1]} + Z_2^{[1]}Z_2^{[1]}\right)\Delta^{[0]}.
\end{aligned}
$$

Evaluation of partonic TMD

- Simplest way to find (leading) coefficient function is to calculate partonic matrix element.
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UV renormalization

$$D^{[0]} = \Delta^{[0]},$$
$$D^{[1]} = \Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2} + \left(Z_q^{[1]} - Z_2^{[1]}\right)\Delta^{[0]},$$
$$D^{[2]} = \Delta^{[2]} - \frac{S^{[1]}\Delta^{[1]}}{2} + \frac{3S^{[1]}S^{[1]}\Delta^{[0]}}{8} - \frac{S^{[2]}\Delta^{[0]}}{2} + \left(Z_D^{[1]} - Z_2^{[1]}\right)\left(\Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2}\right)$$
$$+ \left(Z_D^{[2]} - Z_2^{[2]} - Z_2^{[1]}Z_D^{[1]} + Z_2^{[1]}Z_2^{[1]}\right)\Delta^{[0]}.$$

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$$D^{[0]} = \Delta^{[0]},$$

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$$D^{[2]} = \Delta^{[2]} - \frac{S^{[1]}\Delta^{[1]}}{2} + \frac{3S^{[1]}S^{[1]}\Delta^{[0]}}{8} - \frac{S^{[2]}\Delta^{[0]}}{2} + \left(Z_D^{[1]} - Z_2^{[1]}\right)\left(\Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2}\right)$$

$$+ \left(Z_D^{[2]} - Z_2^{[2]} - Z_2^{[1]}Z_D^{[1]} + Z_2^{[1]}Z_2^{[1]}\right)\Delta^{[0]}.$$

rapidity renormalization

Evaluation of partonic TMD

- Simplest way to find (leading) coefficient function is to calculate partonic matrix element.
- For $n = 0$ we can set parton on-mass-schell $p^2 = 0$.

UV renormalization

$$D^{[0]} = \Delta^{[0]},$$

$$D^{[1]} = \Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2} + \left(Z_q^{[1]} - Z_2^{[1]}\right)\Delta^{[0]},$$

$$D^{[2]} = \Delta^{[2]} - \frac{S^{[1]}\Delta^{[1]}}{2} + \frac{3S^{[1]}S^{[1]}\Delta^{[0]}}{8} - \frac{S^{[2]}\Delta^{[0]}}{2} + \left(Z_D^{[1]} - Z_2^{[1]}\right)\left(\Delta^{[1]} - \frac{S^{[1]}\Delta^{[0]}}{2}\right)$$

$$+ \left(Z_D^{[2]} - Z_2^{[2]} - Z_2^{[1]}Z_D^{[1]} + Z_2^{[1]}Z_2^{[1]}\right)\Delta^{[0]}.$$

MIX

rapidity renormalization

All rapidity and UV divergences cancel
only collinear remains
(checked at NNLO)