Nikhef Topical Graduate Lectures

Symmetries (and their breaking) in the Standard Model

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Overview

In these lectures I will briefly review the symmetries of the fundamental interactions, starting from abelian gauge theories (QED) and then moving to non-abelian theories (QCD and the weak interactions). I will also explain how to break symmetries within the Standard Model, with emphasis on the Higgs mechanism. The outline of the lectures is:

- Abelian gauge theories: QED.
- Non-abelian gauge theories: QCD and the weak interactions.
- Spontaneous symmetry breaking and the Higgs mechanism.

There will be two 1.5h lectures, the first on *Wednesday 29/03/2017* from 14.00 to 15.30 and the second on *Thursday 30/03/2017* from 11.00 to 12.30.

1 The symmetries of electromagnetism

Global symmetries in QED The quantum field theory (QFT) that describes the electromagnetic interactions is *Quantum Electrodynamics* (QED). The QED Lagrangian has the following form:

$$\mathcal{L}_{\rm QED} = \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{1.1}$$

where for simplicity we have assumed a single fermion of mass m and electric charge e - the addition of more fermions with different masses and charges is trivial. In this Lagrangian, the covariant derivative is defined as

$$D_\mu \equiv \partial_\mu + ieA_\mu\,, \tag{1.2}$$

with e the electric charge of this fermion, and the field strength tensor is the one that you might have seen from the covariant description of Maxwell's equations:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{1.3}$$

and where the four-vector field A_μ represents the photon.

The QED Lagrangian Eq. (1.1) has a number of symmetries. By symmetries, we mean *transformations of the field content of the theory*, in this case the fermion field ψ and the photon field A_μ, that leave the *Lagrangian invariant*.[1] Why these symmetries are so important? Because we know from *Noether's Theorem* that for each continous and differentiable global symmetry of the Lagrangian that describes a physical theory, there is an *associated conservation law*. This theorem states that for a Lagrangian that depends on a number of fields and their first derivatives,

$$\mathcal{L}(\psi, \partial_\nu\psi, A_\mu, \partial_\nu A_\nu)\,, \tag{1.4}$$

for each infinitesimal transformation of the fields that leaves the Lagrangian invariant, there will be a *conserved current*.

Let us give an specific example of how a global symmetry of the QED Lagrangian leads to a conservation law. From the QED Lagrangian, Eq. (1.1), we see that it is invariant if we rescale the fermion fields as

$$\psi(x) \to \psi'(x) = e^{i\alpha}\psi(x) \simeq (1+i\alpha)\,\psi = \psi + i\alpha\psi\,, \tag{1.5}$$

where α is a constant, as we are assuming that the transformation is infinitesimal. This transformation does not affect the photon field. This is called a $U(1)$ *transformation*, which corresponds to a rotation a two dimensional space. Note that this is a rotation in the internal field space, as opposed to the standard space-time rotation. In this case, Noether theorem tells us that as a consequence of the U(1) global symmetry of

[1] To be more precise, is the action $S \sim \int d^4x\mathcal{L}$ that should be invariant, so the Lagrangian can change up by a total derivative.

the QED Lagrangian, there will be a conserved current will given by:

$$j^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} \delta \psi \,, \tag{1.6}$$

where $\delta\psi = i\alpha\psi$ is the fermion field modification corresponding to an infinitesimal U(1) transformation. Therefore, given that we have from the Lagrangian that

$$\frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} = \bar{\psi} i \gamma^\mu \,, \tag{1.7}$$

we find that the conserved current is

$$j^\mu = -\alpha \bar{\psi} i \gamma^\mu \psi \,, \tag{1.8}$$

which is nothing but the *electric current associated to a charged fermion.* Noether theorem guarantees that this current is conserved, and indeed it is easy to check that $\partial_\mu j^\mu = 0$. It can also be shown that the integral over the time-like part of the current corresponds to the total electric charge: $Q = \int d^3 x j^0$. Therefore, invariance of the Lagrangian over U(1) transformations implies the conversation of the electric charge in QED. Note that the electric current is conserved irrespective of the specific value (or sign) of the electric charge.

This holds true in any physical theory: symmetries are important because in many cases they have *associated conservation laws.* In the specific case of the QED Lagrangian, we have the a number of symmetries with the associated conservation laws, including

- Invariance under spatial rotations → conservation of angular momentum.
- Invariance under time translations → conservation of energy.
- Invariance under spatial translations → conservation of momentum.

In addition to *global symmetries*, there is another type of symmetries that can be present in quantum field theories, namely *local symmetries*, where the size of the transformation is different for each space-time coordinate x.

Local symmetries in QED We have seen that QED is invariant under global U(1) rotations of the fermion fields, and that this has associated conservation of the electric four-current. However, there is a deeper symmetry, and it can be shown that the U(1) invariance also holds if the transformation is *local* as opposed to global. This symmetry under local rotations is know as the *gauge symmetry of the theory.*

Under a local U(1) rotation, fermions transform as

$$\psi(x) \to \psi'(x) = e^{i\phi(x)} \psi(x) \,, \tag{1.9}$$

which is similar to the global U(1) symmetry for fermions that we saw before, Eq. (1.5) with the very important difference that now the amount by which the field is rotated depends on the space-time point x. When the U(1) rotation is made local, also the vector potential A_μ needs to be modified by an additive derivate of the form

$$A^\mu \to A^{'\mu} = A^\mu + \partial^\mu \chi(x) \,, \tag{1.10}$$

where in principle $\chi(x)$ is a different space-time function. Note that the kinetic term of the photon Lagrangian, $F_{\mu\nu}F^{\mu\nu}$ is invariant under this transformations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu \left(A_\nu + \partial_\nu \chi(x)\right) - \partial_\nu \left(A_\mu + \partial_\mu \chi(x)\right) = F_{\mu\nu} \,, \tag{1.11}$$

because partial derivatives commute.

However, since now the U(1) transformation is local, it is not obvious a priori that the total Lagrangian will be invariant, since the fermion kinetic term involves a first derivative. To check that invariance is still maintained, one can see that the covariant derivative transforms as

$$D_\mu \psi = \left(\partial_\mu + ieA_\mu\right)\psi \to \left(\partial_\mu + ieA_\mu + ie\partial_\mu \chi(x)\right) e^{i\phi(x)}\psi = e^{i\phi(x)} D_\mu \psi \,, \tag{1.12}$$

where we have used the freedom of local transformations to fix $\chi = -\phi/e$. Hence the name of the *covariant* derivative: it transforms in the same way as the fermion field itself.

Indeed it is not a coincidence that the specific form of the interaction between the fermion and the photon fields,

$$\mathcal{L}_{\rm int} \sim ie\bar{\psi}\gamma^\mu A_\mu \psi \,, \tag{1.13}$$

is such that invariance under local U(1) transformations is satisfied. Requesting that a physical theory is invariant under a specific set of symmetries, global and local, basically determines completely the theory once the particle content is specified. We will now see how this is the same for other fundamental interactions, which are characterized by less simple internal symmetry groups.

2 The symmetries of the strong interactions

As compared to QED, the main difference in the QCD case is the existence of *new internal quantum number, color*. The strong interactions are invariant under a specific type of transformations of the color content of the fields of the theory which as we have seen before will have associated conserved quantities, in this case the *color charge*. While in QED the symmetry was U(1), which is an abelian group, the local symmetry group of QCD is $SU(3)$, the group of specially unitary transformations of degree $n = 3$. This group is defined by all $n \times n$ unitary matrices

$$U^T U = \mathbb{1} \tag{2.1}$$

which have determinant equal to 1, det U = 1. The fact that the gauge invariance of QCD is under a non-abelian group has important consequences, as we will see now.

In the *fundamental representation*, a suitable choice of generators for $SU(3)$ are the Gell-Mann matrices, which are hermitian and traceless,

$$t^A \equiv \frac{1}{2}\lambda^A \,, \quad A = 1, \ldots, 8 \,, \tag{2.2}$$

and which obey the commutation relations of the group's Lie algebra

$$\left[t^A, t^B\right] = if^{ABC}t^C \,, \quad A, B, C = 1, \ldots, 8 \,, \tag{2.3}$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ,$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .$$

Figure 2.1: The Gell-Mann matrices for the fundamental representation of $SU(3)$.

with f^{ABC} the structure constants of $SU(3)$, namely

$$f_{123} = 1 \,, \tag{2.4}$$

$$f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \,, \tag{2.5}$$

$$f_{458} = f_{678} = \frac{\sqrt{3}}{2} \,, \tag{2.6}$$

and the corresponding permutations. All other structure constants are zero. The Gell-Mann matrices for the fundamental representation of $SU(3)$ are summarized in Fig. 2.1. A color transformation is a complex rotation in a three-dimensional space, which mixes the color content of the fields in the theory.

In QED, the fact that the gauge group is U(1) implies that there is a single type of electric charge and a single type of gluon. In QCD instead, since the gauge group is SU(3), we have *three* different color charges and *eight* different types of gluons. Another important difference, as we will see below, is that in QED the photon is charge neutral but in QCD the gluons are *charged under color*.

Hadron structure and SU(3) Let us study the implications of SU(3) symmetry in the hadron structure. Recall that this SU(3) transformation is a rotation in the internal color space of the fields of the theory. Let us define the quark wave-function as follows:

$$\psi_{i\leftarrow \text{color}}^{(f)\leftarrow \text{flavor}} \,, \tag{2.7}$$

which has two indices: a flavor index (like in QED) and a color index (genuinely new feature of QCD). The color index can take any value from 1 to N_c, the number of colors in the theory. In QCD, quarks transform in the fundamental representation of $SU(3)$,

$$\psi_i^{(f)} \to \psi_i^{(f)'} = U_{ij}(x)\psi_j^{(f)} = \exp\left(i\theta^a(x)\, t^a_{ij}\right) \psi_j^{(f)} \,, \tag{2.8}$$

which leaves the flavor degrees of freedom unaffected.

Given that all observed strongly-interesting particles are color-singlet, the total color charge of physical states must be zero. To see this, consider the the wave function of a quark-antiquark bound state, known as *mesons*, constructed as

$$\sum_i^{N_c} \psi_i^{*(f)} \psi_i^{(f')} \,, \tag{2.9}$$

which can be shown to be invariant under a and under a $SU(3)$ transformation

$$\sum_i^{N_c} \left(\sum_j^{N_c} U_{ij}^* \psi_j^{*(f)} \right) \left(\sum_k^{N_c} U_{ik} \psi_k^{(f)} \right) = \sum_{kj} \left(\sum_i U_{ji}^T U_{ik} \right) \psi_j^{*(f)} \psi_k^{(f')} = \sum_k \psi_k^{*(f)} \psi_k^{(f')} \tag{2.10}$$

using the unitarity properties of the $SU(3)$ matrices. The same property must also hold for particles such as protons and neutrons, which are composed by three quarks (these hadrons are known as *baryons*). This can be achieved if the wave function of three-quark states is constructed as follows, using the antisymmetric tensor:

$$\sum_{ijk}^{N_c} \epsilon^{ijk} \psi_i^{(f)} \psi_j^{(f')} \psi_k^{(f'')} \,, \tag{2.11}$$

This can be shown to be color singlet using the relation

$$\sum_{ijk} \epsilon^{ijk} U_{ii'} U_{jj'} U_{kk'} = [\det \mathrm{U}] \; \epsilon^{i'j'k'} \; . \tag{2.12}$$

and the fact that the $SU(3)$ matrices are unitary. Therefore, baryons are also color singlet, in agreement with empirical observations. Note that the wave-function of the baryons Eq. (2.12) explains why hadrons like Δ^{++}, composed by three quarks of the same flavour (in this case three up quarks) are not forbidden by Pauli's exclusion principle: the exchange of any two of the identical quarks will change the sign of the baryon wave function since $\epsilon^{ijk} = -\epsilon^{jik}$, as requested for a fermion.

The QCD Lagrangian and its symmetries The Lagrangian of the theory of the strong interactions, *Quantum Chromodynamics* (QCD), has formally a similar structure as compared to the QED one, Eq. (1.1), though now the invariance is with respect to the non-abelian group $SU(3)$, and this leads to a number of striking differences between the two theories. The QCD Lagrangian is the following:

$$\mathcal{L}_{\rm QCD} = \sum_f \overline{\psi}_i^{(f)} \left(i\gamma_\mu D_{ij}^\mu - m_f \delta_{ij} \right) \psi_j^{(f)} - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} \,, \tag{2.13}$$

where a is now a color index that runs from 1 to $N_c - 1 = 8$, i, j are color indices from 1 to $N_c = 3$, since quarks transform in the fundamental representation of $SU(3)$, and we are assuming N_f massive fermions now. Unlike fermions, which transform in the *fundamental* representation of SU(3), gluons transform in the *adjoint* representation of SU(3). In QCD, the covariant derivative is given by a similar expression as in the case of QED

$$D_{ij}^\mu = \partial^\mu \delta_{ij} + i g_s t_{ij}^a A_a^\mu \,, \tag{2.14}$$

where now g_s is the *strong coupling constant*, and the gluon field replaces the photon field that we had in QED. Note that in QCD the covariant derivative has a *matrix structure*: it is a matrix acting on fermions in the fundamental representation. In particular, the action of the covariant derivative mixes the color degrees of freedom, and this has important physical consequences. In QCD, the field strength tensor is defined as

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu \,, \tag{2.15}$$

with f_{abc} the structure constants of $SU(3)$ introduced above. Note the additional term, bilinear in the gluon fields, which is absent in the QED field-strength tensor Eq. (1.3). The presence of this term is a direct consequence of the fact that SU(3) is a *non-abelian* symmetry group. It has a very important consequence: even in the absence of fermions, the QCD Lagrangian contains terms of the form

$$\mathcal{L}_{\rm gluons} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a = -g_s (\partial_\mu A^a_\nu) f_{abc} A^b_\mu A^c_\nu + \dots \,, \tag{2.16}$$

which correspond to *interaction terms for the gluon fields*: in QCD the gluons are charged and thus can interact among themselves, even in the absence of any fermions. This implies than a theory of free gluons is much more interesting than a theory of free photons, which is essentially trivial.

Now, the fermion sector of the Lagrangian must be independently gauge invariant, since in principle there are n_f fermion fields in the theory, each with a different mass. To achieve this, as in the case of QED, the covariant derivative must transform as the quark field itself

$$D^\mu_{ij} \psi_j \to \left(D^\mu_{ij} \psi_j \right)' = U_{ik}(x) D^\mu_{kj} \psi_j \,, \tag{2.17}$$

which is indeed the case since under a SU(3) transformation the gluon field changes as

$$t^a A^\mu_a \to t^a A^{'\mu}_a = U(x) t^a A^\mu_a U^{-1}(x) + \frac{1}{g_s} \left(\partial^\mu U(x) \right) U^{-1}(x) \,. \tag{2.18}$$

where note that the SU(3) gauge transformation acts on the product $t^a A^\mu_a$, rather than on the gluon field itself. This gauge transformation seems rather different than the corresponding QED one Eq. (1.10). However, it looks more involved because SU(3) transformations are 3×3 matrices than in general do not commute among them, as opposed to the complex numbers of U(1) that always commute. Indeed, if in Eq. (2.18) we assume that we have a U(1) symmetry group instead of SU(3), with $U = e^{i\phi(x)}$, we get

$$A^\mu \to A^{'\mu} = A^\mu_a + \frac{i}{g_s} \partial^\mu \phi(x) \,, \tag{2.19}$$

which is the QED gauge transformation.

An important difference between QED and QCD is that the field-strength tensor itself is not gauge invariant. To show this, a useful relation is provided by the commutator between two covariant derivatives

$$[D_\mu, D_\nu] = i g_s \, t^a \, F^a_{\mu\nu} \,, \tag{2.20}$$

which can be derived from the definition of the covariant derivative acting on a fermion field. Now, transformation law of the field strength tensor under $SU(3)$ transformations will be given by

$$t^a F^a_{\mu\nu} \to t^a F^{'a}_{\mu\nu} = U(x) \, t^a F^a_{\mu\nu} \, U^{-1}(x) \,, \tag{2.21}$$

to is it in general not invariant under gauge transformations. However, the product $F^a_{\mu\nu} F^{\mu\nu}_a$ *is* invariant under $SU(3)$ transformations, and hence the gluon sector of the Lagrangian is also gauge invariant. We can use the following property of $SU(3)$

$$\mathrm{Tr}\,\left[t^a t^b\right] = \frac{1}{2}\delta^{ab}\,, \tag{2.22}$$

and therefore we can have the following

$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a = -\frac{1}{2}F^a_{\mu\nu}\,F^{\mu\nu,b}\mathrm{Tr}\,\left[t^a t^b\right] = -\frac{1}{2}\mathrm{Tr}\,\left[F^a_{\mu\nu}\,t^a\,F^{\mu\nu,b}\,t^b\right]\,, \tag{2.23}$$

which, using the transformation law Eq. (2.21), can be shown to be invariant under $SU(3)$ transformations due to the cyclic properties of the trace. Therefore, the purely gluonic piece of the QCD Lagrangian is gauge invariant, even if the individual field-strength tensor is not.

As in the case of QED, the role of symmetries, and in particular of the SU(3) gauge symmetry, in fixing the structure of the strong interactions is extremely important. We now discuss another important symmetry of the strong interactions, which despite being only *approximate* has very important phenomenological consequences.

Chiral symmetry In addition to the *exact* symmetries of the QCD Lagrangian, there is also an important *approximate* symmetry, which is chiral symmetry. Strong interactions are found experimentally to behave very similar for particles, like protons and neutrons, that arise in the same *isospin multiplet.* Isospin is an *approximate global $SU(2)$ symmetry* which relates the up and the down content of hadrons, and that arises from the fact that $m_u \simeq m_d$. In the quark model, the isospin content of hadrons is defined as follows

$$I_3 = \frac{1}{2}\left[(n_u - n_{\bar{u}}) - (n_d - n_{\bar{d}})\right]\,, \tag{2.24}$$

so that we have that I_3 is 1/2 for protons and -1/2 for neutrons, members of the same isospin multiplet. Formally, an isospin transformation acts on the quark field as a unitary constant $SU(2)$ matrix

$$\psi_i^{(f)} \to \sum_{f'} U^{ff'}\psi_i^{(f')}\,, \tag{2.25}$$

which looks superficially similar to the color $SU(3)$ transformations, but they should not be mixed since this is a global transformation (not a local one as in the case of gauge symmetries) and it leaves the color indices unchanged.

We want to study under which conditions the fermion sector of the QCD Lagrangian is invariant under isospin transformations, so let us now separate the up and down fermions from all other fermions

$$\begin{aligned}\mathcal{L}_{\mathrm{QCD}} &= \overline{\psi}_i^{(u)}\left(i\gamma_\mu D^\mu_{ij} - m_u\delta_{ij}\right)\psi_j^{(u)} + \overline{\psi}_i^{(d)}\left(i\gamma_\mu D^\mu_{ij} - m_d\delta_{ij}\right)\psi_j^{(d)} \\ &+ \sum_{f,f\neq u,d}\overline{\psi}_i^{(f)}\left(i\gamma_\mu D^\mu_{ij} - m_f\delta_{ij}\right)\psi_j^{(f)} - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a\,.\end{aligned} \tag{2.26}$$

Restricting the fermion sector to only up and down quarks, the isospin transformed QCD Lagrangian reads

$$\sum_{f',f''}\sum_f\left(U^T_{f'f}U_{ff''}\right)\overline{\psi}_i^{(f')}\left(i\gamma_\mu D^\mu_{ij} - m_f\delta_{ij}\right)\psi_j^{(f'')}\,. \tag{2.27}$$

Therefore, from this equation we see that the QCD Lagrangian will be invariant if $m_u = m_d$, and in particular this also includes the $m_d = m_u = 0$ scenario. Experimentally, we know that the up and down quark masses are much smaller than the typical scale of the QCD interactions, $m_{u,d} \ll \Lambda_S$. In the limit of quark masses vanishing, we can separate *left-handed and right-handed fermion chiralities* and the Lagrangian will be separately invariant for the two components

$$\psi = \psi_R + \psi_L \,, \qquad \psi_L = \frac{1}{2}(1-\gamma_5)\psi \,, \qquad \psi_R = \frac{1}{2}(1+\gamma_5)\psi \,, \tag{2.28}$$

$$\sum_f \left(\overline{\psi}_R^{(f)} \left(i\gamma_\mu D^\mu \right) \psi_R^{(f)} + \overline{\psi}_L^{(f)} \left(i\gamma_\mu D^\mu \right) \psi_L^{(f)} \right) \,. \tag{2.29}$$

As a result of this invariance, *chirality is be a conserved quantum number* for massless fermions.

However, we know that QCD is not chirally invariant even in the $m_u = m_d = 0$ limit, and this arises because of the spontaneous breaking of chiral symmetry. Spontaneous breaking of a symmetry occurs when the symmetry group of the solutions of a theory is dynamically generated to be less than the symmetry of the original Lagrangian. In the case of QCD, it is known that the vacuum has a *non-zero expectation value* of the light quark operator

$$\langle 0|\bar{q}q|0\rangle = \left\langle 0|\bar{u}u + \bar{d}d|0\right\rangle \simeq (250 \text{ MeV})^3 \tag{2.30}$$

which breaks chiral symmetry, and is responsible for most of the hadron masses. From Goldstone theorem, we expect that the mass of the pion would vanish exactly if the up and down quarks were exactly massless, since there are the Goldstone bosons of an spontaneously broken symmetry. The fact that the pion mass is relatively small, $m_\pi \simeq 130$ MeV, but non-zero is a consequence of the fact that m_u and m_d are small but non-zero.

An important consequence of spontaneous chiral symmetry breaking is that it implies that the QCD Lagrangian is responsible for around 90% of the proton mass, and thus responsible for around 90% of the total mass of the visible universe. In other words, even if the Higgs boson did not exist, the total mass content of the universe would be not that different! On the other hand, the formation of stable atoms would not be possible in that case, so it would be a rather dull Universe.

3 Spontaneous symmetry breaking

The weak interactions seem to be very different than QED. Not only for the fact that they *maximally violate parity invariance*, also because of the fact that as opposed to QED, which is a long range interactions, the range of the weak interaction is much shorter, distances of around the size of the atomic nuclei. It would seem that the "photons" of the weak interaction had a very large mass, explaining their short range, but naively this seems to be incompatible with gauge invariance. As we will show now, it is still possible to have a quantum field theory of the weak interactions with massive gauge bosons, using a different way of breaking the gauge symmetry.

In the Standard Model of particle physics, the Higgs mechanism is responsible for giving mass to the W and Z bosons without *explicitly breaking* the gauge invariance of the theory, which would make them internally inconsistent. Here we present a simpler version of the Higgs mechanism, the *abelian case*, which illustrates how it would be possible to give *mass to the photon* while preserving gauge invariance of electromagnetism.

This is the so-called *Abelian Higgs mechanism.*

In classical and in quantum electromagnetism the photon is *massless*, hence it propagates at the speed of light. If we would like to give a mass to the photon, the simplest option would be to add a *explicit mass term* of the Lagrangian of the form

$$\mathcal{L} \in \frac{1}{2}m^2 A_\mu A^\mu \,, \tag{3.1}$$

which would result in a term of the form $\sim m^2 A_\mu$ added to Maxwell's equations, that is, a non-zero mass for the photon:

$$\partial_\mu F^{\mu\nu} - \frac{1}{2}m^2 A^\nu = 0 \,. \tag{3.2}$$

Why we can be confident that in electromagnetism the photon is really massless, $m = 0$, and it does not perhaps have a tiny mass consistent with all experimental constraints? Because a term of the form of Eq. (1.10) in the Lagrangian on the theory *does not satisfy the gauge invariance requirements.* This can be easily shown, since if I now apply a transformation of the photon field of the form Eq. (1.10) then the Lagrangian is not invariant. Indeed, under a gauge transformation

$$m^2 A_\mu A^\mu \to m^2 \left(A_\mu + \partial_\mu \eta\right)\left(A^\mu + \partial^\mu \eta\right) = m^2 A_\mu A^\mu + 2A_\mu \partial^\mu \eta + \left(\partial_\mu \eta\right)^2 \,, \tag{3.3}$$

which is different to the original expression, and thus not gauge invariant. Therefore, a photon mass term such as Eq. (3.1) is not acceptable since it breaks one of the *basic symmetries of electromagnetism*, gauge symmetry (which is moreover essential to make sense of the theory at the quantum level). So Eq. (3.1) does not seem a good option to give a mass to the photon.

While in electromagnetism the photon is indeed massless, the Standard Model of particle physics has other forces, in particular the strong and the weak force, and in the latter case experimental evidence implies that the *bosons of the weak interaction are massive* (hence the force is short-ranges). On the other hand, we also know that the weak force can be described by a gauge symmetry. So how it is possible to satisfy at the same time this two requirements, that is, having a theory with massive gauge bosons without breaking gauge invariance?

However, there is another way to add a mass to the photon and still satisfy the constraints from gauge symmetry. Let us consider a version of QED which contains only photons but no fermions, and let us introduce a *complex scalar field* ϕ which is charged under electromagnetism, in order words, it has a non-zero coupling to the photon. We also require that this scalar field exhibits *self-interactions*, that is, that it couples to itself. To maintain gauge invariance, the photon-scalar coupling must take place via covariant derivative, and then the Lagrangian of the theory reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(D_\mu \phi\right)^\dagger \left(D_\mu \phi\right) - V(\phi^\dagger \phi) \,, \tag{3.4}$$

where the covariant derivative is defined as

$$D_\mu = \partial_\mu - ieA_\mu \,, \tag{3.5}$$

which takes its name because it exhibits the same properties under a gauge transformation than the field it acts upon, here the scalar field ϕ. Indeed, it is possible to check that under a $U(1)$ gauge transformation,

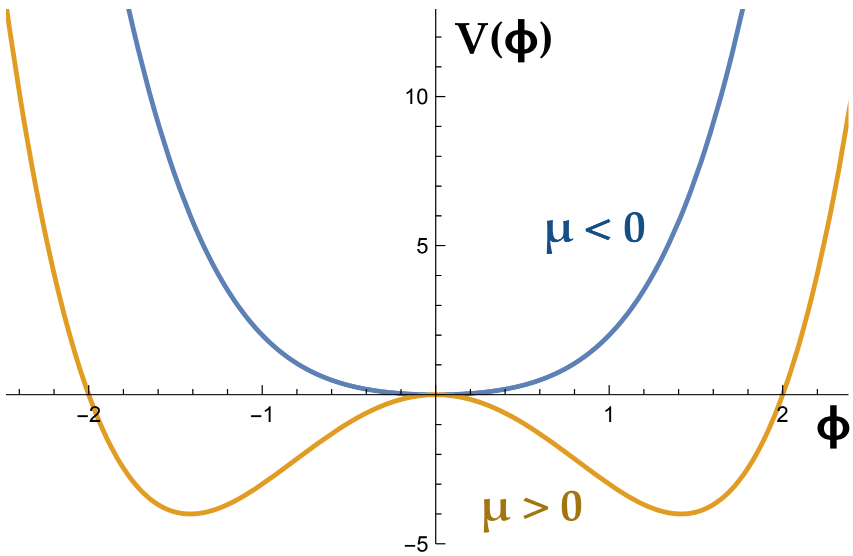

Figure 3.1: The Higgs potential Eq. (3.8) for two different possibilities for the sign of μ, either positive or negative.

while the photon field transforms as in Eq. (1.10), the scalar field transforms as

$$\phi(x) \to \exp(ie\eta(x))\phi(x)\,, \tag{3.6}$$

and therefore the covariant derivative itself transforms as the field it acts upon,

$$D_\mu\phi \to \exp(ie\eta(x))D_\mu\phi\,, \tag{3.7}$$

as was the case for fermion fields. You can check that the full Lagrangian Eq. (3.4) is invariant under gauge transformations of the $U(1)$ type, as required. The fact that the electric charge e appears in the gauge transformation of the scalar field ϕ Eq. (3.6) is a consequence of the fact that this field is charged under electromagnetism (else it could not couple to the photon).

A crucial aspect of the Higgs mechanism is that in the Lagrangian Eq. (3.4) we have introduced a *potential* for the scalar field ϕ, which we choose to have the following shape:

$$V(\phi^\dagger\phi) \equiv -\mu\,\phi^\dagger\phi + \lambda\,(\phi^\dagger\phi)^2\,. \tag{3.8}$$

One can check that the Lagrangian Eq. (3.4) is invariant under the gauge transformations of electromagnetism, taking into account that the photon obeys Eq. (1.10) and for the scalar field we have Eq. (3.6). Note also that any potential for the scalar field that depends on them only though the combination $\phi^\dagger\phi$ would likewise be a consistent choice respecting gauge invariance.

In Fig. 3.1 we represent the Higgs potential Eq. (3.8) for two different possibilities for the sign of μ, either positive or negative. We see that the shape of the potential is quite different depending on the sign, in particular for $\mu > 0$ the classical minimum of the potential corresponds to a non-zero value of the scalar field ϕ. As we now show, this particular feature of the classical potential is what makes possible to *spontaneously* generate a mass for the photon. On the other hand, for $\mu \leq 0$ the minimum value of the potential (which corresponds to the *vacuum state* of the theory) is the one where the scalar field ϕ vanishes.

Classically, we know that the vacuum of the theory (that is, the state with the smallest total energy) will be the one for which the potential $V(\phi)$ has a minimum. Then, imposing this condition

$$\frac{\partial V(\phi)}{\partial \phi} = 0\,, \tag{3.9}$$

we find two possibilities for the vacuum state of our theory:

- For $\mu < 0$, we find that the state of minimum energy of the theory is that where the field ϕ vanishes, $\langle \phi \rangle = 0$. In this case, the resulting theory is the standard classical electrodynamics with a massless photon coupled to a charged scalar particle.

- for $\mu > 0$ instead, the state with minimum energy is such that $\langle \phi \rangle \neq 0$, and the scalar field will acquire a vacuum expectation value (VEV) $\langle \phi \rangle = \sqrt{\mu^2/2\lambda} \equiv v/\sqrt{2}$. In this case gauge symmetry will be *spontaneously broken*, due to the fact that the vacuum (preferred configuration) is not invariant under a gauge transformation.

  For instance, note that either $\langle \phi \rangle = +\sqrt{\mu^2/2\lambda}$ or $\langle \phi \rangle = +\sqrt{\mu^2/2\lambda}$ are equally good solutions for Eq. (3.9) (as follows from the $U(1)$ rotational invariant of the theory), however only one of the two options can actually be implemented in nature, breaking thus the original gauge invariance.

Therefore, when the gauge symmetry of electromagnetism is spontaneously broken, we can parametrize the scalar field ϕ in terms of its VEV as follows

$$\phi = \frac{v+h}{\sqrt{2}} \exp\left(i\frac{Z}{v} \right)\,, \tag{3.10}$$

where h would be the *Higgs boson* of spontaneously broken electromagnetism. In other words, the Higgs boson h corresponds to the excitations of the Higgs field with respect to its vacuum expectation value, and Z is called a Goldstone boson (it is required to maintain the degrees of freedom). The field Z is also real scalar fields without VEV, which can be eliminated via a field redefinition and that eventually will reappear as the longitudinal component of the (now massive) photon.

Here, it is important not to mistake the *Higgs field*, responsible for EW symmetry breaking, with the *Higgs boson*, which represents the *excitations of the Higgs field around its vacuum expectation value*. From Eq. (3.10) we see that $\langle h \rangle = 0$, and thus the Higgs field h now admits a particle-like interpretation (since no particles means no energy).

Using the parametrization Eq. (3.10) in the Lagrangian Eq. (3.4), in particular in the covariant derivative that governs the interaction between the photon A_μ and the Higgs field ϕ, we find that now that the following term appears

$$\mathcal{L} \in +\frac{e^2 v^2}{2} A_\mu A^\mu\,, \tag{3.11}$$

and therefore the photon has now acquired a mass, $m_A = ev$, proportional to both its coupling e with the scalar field and to the VEV of the Higgs potential. This has been achieved respecting gauge invariance at a fundamental level, but choosing a vacuum that is instead not gauge invariant.

It is possible to check that, after spontaneously symmetry breaking, *the number of degrees of freedom* of our theory continues to be the same as before: now we have two real scalar fields, h and Z, to be compared with the starting complex field ϕ. Physically, we can say that the two degrees of freedom of the field ϕ after symmetry breaking result in the real single-particle field h and the longitudinal component of the massive photon A_μ.

In the full Standard Model, the Higgs mechanism works in a similar way but the symmetry group of electroweak theory is more complex than that of electromagnetism. In particular, after EWSB the residual symmetry of the theory is $SU(2)_L \otimes U(1)_Q$. A major difference with respect to the Abelian case is that the SM gauge group is *non-abelian*, which ultimately leads to the presence of *self-interactions between gauge bosons* (unlike in EM, where the photon does not couple to itself).

4 The electroweak interactions

As in the case of QED and QCD, electroweak interactions can be described by a renormalizable QFT with a Lagrangian which is invariant under a specific type of gauge transformations. In particular, electroweak interactions are invariant under the $SU(2)_L \otimes U(1)$ gauge group. In this group, the first subgroup corresponds to the *weak isospin* quantum number and the second subgroup to the *weak hypercharge* quantum numbers.

The weak interactions. The first theory of the weak interaction was proposed by Enrico Fermi, who suggested that β decay, the process by which a proton or a neutron in an atom transforms into a neutron or a proton plus an electron and a neutrino,

$$p \to n + e^+ + \nu_e \,, \tag{4.1}$$

is mediated by a contact (very short range) interaction, which can be modeled by a Lagrangian

$$\mathcal{L} = \frac{G_F}{\sqrt{2}} \psi_{\nu_e} \gamma^\mu (1-\gamma^5) \psi_{e^+} \psi_p \gamma_\mu (1-\gamma^5) \psi_n \,, \tag{4.2}$$

that is, a four fermion interaction.

Actually, in the original proposal from Fermi, the four-fermion interaction was the product of two vector couplings, similar to the case of the electromagnetic interaction. However, it was realized later that weak interactions violate maximally parity, and therefore an axial current component also needs to be introduced. The structure of the Fermi effective Lagrangian, supplemented by the $V - A$ theory considerations, implies that the electroweak interaction is a chiral theory which involves only the couplings between left-handed fermions. Indeed, the projector

$$P_L \equiv \frac{1-\gamma^5}{2} \,, \tag{4.3}$$

projects on the left-handed component on the fermion field upon which it acts. This was quite surprising, that one of the fundamental interactions of nature made a distinction between *left* and *right*, but the experiments of Chien-Siung Wu in the 60s clearly demonstrated that *parity* was not a fundamental symmetry of nature.

Since the weak interaction was of very close range, it was unclear how to describe it using a similar mathematical language as that of QED. However, it was realized that the weak interaction can be described as mediated massive gauge bosons, the W and Z, without being in conflict with the gauge symmetry requirements, provided that the gauge symmetry is *spontaneously broken*. This very important concept, which was already known from condensed matter systems, allowed to derive a quantum field theory of the weak interaction, as we now show.

To realize the fact that the weak interactions are of short range (and thus need to be mediated by a massive gauge boson) we need to add something else into our theory, and this is the Higgs field. We showed before the same physical processes at a simpler setup, with the *abelian Higgs mechanism*.

The electroweak Lagrangian Let us study how the same mechanism works in the full electroweak theory. Let us start by assuming that we have a purely bosonic theory, and write down a Lagrangian which exhibits a specific gauge invariance, in analogy with QED and QCD. In the case of the electroweak interactions, the gauge group that ultimately leads to the correct phenomenology is $SU(2)_L \otimes U(1)$, leading to four electroweak gauge bosons. This Lagrangian will now have three massless bosons W_μ^i, associated to the weak isospin subgroup, and one more boson, B_μ, charged under the weak hypercharge subgroup. As for the photon and gluon, these bosons are massless due to gauge invariance constraints. The resulting Lagrangian will be

$$\mathcal{L} = -\frac{1}{4} W^{i,\mu\nu} W^i_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \,, \tag{4.4}$$

with $i = 1, 2, 3$, and where the corresponding field strength tensors have the familiar expressions

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g_W \epsilon^{ijk} W^j_\mu W^k_\nu \,, \tag{4.5}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \,, \tag{4.6}$$

with g_W the coupling associated to the weak isospin interaction, and where we have used that the structure constants of $SU(2)_L$ are the totally antisymmetric tensor ϵ^{ijk}. It can be easily checked, using the same techniques as in the case of the QCD Lagrangian, that Eq. (4.4) is gauge invariant. The fact that $SU(2)_L$ is *non abelian* leads to the self-interactions of the W^i_μ bosons, as was the case for the gluons in QCD.

The coupling of the gauge fields to matter, as usual, will be through the covariant derivative, which for the case of the $SU(2)_L \otimes U(1)$ gauge group reads

$$D^\mu = \delta_{ij}\partial^\mu + i g_W \left(T \cdot W^\mu\right)_{ij} + iY \delta_{ij} g'_W B^\mu \,, \tag{4.7}$$

where g'_W is the weak hypercharge coupling and Y is the value of the so-called *weak hypercharge* of a given matter particle. The matrices T are a suitable representation of the $SU(2)$ algebra. The $SU(2)$ algebra is defined by

$$\left[T^i, T^j\right] = i\epsilon^{ijk} T^k \,, \tag{4.8}$$

and typically one defines the combination

$$T^\pm = T^1 \pm iT^2 \,. \tag{4.9}$$

The Lagrangian Eq. (4.4) describe a long-range non-abelian interaction. As usual, mass terms are forbidden due to gauge invariance.

The Higgs mechanism and EW symmetry breaking in the SM. In the Standard Model, the electroweak $SU(2)_L \otimes U(1)$ gauge symmetry group is spontaneously broken by the *Higgs mechanism*. This requires to add to our theory a new *complex doublet of scalar fields*,

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} , \tag{4.10}$$

which transform as a doublet of $SU(2)_L$ and have weak hypercharge value of $Y = 1/2$.[2] It is possible to couple the Higgs field ϕ to the massless electroweak Lagrangian while preserving gauge symmetry by means of the same covariant derivative that was introduced in Eq. (4.7), namely

$$\begin{aligned} \mathcal{L} \;\; = \;\; & \left(\partial^\mu \phi^\dagger + i g_W \left(T \cdot W^\mu\right) \phi^\dagger + i\frac{1}{2}\delta_{ij} g'_W B^\mu \phi^\dagger \right) \cdot \\ & \left(\partial_\mu \phi - i g_W \left(T \cdot W_\mu\right) \phi - i\frac{1}{2}\delta_{ij} g'_W B^\mu \phi \right) - \mathcal{V}\left(\phi^\dagger \phi\right) \end{aligned} \tag{4.11}$$

where for simplicity the isospin indices have been left implicit. It is easy to check explicitly that the above Lagrangian is invariant under $SU(2)_L \otimes U(1)$ transformations.

A remarkable feature of Eq. (4.11) is the presence of a potential term for the scalar field. Other than the fact that it can depend only on the product $\phi^\dagger\phi$, to enforce gauge symmetry, there is no other restriction on its form. In the Standard Model, the Higgs potential takes the following value

$$\mathcal{V}\left(\phi^\dagger \phi\right) = \lambda \left(\phi^\dagger \phi\right)^2 - \mu^2 \left(\phi^\dagger \phi\right) . \tag{4.12}$$

There are some interesting feature of this potential:

- the sign of the mass term is different as the one that would be used for example in the $\lambda\phi^4$ scalar theory. This causes the field to have a *vacuum expectation value* $\langle\phi\rangle \neq 0$ at the minimum.
- the λ accounts for the self-interactions of the scalar field ϕ with itself.

Therefore $\lambda, \mu^2 > 0$, this potential exhibits degenerate minima for values of the field ϕ which are different from zero.

In order to better represent the phenomenon of electroweak symmetry breaking, it useful to make explicit the four real components of the Higgs doublet as follows:

$$\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{4.13}$$

so that the inner product of Higgs fields reads

$$\phi^\dagger \phi = \sum_{i=1}^{4} \phi_i^2 , \tag{4.14}$$

which is of course invariant under four dimensional rotations. By minimizing the classical potential, we find

[2]As we have seen in the Abelian example, the scalar fields that break EWS need to be charged under the same gauge groups as those which are spontaneously broken.

that the space of minima is degenerate, and defined by the condition

$$|\phi| = \sqrt{\frac{\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}}\,, \tag{4.15}$$

and transformation of the Higgs field that satisfy this condition therefore have no energy costs associated.

Since all minima that satisfy the above equation are equivalent, it is possible to select as a particular choice,

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{4.16}$$

This is called a *vacuum expectation value*, or vev. While this choice is arbitrary, we should ensure that it is invariant under residual $U(1)$ transformations, since these will correspond to electromagnetic interactions, which are not spontaneously broken. A basic requirement to formulate electroweak theory is that the QED U(1) gauge symmetry is respected and not broken. As we will see, this fixes the combination between the original bosons W^i_μ and B_μ that lead to the photon field A_μ.

To realize that with this choice, transformations generated by the specific combination $T^3 + Y$ need to leave the vacuum expectation value invariant, that is,

$$\left(T^3 + Y\right)\langle\phi\rangle = 0\,, \tag{4.17}$$

This combination is the single unbroken generator that is associated with the *electric charge*

$$Q \equiv T^3 + Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.18}$$

which arises because

$$T^3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.19}$$

since the Higgs boson has $Y = 1/2$ weak hypercharge. The electric charge operator Eq. (4.18) applied to the vev Eq. (4.16) can be seen to satisfy Eq. (4.17). The specific combination of group generators that leaves the vacuum invariant has associated a conserved quantity, and in this case this is the electric charge, as it should since $U(1)_Q$ is unbroken in the SM.

The specific choice of vacuum expectation value (vev) that we have adopted breaks the $SU(2)_L \otimes U(1)$ symmetry, since we have identified a preferred direction in the internal space. This can be easily seen: under a generic $SU(2)$ transformation, the vev Eq. (4.16) would be transformed into a different vacuum expectation value. The Higgs mechanism is the paradigmatic example of *spontaneous symmetry breaking*: the underlying theory respects the symmetry but the ground state does not. As we now show, this allows to generate a mass for the massive weak bosons without breaking gauge symmetry.

At this point we need to reparametrize the scalar field in order to be able to represent the fluctuations with respect to the vacuum expectation value of the Higgs field. The physical manifestation of these fluctuations will correspond to the Higgs boson. We can introduce the following parametrization

$$\phi = U^{-1}(\xi) \begin{pmatrix} 0 \\ (h+v)/\sqrt{2} \end{pmatrix} \tag{4.20}$$

where we have introduced the following unitary matrix

$$U(\xi) \equiv \exp\left(-i\frac{T\cdot\xi}{v}\right) . \tag{4.21}$$

Now we have four degrees of freedom, three in ξ and one in h, corresponding to the four original degrees of freedom of the complex scalar doublet. Is clear that if now we set to zero these field fluctuations we reproduce the original vev value. Note also that Eq. (4.21) has the same form as a $SU(2)_L$ gauge transformation. But since our theory is gauge invariant, we are free to perform arbitrary gauge transformations without modifying the physics of our theory. So therefore we are allowed to perform the following $SU(2)_L$ gauge transformation of the Higgs field

$$\begin{aligned} \phi &\to U(\xi)\phi\,, \\ T\cdot W^\mu &\to UT\cdot W^\mu U^{-1} + \frac{i}{g_W}\left(\partial^\mu U\right)U^{-1}\,, \end{aligned} \tag{4.22}$$

the second equation is the corresponding gauge transformation as for the gluon gauge transformation under the color group $SU(3)$. Upon this gauge transformation, the ξ_i degrees of freedom no longer appear in the Higgs Lagrangian, since we cancel them from ϕ and the rest of the electroweak Lagrangian is invariant under gauge transformations. They will re-appear latter as the longitudinal models of the massive gauge bosons. This gauge is known as the *unitary gauge*.

Following the Higgs field redefinition, and in the unitary gauge, the Higgs Lagrangian now reads

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu h \partial^\mu h - \mathcal{V}\left((v+h)^2/2\right) \\ &+ \frac{(v+h)^2}{8}\chi^\dagger\left(2g_W\left(T\cdot W_\mu\right) + g'_W B_\mu\right)\left(2g_W\left(T\cdot W^\mu\right) + g'_W B^\mu\right)\chi\,, \end{aligned} \tag{4.23}$$

where we have defined $\chi \equiv (0,1)$, a unit vector in the direction of the vev. In addition to Eq. (4.23), we also have in the theory the kinematic massless gauge boson term, Eq. (4.4), which are left unaffected by either adding the Higgs field to the theory or by the gauge transformation.

Let us take a closer look at the consequences of spontaneous symmetry breaking. In the Higgs Lagrangian Eq. (4.23), we can consider only the terms that are quadratic in the vector boson fields. Using the expressions for the generators of $SU(2)$,

$$T_1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 1/2 & 1 \\ 0 & -1/2 \end{pmatrix}, \tag{4.24}$$

we obtain the following result

$$\mathcal{L} \subset \frac{v^2}{8}\left[\left(g_W W^3_\mu - g'_W B_\mu\right)\left(g_W W^{\mu,3} - g'_W B^\mu\right)\right] + 2g_W^2 W^-_\mu W^{\mu,+}\,, \tag{4.25}$$

where we have defined the following combination of the W gauge boson fields:

$$W_\mu^\pm \equiv \frac{W_\mu^1 \mp i W_\mu^2}{\sqrt{2}} \,. \tag{4.26}$$

Now, we want to express the quadratic terms of the Lagrangian so they they are diagonal in the fields, since p physical fields propagate independently, and therefore quadratic terms should be diagonal in the Lagrangian. Note also that the two fields B and W^3 are electrically neutral, since they vanish under the action of the electric charge operator Q. Therefore, it is possible to rotate the electrically neutral fields into new fields which are diagonal in the Lagrangian, and thus that propagate independently, using the following condition:

$$\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \,, \tag{4.27}$$

where the degree of mixing of the two fields is determined by the ratio of the coupling constants of the $SU(2)$ weak isospin and $U(1)$ weak hypercharge,

$$\sin^2\theta_W \equiv \frac{g_W'^2}{g_W'^2 + g_W^2} \,, \tag{4.28}$$

and where θ_W is known as the *Weinberg angle*. It is now easy to see how the quadratic terms of the Lagrangian transform under the rotation Eq. (4.27) and we get that the term quadratic in vector boson fields reads now

$$\mathcal{L} = \frac{v^2 g_W^2}{4} W_\mu^- W^{\mu,+} + \frac{\left(g_W'^2 + g_W^2\right)^2 v^2}{8} Z_\mu Z^\mu \,, \tag{4.29}$$

and there is no mass term for the photon A_μ as expected since we want to recover QED. Therefore, the Higgs mechanism allows to give masses to the physical W and Z fields

$$M_W = \frac{1}{2} v g_W \,, \quad M_Z = \frac{1}{2} v \sqrt{g_W^2 + (g_W')^2} \,. \tag{4.30}$$

The result is the net effect of electroweak symmetry breaking, which can be summarized as

- We start with a QFT of gauge fields invariant under $SU(2)_L \otimes U(1)$. Gauge fields are massless due to gauge invariance.
- We have added a complex scalar doublet ϕ, which couples to the W, B fields though the covariant derivative as requested by gauge invariance.
- This field ϕ has a potential which exhibits degenerate minima for $\langle\phi\rangle \neq 0$. Choosing a particular direction in the space of minima, and expanding the field around it, we obtain terms which are quadratic in the W, B fields.
- Diagonalizing the propagators, we find that of the two rotated fields, one, the photon A_μ is massless and the other, the Z boson, acquires a mass $M_Z = v\sqrt{g_W^2 + g_W'^2}/2$.
- The terms in the Lagrangian than contain the photon field A_μ are nothing but the standard QED Lagrangian, illustrating how electroweak theory unifies the weak and electromagnetic interactions.

Of the initial four components of the complex doublet scalar ϕ, one corresponds now to the Higgs boson h (a real scalar field), while the other three are substituted by the longitudinal modes of the now massive gauge bosons. If we take the Lagrangian, consider only the Higgs sector, and expand the potential, we find the following terms

$$\mathcal{L} = \frac{1}{2}\partial_\mu h \partial^\mu h - \mu^2 h^2 - \lambda v h^3 - \frac{1}{4}\lambda h^4 \,, \tag{4.31}$$

and therefore the mass of the Higgs boson is $M_h = \sqrt{2}\mu = \sqrt{2\lambda}v$, as can be read from the kinetic term. Since the mass of the W and Z bosons is known, v has been known for a long time, and the only other free parameter related to the Higgs sector is μ, its mass. Once the Higgs mass is measured, there are no other free parameters in the electroweak Lagrangian.

Mass generation for fermions and the Yukawa coupling The previous discussion was restricted to the purely gauge electroweak sector of the Standard Model Lagrangian, as we have shown how the weak vector bosons can acquire a mass. We now discuss how the same Higgs mechanism provides the mass for the fermionic fields, and what this implies for the Yukawa coupling between the Higgs boson and all massive fermions. The key requirement is that the complete Standard Model Lagrangian must be invariant under $SU(2)_L \otimes U(1)$. As we will show now, this implies that any explicit mass term in the Lagrangian is forbidden, and thus one needs to generate a mass through spontaneous symmetry breaking as well. In terms of its left-handed and right-handed components, we can write the Dirac Lagrangian as

$$\psi = \psi_R + \psi_L \,, \qquad \psi_L = \frac{1}{2}(1-\gamma_5)\psi \,, \qquad \psi_R = \frac{1}{2}(1+\gamma_5)\psi \,, \tag{4.32}$$

$$\sum_f \left(\overline{\psi}_R^{(f)} \left(i\gamma_\mu D^\mu\right) \psi_R^{(f)} + \overline{\psi}_L^{(f)} \left(i\gamma_\mu D^\mu\right) \psi_L^{(f)} + m_f \overline{\psi}_R^{(f)} \psi_L^{(f)} + m_f \overline{\psi}_L^{(f)} \psi_R^{(f)} \right) . \tag{4.33}$$

Since $SU(2)_L$ gauge transformations act only on left-handed fermion fields, the mass term in the Lagrangian is forbidden by gauge symmetry.

In order to satisfy gauge invariance, bosonic fields must couple to the matter (fermion) fields using the covariant derivative. Before electroweak symmetry breaking, the electroweak Lagrangian with fermions contains

$$\mathcal{L} \subset \bar{\psi}_R \left(\gamma^\mu \partial_\mu + i g_W' Y_R \gamma^\mu B_\mu\right) \psi_R + \bar{\psi}_L \left(\gamma^\mu \partial_\mu + i g_W \gamma^\mu T \cdot W_\mu + i g_W' Y_L \gamma^\mu B_\mu\right) \psi_L \,, \tag{4.34}$$

where we have used the fact that $SU(2)_L$ rotations only affect left handed fermions, and that weak hypercharge is different for left handed and right handed fermions. Note that in addition one has the coupling with the photon field A_μ, which is the same as in QED. The values of the weak hypercharge of right and left-handed fermions are fixed by their electric charge, $Q = T^3 + Y$, and for the SM matter particles their values are summarized in Table 1. Before electroweak symmetry breaking, all the SM fermions are massless, since a explicit mass term would violate $SU(2)_L \otimes U(1)$ symmetry. This allows to treat separately left-handed and right-handed fermions in terms of their weak charges.

As we see in Table 1, left-handed fermions correspond to a $SU(2)_L$ doublet, while right-handed quarks are singlets, and there are no right-handed neutrinos (at least in the SM). Let us now show how the Higgs mechanism, which gave mass to W, Z bosons, can also give mass to the fermions. Let us now add to the

Fermion	T_L^3	Y_L	T_R^3	Y_R	Q_f
up quark	1/2	1/6	0	2/3	2/3
down quark	1/2	1/6	0	2/3	2/3
electron neutrino	1/2	-1/2	-	-	0
electron	-1/2	-1/2	-	-1	-1

Table 1: The weak isospin and weak hypercharge for the first family of right- and left-handed fermions in the Standard Model. The same assignments hold for the second and third families.

weak Lagrangian the so-called *Yukawa interaction* terms, of the form

$$\mathcal{L} \subset \sum_f g_f \bar{\psi}_f \psi_f \phi \,, \tag{4.35}$$

where the sum runs over all the massive fermions in the SM. This term is explicitly gauge invariant. Why Eq. (4.35) is gauge invariant, while a explicit mass term for fermions in the weak Lagrangian is not allowed instead? The answer lies on the chiral nature of the $SU(2)_L$ gauge transformations.

Now, as we have shown, after electroweak symmetry breaking an in the unitary gauge, the Higgs field ϕ can be expressed in terms of the Higgs boson h and the vacuum expectation value v:

$$\phi = \frac{1}{\sqrt{2}} \left(\begin{array}{c} 0 \\ h+v \end{array} \right) \,, \tag{4.36}$$

and if we substitute in Eq. (4.35) we find that fermions acquire a mass of the form

$$m_f = g_f v/\sqrt{2} \,. \tag{4.37}$$

In addition, the Yukawa interaction term Eq. (4.35) also dictates the strength of the interaction between the Higgs boson and the fermions, since now the Lagrangian contains

$$\mathcal{L} \subset \sum_f \left(\frac{\sqrt{2} m_f}{v} \right) \bar{\psi}_f \psi_f h \,, \tag{4.38}$$

so therefore the interaction between the Higgs boson and the SM fermions is proportional to the fermion mass.