Nicolas BORGHINI

Three lectures:

- Overview on the dynamics of relativistic perfect fluids
- Application of perfect relativistic fluid dynamics to the description of high-energy nucleus-nucleus collisions
- Beyond perfect relativistic fluid dynamics

Three lectures:

Overview on the dynamics of relativistic perfect fluids

phenomenological approach

- reminder(?) on non-relativistic fluid dynamics ("hydrodynamics")
- fundamental equations of perfect relativistic fluid dynamics
- an example of solution
- Application of perfect relativistic fluid dynamics to the description of high-energy nucleus-nucleus collisions
- Beyond perfect relativistic fluid dynamics

Three lectures:

- Overview on the dynamics of relativistic perfect fluids
- Application of perfect relativistic fluid dynamics to the description of high-energy nucleus-nucleus collisions

phenomenology & experiment

- Iluid dynamical modeling of the fireball created in the collisions
- yields a surprisingly good rendering of some measurements
- In at the cost of introducing elements that are not within perfect hydrodynamics
- Beyond perfect relativistic fluid dynamics

Three lectures:

- Overview on the dynamics of relativistic perfect fluids
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theoretical approaches

many ideas: dissipative fluid dynamics / including hydrodynamical fluctuations / or "non-hydro modes"... r out-of-equilibrium physics

... coming from the underlying "microscopic" physics (transport) or invoking strong-coupling scenarios

Questions are welcome at any time!

Lecture I:

Overview on the dynamics of relativistic perfect fluids

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- reminder(?) on non-relativistic fluid dynamics ("hydrodynamics")
- fundamental equations of perfect relativistic fluid dynamics
- an example of solution

(possibly in lecture II)

Application of perfect relativistic fluid dynamics to the description of high-energy nucleus-nucleus collisions

Beyond perfect relativistic fluid dynamics

Elements of fluid dynamics...

General references:

• A few textbooks aimed at physicists:

Landau & Lifshitz, vol.6: "Fluid Mechanics"

(L&L-style..., a short chapter on relativistic hydro)

• Guyon, Hulin, Petit, Mitescu: "Physical Hydrodynamics"

(more phenomenological, only non-relativistic fluid dynamics)

• Rezzola & Zanotti, "Relativistic Hydrodynamics"

(by astrophysicists with an interest in numerical fluid dynamics)

 General Relativity / Cosmology textbooks often contain a chapter on relativistic fluid dynamics

> Weinberg, "Gravitation and Cosmology" Misner, Thorne, Wheeler, "Gravitation"

Elements of fluid dynamics...

- Review articles
 - Andersson & Comer, "Relativistic fluid dynamics: Physics for many scales"
 <u>arXiv:gr-qc/0605010</u>
 - Romatschke, "New developments in relativistic viscous hydrodynamics" <u>arXiv:0902.3663</u>
 - Ollitrault, "Relativistic hydrodynamics for heavy ion collisions" <u>arXiv:0708.2443</u>
 - ... and many others
- Online lecture notes
 - a few chapters in Blandford & Thorne, "Applications of Classical Physics" (will soon become a book; use your favorite search engine...)
 - N.B. @ <u>http://www.physik.uni-bielefeld.de/~borghini/Teaching/Hydrodynamics</u> (sorry for th<u>e</u> lack of modesty)

Topical Lectures, NIKHEF, June 15-17, 2015

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Lecture I:

Overview on the dynamics of relativistic perfect fluids

phenomenological approach

- reminder(?) on non-relativistic fluid dynamics (``hydrodynamics")
- fundamental equations of perfect relativistic fluid dynamics
- an example of solution
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Classical definition:

A "fluid" is a continuous medium that keeps on deforming as long as it is subject to tangential forces ("shear stresses" \neq normal stresses): gas, liquid, plasma...

≠ deformable solid (elastic / plastic), which will reach an equilibrium

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Continuous medium?

• for the mathematician, this makes life easier

🖛 "differentiable medium" is even better

for the physicist, this is a model, thus open to discussion if need be
 atoms exist, don't they?
 Well, so do fields, so who knows...

Classical definition:

A "fluid" is a continuous medium that keeps on deforming as long as it is subject to tangential forces ("shear stresses" \neq normal stresses):

Idea: decompose the contact force — exerted by the neighboring fluid element(s) or by a wall / obstacle — per unit surface into a normal and a tangential part:



Classical definition:

A "fluid" is a continuous medium that keeps on deforming as long as it is subject to tangential forces ("shear stresses" \neq normal stresses):

In a fluid at rest, the contact forces are normal! (hydrostatic) pressure forces

One introduce a pressure field $\mathcal{P}(t, \vec{r})$ at each point of the fluid

will quickly be continuous, differentiable... at least by parts

The "degrees of freedom" = dynamical variables

- A "fluid" is a continuous medium characterized by its
- mass density $ho(t, ec{r})$
- pressure $\mathcal{P}(t, ec{r})$
- local flow velocity $\vec{\mathbf{v}}(t,\vec{r})$
- energy density $e(t, \vec{r})$
- ... further dynamical fields?
- at each time and position.
 - rall of them are assumed to behave smoothly, at least by parts

These "mesoscopic" quantities deserve a better definition, in terms of more microscopic ones... Will come later!



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Model of a continuous medium

... the hidden physical assumptions

System (of particles / fields) can be divided in thought in cells that fulfill two contradictory conditions:

they must be large enough

they must be small enough

Model of a continuous medium

... the hidden physical assumptions

System (of particles / fields) can be divided in thought in cells that fulfill two contradictory conditions:

 they must be large enough that quantities defined as averages over their content (e.g.: mass density, average particle velocity...) have small fluctuations

IF cell size \gg mean free path ℓ_{mfp}

 they must be small enough to remain statistically homogeneous, i.e. the system "mesoscopic" properties do not vary too much over the cell

IF cell size \ll scale L of macroscopic gradients

Model of a continuous medium

an important dimensionless number

Two length scales:

- mean free path* $\ell_{\rm mfp}$
- size L over which macroscopic fields vary ($\sim 1/|\partial_{ec r}|$)

Image: The description as a continuous medium is meaningful iff

Knudsen number
$${
m Kn}\equiv {\ell_{
m mfp}\over L}\ll 1$$

necessary consistency check (easily written... non-trivial!)

*strictly speaking, is well defined only for "dilute" systems

The "degrees of freedom" = dynamical variables

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at each time and position.

We now need equations for these various fields!

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The "degrees of freedom" = dynamical variables

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- energy density $e(t, \vec{r})$
- ... further dynamical fields?
- at each time and position.

IF We now need equations for these various fields!

Complicated-looking(?) equations, yet with a clear physical meaning!

The dynamical equations of motion

Invoke the most basic laws of physics: conservation equations!

- mass / particle number conservation
- momentum conservation (or more generally, Newton's 2nd law)
- energy conservation

each of which are expressed locally.

The dynamical equations of motion

Invoke the most basic laws of physics: conservation equations!

- mass / particle number conservation (equivalent since ho=nm)
- momentum conservation (or more generally, Newton's 2nd law)
- energy conservation

each of which are expressed locally.

"Continuity equation":

$$\frac{\partial \rho(t,\vec{r})}{\partial t} + \vec{\nabla} \cdot \left[\rho(t,\vec{r}) \,\vec{\mathsf{v}}(t,\vec{r}) \right] = 0$$

$$\frac{\partial \mathbf{n}(t,\vec{r})}{\partial t} + \vec{\nabla} \cdot \left[\mathbf{n}(t,\vec{r})\,\vec{\mathbf{v}}(t,\vec{r})\right] = 0$$

(Remember charge conservation in electrodynamics!)

The dynamical equations of motion

Invoke the most basic laws of physics: conservation equations!

- mass / particle number conservation (equivalent since $ho= \pi m$)
- momentum conservation (or more generally, Newton's 2nd law)
- energy conservation

each of which are expressed locally.

"Continuity equation":

 $\partial_t \rho(t, \vec{r}) + \partial_i \left[\rho(t, \vec{r}) \,\mathbf{v}^i(t, \vec{r}) \right] = 0$ $\partial_t n(t, \vec{r}) + \partial_i \left[n(t, \vec{r}) \,\mathbf{v}^i(t, \vec{r}) \right] = 0$

with a sum over repeated indices (here i=1,2,3)

Rem.: in non-Cartesian coordinates, replace partial by covariant derivatives

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Non-relativistic fluid dynamics The dynamical equations of motion

Invoke the most basic laws of physics: conservation equations!

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- momentum conservation (or more generally, Newton's 2nd law)
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 each of which are expressed locally.

Euler equation:

$$\rho(t,\vec{r}) \Big[\partial_t \vec{\mathsf{v}}(t,\vec{r}) + \big[\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla} \big] \vec{\mathsf{v}}(t,\vec{r}) \Big] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \vec{f}_{\nu}(t,\vec{r})$$

The dynamical equations of motion

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local fluid acceleration

pressure forces volume forces (gravity...)

The dynamical equations of motion

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- energy conservation
 each of which are expressed locally.

Euler equation:

component-wise:

$$\rho(t,\vec{r}) \left[\partial_t \mathsf{v}^i(t,\vec{r}) + \mathsf{v}^j(t,\vec{r}) \partial_j \mathsf{v}^i(t,\vec{r}) \right] = -\partial^i \mathcal{P}(t,\vec{r}) + (f_{\nu})^i(t,\vec{r})$$

Non-relativistic fluid dynamics The dynamical equations of motion

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or Navier-Stokes equation:

$$\rho(t,\vec{r}) \Big[\partial_t \vec{\mathsf{v}}(t,\vec{r}) + \big[\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla} \big] \vec{\mathsf{v}}(t,\vec{r}) \Big] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \vec{f}_{\text{visc.}}(t,\vec{r}) + \vec{f}_{\nu}(t,\vec{r})$$

 $\vec{f}_{\rm visc.}(t,\vec{r}) \equiv \eta \vec{\nabla}^2 \vec{\mathsf{v}}(t,\vec{r}) + \left(\zeta + \frac{\eta}{3}\right) \vec{\nabla} \left[\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r})\right]$

with

Non-relativistic fluid dynamics The dynamical equations of motion

Invoke the most basic laws of physics: conservation equations!

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 each of which are expressed locally.

Euler equation: $\rho(t,\vec{r}) \left[\partial_t \vec{v}(t,\vec{r}) + \left[\vec{v}(t,\vec{r}) \cdot \vec{\nabla} \right] \vec{v}(t,\vec{r}) \right] = -\vec{v} \mathcal{P}(t,\vec{r}) + \vec{f}_{\nu}(t,\vec{r})$ or Navier–Stokes equation: $\rho(t,\vec{r}) \left[\partial_t \vec{v}(t,\vec{r}) + \left[\vec{v}(t,\vec{r}) \cdot \vec{\nabla} \vec{v}(t,\vec{r}) \right] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \vec{f}_{\text{visc.}}(t,\vec{r}) + \vec{f}_{\nu}(t,\vec{r})$ with $\vec{f}_{\text{visc.}}(t,\vec{r}) \equiv \eta \vec{\nabla}^2 \vec{v}(t,\vec{r}) + \left(\zeta + \frac{\eta}{3} \right) \vec{\nabla} \left[\vec{\nabla} \cdot \vec{v}(t,\vec{r}) \right]$

The dynamical equations of motion

- Momentum conservation (or more generally, Newton's 2nd law) is expressed locally by...
 - the Euler equation

 $\rho(t,\vec{r}) \Big[\partial_t \vec{\mathsf{v}}(t,\vec{r}) + \big[\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla} \big] \vec{\mathsf{v}}(t,\vec{r}) \Big] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \vec{f}_{\nu}(t,\vec{r})$

if the fluid is "perfect" (or "ideal")

• the Navier-Stokes equation

 $\rho(t,\vec{r}) \Big[\partial_t \vec{\mathsf{v}}(t,\vec{r}) + \big[\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla} \big] \vec{\mathsf{v}}(t,\vec{r}) \Big] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \vec{f}_{\text{visc.}}(t,\vec{r}) + \vec{f}_{\vec{\nu}}(t,\vec{r})$ IF if the fluid is ``Newtonian''

• the Burnett / super-Burnett equation... ... if the fluid is...
not kidding!

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may be seen as properties of specific flows (more later)

Perfect / ideal fluids:

... are such that there are no dissipative effects in them:

reither shear stresses (friction) nor heat conduction

I™ "dry water"

(Feynman, Lectures on Physics vol.II, chap.40)

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Perfect / ideal fluids:

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\Leftrightarrow

At each point in the fluid, the properties as seen by an observer at rest w.r.t. the fluid—i.e. comoving with it—are (locally) isotropic.

Conversely, "anisotropic fluids" are not perfect. (So what?)

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\Leftrightarrow

At each point in the fluid, the properties as seen by an observer at rest w.r.t. the fluid—i.e. comoving with it—are (locally) isotropic.

In particular, the momentum flux-density tensor $T^{ij}(t, \vec{r})$ is isotropic:

 \blacksquare amount of *i*-th component of momentum transported in direction *j*

Conversely, "anisotropic fluids" are not perfect. (So what?)

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may be seen as properties of specific flows (more later)

Perfect / ideal fluids:

... have a locally isotropic momentum flux-density tensor:

(amount of i-th component of momentum transported in direction j)

$$T^{ij}(t,\vec{r}) = \underbrace{P(t,\vec{r})\delta^{ij}}_{\text{thermal transport}} + \underbrace{\rho(t,\vec{r})\mathbf{v}^{i}(t,\vec{r})\mathbf{v}^{j}(t,\vec{r})}_{\text{convective transport}}$$

Rems.:

- $T^{ij}(t, \vec{r})$ is symmetric.
- In non–Cartesian coordinates, replace δ^{ij} by the inverse metric tensor $g^{ij}(t, \vec{r})$.

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may be seen as properties of specific flows (more later)

Perfect / ideal fluids:

... have a locally isotropic momentum flux-density tensor:

(amount of i-th component of momentum transported in direction j)

$$T^{ij}(t,\vec{r}) = \underbrace{\mathcal{P}(t,\vec{r})\delta^{ij}}_{\text{thermal}} + \underbrace{\rho(t,\vec{r})\mathbf{v}^{i}(t,\vec{r})\mathbf{v}^{j}(t,\vec{r})}_{\text{convective transport}}$$

IF only normal surface forces (no tangential stresses): pressure

Rems.:

- $T^{ij}(t,\vec{r})$ is symmetric.
- In non–Cartesian coordinates, replace δ^{ij} by the inverse metric tensor $g^{ij}(t, \vec{r})$.
may be seen as properties of specific flows (more later)

• Dissipative fluids:

... have a possibly anisotropic momentum flux-density tensor:

w because they admit dissipative currents



What is the form of the viscous stress tensor π^{ij} ?

models (many!)

may be seen as properties of specific flows (more later)

Newtonian fluids:

... have an anisotropic momentum flux-density tensor T^{ij} including a viscous stress tensor π^{ij} depending linearly on the 1st-order spatial derivatives of the flow velocity:

$$T^{ij}(t,\vec{r}) = \mathcal{P}(t,\vec{r})\delta^{ij} + \rho(t,\vec{r})\mathbf{v}^{i}(t,\vec{r})\mathbf{v}^{j}(t,\vec{r}) + \pi^{ij}(t,\vec{r})$$

with $\pi^{ij}(t,\vec{r}) \equiv -\eta(t,\vec{r}) \left[\partial^j \mathbf{v}^i(t,\vec{r}) + \partial^i \mathbf{v}^j(t,\vec{r}) - \frac{2}{3} \delta^{ij} \vec{\nabla} \cdot \vec{\mathbf{v}}(t,\vec{r}) \right] - \zeta(t,\vec{r}) \delta^{ij} \vec{\nabla} \cdot \vec{\mathbf{v}}(t,\vec{r})$

where shear viscosity η & bulk viscosity ζ are independent of \vec{v} .

transport coefficients

may be seen as properties of specific flows (more later)

Newtonian fluids:

... have an anisotropic momentum flux-density tensor T^{ij} including a viscous stress tensor π^{ij} depending linearly on the 1st-order spatial derivatives of the flow velocity:

$$T^{ij}(t,\vec{r}) = \mathcal{P}(t,\vec{r})\delta^{ij} + \rho(t,\vec{r})\mathbf{v}^{i}(t,\vec{r})\mathbf{v}^{j}(t,\vec{r}) + \pi^{ij}(t,\vec{r})$$

with
$$\pi^{ij}(t,\vec{r}) \equiv -\eta(t,\vec{r}) \left[\partial^{j} \mathbf{v}^{i}(t,\vec{r}) + \partial^{i} \mathbf{v}^{j}(t,\vec{r}) - \frac{2}{3} \delta^{ij} \vec{\nabla} \cdot \vec{\mathbf{v}}(t,\vec{r}) \right] - \zeta(t,\vec{r}) \underbrace{\delta^{ij} \vec{\nabla} \cdot \vec{\mathbf{v}}(t,\vec{r})}_{\propto \delta^{ij} !}$$
 symmetric, traceless

where shear viscosity η & bulk viscosity ζ are independent of \vec{v} . transport coefficients

may be seen as properties of specific flows (more later)

Newtonian fluids:

... have a shear viscosity η and a bulk viscosity ζ .

modify the surface forces:

Normal force: with an "effective pressure"

$$\begin{aligned} \mathcal{P}(t,\,\vec{r}) - \left[\zeta(t,\vec{r}) - \frac{2}{3}\eta(t,\,\vec{r})\right] \underbrace{\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\,\vec{r})}_{\text{local expansion rate}} \\ \text{local expansion rate}_{\text{of the fluid}} \end{aligned}$$

Tangential force: friction

"wet water" (Feynman, Lectures on Physics vol.II, chap.41)

... on a surface in the xy-plane, with the velocity along x, and a velocity gradient along the z direction:

 $F_x \propto \eta \, \partial_z \mathbf{v}_x$

may be seen as properties of specific flows (more later)

Newtonian fluids:

... have an anisotropic momentum flux-density tensor T^{ij} including a viscous stress tensor π^{ij} depending linearly on the 1st-order spatial derivatives of the flow velocity \vec{v} :

mer define "first-order dissipative hydrodynamics"

may be seen as properties of specific flows (more later)

Newtonian fluids:

... have an anisotropic momentum flux-density tensor T^{ij} including a viscous stress tensor π^{ij} depending linearly on the 1st-order spatial derivatives of the flow velocity \vec{v} :

refine "first-order dissipative hydrodynamics"

Dissipative fluids:

... with a momentum flux-density tensor with a viscous stress tensor depending linearly on the 2nd-order spatial derivatives of \vec{v} IF define "second-order dissipative hydrodynamics"

may be seen as properties of specific flows (more later)

Newtonian fluids:

... have an anisotropic momentum flux-density tensor T^{ij} including a viscous stress tensor π^{ij} depending linearly on the 1st-order spatial derivatives of the flow velocity \vec{v} :

mer define "first-order dissipative hydrodynamics"

• Dissipative fluids:

... with a momentum flux-density tensor with a viscous stress tensor depending linearly on the 2nd-order spatial derivatives of \vec{v} Im define "second-order dissipative hydrodynamics"

and so on... You can also devise other models (see e.g. Lecture III)



The dynamical equations of motion

- Momentum conservation (or more generally, Newton's 2nd law) is expressed locally by...
 - the Euler equation $\rho(t,\vec{r}) \left[\partial_t \vec{v}(t,\vec{r}) + \left[\vec{v}(t,\vec{r}) \cdot \vec{\nabla} \right] \vec{v}(t,\vec{r}) \right] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \vec{f}_{\nu}(t,\vec{r})$

for perfect/ideal fluids

- the Navier-Stokes equation $\rho(t,\vec{r}) \left[\partial_t \vec{v}(t,\vec{r}) + \left[\vec{v}(t,\vec{r}) \cdot \vec{\nabla} \right] \vec{v}(t,\vec{r}) \right] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \vec{f}_{\text{visc.}}(t,\vec{r}) + \vec{f}_{\psi}(t,\vec{r}) \right]$ for Newtonian fluids
 - the Burnett / super-Burnett equation... Image 2nd / 3rd order hydro

The dynamical equations of motion

- Momentum conservation (or more generally, Newton's 2nd law):
 - the Euler equation (perfect / ideal hydro) $\rho(t,\vec{r}) \left[\partial_t \vec{v}(t,\vec{r}) + \left[\vec{v}(t,\vec{r}) \cdot \vec{\nabla} \right] \vec{v}(t,\vec{r}) \right] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \vec{f}_{\nu}(t,\vec{r})$
 - the Navier–Stokes equation (1st order dissipative hydro) $\rho(t,\vec{r}) \left[\partial_t \vec{\mathsf{v}}(t,\vec{r}) + \left[\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla} \right] \vec{\mathsf{v}}(t,\vec{r}) \right] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \vec{f}_{\text{visc.}}(t,\vec{r}) + \vec{f}_{\psi}(t,\vec{r})$ $\vec{f}_{\text{visc.}}(t,\vec{r}) \equiv \eta \vec{\nabla}^2 \vec{\mathsf{v}}(t,\vec{r}) + \left(\zeta + \frac{\eta}{3} \right) \vec{\nabla} \left[\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) \right]$
 - the Burnett / super-Burnett equations... (2nd / 3rd order hydro) are reformulations of $\partial_t \left[\rho(t, \vec{r}) \mathbf{v}^i(t, \vec{r}) \right] + \partial_j T^{ij}(t, \vec{r}) = (f_{\nu})^i(t, \vec{r})$

with constant transport coefficients in the dissipative case Topical Lectures, NIKHEF, June 15–17, 2015 N.Borghini — I–24/58

Universität Bielefeld

The dynamical equations of motion

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 - the Euler equation (perfect / ideal hydro) $\rho(t,\vec{r}) \left[\partial_t \vec{v}(t,\vec{r}) + \left[\vec{v}(t,\vec{r}) \cdot \vec{\nabla} \right] \vec{v}(t,\vec{r}) \right] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \vec{f}_{\nu}(t,\vec{r})$
 - the Navier–Stokes equation (1st order dissipative hydro) $\rho(t,\vec{r}) \Big[\partial_t \vec{\mathsf{v}}(t,\vec{r}) + \big[\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla} \big] \vec{\mathsf{v}}(t,\vec{r}) \Big] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \vec{f}_{\text{visc.}}(t,\vec{r}) + \vec{f}_{\vec{\nu}}(t,\vec{r})$ $\vec{f}_{\text{visc.}}(t,\vec{r}) \equiv \eta \vec{\nabla}^2 \vec{\mathsf{v}}(t,\vec{r}) + \left(\zeta + \frac{\eta}{3}\right) \vec{\nabla} \big[\vec{\nabla} \cdot \vec{\mathsf{v}}(t,\vec{r}) \big]$
- the Burnett / super-Burnett equations... (2nd / 3rd order hydro) are reformulations of $\partial_t \left[\rho(t, \vec{r}) v^i(t, \vec{r}) \right] + \partial_j T^{ij}(t, \vec{r}) = (f_v)^i(t, \vec{r})$ momentum density momentum flux density with constant transport coefficients in the dissipative case Topical Lectures, NIKHEF, June 15-17, 2015

a second important dimensionless number

Write down the Navier-Stokes equation in the simpler case $\vec{\nabla} \cdot \vec{v} = 0$ & in the absence of volume forces:

$$\rho(t,\vec{r}) \Big[\partial_t \vec{\mathsf{v}}(t,\vec{r}) + \big[\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla} \big] \vec{\mathsf{v}}(t,\vec{r}) \Big] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \eta \vec{\nabla}^2 \vec{\mathsf{v}}(t,\vec{r})$$

"incompressible Navier-Stokes equation"



a second important dimensionless number

Write down the Navier-Stokes equation in the simpler case $\vec{\nabla} \cdot \vec{v} = 0$ & in the absence of volume forces:

$$\rho(t,\vec{r}) \Big[\partial_t \vec{\mathsf{v}}(t,\vec{r}) + \big[\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla} \big] \vec{\mathsf{v}}(t,\vec{r}) \Big] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \eta \vec{\nabla}^2 \vec{\mathsf{v}}(t,\vec{r})$$

"incompressible Navier-Stokes equation"

Solve it... become rich!

(Millenium Clay problem...)



a second important dimensionless number

Write down the Navier-Stokes equation in the simpler case $\vec{\nabla} \cdot \vec{v} = 0$ & in the absence of volume forces:

$$\rho(t,\vec{r}) \Big[\partial_t \vec{\mathsf{v}}(t,\vec{r}) + \big[\vec{\mathsf{v}}(t,\vec{r}) \cdot \vec{\nabla} \big] \vec{\mathsf{v}}(t,\vec{r}) \Big] = -\vec{\nabla} \mathcal{P}(t,\vec{r}) + \eta \vec{\nabla}^2 \vec{\mathsf{v}}(t,\vec{r})$$

Divide lengths, flow velocity & pressure by typical values (depend on the fluid and flow under consideration!)

ryields dimensionless equation

$$\partial_{t^{*}}\vec{\mathsf{v}}^{*}(t^{*},\vec{r}^{*}) + \left[\vec{\mathsf{v}}^{*}(t^{*},\vec{r}^{*})\cdot\vec{\nabla}^{*}\right]\vec{\mathsf{v}}^{*}(t^{*},\vec{r}^{*}) = -\frac{\vec{\nabla}^{*}\mathcal{P}^{*}(t^{*},\vec{r}^{*})}{\rho} + \frac{\eta}{\rho\,\mathsf{v}_{c}\,L_{c}}\vec{\nabla}^{*2}\vec{\mathsf{v}}^{*}(t^{*},\vec{r}^{*})$$

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a second important dimensionless number

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Divide lengths, flow velocity & pressure by typical values (depend on the fluid and flow under consideration!)

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$$\partial_t \vec{\mathbf{v}} + \left(\vec{\mathbf{v}} \cdot \vec{\nabla}\right) \vec{\mathbf{v}} = -\frac{\vec{\nabla} \mathcal{P}}{\rho} + \frac{\eta}{\rho \, \mathbf{v}_c \, L_c} \vec{\nabla}^2 \vec{\mathbf{v}}$$

involving the

Reynolds number
$$\operatorname{Re} \equiv \frac{\rho v_c L_c}{\eta}$$

a second important dimensionless number

The

Reynolds number
$$\operatorname{Re} \equiv \frac{\rho v_c L_c}{\eta}$$

measures the importance of viscous effects in the flow:

- for Re « 1: viscous flow
- for Re >> 1: inviscid ("ideal") flow...

► You may probably describe the fluid as perfect (not everywhere...)

Invoke the most basic laws of physics: conservation equations!

- mass / particle number conservation (equivalent since $ho=\pi m$)
- momentum conservation (or more generally, Newton's 2nd law)
- energy conservation

each of which are expressed locally.

for a perfect fluid (I omit the variables and drop the volume forces...):

$$\partial_t \left(\underbrace{e + \frac{1}{2}\rho \vec{v}^2}_{\text{energy density}} + \vec{\nabla} \cdot \left[\underbrace{\left(e + \mathcal{P} + \frac{1}{2}\rho \vec{v}^2 \right) \vec{v}}_{\text{energy flux density}} \right] = 0$$

component-wise:

$$\partial_t \left(e + \frac{1}{2}\rho \vec{\mathsf{v}}^2 \right) + \partial_i \left[\left(e + \mathcal{P} + \frac{1}{2}\rho \vec{\mathsf{v}}^2 \right) \mathsf{v}^i \right] = 0$$

Invoke the most basic laws of physics: conservation equations!

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for a Newtonian fluid (I omit the variables and drop the volume forces...):

$$\begin{split} \partial_t \left(e + \frac{1}{2}\rho \,\vec{\mathsf{v}}^2 \right) + \vec{\nabla} \cdot \left\{ \left(e + \mathcal{P} + \frac{1}{2}\rho \,\vec{\mathsf{v}}^2 \right) \vec{\mathsf{v}} - \kappa \vec{\nabla} T \\ &- \left[\zeta - \frac{2\eta}{3} \right] (\vec{\nabla} \cdot \vec{\mathsf{v}}) \,\vec{\mathsf{v}} \\ &- \eta \left[\left(\vec{\mathsf{v}} \cdot \vec{\nabla} \right) \vec{\mathsf{v}} + \vec{\nabla} \left(\frac{\vec{\mathsf{v}}^2}{2} \right) \right] \right\} = 0 \end{split}$$
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$$\partial_{t} \left(e + \frac{1}{2}\rho \vec{v}^{2} \right) + \vec{\nabla} \cdot \left\{ \left(e + \mathcal{P} + \frac{1}{2}\rho \vec{v}^{2} \right) \vec{v} - (\kappa \vec{\nabla} T) \right\} + heat transport \kappa heat conductivity diffusive energy transport by the viscous forces
$$\begin{cases} -\left[\zeta - \frac{2\eta}{3}\right] (\vec{\nabla} \cdot \vec{v}) \vec{v} \\ -\eta \left[(\vec{v} \cdot \vec{\nabla}) \vec{v} + \vec{\nabla} \left(\frac{\vec{v}^{2}}{2} \right) \right] \right\} = 0 \end{cases}$$
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dynamical variables and equations

A fluid is characterized at each time and position by its

- mass density $\rho(t, \vec{r})$ / particle number $n(t, \vec{r})$ (equivalent!)
- pressure $\mathcal{P}(t,ec{r})$
- local flow velocity $\vec{\mathbf{v}}(t,\vec{r})$
- energy density $e(t, \vec{r})$

and possibly transport coefficients (η , ζ , κ ...): material properties!

These are governed by local expressions of conservation equations

- mass / particle number conservation
- momentum conservation
- energy conservation

dynamical variables and equations

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Image: 6 dynamical fields

These are governed by local expressions of conservation equations

- mass / particle number conservation
- momentum conservation
- energy conservation

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dynamical variables and equations

6 dynamical fields $-\rho(t, \vec{r})$ or $n(t, \vec{r})$, $\vec{v}(t, \vec{r})$, $P(t, \vec{r})$, and $e(t, \vec{r}) - yet$ only 5 coupled (conservation) equations relating them to each other.

is one more equation needed!

 One possible easy way out: only investigate fluid motions with a given kinematic constraint
 ➡ steady flows (∂_t=0), incompressible flows (∇ · v = 0), irrotational / potential flows (∇ × v = 0: no vorticity)...

Nature is not always that nice!

dynamical variables and equations

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 ➡ steady flows (∂_t=0), incompressible flows (∇ · v = 0), irrotational / potential flows (∇ × v = 0: no vorticity)... Nature is not always that nice!
- More general, yet not innocent:

There exists a relation between internal energy density, pressure, and particle number, "the" equation of state (EoS).

[= a combination of the thermal & mechanical EoS e(n,T) & $\mathcal{P}(n,T)$]

dynamical variables and equations

An equation of state is indeed a relation between e, \mathcal{P} & n or ρ .

Yet its use presupposes that the fluid is at thermodynamic equilibrium (or more precisely, at local thermodynamic equilibrium at each point & instant).

strong assumption!

dynamical variables and equations

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Yet its use presupposes that the fluid is at thermodynamic equilibrium (or more precisely, at local thermodynamic equilibrium at each point & instant).

strong assumption!

Actually already hidden in the transport coefficients (η , ζ , κ ...):

material properties—like the equation(s) of state;

• that quantify dissipative currents describing the linear response of the system to (small) departures from thermodynamic equilibrium.

re well-defined close to thermodynamic equilibrium.

But now, we may use the whole bunch of thermodynamic relations.

Thermodynamics

Differential of internal energy U:

$\mathrm{d} U = -\mathcal{P} \mathrm{d} \mathcal{V} + T \mathrm{d} S + \mu \mathrm{d} N$

 ${\mathcal P}$ pressure, ${\mathcal V}$ volume, T temperature, S entropy, μ chemical potential; N is the number of particles.

➡ this number is not conserved in a relativistic system: should be replaced by a conserved quantum number (e.g., baryon number).

 \blacktriangleright In a relativistic system, U also includes the mass energy of the constituents.

Internal energy: $U = -\mathcal{P}\mathcal{V} + TS + \mu N$ (1)

Gibbs-Duhem relation: $\mathcal{V} d\mathcal{P} = S dT + N d\mu$

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Thermodynamics

In fluid dynamics, the useful quantities are rather the densities:

- internal energy density e = U/V,
- entropy density $s = S/\mathcal{V}$,
- (baryon) number density $n = N/\mathcal{V}$.

Image: Eq.(1) gives $e = -P + Ts + \mu n$ Gibbs-Duhem becomes $dP = s dT + n d\mu$ leading to: $de = T ds + \mu dn$

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Thermodynamics and fluid dynamics

Using the local Gibbs-Duhem relation $dP = s dT + n d\mu$, one can study the behavior of entropy in moving fluids:

 in perfect fluids, the continuity equation, Euler equation, & energy conservation equation automatically lead to entropy conservation

 $\partial_t s(t, \vec{r}) + \vec{\nabla} \cdot \left[s(t, \vec{r}) \,\vec{\mathsf{v}}(t, \vec{r}) \right] = 0$

• in turn, in Newtonian fluids, the continuity, Navier-Stokes, & energy conservation equations lead, if the transport coefficients η , ζ , κ are taken to be positive, to the production of entropy

 $\partial_t s(t, \vec{r}) + \vec{\nabla} \cdot \left[s(t, \vec{r}) \, \vec{\mathsf{v}}(t, \vec{r}) \right] \ge 0$

which makes sense, since these coefficients characterize dissipative currents.

... describes the motion of continuous media in local thermodynamic equilibrium at each $t \& \vec{r}$, using a set of 6 equations:

 5 dynamical relations, expressing the local conservations of mass (or particle number), momentum, & energy

take different forms in perfect / 1st-order / 2nd-order hydro

• and an equation of state.

These equations govern the coupled evolutions of

- mass density ρ (or equivalently particle number density *n*),
- internal energy density e,
- pressure ${\mathcal P}$,
- flow velocity \vec{v} .

Besides its equation of state, the fluid is characterized by transport coefficients (η , ζ , κ ...).

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Elements of fluid dynamics for the modeling of heavy-ion collisions

Lecture I:

Overview on the dynamics of relativistic perfect fluids

phenomenological approach

reminder(?) on non-relativistic fluid dynamics ("hydrodynamics")

fundamental equations of perfect relativistic fluid dynamics

an example of solution

Application of perfect relativistic fluid dynamics to the description of high-energy nucleus-nucleus collisions

Beyond perfect relativistic fluid dynamics

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- ... describes the motion of continuous media in local thermodynamic equilibrium at each t & \vec{r} , using a set of 6 equations: 5 + N_f equations •4 t N_f dynamical relations •4 t N_f dynamical relations, expressing the local conservations of mass (or quantum numbers), momentum, & energy related the different forms in perfect / 1st-order / 2nd-order hydro
- and an equation of state.

These equations govern the coupled evolutions of

- mass density ρ (or equivalently particle number density π), quantum densities n_a
- internal energy density e,
- pressure ${\mathcal P}$,
- flow velocity \vec{v} . "repackaged" within a 4-velocity

Besides its equation of state, the fluid is characterized by transport coefficients (η , ζ , κ ...).

... describes the motion of continuous media in local thermodynamic equilibrium at each t & \vec{r} , using a set of $\frac{1}{2}$ equations: 5 + N_f equations 4 t Nf. dynamical relations, expressing the local conservations of mass (or quantum numbers), momentum, & energy take different forms in perfect / 1st-order / 2nd-order hydro



- mass density p (or equivalently particle number density n)
- internal energy density e,
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Besides its equation of state, the fluid is characterized by transport coefficients (η , ζ , κ ...).

Relativistic fluid dynamics

dynamical variables and equations

A relativistic fluid is a continuous medium characterized by currents:

- a quantum number 4-current $N_a(x)$, with components $N_a^{\mu}(x)$, for each conserved quantum number a, such that (in Minkowski coordinates)
 - $N_a^0(\mathbf{x})$ is the local density of quantum number a, and
 - the $N_a^i(x)$ are the components of the local flux density of a;

Greek resp. Latin indices run from 0 resp. 1 to 3.
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 - $N_a^0(\mathbf{x})$ is the local density of quantum number a, and
 - the $N_a^i(x)$ are the components of the local flux density of a;
- an energy-momentum tensor T(x), with components $T^{\mu\nu}(x)$, such that
 - $T^{00}(x)$ is the local energy density;
 - $T^{0j}(x)$ is the density of the j^{th} component of momentum;
 - the $T^{i0}(x)$ are the components of the energy flux density;
 - the $T^{ij}(x)$ are the components of the momentum flux-density.

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Relativistic fluid dynamics

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A relativistic fluid is a continuous medium characterized by currents:

• a quantum number 4-current $N_a(x)$, with components $N_a^{\mu}(x)$, for each conserved quantum number a, whose conservation equation reads

 $\left[\partial_{\mu}N_{a}^{\mu}(\mathbf{x})=0\right]$

• an energy-momentum tensor T(x), with components $T^{\mu\nu}$ (x), whose conservation equation reads

$$\partial_{\mu}T^{\mu\nu}(\mathbf{x}) = 0$$

Rem.: in non-Minkowski coordinates, replace partial by covariant derivatives: $\partial_{\mu} \rightarrow d_{\mu}$

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Rem.: in non-Minkowski coordinates, replace partial by covariant derivatives: $\partial_{\mu} \rightarrow d_{\mu}$ Topical Lectures, NIKHEF, June 15–17, 2015 N.Borghini — I-39/58

In perfect/ideal fluids:

... one can at each point x find a reference frame LR(x), such that an observer at rest in LR(x) sees the instantaneous local properties of the fluid as isotropic, i.e. (in Minkowski coordinates)

the spatial components of every quantum number 4-current N_a(x) vanish in LR(x):

$$N_a^{\mu}(\mathbf{x})\big|_{\mathrm{LR}(\mathbf{x})} = \begin{pmatrix} n_a(\mathbf{x})\\ \vec{0} \end{pmatrix}$$

• in the energy-momentum tensor T(x), all $T^{i0}(x)$ and $T^{0j}(x)$ vanish in LR(x), while $T^{ij}(x)$ is diagonal:

$$T^{\mu\nu}(\mathbf{x})\big|_{\mathrm{LR}(\mathbf{x})} = \begin{pmatrix} \epsilon(\mathbf{x}) & 0 & 0 & 0 \\ 0 & \mathcal{P}(\mathbf{x}) & 0 & 0 \\ 0 & 0 & \mathcal{P}(\mathbf{x}) & 0 \\ 0 & 0 & 0 & \mathcal{P}(\mathbf{x}) \end{pmatrix}$$

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In perfect / ideal fluids:

... one can at each point x find a reference frame LR(x), such that an observer at rest in LR(x) sees the instantaneous local properties of the fluid as isotropic.

The 4-velocity u(x) of that observer with respect to an observer at rest in another reference frame \mathcal{R} defines the flow 4-velocity of the fluid w.r.t. \mathcal{R} .

That is, the flow 4-velocity u(x) — which is timelike and normalized to -1, i.e. $[u(x)]^2 = -1$ — has in the "local rest frame" LR(x) at x the Minkowski components

$$u^{\mu}(\mathbf{x})\big|_{\mathrm{LR}(\mathbf{x})} = \begin{pmatrix} 1\\ \vec{0} \end{pmatrix}$$

As is now clear, the metric with signature (-,+,+,+) will be used.

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In perfect / ideal fluids:

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The 4-velocity u(x) of that observer with respect to an observer at rest in another reference frame \mathcal{R} defines the flow 4-velocity of the fluid w.r.t. \mathcal{R} .

Denoting by $\vec{v}(x)$ the corresponding 3-velocity w.r.t. \mathcal{R} and by $\gamma(x)$ the associated Lorentz factor, the components of the 4-velocity in \mathcal{R} read

$$u^{\mu}(\mathbf{x})|_{\mathcal{R}} = \begin{pmatrix} \gamma(\mathbf{x}) \\ \gamma(\mathbf{x})\vec{\mathbf{v}}(\mathbf{x}) \end{pmatrix} \quad \text{with} \quad \gamma(\mathbf{x}) \equiv \frac{1}{\sqrt{1-\vec{\mathbf{v}}(\mathbf{x})^2}}$$

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In the local frame LR(x) the quantum number 4-current $N_a(x)$ and energy-momentum tensor T(x), have the simple expressions

$$N_{a}^{\mu}(\mathbf{x})\big|_{\mathrm{LR}(\mathbf{x})} = \begin{pmatrix} n_{a}(\mathbf{x}) \\ \vec{0} \end{pmatrix} \qquad T^{\mu\nu}(\mathbf{x})\big|_{\mathrm{LR}(\mathbf{x})} = \begin{pmatrix} \epsilon(\mathbf{x}) & 0 & 0 & 0 \\ 0 & \mathcal{P}(\mathbf{x}) & 0 & 0 \\ 0 & 0 & \mathcal{P}(\mathbf{x}) & 0 \\ 0 & 0 & 0 & \mathcal{P}(\mathbf{x}) \end{pmatrix}$$

involving local densities $n_a(x)$ for quantum number $a \& \epsilon(x)$ for energy, and pressure $\mathcal{P}(x)$.

In an arbitrary reference frame and system of coordinates, they are given by

 $N_a^{\mu}(\mathbf{x}) = n_a(\mathbf{x})u^{\mu}(\mathbf{x})$

 $T^{\mu\nu}(\mathbf{x}) = \left[\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})\right] u^{\mu}(\mathbf{x}) u^{\nu}(\mathbf{x}) + \mathcal{P}(\mathbf{x})g^{\mu\nu}(\mathbf{x})$

(identities between two 4-vectors / tensors valid in one reference frame, thus in any.) Topical Lectures, NIKHEF, June 15-17, 2015 N.Borghini — I-43/58

The components of the quantum number 4-current $N_a(x)$ and energymomentum tensor T(x) of a perfect fluid are given by

 $N_a^{\mu}(\mathbf{x}) = \mathbf{n}_a(\mathbf{x})u^{\mu}(\mathbf{x})$

 $T^{\mu\nu}(\mathbf{x}) = \left[\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})\right] u^{\mu}(\mathbf{x}) u^{\nu}(\mathbf{x}) + \mathcal{P}(\mathbf{x})g^{\mu\nu}(\mathbf{x})$

in terms of the flow 4-velocity u(x).

Introducing the tensor $\Delta^{\mu\nu}(x) \equiv g^{\mu\nu}(x) + u^{\mu}(x)u^{\nu}(x)$, which projects on the 3-space orthogonal to the 4-velocity [check!], the latter may be recast as

$$T^{\mu\nu}(\mathbf{x}) = \epsilon(\mathbf{x})u^{\mu}(\mathbf{x})u^{\nu}(\mathbf{x}) + \mathcal{P}(\mathbf{x})\Delta^{\mu\nu}(\mathbf{x})$$

The components of the quantum number 4-current $N_a(x)$ and energymomentum tensor T(x) of a perfect fluid are given by

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$$T^{\mu\nu}(\mathbf{x}) = \epsilon(\mathbf{x})u^{\mu}(\mathbf{x})u^{\nu}(\mathbf{x}) + \mathcal{P}(\mathbf{x})\Delta^{\mu\nu}(\mathbf{x})$$

in terms of the flow 4-velocity u(x) and the projector orthogonal to it. One easily checks [exercise!] the identities

$$n_{a}(\mathbf{x}) = \frac{N_{a}^{\mu}(\mathbf{x})u_{\mu}(\mathbf{x})}{u^{\nu}(\mathbf{x})u_{\nu}(\mathbf{x})} \quad , \quad \epsilon(\mathbf{x}) = u_{\mu}(\mathbf{x})T^{\mu\nu}(\mathbf{x})u_{\nu}(\mathbf{x}) \quad , \quad \mathcal{P}(\mathbf{x}) = \frac{1}{3}\Delta_{\mu\nu}(\mathbf{x})T^{\mu\nu}(\mathbf{x})$$

valid in reference frame and system of coordinates.

These relations show that $n_a(x)$, $\epsilon(x)$, $\mathcal{P}(x)$ are scalar fields.

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dynamical variables and equations

In a perfect relativistic fluid the conserved currents are:

• N_f quantum number 4-currents N_a(x), with components $N_a^{\mu}(x)$, whose conservation equation read

$$\partial_{\mu}N^{\mu}_{a}(\mathbf{x}) = 0$$
 with $N^{\mu}_{a}(\mathbf{x}) = n_{a}(\mathbf{x})u^{\mu}(\mathbf{x})$

• the energy-momentum tensor T(x), with components $T^{\mu\nu}$ (x), whose conservation equation reads

$$\partial_{\mu}T^{\mu\nu}(\mathbf{x}) = 0 \qquad \text{with} \quad T^{\mu\nu}(\mathbf{x}) = \epsilon(\mathbf{x})u^{\mu}(\mathbf{x})u^{\nu}(\mathbf{x}) + \mathcal{P}(\mathbf{x})\Delta^{\mu\nu}(\mathbf{x})$$

5 (= $\epsilon(x)$, $\mathcal{P}(x)$, only 3 components of u(x)) + N_f fields, 4 + N_f equations requation of state to close the system of equations!

dynamical variables and equations

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the energy-moment equation reads with
$$N^{\mu}(\mathbf{x}) = n_a(\mathbf{x})u^{\mu}(\mathbf{x})$$

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 $N^{\mu}(\mathbf{x}) = n_a(\mathbf{x})u^{\mu}(\mathbf{x})$

$$\partial_{\mu}T^{\mu\nu}(\mathbf{x}) = 0 \qquad \text{with} \quad T^{\mu\nu}(\mathbf{x}) = \epsilon(\mathbf{x})u^{\mu}(\mathbf{x})u^{\nu}(\mathbf{x}) + \mathcal{P}(\mathbf{x})\Delta^{\mu\nu}(\mathbf{x})$$

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a quick example

Consider a flow with small velocity. To first order in the latter:

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & (\epsilon + \mathcal{P})\mathbf{v}^1 & (\epsilon + \mathcal{P})\mathbf{v}^2 & (\epsilon + \mathcal{P})\mathbf{v}^3 \\ (\epsilon + \mathcal{P})\mathbf{v}^1 & \mathcal{P} & 0 & 0 \\ (\epsilon + \mathcal{P})\mathbf{v}^2 & 0 & \mathcal{P} & 0 \\ (\epsilon + \mathcal{P})\mathbf{v}^3 & 0 & 0 & \mathcal{P} \end{pmatrix}$$

so that the energy-momentum conservation equation reads

$$\partial_{\mu}T^{\mu0} = 0$$
 $\partial_{t}\epsilon + \vec{\nabla} \cdot \left[(\epsilon + \mathcal{P})\vec{\mathsf{v}} \right] = 0$ (2)

$$\partial_{\mu}T^{\mu j} = 0 \qquad \qquad \partial_t \left[(\epsilon + \mathcal{P})\vec{\mathsf{v}} \right] + \vec{\nabla}\mathcal{P} = \vec{\mathsf{0}}$$
 (3)

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a quick example

Consider small adiabatic perturbation (sound wave!) of a uniform and steady flow: $\int \epsilon(\mathbf{x}) = \epsilon_0 + \delta \epsilon(\mathbf{x})$

$$\begin{cases} \mathcal{P}(\mathbf{x}) = \mathcal{P}_0 + \delta \mathcal{P}(\mathbf{x}) \\ \mathcal{P}(\mathbf{x}) = \mathcal{P}_0 + \delta \mathcal{P}(\mathbf{x}) \end{cases}$$

a quick example

Consider small adiabatic perturbation (sound wave!) of a uniform and steady flow:

$$\begin{cases} \epsilon(\mathbf{x}) = \epsilon_0 + \delta \epsilon(\mathbf{x}) \\ \mathcal{P}(\mathbf{x}) = \mathcal{P}_0 + \delta \mathcal{P}(\mathbf{x}) \end{cases}$$

Linearizing the equations of motion (2,3) yields:

• from (2): $\partial_t($

$$(\delta\epsilon) + (\epsilon_0 + \mathcal{P}_0)\vec{\nabla}\cdot\vec{\mathbf{v}} = 0 \tag{4}$$

 $(\epsilon_0 + \mathcal{P}_0)\partial_t \vec{\mathsf{v}} + \vec{\nabla}\delta\mathcal{P} = \vec{0}$ (5) • from (3):

a quick example

Consider small adiabatic perturbation (sound wave!) of a uniform and steady flow: $\int \epsilon(x) = \epsilon_0 + \delta \epsilon(x)$

$$\begin{cases} \epsilon(\mathbf{x}) = \epsilon_0 + \delta \epsilon(\mathbf{x}) \\ \mathcal{P}(\mathbf{x}) = \mathcal{P}_0 + \delta \mathcal{P}(\mathbf{x}) \end{cases}$$

Linearizing the equations of motion (2,3) yields:

- from (2): $\partial_t (\delta \epsilon) + (\epsilon_0 + \mathcal{P}_0) \vec{\nabla} \cdot \vec{v} = 0$ (4)
- from (3): $(\epsilon_0 + \mathcal{P}_0)\partial_t \vec{v} + \vec{\nabla}\delta \mathcal{P} = \vec{0}$ (5)

Defining
$$c_s^2 \equiv \left(\frac{\partial P}{\partial \epsilon}\right)_{|\frac{s}{n}}$$
, eqs. (4) & (5) lead to
 $\partial_t^2(\delta \epsilon) - c_s^2 \vec{\nabla}^2(\delta \epsilon) = 0$

wave equation, c_s speed of sound.

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requation of state $\mathcal{P}(\epsilon)$!

"equation of state"

Since the speed of sound c_s is given by the relation

$$c_s^2 \equiv \left(\frac{\partial \mathcal{P}}{\partial \epsilon}\right)_{\mid}$$

the equation of state when there are no relevant conserved quantum number is precisely the relation between pressure and energy density. Simple examples (for analytical calculations) are

- for massless particles (living in 1+3 dimensions): $\epsilon = 3P$
- massive particles (in cosmology): P = 0 ("dust")
- "vacuum" (in cosmology): $P = -\epsilon$

For heavy-ion collisions, compact astrophysical objects ($n \neq 0$), precision cosmology: more complicated forms!

Elements of fluid dynamics for the modeling of heavy-ion collisions

Lecture I:

Overview on the dynamics of relativistic perfect fluids

phenomenological approach

- reminder(?) on non-relativistic fluid dynamics ("hydrodynamics")
- Indamental equations of perfect relativistic fluid dynamics
- an example of solution

Application of perfect relativistic fluid dynamics to the description of high-energy nucleus-nucleus collisions

Beyond perfect relativistic fluid dynamics

Relativistic perfect fluid dynamics an example of solution

The dynamical equations of perfect relativistic fluid read

• N_f times
$$\partial_{\mu} N_{a}^{\mu}(\mathbf{x}) = 0$$
 with $N_{a}^{\mu}(\mathbf{x}) = n_{a}(\mathbf{x})u^{\mu}(\mathbf{x})$
• and $\partial_{\mu} T^{\mu\nu}(\mathbf{x}) = 0$ with $T^{\mu\nu}(\mathbf{x}) = \epsilon(\mathbf{x})u^{\mu}(\mathbf{x})u^{\nu}(\mathbf{x}) + \mathcal{P}(\mathbf{x})\Delta^{\mu\nu}(\mathbf{x})$

Projecting the energy-momentum conservation equation parallel and perpendicular to the 4-velocity yields:

 $u^{\mu}(\mathbf{x})\partial_{\mu}\epsilon(\mathbf{x}) + \left[\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})\right]\partial_{\mu}u^{\mu}(\mathbf{x}) = 0$ $\left[\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})\right]u^{\mu}(\mathbf{x})\partial_{\mu}u^{\nu}(\mathbf{x}) + \Delta^{\mu\nu}(\mathbf{x})\partial_{\mu}\mathcal{P}(\mathbf{x}) = 0$

Rem.: in non-Minkowski coordinates, replace partial by covariant derivatives: $\partial_{\mu} \rightarrow d_{\mu}$

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Relativistic perfect fluid dynamics an example of solution

The dynamical equations of perfect relativistic fluid read

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Convenient notation:

 $\nabla^{\nu} \equiv \Delta^{\mu\nu}(\mathbf{x})\partial_{\mu}$

Rem.: in non-Minkowski coordinates, replace partial by covariant derivatives: $\partial_{\mu} o d_{\mu}$

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an example of solution

Quantum-number-free perfect fluid, $rac{n_a(x)}{t} = 0$ in one-dimensional motion along the z-axis with the 3-velocity $v^z = \frac{z}{t}$ for z < ct and $t > t_0$.

First(?) discussed by R.Hwa (1974); made popular by J.D.Bjorken (1983)

PHYSICAL REVIEW D

VOLUME 27, NUMBER 1

1 JANUARY 1983

Highly relativistic nucleus-nucleus collisions: The central rapidity region

J. D. Bjorken

Fermi National Accelerator Laboratory, * P.O. Box 500, Batavia, Illinois 60510 (Received 13 August 1982)

The space-time evolution of the hadronic matter produced in the central rapidity region in extreme relativistic nucleus-nucleus collisions is described. We find, in agreement with previous studies, that quark-gluon plasma is produced at a temperature $\geq 200-300$ MeV, and that it should survive over a time scale ≥ 5 fm/c. Our description relies on the existence of a flat central plateau and on the applicability of hydrodynamics.

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N.Borghini – I-52/58

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7. Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region

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References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote KEK scanned document ; ADS Abstract Service; Fermilab Library Server (fulltext available); Phys. Rev. D Server; Link to Fulltext

Detailed record - Cited by 2316 records 1000+ as of June 7, 2015

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An example of relativistic flow: "Bjorken flow"

Quantum-number-free perfect fluid, $rac{n_a(x)}{t} = 0$ in one-dimensional motion along the z-axis with the 3-velocity $v^z = \frac{z}{t}$ for z < ct and $t > t_0$.

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One-dimensional flow along the *z*-axis with the 3-velocity $v^z = \frac{z}{t}$ for z < ct and $t > t_0$.

For 4-velocity*
$$u^{\mu}(\mathbf{x}) = \begin{pmatrix} \gamma(\mathbf{x}) \\ \gamma(\mathbf{x})\mathbf{v}^{z}(\mathbf{x}) \end{pmatrix}$$
 with $\gamma(\mathbf{x}) = \frac{1}{\sqrt{1 - \mathbf{v}^{z}(\mathbf{x})^{2}}}$

*only the non-trivial components are shown

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$$= \frac{t}{\sqrt{t^{2} - z^{2}}}$$

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One-dimensional flow along the *z*-axis with the 3-velocity $v^z = \frac{z}{t}$ for z < ct and $t > t_0$.

Solution
$$t > t_0$$
.
4-velocity* $u^{\mu}(\mathbf{x}) = \begin{pmatrix} \frac{t}{\sqrt{t^2 - z^2}} \\ \frac{z}{\sqrt{t^2 - z^2}} \end{pmatrix}$

*only the non-trivial components are shown

One-dimensional flow along the z-axis with the 3-velocity $v^z = \frac{z}{t}$ for z < ct and $t > t_0$.

For 4-velocity*
$$u^{\mu}(\mathbf{x}) = \begin{pmatrix} \overline{\sqrt{t^2 - z^2}} \\ \frac{z}{\sqrt{t^2 - z^2}} \end{pmatrix}$$

Milne coordinates

$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} \end{cases} \Leftrightarrow \begin{cases} t = \tau \cosh \varsigma \\ z = \tau \sinh \varsigma \end{cases}$$

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One-dimensional flow along the *z*-axis with the 3-velocity $v^z = \frac{z}{t}$ for z < ct and $t > t_0$.

For 4-velocity*
$$u^{\mu}(\mathbf{x}) = \begin{pmatrix} \frac{\iota}{\sqrt{t^2 - z^2}} \\ \frac{z}{\sqrt{t^2 - z^2}} \end{pmatrix} = \begin{pmatrix} \cosh \varsigma \\ \sinh \varsigma \end{pmatrix}$$

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Two possibilities:

- work directly in Milne coordinates $(x^{0'}, x^{1'}) = (\tau, \varsigma) \models u^{\mu'}(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- stay with Minkowski coordinates 🖛 simple derivatives

*only the non-trivial components are shown

$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} \end{cases} \Leftrightarrow \begin{cases} t = \tau \cosh \varsigma \\ z = \tau \sinh \varsigma \end{cases}$$

One easily shows [check!] that the equation of motion "along u(x)" $u^{\mu}(x)\partial_{\mu}\epsilon(x) + \left[\epsilon(x) + \mathcal{P}(x)\right]\partial_{\mu}u^{\mu}(x) = 0$

becomes

$$\partial_{\tau} \epsilon(\mathbf{x}) + \frac{\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})}{\tau} = 0$$

or equivalently $\partial_{\tau} [\tau \epsilon(\mathbf{x})] = - \mathcal{P}(\mathbf{x})$

which simply relates the change in the energy in a comoving volume (proportional to τ) to the work of pressure forces...

$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} \end{cases} \Leftrightarrow \begin{cases} t = \tau \cosh \varsigma \\ z = \tau \sinh \varsigma \end{cases}$$

In turn, the equation of motion "perpendicular to u(x)" $\big[\epsilon(x) + \mathcal{P}(x)\big]u^{\mu}(x)\partial_{\mu}u^{\nu}(x) + \nabla^{\nu}\mathcal{P}(x) = 0$

yields [check!]

$$\partial_{\varsigma} \mathcal{P}(\mathsf{x}) = 0$$

i.e. pressure — and, invoking the equation of state, energy density — is independent of space-time rapidity.

boost invariance!

Coming back to $\partial_{\tau}\epsilon(\mathbf{x}) + \frac{\epsilon(\mathbf{x}) + \mathcal{P}(\mathbf{x})}{\tau} = 0$

inserting the equation of state $\mathcal{P}(\mathbf{x}) = c_s(\mathbf{x})^2 \epsilon(\mathbf{x})$ yields $\partial_{\tau} \epsilon(\mathbf{x}) + \left[1 + c_s(\mathbf{x})^2\right] \frac{\epsilon(\mathbf{x})}{\tau} = 0$

If the speed of sound is constant, this leads to

$$\epsilon(\mathbf{x}) \propto \frac{1}{\tau^{1+c_s^2}} \qquad \qquad \mathcal{P}(\mathbf{x}) \propto \frac{1}{\tau^{1+c_s^2}}$$

In turn, the entropy conservation equation $\partial_{\mu}[s(\mathbf{x})u^{\mu}(\mathbf{x})] = 0$ becomes $\partial_{\tau}s(\mathbf{x}) + \frac{s(\mathbf{x})}{\tau} = 0$

 $s(\mathbf{x}) \propto \frac{1}{\tau}$

leading at once to

Using now $\epsilon(\mathbf{x}) \propto \frac{1}{\tau^{1+c_s^2}}$, $\mathcal{P}(\mathbf{x}) \propto \frac{1}{\tau^{1+c_s^2}}$ and the relation $\epsilon + \mathcal{P} = Ts$, one arrives at $T(\mathbf{x}) \propto \frac{1}{\tau^{c_s^2}}$

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We have found $\epsilon(x) \propto \frac{1}{\tau^{1+c_s^2}}$, $\mathcal{P}(x) \propto \frac{1}{\tau^{1+c_s^2}}$, $s(x) \propto \frac{1}{\tau}$ and indirectly $T(x) \propto \frac{1}{\tau^{c_s^2}}$

For an ultrarelativistic gas $\epsilon \propto T^4$ (Stefan-Boltzmann!), $\mathscr{P} \propto T^4$ $s \propto T^3$ (remember $\epsilon + \mathscr{P} = Ts$) and $c_s^2 = \frac{1}{3}$... Everything is OK!

We have found $\epsilon(x) \propto \frac{1}{\tau^{1+c_s^2}}$, $\mathcal{P}(x) \propto \frac{1}{\tau^{1+c_s^2}}$, $s(x) \propto \frac{1}{\tau}$ and indirectly $T(x) \propto \frac{1}{\tau^{c_s^2}}$

For an ultrarelativistic gas $\epsilon \propto T^4$ (Stefan-Boltzmann!), $\mathscr{P} \propto T^4$ s $\propto T^3$ (remember ϵ Questions? Everything is OK!